On isentropic solutions of first-order quasilinear equations [1]

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Abstract

Conditions of existence of nontrivial isentropic solutions to first order conservation laws are found. Applications are given to the problem on functional dependence for partial derivatives of a smooth function of two variables. In particular, we find conditions on the function φ, which are necessary for existence of nontrivial C^1-smooth solutions for the equation $\frac{\partial v}{\partial t} = \varphi(\frac{\partial v}{\partial x})$. This paper is the corrected and extended version of the preprint [14].

Introduction. The present paper arises in connection with the following problem on functional dependence of derivatives for a smooth function of two variables. Let $v = v(t, x)$ be a smooth nonlinear function in a domain $\Omega \subset \mathbb{R}^2$ such that

$$v_t = \varphi(v_x) \quad \text{in } \Omega, \tag{1}$$

where $\varphi(u)$ is a continuous function. The question is about properties of $\varphi(u)$. The derivatives v_t, v_x are only continuous and by the first look relation (1) can not give more than the natural property of continuity for $\varphi(u)$. But it turns out that the function $\varphi(u)$ must satisfy the strong enough condition (*) in Theorem 7 below and, in particular, $\varphi(u)$ is twice (!) differentiable a.e. on the set $v_x(\Omega)$. To solve the problem under consideration we make use of results of the theory of isentropic solutions for first-order quasilinear equations, introduced in the thesis [11]. These results are contained in § 1,2 below and, by the authors' opinion, can be intresting by themselfes.

Remind firstly some notions and known results of the theory of generalized entropy solutions to the Cauchy problem for a first order quasilinear equation

$$u_t + \mathrm{div}_x \varphi(u) = 0, \tag{2}$$

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$\varphi = (\varphi_1, \dots, \varphi_n)$, $u = u(t,x)$, $(t,x) \in \Pi_T = (0,T) \times \mathbb{R}^n$, $0 < T \le +\infty$, with initial condition

$$u(0,x) = u_0(x) \in L^\infty(\mathbb{R}^n). \tag{3}$$

We suppose that the flux vector $\varphi(u)$ is only continuous: $\varphi(u) \in C(\mathbb{R}, \mathbb{R}^n)$.

Definition 1 (S.N. Kruzhkov, [5, 6]). A bounded measurable function $u = u(t,x) \in L^\infty(\Pi_T)$ is called a generalized entropy solution (briefly - g.e.s.) to the Cauchy problem (2), (3) if

a) $\forall k \in \mathbb{R}$

$$|u-k|_t + \mathrm{div}_x[\mathrm{sign}\,(u-k)(\varphi(u) - \varphi(k))] \le 0 \tag{4}$$

in the sense of distributions on Π_T (in $\mathcal{D}'(\Pi_T)$);

b) $\operatorname*{ess\,lim}_{t \to 0+} u(t,\cdot) = u_0$ in $L^1_{loc}(\mathbb{R}^n)$, i.e. there exists a set $\mathcal{E} \subset (0,T)$ of full Lebesgue measure such that $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0+$, $t \in \mathcal{E}$.

Condition (4) means that for any test function $f = f(t,x) \in C_0^\infty(\Pi_T)$, $f \ge 0$

$$\int_{\Pi_T} \{|u-k| f_t + \mathrm{sign}\,(u-k)(\varphi(u) - \varphi(k), \nabla_x f)\} dtdx \ge 0.$$

Here and below we use the notation $(\cdot,\cdot)$ for the scalar product in $\mathbb{R}^n$. We also use the notation $|\cdot|$ for the Eucleadian norm of a finite-dimensional vector.

Taking in (4) $k = \pm\|u\|_\infty$ one could easily derive that $u_t + \mathrm{div}_x \varphi(u) = 0$ in $\mathcal{D}'(\Pi_T)$ and $u(t,x)$ satisfies equation (2) in the sense of distributions, i.e. it is really a generalized solution (g.s.).

As was shown in [13], the condition a), b) can be described by a single integral inequality, namely (see Proposition 1 in [13]) $u = u(t,x)$ is a g.e.s. of the problem (2), (3) if and only if for all $k \in \mathbb{R}$ and $\forall f = f(t,x) \in C_0^\infty([0,T) \times \mathbb{R}^n)$, $f \ge 0$

$$\int_{\Pi_T} \{|u-k| f_t + \mathrm{sign}\,(u-k)(\varphi(u) - \varphi(k), \nabla_x f)\} dtdx + \int_{\mathbb{R}^n} |u_0(x) - k| f(0,x) dx \ge 0. \tag{5}$$

As is known (see [11]-[12]) a g.e.s. of the problem (2), (3) always exists but, in the case under consideration when the flux vector is only continuous, can be nonunique if number of spatial variables $n > 1$. The corresponding examples were firstly constructed in [11, 7, 8], where some sufficient conditions of uniqueness were found as well (in particular these conditions are always satisfied for $n = 1$ and a g.e.s. is unique in this case). Remark also that for the Lipschitz continuous flux vector uniqueness of g.e.s. is well-known (see [5, 6]).

We study here only bounded g.e.s. (unlike isentropic solutions below) because in a wider class of locally bounded g.e.s. the Cauchy problem seems to be ill-posed (see, for instance, [2]).

§ **1. Isentropic solutions.**

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open connected set (domain).

Definition 2 ([11]). A function $u(t,x) \in L^\infty_{loc}(\Omega)$ is called isentropic solution (briefly - i.s.) of (2) in the domain Ω if $\forall k \in \mathbb{R}$

$$|u-k|_t + \mathrm{div}_x[\mathrm{sign}\,(u-k)(\varphi(u)-\varphi(k))] = 0 \ \text{ in } \mathcal{D}'(\Omega), \tag{6}$$

that is entropy condition (4) is fulfilled in $\mathcal{D}'(\Omega)$ with equality sign.

Remark 1. As follows from the definition, $u = u(t,x)$ is an i.s. of the equation (2) in Ω if and only if $-u$ is an i.s. in Ω for the equation $u_t - \mathrm{div}_x\varphi(-u) = 0$.

Notice that in [11] more general isentropic measure valued solutions are considered.

Condition (6) can be rewritten in more convenient form. We have the following

Proposition 1. *Condition* (6) *is equivalent to any of following ones:*

1) $\forall k \in \mathbb{R} \quad \max(u,k)_t + \mathrm{div}_x\varphi(\max(u,k)) = 0$ *in* $\mathcal{D}'(\Omega)$;

2) $\forall k \in \mathbb{R} \quad \min(u,k)_t + \mathrm{div}_x\varphi(\min(u,k)) = 0$ *in* $\mathcal{D}'(\Omega)$.

Proof. Observe firstly that

$$\begin{aligned} 2\max(u,k) = |u-k| + (u+k),\ 2\varphi(\max(u,k)) = \\ (\varphi(u)-\varphi(k))\mathrm{sign}\,(u-k) + (\varphi(u)+\varphi(k)); \end{aligned} \tag{7}$$

$$\begin{aligned} 2\min(u,k) = -|u-k| + (u+k),\ 2\varphi(\min(u,k)) = \\ -(\varphi(u)-\varphi(k))\mathrm{sign}\,(u-k) + (\varphi(u)+\varphi(k)). \end{aligned} \tag{8}$$

Then, the equality

$$u_t + \mathrm{div}_x\varphi(u) = 0 \ \text{ in } \mathcal{D}'(\Omega)$$

follows from (6) as well as from each of conditions 1), 2) (in the limit as $k \to \pm\infty$). This, together with equalities (7), (8), proves equivalence condition (6) to conditions 1), 2). The proof is complete.

In fact, Proposition 1 is a particular case of the following result:

Theorem 1. *If* $u_1 = u_1(t,x)$, $u_2 = u_2(t,x)$ *are i.s. of equation* (2) *in the domain* Ω *then the functions* $\max(u_1,u_2)$, $\min(u_1,u_2)$ *are also i.s. of this equation in* Ω.

Proof. By condition 1) in Proposition 1 for any $k \in \mathbb{R}$

$$\max(u,k)_t + \mathrm{div}_x\varphi(\max(u,k)) = 0 \ \text{ in } \mathcal{D}'(\Omega). \tag{9}$$

Now we apply the method of "doubling variables" developed by S.N. Kruzhkov in papers [5, 6]. Consider the integral equality corresponding to (9) with $u = u_1(t,x)$, in which the test function $f(t,x;\tau,y) \in C_0^\infty(\Omega \times \Omega)$, and $k = \max(u_2(\tau,y),c)$, $c \in \mathbb{R}$. After integrating on $\Omega(\tau,y)$ this equality implies that

$$v_t + \mathrm{div}_x\varphi(v) = 0 \text{ in } \mathcal{D}'(\Omega \times \Omega), \tag{10}$$

$v = v(t,x;\tau,y) = \max(u_1(t,x),u_2(\tau,y),c)$. Changing places of function u_1, u_2 and variables (t,x), (τ,y) we derive the similar equality

$$v_\tau + \mathrm{div}_y\varphi(v) = 0 \ \text{ in } \mathcal{D}'(\Omega \times \Omega). \tag{11}$$

Combining (10) and (11) we obtain that

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\right) v + (\mathrm{div}_x + \mathrm{div}_y)\varphi(v) = 0 \text{ in } \mathcal{D}'(\Omega \times \Omega). \tag{12}$$

Choose a function $\beta(t,x) \in C_0^\infty(\mathbb{R}^{n+1})$, $\beta(t,x) \ge 0$, $\int \beta(t,x)dtdx = 1$ and set $\delta_\nu(t,x) = \nu^{n+1}\beta(\nu t,\nu x)$, $\nu \in \mathbb{N}$. Obviously, the sequence $\delta_\nu(t,x)$ converges to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R}^{n+1})$ as $\nu \to \infty$.

Let $f(t,x) \in C_0^\infty(\Omega)$, $\nu \in \mathbb{N}$, and $g(t,x;\tau,y) = f(t,x)\delta_\nu(t-\tau,x-y)$. Clearly, for large enough $\nu \in \mathbb{N}$ the function $g(t,x;\tau,y) \in C_0^\infty(\Omega \times \Omega)$ and $\mathrm{supp}\, g \subset K \times K$, where $K \subset \Omega$ is a compact set. Applying equality (12) to the test function g we derive the integral equality:

$$\int_{\Omega(t,x)}\int_{\Omega(\tau,y)} \{vf_t + (\varphi(v),\nabla_x f)\}\delta_\nu(t-\tau,x-y)d\tau dydtdx = 0. \tag{13}$$

Let $w = w(t,x) = v(t,x;t,x) = \max(u_1(t,x), u_2(t,x), c)$,

$$I_\nu(t,x) = \int_{\Omega(\tau,y)} \{v f_t + (\varphi(v), \nabla_x f)\}\delta_\nu(t-\tau, x-y)d\tau dy;$$

$$I(t,x) = w(t,x)f_t(t,x) + (\varphi(w(t,x)), \nabla_x f(t,x)) =$$
$$\int_{\Omega(\tau,y)} \{w f_t + (\varphi(w), \nabla_x f)\}\delta_\nu(t-\tau, x-y)d\tau dy.$$

Further, $|v(t,x;\tau,y) - w(t,x)| \le |u_2(\tau,y) - u_2(t,x)|$ and for a.e. $(t,x,\tau,y) \in K \times K$ $\quad |\varphi(v(t,x;\tau,y)) - \varphi(w(t,x))| \le \omega(|u_2(\tau,y) - u_2(t,x)|)$, where $\omega(r) = \sup\{\ |\varphi(u) - \varphi(v)| \mid u,v \in [-M,M], |u-v| < r\ \}$ is a continuity modulus of the vector $\varphi(u)$ on the segment $[-M,M]$ with $M = \max(\|u_1\|_{L^\infty(K)}, \|u_2\|_{L^\infty(K)})$. Therefore, we derive the estimate

$$|I_\nu(t,x) - I(t,x)| \le C_f \int_\Omega \Big[|u_2(\tau,y) - u_2(t,x)| +$$
$$\omega(|u_2(\tau,y) - u_2(t,x)|)\Big]\delta_\nu(t-\tau, x-y)d\tau dy,\ C_f = \text{const},$$

which implies that $I_\nu(t,x) \to I(t,x)$ as $\nu \to \infty$ on the set of full measure consisting of Lebesgue points of the function $u_2(t,x)$. By boundedness of u_1, u_2 on the compact K we have also the estimate $\|I_\nu\|_\infty \le$ const. Applying the Lebesgue theorem on convergence under an integral sign we derive the relation

$$\int_\Omega I_\nu(t,x)dtdx \underset{\nu\to\infty}{\to} \int_\Omega I(t,x)dtdx$$

and, since $\int_\Omega I_\nu(t,x)dtdx = 0$ in veiw of (13), $\int_\Omega I(t,x)dtdx = 0$ that is

$$\int_\Omega [w(t,x)f_t(t,x) + (\varphi(w(t,x)), \nabla_x f(t,x))]dtdx = 0.$$

The latter equality together with arbitrariness of $f \in C_0^\infty(\Omega)$ means that $w_t + \mathrm{div}_x \varphi(w) = 0$ in $\mathcal{D}'(\Omega)$, i.e. for $p = p(t,x) = \max(u_1, u_2)$

$$\max(p,c)_t + \mathrm{div}_x \varphi(\max(p,c)) = 0 \ \text{ in } \mathcal{D}'(\Omega)\ \forall c \in \mathbb{R}.$$

By Proposition 1 $\quad p = \max(u_1, u_2)$ is an i.s. of equation (2) in Ω. At last, with regard to the equality $\min(u_1,u_2) = -\max(-u_1,-u_2)$ and Remark 1, $\min(u_1,u_2)$ is an i.s. of (2) as well. The proof is complete.

Below, in the proof of Theorem 4 we shall use the following

Proposition 2. *Let $u(t,x) \in L^\infty_{loc}(\Pi_T)$ be an i.s. of equation (2), $-\infty \le m = \operatorname{ess\,inf} u(t,x)$, $\operatorname{ess\,sup} u(t,x) = M \le +\infty$. Then there exists a set $E \subset (0,T)$ of full measure such that the map $t \to u(t,\cdot)$ is continuous on this set in topology of $L^1_{loc}(\mathbb{R}^n)$ and $\forall t \in E$*

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^n} u(t,x) = m, \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} u(t,x) = M.$$

Proof. Firstly suppose that the i.s. $u(t,x)$ is bounded: $u(t,x) \in L^\infty(\Pi_T)$. Choose, in the same way as in the proof of Theorem 1, the function $\beta(t) \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\beta \subset [0,1]$, $\beta(t) \ge 0$, $\int \beta(t)dt = 1$, and set for $\nu \in \mathbb{N}$ $\delta_\nu(t) = \nu\beta(\nu t)$, $\theta_\nu(t) = \int_0^t \delta_\nu(s)ds$, so that $\theta'_\nu(t) = \delta_\nu(t)$. It is clear that the sequence $\delta_\nu(t)$ converges to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R})$ as $\nu \to \infty$, while the sequence $\theta_\nu(t)$ converges pointwise to the Heaviside function $\theta(t) = \begin{cases} 0 & , \quad t \le 0, \\ 1 & , \quad t > 0. \end{cases}$

Let $t_0 \in (0,T)$, $f = f(t,x) \in C_0^\infty(\Pi_T)$. Applying the relation (6) to the test function $f\theta_\nu(t-t_0)$, we obtain after simple transformations that $\forall k \in \mathbb{R}$

$$\int_0^T \left(\int_{\mathbb{R}^n} |u(t,x)-k| f(t,x)dx\right) \delta_\nu(t-t_0)dt +$$
$$\int_{\Pi_T} \{|u-k|f_t + \operatorname{sign}(u-k)(\varphi(u)-\varphi(k), \nabla_x f)\}\theta_\nu(t-t_0)dtdx = 0. \quad (14)$$

Let $S \subset C_0^\infty(\Pi_T)$ be a countable dense (with respect to standard locally convex topology in $C_0^\infty(\Pi_T)$) set and $E \subset (0,T)$ be a set of common Lebesgue points of the functions

$$t \to \int_{\mathbb{R}^n} |u(t,x)-k| f(t,x)dx,$$

with $f \in S$ and rational $k \in \mathbb{Q}$. Since the sets S and $\mathbb{Q}$ are countable the set E has full Lebesgue measure on $(0,T)$. For $t_0 \in E$ (14) implies in the limit as $\nu \to \infty$ that

$$\int_{\mathbb{R}^n} |u(t_0,x)-k| f(t_0,x)dx + \int_{\Pi_T} \{|u-k|f_t +$$
$$\operatorname{sign}(u-k)(\varphi(u)-\varphi(k), \nabla_x f)\}\theta(t-t_0)dtdx = 0. \quad (15)$$

Since the sets $S \subset C_0^\infty(\Pi_T)$ and $\mathbb{Q} \subset \mathbb{R}$ are dense while the integrals in (15) depend on f and k continuously then the relation (15) holds for all $f \in C_0^\infty(\Pi_T)$ and $k \in \mathbb{R}$. In particular, $u(t,x)$ satisfies (5) with replacement Π_T by $(t_0, T) \times \mathbb{R}^n$, and $u_0(x)$ by $u(t_0, x)$. Therefore $u(t,x)$ is a g.e.s. of the Cauchy problem in the indicated layer with initial function $u(t_0, x)$, and by initial condition b)

$$\operatorname*{ess\,lim}_{t \to t_0+} u(t, \cdot) = u(t_0, \cdot) \ \text{ in } L^1_{loc}(\mathbb{R}^n). \tag{16}$$

Further, change the direction of "time" and consider the function $u(T-t, -x)$. As easy to see, this function is also an i.s. of the equation (2), and it satisfies the relation like (16), which means that for $t_0 \in E$

$$\operatorname*{ess\,lim}_{t \to t_0-} u(t, \cdot) = u(t_0, \cdot) \ \text{ in } L^1_{loc}(\mathbb{R}^n).$$

Thus, $\forall t_0 \in E$

$$\operatorname*{ess\,lim}_{t \to t_0} u(t, \cdot) = u(t_0, \cdot) \ \text{ in } L^1_{loc}(\mathbb{R}^n). \tag{17}$$

As easily follows from (17), $\forall t_0 \in E$

$$\lim_{t \to t_0, t \in E} u(t, \cdot) = u(t_0, \cdot) \ \text{ in } L^1_{loc}(\mathbb{R}^n),$$

and the map $t \to u(t, \cdot)$ is continuous in topology of $L^1_{loc}(\mathbb{R}^n)$. For $t_0 \in E$ the functions $u_+(t,x) = u(t+t_0, x)$ and $u_-(t,x) = u(t_0 - t, -x)$ are bounded i.s. (and, therefore, g.e.s.) of the problem (2), (3) in the layers Π_{T-t_0} and Π_{t_0} respectively with initial data $u(t_0, x)$ and $u(t_0, -x)$. Denote

$$m_0 = \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} u(t_0, x) = \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} u(t_0, -x),$$
$$M_0 = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} u(t_0, x) = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} u(t_0, -x).$$

Then, by the maximum principle (see [13, 12]) $m_0 \le u_+(t,x) \le M_0$ and $m_0 \le u_-(t,x) \le M_0$ a.e. in Π_{T-t_0} and Π_{t_0} respectively, i.e. $m_0 \le u(t,x) \le M_0$ a.e. in Π_T. Therefore

$$m_0 \le m = \operatorname{ess\,inf} u(t,x) \le \operatorname{ess\,sup} u(t,x) = M \le M_0$$

and since inverse inequalities $m \le m_0$, $M_0 \le M$ are obvious (with regard to continuity of $u(t,\cdot)$ in $L^1_{loc}(\mathbb{R}^n)$ for $t \in E$), then $\forall t_0 \in E$

$$m_0 = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} u(t,x) = m, \quad M_0 = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} u(t,x) = M,$$

as required. The Proposition is proved in the case of a bounded $u(t,x)$.

In the general case $u(t,x) \in L^\infty_{loc}(\Pi_T)$ consider "cut-off" functions $u_r(t,x) = \max(-r, \min(u(t,x), r))$, $r \in \mathbb{N}$. By Theorem 1 the functions $u_r(t,x)$ are i.s. of the equation (2), and since these i.s. are bounded they satisfy the conclusions of Proposition 2 on some sets E_r of full measure. To complete the proof it only remains to set $E = \bigcap\limits_{r\in\mathbb{N}} E_r$ and pass to the limit as r tends to infinity.

In linear case when $\varphi(u) = (c,u)$, $c \in \mathbb{R}^n$ any g.e.s. of the equation (2) is an i.s. This easily follows from the fact that a generalized solution is constant on any characteristic $x - ct = \text{const}$. For nonlinear equation (2) existence of nontrivial i.s. imposes strong enough restrictions on character of nonlinearity (see § 2 below).

In the case when the flux functions are Lipschitz i.s. of the Cauchy problem for equation (2) can be easily described. More exactly, suppose that $u_0(x) \in [a,b]$ a.e. in the open ball $V_r = \{ x \in \mathbb{R}^n \mid |x| < r \}$, $0 < r \le \infty$ and

$$|\varphi(u) - \varphi(v)| \le N|u - v| \quad \forall u, v \in [a,b], \tag{18}$$

$\Omega = \Omega_r$ is a cone $\{ (t,x) \in \Pi_T \mid |x| + Nt < r \}$ (remark that $\Omega = \Pi_T$ for $r = \infty$). Since the flux vector satisfies the Lipschitz condition (18) the vector of generalized derivatives $\varphi'(u)$ lies in $L^\infty([a,b])$, and $|\varphi'(u)| \le N$. Define for $(t,x) \in \Omega$

$$u^+(t,x) = \inf\{ u \in [a,b] \mid v > u_0(x - \varphi'(v)t) \ \text{ for a.e. } v \in (u,b) \},$$
$$u^-(t,x) = \sup\{ u \in [a,b] \mid v < u_0(x - \varphi'(v)t) \ \text{ for a.e. } v \in (a,u) \}.$$

Observe that by the estimate $|\varphi'(v)| \le N$ we have $x - \varphi'(v)t \in V_r$ for $(t,x) \in \Omega$. Let the function $u(t,x) \in L^\infty(\Omega)$ be such that $u(t,x) \in [a,b]$ a.e. in Ω and $u(0,x) = u_0(x)$ in the sense of relation

$$\operatorname*{ess\,lim}_{t\to 0} \int_{|x|<r-Nt} |u(t,x) - u_0(x)|dx = 0. \tag{19}$$

Then we have the following

Theorem 2. *A function $u(t,x)$ is an i.s. of the equation (2) in the domain Ω if and only if $u^+(t,x) = u^-(t,x) = u(t,x)$ a.e. in Ω.*

Proof. Define for $(t,x,v) \in \Omega \times (a,b)$ $U(t,x,v) = \theta(v - u(t,x))$, $U_0(x,v) = \theta(v - u_0(x))$, where $\theta(v) = \begin{cases} 1 & , \quad v > 0 \\ 1/2 & , \quad v = 0 \\ 0 & , \quad v < 0 \end{cases}$. As is easily verified, for $k \in \mathbb{R}$, $u = u(t,x) \in [a,b]$

$$|u - k| = -\int_a^b \operatorname{sign}(v-k) U(t,x,v)dv + |b-k|,$$

$$\operatorname{sign}(u-k)(\varphi(u) - \varphi(k)) =$$
$$-\int_a^b \operatorname{sign}(v-k)\varphi'(v)U(t,x,v)dv + \operatorname{sign}(b-k)(\varphi(b) - \varphi(k)).$$

Therefore for $f = f(t,x) \in C_0^\infty(\Omega)$

$$\int_\Omega [|u-k|f_t + \operatorname{sign}(u-k)(\varphi(u) - \varphi(k), \nabla_x f)]dtdx =$$
$$-\int_a^b \operatorname{sign}(v-k)\left(\int_\Omega U(t,x,v)[f_t + (\varphi'(v), \nabla_x f)]dtdx\right)dv. \tag{20}$$

We take into account that values $c_0 = |b-k|$ and $c = \operatorname{sign}(b-k)(\varphi(b) - \varphi(k)) \in \mathbb{R}^n$ do not depend on (t,x) and therefore $\int_\Omega [c_0 f_t + (c, \nabla_x f)]dtdx = 0$. Besides, as follows from (19)

$$\operatorname*{ess\,lim}_{t\to 0} \int_a^b \int_{|x|<r-Nt} |U(t,x,v) - U_0(x,v)|dxdv = 0. \tag{21}$$

For

$$I_f(v) = \int_\Omega U(t,x,v)[f_t + (\varphi'(v), \nabla_x f)]dtdx$$

relation (20) shows that $u(t,x)$ is an i.s. of (2) in Ω if and only if $\forall k \in \mathbb{R}, f \in C_0^\infty(\Omega)$

$$\int_a^b \operatorname{sign}(v-k) I_f(v)dv = 0,$$

i.e. $I_f(v) = 0$ a.e. on (a,b). Choose a countable dense (with respect to usual locally convex topology in $C_0^\infty(\Omega)$) set $S \subset C_0^\infty(\Omega)$. Then the set $E = \{\ v \in (a,b) \mid I_f(v) = 0\ \forall f \in S\ \}$ has full measure. Since S is dense in

$C_0^\infty(\Omega)$ and for fixed $v \in E$ the map $f \to I_f(v)$ is continuous on $C_0^\infty(\Omega)$ then $I_f(v) = 0\ \forall f \in C_0^\infty(\Omega)$ that is for $v \in E$

$$U_t + (\varphi'(v), \nabla_x U) = 0 \tag{22}$$

in $\mathcal{D}'(\Omega)$. Thus, $U(t,x,v)$ is a g.s. of the linear equation (22). Therefore it is constant on characteristics $x - \varphi'(v)t = \text{const}$. The latter means that $U(t,x,v) = g(x - \varphi'(v)t, v)$ for a.e. $(t,x) \in \Omega$, where $g(\cdot, v)$ is some bounded measurable function on V_r. Then, as follows from the limit relation (21), $g(x,v) = U_0(x,v)$ a.e. on $V_r \times (a,b)$. Hence

$$U(t,x,v) = U_0(x - \varphi'(v)t, v) = \theta(v - u_0(x - \varphi'(v)t)) \text{ a.e. on } \Omega \times (a,b). \tag{23}$$

Thus, the requirement that $u(t,x)$ is an i.s. of (2) reduces to equality (23). The latter in turns means that for a.e. $(t,x) \in \Omega$

$$v > u_0(x - \varphi'(v)t) \text{ for a.e. } v > u(t,x);\ v < u_0(x - \varphi'(v)t) \text{ for a.e. } v < u(t,x)$$

that is $u^+(t,x) = u^-(t,x) = u(t,x)$ a.e. in Ω. The proof is complete.

§ 2. The case of one spatial variable.

Consider the case when $n = 1$ and the equation (2) has the form

$$u_t + \varphi(u)_x = 0, \tag{24}$$

$u = u(t,x)$, $(t,x) \in \Omega \subset \mathbb{R}^2$.

In this case the following sufficent condition of isentropicity takes place:

Theorem 3. *Let $u(t,x) \in C(\Omega)$ be a continuous g.s. of the equation (24) in the domain $\Omega \subset \mathbb{R}^2$. Then $u(t,x)$ is an isentropic solution.*

For the proof of Theorem 3 we need the following simple technical lemma.

Lemma 1. *Let $\varphi(u) \in C([a,b])$ be a continuous and nonlinear function on the segment $[a,b]$. Then there exists $c \in \mathbb{R}$ and $u_0 \in (a,b)$ being a strict extremum point of $\varphi(u) - cu$ on $[a,b]$.*

Proof. Denote $h(c,u) = \varphi(u) - cu$, $c^* = (\varphi(b) - \varphi(a))/(b-a)$. Then, by the nonlinearity condition, $h(c^*,u)$ differs from the constant $h(c^*,a) = h(c^*,b)$. Assume for definiteness that $\exists u \in (a,b)\quad h(c^*,u) < h(c^*,a) = h(c^*,b)$. By continuity there exists $\delta > 0$ such that

$$\min_{u \in [a,b]} h(c,u) < \min(h(c,a), h(c,b))\ \forall c \in (c^* - \delta, c^* + \delta). \tag{25}$$

Remark next that the condition $h(c,u_0)=\min\limits_{u\in[a,b]}h(c,u)$ can be rewritten in the form $\varphi(u)\geq l(u)=c(u-u_0)+\varphi(u_0)\ \forall u\in[a,b]$, which implies that $\varphi^*(u)\geq c(u-u_0)+\varphi(u_0)\geq c(u-u_0)+\varphi^*(u_0)\ \forall u\in[a,b]$, where $\varphi^*(u)$ is a convex envelope of $\varphi(u)$ on $[a,b]$. In particular, taking $u=u_0$ we derive that $\varphi(u_0)=\varphi^*(u_0)$. Thus, the condition

$$\varphi(u_0)-cu_0=\min_{u\in[a,b]}(\varphi(u)-cu) \tag{26}$$

implies that

$$\varphi^*(u_0)-cu_0=\min_{u\in[a,b]}(\varphi^*(u)-cu).$$

Therefore, if u_1,u_2 are minimum points of the function $\varphi(u)-cu$, $u_1<u_2$ then these points are also minimum points of $\varphi^*(u)-cu$. Since the function $\varphi^*(u)$ is convex then the set of minimum points for the function $\varphi^*(u)-cu$ contains the segment $[u_1,u_2]$. This means that the function $\varphi^*(u)$ is linear on (u_1,u_2) and has a slope c. Obviously, the indicated intervals (u_1,u_2) do not intersect for different c, hence their set is at most countable. Clearly, the same is true for the corresponding set of slopes S consisting of values $c\in\mathbb{R}$ such that the function $\varphi(u)-cu$ has more than one minimum point. Therefore, we can find a value $c\in(c^*-\delta,c^*+\delta)\setminus S$, for which there is a unique point $u_0\in[a,b]$ such that (26) is satisfied. Besides, as follows from (25), $u_0\neq a$, $u_0\neq b$ and $u_0\in(a,b)$. By construction, $\forall u\in[a,b]$, $u\neq u_0$ $\varphi(u)-cu>\varphi(u_0)-cu_0$ and u_0 is a strict minimum point of the function $\varphi(u)-cu$. The proof is complete.

Proof of Theorem 3.

By Proposition 1 we need to prove that $\forall k\in\mathbb{R}$, $f\in C_0^\infty(\Omega)$

$$\int_\Omega[\max(u,k)f_t+\varphi(\max(u,k))f_x]dtdx=0. \tag{27}$$

Obviously, it is sufficient to verify the relation (27) for a dense set of values k. Introduce two suitable sets, the union of which is dense.

K_1 consists of points k, in vicinity of which the function $\varphi(u)$ is linear, i.e. $\exists\delta>0$, $c\in\mathbb{R}$ $\varphi(u)\equiv\varphi(k)+c(u-k)$ for $|u-k|<\delta$.

K_2 consists of points k such that for some values $c\in\mathbb{R}$ and $\delta>0$, k is a point of strict extremum of the function $\varphi(u)-cu$ on the interval $|u-k|<\delta$.

As it directly follows from Lemma 1, a union $K_1\cup K_2$ is dense in $\mathbb{R}$.

Fix k belonging to one of the indicated sets. Adding to the functions u and $\varphi(u)$ appropriate constant we can assume, without loss of generality, that

$$k = \varphi(k) = 0.$$

After linear change $x \to x - ct$ in the equation (24) the flux function $\varphi(u)$ is replaced by $\varphi(u) - cu$, and the corresponding map $u(t,x) \to u(t, x - ct)$ obviously conserves classes of generalized and isentropic solutions. Therefore we can assume that the slope c in the definition of $k \in K_1 \cup K_2$ equals to zero.

It is sufficient to verify the condition (27) locally, i.e. that any point $y \in \Omega$ has a neighbourhood $U(y)$ such that $\forall f \in C_0^\infty(U(y))$ the equality (27) is fulfilled.

By localization of our problem we can also assume that the domain Ω is a square, say $\Omega = (0,1)^2$, and $u(t,x) \in C(\bar{\Omega})$, $\bar{\Omega} = \mathrm{Cl}\,\Omega = [0,1]^2$; $|u(t,x)| < \delta$, $|\varphi(u(t,x))| < \delta$ for all $(t,x) \in \Omega$, where δ is the parameter in the definition of sets K_1–K_2.

The case corresponding to the set K_1 is trivial, in this case $\varphi(u) \equiv \mathrm{const}$ on the image of $u(t,x)$ (let us remind that $c = 0$) and the equation (24) has a simple form $u_t = 0$.

Consider the case $k \in K_2$. Suppose for definiteness that $k = 0$ is a point of strict local minimum of $\varphi(u)$, i.e. $\varphi(u) < \varphi(0) = 0$ for $0 < |u| < \delta$. By (24) there exists a smooth function $v : \bar{\Omega} \to \mathbb{R}$ (potential), $v \in C^1(\bar{\Omega})$ determined by equalities $v_x(t,x) = u(t,x)$, $v_t(t,x) = -\varphi(u(t,x))$ for $(t,x) \in \bar{\Omega}$. By the condition

$$v_t(t,x) = -\varphi(u(t,x)) \geq 0 \quad \text{on } \Omega \tag{28}$$

the function $v(t,x)$ increases with respect to the coordinate t.

We shall need one consequence of the coarea formula see, for instance, Theorem 3.2.11 in [1].

Theorem (coarea formula). *Let $v : \Omega \to \mathbb{R}$ be a Lipschitz function in the bounded domain $\Omega \subset \mathbb{R}^2$, and $B \subset \mathbb{R}^2$ be a Borel set. Then for almost each values $y \in \mathbb{R}$ the inverse image $v^{-1}(y)$ is 1-σ-rectifiable set (i.e. the set $v^{-1}(y)$ can be covered by a countable family of Lipschitz curves). Moreover, length $\mathcal{H}^1(v^{-1}(y)) \neq \infty$ (i.e. one-dimensional Haussdorf measure of $v^{-1}(y)$ is finite) and*

$$\int_B |\nabla v(t,x)|\, dtdx = \int_{\mathbb{R}} \mathcal{H}^1(v^{-1}(y) \cap B)\, dy.$$

In particular, for the set $S = \{(t,x) \in \Omega \mid v_x(t,x) = v_t(t,x) = 0\}$ the following ststement is true: for a.e. $y \in \mathbb{R}$ *length of the inverse image* $v^{-1}(y)$ *is finite, and length of the intersection* $S \cap v^{-1}(y)$ *equals to zero, that is*

$$\mathcal{H}^1(v^{-1}(y)) < \infty, \quad \mathcal{H}^1(v^{-1}(y) \cap S) = 0. \tag{29}$$

Besides, as follows from the classical Morse-Sard theorem (see, for example, Theorem 1.3 in [3]) and equivalence of conditions $v_x(t,x) = 0$ and $v_t(t,x) = 0$, a.e. values $y \in \mathbb{R}$ are regular (in classical sense) values of $v(t,x)$ on the boundary $\partial\Omega$, i.e.

$$v_x(t,x) \neq 0, \ v_t(t,x) \neq 0 \ \forall (t,x) \in \partial\Omega \cap v^{-1}(y). \tag{30}$$

The values y that satisfy conditions (29), (30) will be called regular. Denote by Y the set of regular values y such that $v^{-1}(y) \neq \emptyset$. By the coarea Theorem formulated above measure of the difference $v(\Omega)\backslash Y$ equals to 0. If $y \in Y$ then for a fixed x the set $\{ \ t \mid v(t,x) = y \ \}$ can not contain more than one point. Indeed, if $v(t_1,x) = v(t_2,x) = y$, $t_1 < t_2$ then, by monotonicity of $v(t,x)$ with respect to t, $v(t,x) = y$ and $v_t(t,x) = -\varphi(u(t,x)) = 0 \ \forall t \in [t_1,t_2]$. Since $\varphi(u) = 0$ only for $u = 0$ then also $v_x(t,x) = 0 \ \forall t \in [t_1,t_2]$. We see that the set $S \cap v^{-1}(y)$ contains the segment $[t_1,t_2] \times \{x\}$ and therefore $\mathcal{H}^1(v^{-1}(y) \cap S) > 0$. This contradicts to regularity of the value y. Obviously, the set of x such that $\exists t(x) \ v(t(x),x) = y$ is closed and its connected components are segments. If $[a,b]$ is one of such segments then the curve $C_y = \{ \ (t(x),x) \mid x \in [a,b] \ \}$ is a connected component of $v^{-1}(y)$, and the map $\gamma(x) = (t(x),x)$ is a homeomorphism of $[a,b]$ onto C_y, in particular the endpoints $\gamma(a), \gamma(b) \in \partial\Omega$. Remark also that by regularity of y for a.e. $x \in [a,b]$ $(t(x),x) \notin S$ and the function $t(x)$ has a derivative

$$t'(x) = -v_x(t,x)/v_t(t,x) = u(t,x)/\varphi(u(t,x)), \ t = t(x). \tag{31}$$

Show that the function $t(x)$ is strictly monotone. Supposing the contrary one can find values $x_1, x_2 \in (a,b)$, $x_1 < x_2$ such that $t(x_1) = t(x_2) = \tau$. Then, $t(x)$ does not equal to τ identically on $[x_1,x_2]$. Otherwise, $t'(x) = 0$ on (x_1,x_2), which implies that $v_t = v_x = 0$ on the corresponding segment $\{\tau\} \times [x_1,x_2] \in C_y$. Since $\mathcal{H}^1$-measure of this segment is positive this contradicts to regularity of y.

Thus, $\exists \bar{x} \in (x_1,x_2) \ t(\bar{x}) \neq \tau$. Assume for definiteness that $t(\bar{x}) > \tau$.

It is clear that the part of the curve C_y corresponding to $x \in [x_1, x_2]$ lies in Ω (otherwise, $\exists x \in (a,b)$ $(t(x), x) \in \partial\Omega$, and, as is easily verified, in this case $v_t = v_x = 0$ at the point $(t(x), x)$, which contradicts to regularity of y). Therefore, there is a small enough $h > 0$ such that $h < t(\bar{x}) - \tau$ and $(t(x) + h, x) \in \Omega$ for $x \in [x_1, x_2]$.

Then $\alpha = \min\limits_{x \in [x_1, x_2]} v(t(x) + h, x) - y > 0$ and for any $z \in (y, y + \alpha) \cap Y$ one can find the level curve $C_z = \{(t_z(x), x) \mid x \in [x_1, x_2]\} \subset v^{-1}(z)$, where $t(x) < t_z(x) < t(x) + h$.

We define the function

$$g : Y \cap (y, y + \alpha) \ni z \to \max_{x \in [x_1, x_2]} t_z(x).$$

As follows from the condition $h < t(\bar{x}) - \tau$, $\max\limits_{x \in [x_1, x_2]} t_z(x) > t(\bar{x}) > \tau + h > \max(t_z(x_1), t_z(x_2))$, and the maximum in the definition of g is achieved in a point $x_0 \in (x_1, x_2)$. Obviously, $v_x(t_0, x_0) = 0$, where $t_0 = t_z(x_0)$ (in the opposite case $t_z(x)$ is strictly monotone in a neighbourhood of x_0). The also $v_t(t_0, x_0) = 0$. This easily implies that the derivative $g'(z) = +\infty$ for each $z \in Y \cap (y, y + \alpha)$. But the function g strictly increases by its definition, and hence it is a.e. differentiable. The obtained contradiction completes the proof of strict monotonicity of $t(x)$.

Denote $\Omega_+ = \{(t, x) \in \Omega \mid v_x(t, x) > 0\}$, $\Omega_{+0} = \mathrm{Int}\{(t, x) \in \Omega \mid v_x(t, x) \geq 0\}$, $S_0 = \partial\Omega_{+0}$ (the boundary over Ω). Obviously, $\Omega_+ \subset \Omega_{+0}$, $S_0 \subset S$. Let us prove the following property:

(P) For $y \in Y$ the level set $v^{-1}(y)$ can not intersect sets Ω_+ and S_0 simultaneously.

In the opposite case there exists a curve $C_y = \{(t(x), x) \mid x \in [a, b]\}$ such that $C_y \cap \Omega_+ \neq \emptyset$ and $C_y \cap S_0 \neq \emptyset$. Therefore $v_x > 0$ on $C_y \cap \Omega_+$, and since this set is not empty then with regard to (31) $\exists x \in (a, b)$ $t'(x) < 0$. Now, by strict monotonicity of $t(x)$ we conclude that $t(x)$ strictly decreases. In particular, $\tau_2 = t(b) < \tau_1 = t(a)$. Since $C_y \cap S_0 \neq \emptyset$, $S_0 = \partial\Omega_{+0}$ then there is a point $(t_1, x_1) \in (\tau_2, \tau_1) \times (a, b)$ such that $v_x(t_1, x_1) < 0$, $v(t_1, x_1) = z \in Y$ and $z \neq y$. Let $C_z = \{(t_z(x), x) \mid x \in [a_1, b_1]\}$ be a connected component of $v^{-1}(z)$ containing the point (t_1, x_1). Then, by (31) and strict monotonicity of $t_z(x)$, we conclude that $t_z(x)$ strictly increases. Remark also that $t_z(a_1), t_z(b_1) \in \partial\Omega$. The indicated properties of the curves C_y, C_z imply that these curves must intersect. But this is impossible, and the obtained contrudiction proves (P).

Now, we go to the proof of the main relation (27). Let $f(t,x)\in C_0^\infty(\Omega)$, $K=\operatorname{supp} f$. Consider the compact $K_0=K\cap S_0$ and set $B=v(K_0)$, $A=v^{-1}(B)\cap\Omega_+$. By (P) $Y\cap v(A)=\emptyset$ and $v(A)$ has null Lebesgue measure. Then, by coarea formula

$$\int_A |\nabla v(t,x)|\,dtdx=\int_{\mathbb{R}}\mathcal{H}^1(v^{-1}(y)\cap A)\,dy=\int_{v(A)}\mathcal{H}^1(v^{-1}(y)\cap A)\,dy=0$$

and since $|\nabla v(t,x)|>0$ on Ω_+ then $\operatorname{meas}(A)=0$, where we denote by $\operatorname{meas}(A)$ the Lebesgue measure of a measurable set A (this notation $\operatorname{meas}(A)$ is common for all dimensions).

Let $r>0$ and

$$B_r=\{\ y\in\mathbb{R}\mid \operatorname{dist}(y,B)=\inf_{z\in B}|y-z|<r\ \}$$

be a r-neighbourhood of the set B, $A_r=v^{-1}(B_r)\cap\Omega_+$. The set B is closed (as an image of the compact set under a continuous map). Therefore, $B=\bigcap\limits_{r>0}B_r$, which implies that also $A=\bigcap\limits_{r>0}A_r$. Fix $\varepsilon>0$. Since $\operatorname{meas}(A)=0$ then $\exists r>0$ $\operatorname{meas}(A_r)<\varepsilon$. Now, we choose the function $\rho(y)\in C_0^\infty(\mathbb{R})$ such that $0\le\rho(y)\le 1$, $\operatorname{supp}\rho\subset B_r$, $\rho=1$ in the neighbourhood $B_{r/2}$ of the set B and set

$$g=g(t,x)=f(t,x)\rho(v(t,x)),\ \ h=h(t,x)=f(t,x)[1-\rho(v(t,x))].$$

Clearly, $g,h\in C_0^1(\Omega)$, $f=g+h$. Therefore, the integral in the left-hand side of (27) with $k=\varphi(k)=0$ can be rewritten as

$$I=\int_\Omega[u^+f_t+\varphi(u^+)f_x]dtdx=I_1+I_2,$$

where $u^+=\max(u,0)$ and

$$I_1=\int_{\Omega_+}[ug_t+\varphi(u)g_x]dtdx,\ \ I_2=\int_{\Omega_{+0}}[uh_t+\varphi(u)h_x]dtdx.$$

By construction $h=0$ in a neighbourhood of $S_0=\partial\Omega_{+0}$, which means that the resriction of $h(t,x)$ on Ω_{+0} lies in $C_0^1(\Omega_{+0})$. Since $u(t,x)$ is a g.s. of the equation (24) in Ω_{+0} then the second integral $I_2 = 0$. Further, for $v = v(t,x)$, $u = u(t,x)$

$$g_t = f_t\rho(v) + f\rho'(v)v_t = f_t\rho(v) - f\rho'(v)\varphi(u),$$
$$g_x = f_x\rho(v) + f\rho'(v)v_x = f_x\rho(v) + f\rho'(v)u.$$

Substituting these expressions in the integral I_1 and taking into account that $\rho(v) = 0$ out of A_r we obtain the equality

$$I_1 = \int_{\Omega_+} [uf_t + \varphi(u)f_x]\rho(v)dtdx = \int_{A_r} [uf_t + \varphi(u)f_x]\rho(v)dtdx.$$

Thus,

$$|I_1| \le \int_{A_r} [|u||f_t| + |\varphi(u)||f_x|]dtdx \le c\,\mathrm{meas}(A_r) < c\varepsilon,$$

where c is a constant not depending on ε. Then $|I| = |I_1| < c\varepsilon$ and with regard of arbitrariness of $\varepsilon > 0$ we have $I = 0$, i.e. the condition (27) is satisfied. The proof is complete.

It turns out that in the case $\Omega = \Pi_T$, $0 < T \le \infty$ the necessary condition for existence of i.s. is that the flux function $\varphi(u)$ is Lipschitz in a set of values of this solution. More exactly, we have the following

Theorem 4. *Let $u(t,x) \in L^\infty_{loc}(\Pi_T)$ be an i.s. of (24) in the domain $\Omega = \Pi_T$; $-\infty \le m = \operatorname{ess\,inf} u(t,x)$, $M = \operatorname{ess\,sup} u(t,x) \le +\infty$ in Ω. Then $\varphi(u)$ satisfies the Lipschitz condition on any segment $[a,b] \subset (m,M)$.*

Proof. Suppose, in the contrary, that $\varphi(u)$ does not satisfy the Lipschitz condition on some segment $[a,b] \subset (m,M)$. Then there exist sequences $a_n, b_n \in [a,b]$, $a_n < b_n$, $n \in \mathbb{N}$ such that

$$|\varphi(b_n) - \varphi(a_n)| = \max_{u\in[a_n,b_n]} |\varphi(u) - \varphi(a_n)| > n|b_n - a_n|. \tag{32}$$

It is obvious that $b_n - a_n \to 0$ as $n \to \infty$, and after possible extracting a subsequence $a_n \to \alpha$, $b_n \to \alpha$, where $\alpha \in [a,b]$. Denote $C_n = \varphi(b_n) - \varphi(a_n)$ and choose "entropies"

$$p_n(u) = (|u-a_n| - |u-b_n| + b_n - a_n)/(2C_n) = \begin{cases} 0 & , \quad u < a_n \\ (u-a_n)/C_n & , \quad u \in [a_n,b_n] \\ (b_n-a_n)/C_n & , \quad u > b_n \end{cases}$$

with corresponding fluxes

$$q_n(u) = [(\varphi(u) - \varphi(a_n))\mathrm{sign}\,(u - a_n) - (\varphi(u) - \varphi(b_n))\mathrm{sign}\,(u - b_n) +$$
$$\varphi(b_n) - \varphi(a_n)]/(2C_n) = \begin{cases} 0 & , \quad u < a_n \\ (\varphi(u) - \varphi(a_n))/C_n & , \quad u \in [a_n, b_n] \\ 1 & , \quad u > b_n. \end{cases}$$

Then by (6), $\forall n \in \mathbb{N}$

$$p_n(u)_t + q_n(u)_x = 0 \ \text{ in } \mathcal{D}'(\Omega). \tag{33}$$

In view of (32) $0 \le |p_n(u)| \le (b_n - a_n)/|C_n| < 1/n$, $|q_n(u)| \le 1$. After additional extracting of a subsequence we can assume that $q_n(\alpha) \to \beta \in [0, 1]$ as $n \to \infty$. Then $\forall u \in \mathbb{R}$ $q_n(u) \to \theta(u - \alpha)$, where $\theta(v) = \begin{cases} 1 & , \ v > 0 \\ \beta & , \ v = 0 \\ 0 & , \ v < 0 \end{cases}$.

With account of relations above we deduce from (33) in the limit as $n \to \infty$ that

$$\theta(u - \alpha)_x = 0 \ \text{ in } \mathcal{D}'(\Omega).$$

This implies that for a.e. $t \in (0, T)$ $\theta(u(t, \cdot) - \alpha)_x = 0$ in $\mathcal{D}'(\mathbb{R})$. In particular, this equality holds for $t \in E$, where $E \subset (0, T)$ is the set of full measure indicated in Proposition 2. By Proposition 2 $m = \operatorname{ess\,inf} u(t, \cdot)$, $M = \operatorname{ess\,sup} u(t, \cdot)$ fot $t \in E$, and the relation $\alpha \in (m, M)$ contradicts the equality $\theta(u(t, \cdot) - \alpha)_x = 0$ in $\mathcal{D}'(\mathbb{R})$.

The obtained contradiction completes the proof.

It is essential in Theorem above that the domain $\Omega = \Pi_T$. As we show below in the Example, for arbitrary domain Ω Theorem 4 fails. In the general case we have the following

Theorem 5. *Let $u(t, x)$ be an i.s. of the equation (24) in the domain Ω, $m = \operatorname{ess\,inf} u(t, x)$, $M = \operatorname{ess\,sup} u(t, x)$ in Ω. Then there exists a closed (with respect to the interval (m, M)) set F of null Lebesgue measure such that $\varphi(u)$ satisfies the Lipschitz condition locally on the complement $(m, M) \setminus F$.*

Proof. In the case when $u(t, x)$ is constant $m = M$ and the proof is trivial. So assume that $m < M$ and define the set U consisting of values $u \in (m, M)$ such that $\varphi(u)$ is Lipschitz in a neighbourhood of the point u. By construction U is open and $\varphi(u)$ is locally Lipschitz on U. Set $F = (m, M) \setminus U$. Then F is closed in (m, M), and to complete the proof we need only to show that F has a null measure. Let $E = \{ v \in (m, M) \mid \operatorname{meas} u^{-1}(v) = 0 \}$. Since the graph Γ of $u(t,x)$ has null measure then for a.e. $v \in (m, M)$ sections $\Gamma^v = \{ (t,x) \in \Omega \mid (t,x,v) \in \Gamma \} = u^{-1}(v)$ have null measure on Ω, which means that E is a set of full measure on (m, M). It is sufficient to prove that the set $F_1 = F \cap E$ has null measure. The latter in turn may be verified locally, that is we have to establish that $\forall v \in F_1 \ \exists V$, a neighbourhood of v such that $\operatorname{meas}(F_1 \cap V) = 0$.

So, let $v_0 \in F_1$. Consider the sets

$$\Omega^+ = \operatorname{ess} \operatorname{Cl}\{ (t,x) \in \Omega \mid u(t,x) < v_0 \}, \Omega^- = \operatorname{ess} \operatorname{Cl}\{ (t,x) \in \Omega \mid u(t,x) > v_0 \},$$

where an essential closure $\operatorname{ess} \operatorname{Cl}(A)$ of a set A consists of points, any neighbourhood of which intersects A by a set of positive measure. Evidently, $\operatorname{ess} \operatorname{Cl}(A)$ is a closed set. Indeed, its complement is an open set, each point of which has a neighbourhood intersecting A by a set of null measure. Taking into account that $v_0 \in E$, we obtain that the union $\Omega^+ \cup \Omega^- = \Omega$, in the opposite case $u(t,x) = v_0$ a.e. on the nonempty open set $\Omega \setminus (\Omega^+ \cup \Omega^-)$ and $\operatorname{meas} u^{-1}(v_0) > 0$. By connectivity of Ω one can find $(t_0, x_0) \in \Omega^+ \cap \Omega^-$. Then in any neighbourhood V of the point (t_0, x_0) sets

$$\{ (t,x) \in \Omega \mid u(t,x) < v_0 \}, \ \{ (t,x) \in \Omega \mid u(t,x) > v_0 \}$$

have positive measure. Now, we choose the square neighbourhood $V = V_t \times V_x$, where $V_t = (t_0 - \varepsilon, t_0 + \varepsilon)$, $V_x = (x_0 - \varepsilon, x_0 + \varepsilon)$, where $\varepsilon > 0$ is so small that $V \subset \Omega$ and $u(t,x) \in L^\infty(V)$. Set $m_1 = \operatorname{ess\,inf}_V u(t,x)$, $M_1 = \operatorname{ess\,sup}_V u(t,x)$. Then $m \le m_1 < v_0 < M_1 \le M$ and $I = (m_1, M_1)$ is a neighbourhood of the point v_0. We are about to show that $\operatorname{meas}(F_1 \cap I) = 0$. Let $\rho(x) \in C_0^\infty(V_x)$, $\rho(x) \ge 0$, $\rho(x_0) > 0$. Set

$$F(t) = \frac{1}{A} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} u(t,x) \rho(x) dx, \ G(t) = \frac{1}{A} \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} \varphi(u(t,x)) \rho'(x) dx,$$
$$\text{where } A = \int_{x_0 - \varepsilon}^{x_0 + \varepsilon} \rho(x) dx > 0.$$

The functions $F(t)$, $G(t)$ are defined almost everywhere on the interval V_t. In particular, these functions are well defined on the set $J \subset V_t$ of full measure consisting of values t such that for a.e. $x \in V_x$ (t,x) is the Lebesgue point of $u(t,x)$. It is clear that values $t \in J$ are the Lebesgue points of the functions $F(t)$, $G(t)$ as well. From the equality $u_t + \varphi(u)_x = 0$ in $\mathcal{D}'(V)$ it follows that $F'(t) = G(t) \in L^\infty$ in $\mathcal{D}'(V_t)$. In particular, $F(t)$ satisfies the Lipschitz condition on V_t (after possible correction on a set of null measure). Besides, for $t \in J$ there is a classical derivative $F'(t) = G(t)$. Since $F(t)$ is Lipschitz we see that the image $\mathcal{E} = F(V_t \setminus J)$ has null measure. This directly follows from the area formula (see Theorem 3.2.3 in [1])

$$0 = \int_{V_t \setminus J} |F'(t)| dt = \int \mathcal{H}_0(F^{-1}(v) \cap (V_t \setminus J)) dv \geq \mathrm{meas}(\mathcal{E}).$$

Here $\mathcal{H}_0$ is the 0-dimensional Haussdorf measure.

Let $I_1 = I \setminus \mathcal{E}$. If $c \in F_1 \cap I_1$ then the function $\varphi(u)$ does not satisfy the Lipschitz condition in any neighbourhood of c. Hence, there are sequences $a_n, b_n \in [a, b]$, $a_n < b_n$ such that $a_n, b_n \to c$ as $n \to \infty$ and

$$|\varphi(b_n) - \varphi(a_n)| = \max_{u \in [a_n, b_n]} |\varphi(u) - \varphi(a_n)| > n|b_n - a_n|, \ n \in \mathbb{N}.$$

We define the sequence of entropies $p_n(u)$ together with the corresponding sequence of fluxes $q_n(u)$ in the same way as in the proof of Theorem 4, and pass to the limit as $n \to \infty$ in the entropy relation of the form (33). We obtain

$$\theta(v - c)_x = 0 \ \text{ in } \mathcal{D}'(V),$$

where the function $\theta(v)$ is the same as in the proof of Theorem 4. This equality implies that for $t \in J$ one of following relations holds

1) $u(t, x) \geq c$ a.e. in V_x; 2) $u(t, x) \leq c$ a.e. in V_x.

Remark that these relations are equivalent to inequalities $F(t) \geq c$, $F(t) \leq c$ respectively. The both closed sets determined by these inequalities are nonempty, otherwise $c \notin I$. Therefore, one can find a value $\bar{t} \in V_t$ such that $F(\bar{t}) = c$. By the condition $c \notin \mathcal{E}$ we see that $\bar{t} \in J$. Then $u(\bar{t}, x) = c$ a.e. on V_x and $F'(\bar{t}) = G(\bar{t}) = \mathrm{const} \int \rho'(x) dx = 0$. Thus, any value $c \in F_1 \cap I_1$ is critical for $F(t)$, i.e. for $J_0 = J \cap G^{-1}(0)$ the intersection $F^{-1}(c) \cap J_0 \neq \emptyset$. Moreover, $F'(t) = G(t) = 0$ for $t \in J_0$ and by the area formula

$$0 = \int_{J_0} |F'(t)| dt = \int \mathcal{H}_0(F^{-1}(c) \cap J_0) dc \geq \mathrm{meas}(F_1 \cap I_1).$$

Thus, $\mathrm{meas}(F_1 \cap I) = \mathrm{meas}(F_1 \cap I_1) = 0$, as was to be proved.

Theorem 5 is exact in the sense that the set $F \subset (m,M)$ of the "bed" points may be an arbitrary closed set of null measure. Confirm this by the following example, in which the corresponding i.s. is defined and continuous on the closure of a rectangular domain $\Omega = (0,T) \times (0,1)$.

Example. Let $m < M$ and $F \subset (m,M)$ be a closed set of measure zero. Firstly, construct the flux function $\varphi(u) \in C(\mathbb{R})$ such that F is exactly the set of "bed" points of $\varphi(u)$ on (m,M). This means that F is the minimal closed in (m,M) set, such that on its complement $U = (m,M) \setminus F$ the function $\varphi(u)$ is locally Lipschitz. The set U is the union of at most countable family of mutually disjoint intervals. We consider the more comlicated case when this family is countable $U = \bigcup\limits_{i\in\mathbb{N}} (a_i, b_i)$. The case of the finite set of intervals (a_i, b_i) corresponds to the finite set F and is treated similarly with evident simplifications. Set $h_i = b_i - a_i$, $c_i = (a_i + b_i)/2$. Then $\sum\limits_{i=1}^{\infty} h_i = M - m < +\infty$, which yields the existence of a decreasing sequence $\alpha_i > 0$, $i \in \mathbb{N}$ such that $\alpha_i \to 0$ as $i \to \infty$ and

$$T = 8 \sum_{i=1}^{\infty} \frac{h_i}{\alpha_i} < +\infty. \tag{34}$$

Consider for $i \in \mathbb{N}$ nonnegative continuous functions

$$\varphi_i(u) = \begin{cases} \frac{1}{\sqrt{2h_i}} \min\left(\sqrt{u - a_i}, \sqrt{b_i - u}\right) & , \quad u \in (a_i, b_i) \\ 0 & , \quad u \notin (a_i, b_i) \end{cases}.$$

Obviously, $\varphi_i(u)$ satisfies the Lipschitz condition in a neghborhood of any point except a_i, b_i. Besides, we see that $\max \varphi_i(u) = \varphi_i(c_i) = 1/2$. We define $\varphi(u) = \sum\limits_{i=1}^{\infty} \alpha_i \varphi_i(u)$, so that $\varphi(u) = \alpha_i \varphi_i(u)$ on the intervals (a_i, b_i), and $\varphi(u) = 0$ on $\mathbb{R} \setminus U$. Evidently, $0 \le \sum\limits_{i=m}^{\infty} \alpha_i \varphi_i(u) \le \alpha_m/2 \underset{m\to\infty}{\to} 0$ as $m \to \infty$. Therefore, the series determining the function $\varphi(u)$ converges uniformly on $\mathbb{R}$, so this function is continuous. It is also clear that $\varphi(u)$ is locally Lipschitz on U while all points of the set F are "bed". The latter follows from the fact that any neighbourhood V of a point $u \in F$ intersects the set $\bigcup\limits_{i\in\mathbb{N}} \{a_i, b_i\}$ (in the opposite case $V \subset F$, which contradicts to the requirement meas$(F) = 0$) and the Lipschitz condition for $\varphi(u)$ fails on V. Hence, the set of "bed" points of the function $\varphi(u)$ on the interval (m, M) coincides with F.

Now, we go to construction of the solution $u(t, x)$. We consider the step-function $\rho(u)$ on (m, M), which equals to $8/\alpha_i$ on the intervals (a_i, b_i) and zero on the set F. Set $g(u) = \int_m^u \rho(v)dv$. Obviously, $g(u)$ is absolutely continuous (because by condition (34) $\int_m^M \rho(v)dv = T < +\infty$) and strictly increases on the segment $[m, M]$, $g(m) = 0$, $g(M) = T$. Denote $A_i = g(a_i)$, $B_i = g(b_i)$. It is clear that $B_i - A_i = H_i = 8h_i/\alpha_i$. As it is directly verified (taking account of the equalities $\alpha_i H_i = 8h_i$), functions

$$v_i(t, x) = a_i + \frac{\alpha_i(t - A_i)^2}{H_i(x + 1)^2}, \ w_i(t, x) = b_i - \frac{\alpha_i(B_i - t)^2}{H_i(x + 1)^2}$$

are smooth self-similar solutions of the equations

$$u_t + \frac{\alpha_i}{\sqrt{2h_i}}(\sqrt{u - a_i})_x = 0, \ u_t + \frac{\alpha_i}{\sqrt{2h_i}}(\sqrt{b_i - u})_x = 0$$

respectively in closed regions

$$P_i^- = \{ \ (t, x) \mid x \in [0, 1], \ A_i \le t \le A_i + (x + 1)H_i/4 \ \},$$
$$P_i^+ = \{ \ (t, x) \mid x \in [0, 1], \ B_i - (x + 1)H_i/4 \le t \le B_i \ \},$$

where $v_i(t, x) \in [a_i, c_i]$ on P_i^-, $v_i(A_i, x) = a_i$, $v_i(t, x) = c_i$ for $t = A_i + (x + 1)H_i/4$, and $w_i(t, x) \in [c_i, b_i]$ on P_i^+, $w_i(B_i, x) = b_i$, $w_i(t, x) = c_i$ for $t = B_i - (x + 1)H_i/4$. Taking into account that by construction

$$\varphi(u) = \frac{\alpha_i}{\sqrt{2h_i}}\sqrt{u - a_i} \ \text{ for } u \in [a_i, c_i],$$
$$\varphi(u) = \frac{\alpha_i}{\sqrt{2h_i}}\sqrt{b_i - u} \ \text{ for } u \in [c_i, b_i],$$

we conclude that $v_i(t, x)$, $w_i(t, x)$ are smooth solutions of the equation (24).

Next, we consider the function $u_i(t, x) \in [a_i, b_i]$, defined on the closure of the rectangular domain $P_i = (A_i, B_i) \times (0, 1)$ by the equality

$$u_i(t, x) = \begin{cases} v_i(t, x) & , \quad (t, x) \in P_i^-, \\ w_i(t, x) & , \quad (t, x) \in P_i^+, \\ c_i & , \quad (t, x) \in P_i \setminus (P_i^- \cup P_i^+). \end{cases} \tag{35}$$

Observe that $P_i \setminus (P_i^- \cup P_i^+)$ is the triangle

$$P_i \setminus (P_i^- \cup P_i^+) = \{ \ (t,x) \mid x \in [0,1], \ A_i + (x+1)H_i/4 < t < B_i - (x+1)H_i/4 \ \}.$$

The equality (35) determines a piece-wise smooth and continuous up to the boundary of P_i i.s. of the equation (24). The desired i.s. $u(t,x)$ in the whole rectangle Ω is glued from the solutions $u_i(t,x)$. Namely, we set

$$u(t,x) = \begin{cases} u_i(t,x) & , \quad t \in (A_i, B_i), \\ g^{-1}(t) & , \quad t \in g(F). \end{cases} \tag{36}$$

Remark that the inverse function $g^{-1} : [0,T] \to [m,M]$ is well-defined and continuous. In addition, the set $E = g(F)$ is a compact set of measure zero. Indead, by absolute continuity of $g(u)$ we have $\text{meas}(E) = \int_F \rho(u)du = 0$. It is clear that $(0,T) = E \cup \bigcup\limits_{i \in \mathbb{N}} (A_i, B_i)$ and the equality (36) correctly defines the function $u(t,x)$ on whole Ω. Let us show that this function is continuous. Let $h(t) = g^{-1}(t)$. Then $u_i(A_i,x) = h(A_i) = a_i$, $u_i(B_i,x) = h(B_i) = b_i$, which implies estimates $|u_i(t,x) - h(t)| \le b_i - a_i = h_i$, $t \in [A_i, B_i]$. Therefore, for

$$p_m(t,x) = \begin{cases} u_i(t,x) & , \quad t \in (A_i, B_i), \ 1 \le i \le m \\ h(t) & , \quad t \in [0,T] \setminus \bigcup_{i=1}^m (A_i, B_i) \end{cases}$$

$|u(t,x) - p_m(t,x)| \le \max\limits_{i>m} h_i \underset{m\to\infty}{\to} 0$. From the latter relation it follows iniform convergence of the sequence of continuous on $\operatorname{Cl}\Omega$ functions $p_m(t,x)$ to the function $u(t,x)$. Therefore, the limit function $u(t,x)$ is continuous on Ω (and can be extended as a continuous function on $\operatorname{Cl}\Omega$). It only remains to show that $u(t,x)$ is an i.s. of our equation. Fix $k \in \mathbb{R}$ and denote $\eta(u) = |u-k|$, $\psi(u) = \operatorname{sign}(u-k)(\varphi(u) - \varphi(k))$. Then, for any test function $f = f(t,x) \in C_0^\infty(\Omega)$

$$\int_\Omega [\eta(u)f_t + \psi(u)f_x]dtdx = \int_{\Omega \setminus E} [\eta(u)f_t + \psi(u)f_x]dtdx =$$
$$\sum_{i=1}^\infty \int_{P_i} [\eta(u)f_t + \psi(u)f_x]dtdx. \tag{37}$$

Integrating by parts and taking into account that $u = u(t,x)$ is a piecewise smooth i.s., and therefore $\eta(u)_t + \psi(u)_x = 0$ in $\mathcal{D}'(P_i)$, we derive equalities

$$\int_{P_i} [\eta(u)f_t + \psi(u)f_x]dtdx = \int_0^1 \eta(u(B_i,x))f(B_i,x)dx -$$

$$\int_0^1 \eta(u(A_i,x))f(A_i,x)dx = \eta(b_i)q(B_i) - \eta(a_i)q(A_i),$$
$$\text{where } q(t) = \int_0^1 f(t,x)dx \in C_0^\infty((0,T)).$$

From (37), taking account of absolute continuity of $g(u)$ and the condition $\mathrm{meas}(F) = 0$, it follows that

$$\int_\Omega [\eta(u)f_t + \psi(u)f_x]dtdx = \sum_{i=1}^\infty (\eta(b_i)q(B_i) - \eta(a_i)q(A_i)) =$$
$$\sum_{i=1}^\infty \int_{a_i}^{b_i} (\eta(u)q(g(u)))'du = \int_{(m,M)\setminus F} (\eta(u)q(g(u)))'du =$$
$$\int_m^M (\eta(u)q(g(u)))'du = \eta(M)q(T) - \eta(m)q(0) = 0.$$

Thus, $\forall f \in C_0^\infty(\Omega)$
$$\int_\Omega [\eta(u)f_t + \psi(u)f_x]dtdx = 0,$$
i.e. $\eta(u)_t + \psi(u)_x = 0$ in $\mathcal{D}'(\Omega)$. The latter, in view of arbitrarity of $k \in \mathbb{R}$, means that $u(t,x)$ is an i.s. of the equation (24), which completes the proof.

Now, we are going to investigate more fine properties of the flux function $\varphi(u)$, arising from the condition of existence of a nontrivial continuous i.s. The following lemma is needed for the sequel.

Lemma 2. *Let $f(y)$ be a function on $[a,b]$, $y(x) \in C([\alpha,\beta])$, $y([\alpha,\beta]) = [a,b]$, and $E \subset [a,b]$. Suppose that the composition $g(x) = f(y(x))$ has bounded variation on $[\alpha,\beta] \cap y^{-1}(E)$. Then $f(y)$ has bounded variation on E, and also* $\mathrm{Var}\, f(y) \le \mathrm{Var}\, g(x)$.

Proof. Choose values $\alpha_1, \beta_1 \in [\alpha,\beta]$ such that $y(\alpha_1) = a$, $y(\beta_1) = b$ and assume for definiteness that $\alpha_1 < \beta_1$ (the opposite case $\alpha_1 > \beta_1$ reduces to the case under consideration after replacing of $y(x)$ by $y(\alpha + \beta - x)$). Let $y_i \in E$, $i = 1, \ldots, m$ be a family of points, $y_i > y_{i-1}$, $i = 2, \ldots, m$. We set $x_i = \max\{\ x \in [\alpha_1, \beta_1] \mid y(x) = y_i\ \}$. Observe that by continuity of $y(x)$ for all $i > 1$ $\ y_i \in (y_{i-1}, b] \subset y((x_{i-1}, \beta_1])$, which iplies that $x_i > x_{i-1}$, $i = 2, \ldots, m$. By construction $x_i \in y^{-1}(E)$. Then

$$\sum_{i=2}^m |f(y_i) - f(y_{i-1})| = \sum_{i=2}^m |g(x_i) - g(x_{i-1})| \le \mathrm{Var}\, g(x).$$

Taking here the supremum over all finite subset $\{y_i\}$ of the set E we derive that $\operatorname{Var} f(y) \le \operatorname{Var} g(x)$. The proof is complete.

Proposition 3. *Let $n = 1$, $r > 0$; $u_0(x) \in C((-r,r))$ and the image $u_0((-r,r))$ contains an interval (a,b). Assume also that the flux function $\varphi(u)$ satisfies the Lipschitz condition* (18), *and there exists a continuous i.s. $u = u(t,x)$ to the Cauchy problem for equation* (24) *in the trapezoid $\Omega = \Omega_r = \{\ (t,x) \in \Pi_T \mid |x| + Nt < r\ \}$, with initial condition* (19). *Then the derivative $\varphi'(u)$ has (after possible correction on a set of null measure) bounded variation on any closed segment in (a,b).*

Proof. Replace $u(t,x)$ and $u_0(x)$ to $\min(b, \max(u(t,x),a))$ and $\min(b, \max(u_0(x),a))$ respectively (by Theorem 1 after this replacement $u = u(t,x)$ remains to be a continuous i.s. of (24) with initial function $u_0(x)$). Thus, we can assume that $u_0(x), u(t,x) \in [a,b]$.

Let $E \subset (a,b)$ be the set of full measure consisting of Lebesgue points of $\varphi'(u)$, and $[c,d] \subset (a,b)$. Since the function $u_0(x)$ is continuous there exists a segment $J = [\alpha,\beta] \subset (-r,r)$ such that $u_0(J) \supset [c,d]$. Then, for some $h > 0$ the set $\{\ (t,x)\ |\alpha - Nt \le x \le \beta + Nt,\ 0 < t \le h\ \} \subset \Omega$. Let us show that the function $f(y) = y + \varphi'(u_0(y))h$ increases on $J \cap u_0^{-1}(E)$. Assuming the contrary one can find values $y_1, y_2 \in (\alpha,\beta)$, $y_1 < y_2$ such that $u_1 = u_0(y_1), u_2 = u_0(y_2) \in E$, and $f(y_1) > f(y_2)$. The latter means that characteristics $x_2(t) = y_2 + \varphi'(u_2)t$ and $x_1(t) = y_1 + \varphi'(u_1)t$ intersect at $t = t_0$, where

$$t_0 = [y_2 - y_1]/[\varphi'(u_1) - \varphi'(u_2)] \in (0,h). \tag{38}$$

We can also suppose that the function $u_0(y)$ is not constant on neighbourhoods of the points y_1, y_2. Indeed, this property is achieved after replacement y_1, y_2 to y_1', y_2', where values y_1', y_2' are chosen in such way that the segments $[y_1, y_1']$, $[y_2', y_2]$ are maximal among segments $[y_1, z_1]$, $[z_2, y_2]$, where the function $u_0(y)$ takes constant values u_1 and u_2 respectively. After such replacement the numerator of the fraction determining t_0 in equality (38) can only become smaller while the denumerator does not change. Therefore, the condition $t_0 \in (0,h)$ remains true, and our characteristics intersect, as required.

Moving the values y_1 and y_2 we can also achieve the condition that in any neighbourhoods of these points y_1 and y_2 functions $u_0(y) - u_1$ and $u_0(y) - u_2$

change their sign. Indeed, choose $\delta > 0$ such that $\alpha < y_1 - \delta < y_1 + \delta < y_2 - \delta < y_2 + \delta < \beta$ and

$$[y_2 - y_1 + 2\delta]/[\varphi'(u_1) - \varphi'(u_2) - 2\delta] \in (0, h). \tag{39}$$

The function $u_0(y)$ is continuous and is not constant on the interval $I = (y_1 - \delta, y_1 + \delta)$, therefore the interior of its image $V = \operatorname{Int} u_0(I)$ is a nondegenerate interval. Since the graph $\{u = u_0(y)\}$ has null measure on the plane then for a.e. $u \in V$ the inverse image $u_0^{-1}(u)$ is a set of null Lebesgue measure on I. Taking also into account that $u_1 \in E$ is a Lebesgue point of $\varphi'(u)$ we can find a value $\bar{u}_1 \in V \cap E$ such that the set $I_0 = u_0^{-1}(\bar{u}_1)$ has null measure and $|\varphi'(\bar{u}_1) - \varphi'(u_1)| < \delta$.

Further, the sets $I_-, I_+ \subset I$, where respectively $u_0(y) < \bar{u}_1$ and $u_0(y) > \bar{u}_1$, are open, nonempty, and their closures cover I: $I = \operatorname{Cl} I_1 \cup \operatorname{Cl} I_2$ (in the opposite case $u_0(y) = \bar{u}_1$ on a nondegenerate interval). Since I is connective $\exists \bar{y}_1 \in \operatorname{Cl} I_1 \cap \operatorname{Cl} I_2$. Then $u_0(\bar{y}_1) = \bar{u}_1$ and, as follows from the fact that $\bar{y}_1$ is a limit point of the both sets I_-, I_+, the function $u_0(y) - \bar{u}_1$ must change sign in any neighbourhood of $\bar{y}_1$. It is proved similarly that there exists a point $\bar{y}_2 \in (y_2 - \delta, y_2 + \delta)$ such that $\bar{u}_2 = u_0(\bar{y}_2) \in E$, $|\varphi'(\bar{u}_2) - \varphi'(u_2)| < \delta$, and the function $u_0(y) - \bar{u}_2$ changes sign in any neighbourhood of $\bar{y}_2$. We set

$$\bar{t}_0 = [\bar{y}_2 - \bar{y}_1]/[\varphi'(\bar{u}_1) - \varphi'(\bar{u}_2)].$$

Then, in view of (39) $\bar{t}_0 > 0$ and $\bar{t}_0 < [y_2 - y_1 + 2\delta]/[\varphi'(u_1) - \varphi'(u_2) - 2\delta] < h$. Therefore, the characteristics $x = \bar{y}_2 + \varphi'(\bar{u}_2)t$ and $x = \bar{y}_1 + \varphi'(\bar{u}_1)t$ intersect at the point $(\bar{t}_0, \bar{x}_0)$, where $\bar{x}_0 = \bar{y}_1 + \varphi'(\bar{u}_1)\bar{t}_0$. Since $\bar{u}_1 \neq \bar{u}_2$, one of these values is different from $u(\bar{t}_0, \bar{x}_0)$. Suppose for definiteness that $u(\bar{t}_0, \bar{x}_0) \neq \bar{u}_1$. Then two cases are possible: $u(\bar{t}_0, \bar{x}_0) > \bar{u}_1$ and $u(\bar{t}_0, \bar{x}_0) < \bar{u}_1$. We shall assume that $u(\bar{t}_0, \bar{x}_0) > \bar{u}_1$, the other case is treated similarly. By continuity, we claim that $u(\bar{t}_0, x) > \bar{u}_1$ for all x in a small enough neighbourhood $W = (\bar{x}_0 - \varepsilon, \bar{x}_0 + \varepsilon)$ of the point $\bar{x}_0$.

From Theorem 2 it follows that for a.e. $(t, x, v) \in \Omega \times (a, b)$ $\theta(v - u(t, x)) = \theta(v - u_0(x - \varphi'(v)t))$, where the fuction $\theta(v)$ was defined in the proof of Theorem 2. Therefore, the indicated inequality holds on the set of common Lebesgue points of its left and right parts. For $x \in W$ values

$$y = y(x) = x - \varphi'(\bar{u}_1)\bar{t}_0 = x - \bar{x}_0 + \bar{x}_0 - \varphi'(\bar{u}_1)\bar{t}_0 = x - \bar{x}_0 + \bar{y}_1$$

run through the interval $(\bar{y}_1 - \varepsilon, \bar{y}_1 + \varepsilon)$, and since the function $u_0(y) - \bar{u}_1$ changes sign on this interval, $\exists x \in W$ $u_0(y(x)) < \bar{u}_1$.

Taking into account that the functions $u(t,x)$, $u_0(y)$ are continuous, the fact that $\bar{u}_1$ is a Lebesgue point of $\varphi'(v)$, and also inequalities $u(\bar{t}_0,x) > \bar{u}_1$, $u_0(y(x)) < \bar{u}_1$, we claim that $(t_0,x,\bar{u}_1)$ is a Lebesgue point of the functions $\theta(v-u(t,x))$ and $\theta(v-u_0(x-\varphi'(v)t))$. Therefore,

$$0 = \theta(\bar{u}_1 - u(\bar{t}_0,x)) = \theta(\bar{u}_1 - u_0(x-\varphi'(\bar{u}_1)\bar{t}_0)) = 1.$$

The obtained contradiction proves that the function $f(y) = y + \varphi'(u_0(y))h$ increases on the set $J \cap u_0^{-1}(E)$. Thus, $\varphi'(u_0(y)) = (f(y)-y)/h$ has bounded variation as a difference of monotone functions. By Lemma 2 $\varphi'(u)$ is a function of bounded variation on $E \cap [c,d]$ for any segment $[c,d] \subset (a,b)$. In particular, for each $u \in (a,b)$ there are one-sided limits $\varphi'(u+) = \lim\limits_{v\to u+, v\in E} \varphi'(v)$, and the set $\{\ u \in E \mid \varphi'(u) \neq \varphi'(u+)\ \}$ is at most countable. Taking into account that E is aset of full measure on (a,b), we conclude that $\varphi'(u) = \varphi'(u+)$ a.e. on (a,b); from this and the known properties of derivatives for Lipschitz functions follows that there exists the classical derivative $\varphi'(u) = \varphi'(u+)$ at each point of continuity of $\varphi'(u+)$. To complete the proof it only remains to observe that $\varphi'(u+)$ is a function of bounded variation on any segment $[c,d] \subset (a,b)$ (already without exception of a set of null measure).

Remark 2. As follows from the proof of Proposition 3, the function $\varphi'(u)$ has bounded variation on segments $u_0(J)$, $J = [\alpha,\beta] \subset (-r,r)$.

The result of Proposition 3 is easily extended to the case of arbitrary domain $\Omega \subset \mathbb{R}^2$. Namely, we have the following

Theorem 6. *Let $u(t,x)$ be a continuous i.s. of the equation* (24) *in the domain Ω, $(a,b) \subset \operatorname{Int} u(\Omega)$, the flux function $\varphi(u)$ satisfies the Lipschitz condition* (18) *on (a,b). Then the function $\varphi'(u)$ is locally Lipschitz on (a,b).*

Proof. We shall asuume that $u \neq \text{const}$ and $a < b$, the case $a = b$ is trivial. Let $\bar{u} \in (a,b)$. Then sets

$$\Omega^- = \mathrm{Cl}\{\ (t,x) \in \Omega \mid u(t,x) < \bar{u}\ \},\ \Omega^+ = \{\ (t,x) \in \Omega \mid u(t,x) \geq \bar{u}\ \}$$

are closed and nonempty. By connectivity of Ω there is a point $(t_0,x_0) \in \Omega^- \cap \Omega^+$. It is clear that $u(t_0,x_0) = \bar{u}$ and any neighbourhood U of (t_0,x_0) contains points (t,x) such that $u(t,x) < \bar{u}$. Now, we choose a neighbourhood like

$$U = \{\ (t,x) \mid |x-x_0| < r - N|t-t_0|\ \},$$

where $r > 0$ is small so that $U \subset \Omega$. Define $v_0(y) = u(t_0, x_0 + y)$, $y \in (-r, r)$. Then the functions $v = v^+(\tau, y) = u(t_0 + \tau, x_0 + y)$, $v = v^-(\tau, y) = u(t_0 - \tau, x_0 + y)$ are continuous i.s. of equations $v_\tau + \varphi(v)_y = 0$ and $v_\tau - \varphi(v)_y = 0$ respectively in the triangular domain $\{ (\tau, y) \mid \tau > 0, |y| < r - N\tau \}$, and they satisfy the initial condition $v(0, y) = v_0(y)$ in the sense of relation (19). If $v_0(y) \geq \bar{u}$ $\forall y \in (-r, r)$ then, as follows from Theorem 2, $v^\pm(\tau, y) \geq \bar{u}$, whence $u(t, x) \geq \bar{u}$ in U, which contradicts to our choice of the point (t_0, x_0). Thus, $\exists y \in (-r, r)$ $v_0(y) < \bar{u}$ so that one could find values $h_1 > 0$, $\delta \in (0, r)$ so that $[\bar{u} - h_1, \bar{u}] \subset v_0([-r + \delta, r - \delta])$. By Proposition 3 and Remark 2 the generalized derivative $\varphi'(u)$ has bounded variation on $[\bar{u} - h_1, \bar{u}]$.

Taking the point $(t_0, x_0) \in \tilde{\Omega}^- \cap \tilde{\Omega}^+$, where closed sets

$$\tilde{\Omega}^- = \{ (t, x) \in \Omega \mid u(t, x) \leq \bar{u} \},\ \tilde{\Omega}^+ = \mathrm{Cl}\{ (t, x) \in \Omega \mid u(t, x) > \bar{u} \},$$

and, using arguments similar to ones above, we obtain that $\varphi'(u)$ has bounded variation on a set like $[\bar{u}, \bar{u} + h_2]$, $h_2 > 0$. Therefore, $\varphi'(u)$ has bounded variation on the union of the sets $[\bar{u} - h_1, \bar{u}]$ and $[\bar{u}, \bar{u} + h_2]$, i.e. on the segment $[\bar{u} - h_1, \bar{u} + h_2]$. Since the point $\bar{u} \in (a, b)$ is arbitrary we conclude that $\varphi'(u)$ has locally bounded variation on (a, b). The proof is complete.

Corollary. *Under the conditions of Theorem 6 the flux function $\varphi(u)$ is differentiable on (a, b) with exception of some at most countable set, and twice differentiable a.e. on (a, b).*

Proof. By Theorem 6 we may assume that the generalized derivative $\varphi'(u)$ has locally bounded variation on (a, b). Therefore (see, for example, [10]) $\varphi'(u)$ is continuous on (a, b) except some at most countable set of discontinuity points of the first kind and differentiable a.e. on (a, b). This directly implies our statement.

Theorems 4-6 implies the following

Theorem 7. *Let $u(t, x)$ be a continuous g.s. of the equation (24) in the domain Ω, $(a, b) = \operatorname{Int} u(\Omega)$, $a \geq -\infty$, $b \leq +\infty$. Then the following property holds.*

(*) *there exist a closed (relative to (a, b)) set F of null measure such that $\varphi(u)$ satisfies the Lipschitz condition locally on $U = (a, b) \setminus F$, and the derivatives $\varphi'(u)$ has locally bounded variation on U.*

Besides, $F = \emptyset$ if Ω is a strip $(t_1, t_2) \times \mathbb{R}$.

Proof. By Theorem 3 $u(t, x)$ is an i.s. of the equation (24) in Ω. Then, by Theorem 5 there exists a relatively closed set $F \subset (a, b)$ of null measure such that $\varphi(u)$ is locally Lipschitz on $U = (a,b) \setminus F$, and, in particular, it has the generalized derivative $\varphi'(u) \in L^\infty_{loc}(U)$. Besides, by Theorem 4, $F = \emptyset$ if Ω is a strip $(t_1,t_2) \times \mathbb{R}$. Let $\bar{u} \in U$. Choose a neighbourhood $(c,d) \subset U$ of $\bar{u}$ such that $\varphi(u)$ is Lipschitz on $[c,d]$. The function $\tilde{u} = \max(\min(u(t,x),d),c)$ is also a continuous i.s. of (24) in Ω (by Theorem 1) and satisfies the assumptions of Theorem 6. Then, by Theorem 6 the function $\varphi'(u)$ has locally bounded variation on (c,d). Since the point $\bar{u} \in U$ is arbitrary we conclude that $\varphi'(u)$ has locally bounded variation on U. The proof is complete.

We are ready to apply Theorem 7 to the problem on functional dependence of partial derivatives, which was posed in the Introduction. So, let $v = v(t,x)$ be a smooth nonlinear function in the domain $\Omega \subset \mathbb{R}^2$, which satisfies the relation (1): $v_t = \varphi(v_x)$ in Ω, where $\varphi(u)$ is a continuous function. Set $u = v_x \in C(\Omega)$ and differentiate (1) with respect to the variable x in $\mathcal{D}'(\Omega)$. We deduce that

$$u_t - \varphi(u)_x = 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{40}$$

i.e. $u = u(t,x)$ is a continuous g.s. of the equation (40). Let $(a,b) = \operatorname{Int} u(\Omega)$. Then $a < b$ (in the opposite case v_t, v_x are constant) and by Theorem 7 the function $\varphi(u)$ satisfies the condition (*). In particular, $\varphi(u)$ is twice differentiable a.e. in (a,b).

This result gives some information about analytic and geometric properties for images of derivatives of smooth functions of two variables. Remark that some geometric properties for images of derivatives of differentiable (and generally nonsmooth) functions in multidimensional case were studied before, for instance in the papers [9],[4].

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