A CONVERGENT NUMERICAL SCHEME FOR THE CAMASSA–HOLM EQUATION BASED ON MULTIPEAKONS

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Abstract. The Camassa–Holm equation $u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0$ enjoys special solutions of the form $u(x,t) = \sum_{i=1}^n p_i(t)e^{-|x-q_i(t)|}$, denoted multipeakons, that interact in a way similar to that of solitons. We show that given initial data $u|_{t=0} = u_0$ in $H^1(\mathbb{R})$ such that $u - u_{xx}$ is a positive Radon measure, one can construct a sequence of multipeakons that converges in $L^\infty_{\rm loc}(\mathbb{R}, H^1_{\rm loc}(\mathbb{R}))$ to the unique global solution of the Camassa–Holm equation. The approach also provides a convergent, energy preserving nondissipative numerical method which is illustrated on several examples.

1. Introduction

The Camassa–Holm equation (CH) [4, 5]

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0 \tag{1.1}$$

has received considerable attention the last decade. With κ positive it models, see [14], propagation of unidirectional gravitational waves in a shallow water approximation, with u representing the fluid velocity. The Camassa–Holm equation possesses many intriguing properties: It is, for instance, completely integrable and experiences wave breaking in finite time for a large class of initial data. In this article we consider the case $\kappa = 0$ on the real line, that is,

$$u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \tag{1.2}$$

and henceforth we refer to (1.2) as the Camassa–Holm equation.

Local and global well-posedness results as well as results concerning breakdown are proved in [8, 12, 16, 17]. It is known that certain initial data give global solutions, while other classes of initial data experience wave breaking in the sense that u_x becomes unbounded while the solution itself remains bounded. More precisely, the fundamental existence theorem, due to Constantin, Escher and Molinet [8, 9], reads as follows: If $u_0 \in H^1(\mathbb{R})$ and $m_0 := u_0 - u_0''$ is a positive Radon measure, then equation (1.2) has a unique global weak solution $u \in C([0,T), H^1(\mathbb{R}))$ for any T positive with initial data u_0. However, any solution with odd initial data u_0 in $H^3(\mathbb{R})$ such that $u_{0,x}(0) < 0$ blows up in a finite time ([8]).

The Camassa–Holm equation (1.2) exhibits so-called multipeakon solutions (see [5]), i.e., solutions of the form

$$u(x,t) = \sum_{i=1}^n p_i(t)e^{-|x-q_i(t)|}$$

where p_i and q_i are solutions of the following system of ordinary differential equations

$$\begin{aligned} \dot q_i &= \sum_{j=1}^n p_j e^{-|q_i-q_j|}, \\ \dot p_i &= \sum_{j=1}^n p_ip_j \operatorname{sgn}(q_i - q_j)e^{-|q_i-q_j|}. \end{aligned} \tag{1.3}$$

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The main idea in this article is to use multipeakons to approximate solutions of the Camassa–Holm equation. This gives rise to a numerical scheme for which we prove convergence.

In [5], Camassa, Holm, and Hyman use a pseudospectral method to solve (1.2) numerically but they do not study convergence of the method. We have shown in [13] how a particular finite difference scheme converges to the unique global solution in the case with periodic initial data.

The idea of using multipeakons has also been used by Camassa, Huang, and Lee in [3, 7, 6]. In [3], Camassa reformulates equation (1.2) in term of characteristics. The characteristics $q(\xi,t)$ are defined as solutions of the equation

$$q_t(\xi,t) = u(q(\xi,t),t)$$

with initial condition $q(\xi,0) = \xi$. After introducing the auxiliary variable p, which is called momentum, by

$$p(\xi,t) = (u - u_{xx})(q(\xi,t),t)\frac{\partial q}{\partial \xi}(\xi,t),$$

Camassa shows that (1.2) reduces to the following system of partial differential equations

$$\begin{aligned} q_t(\xi,t) &= \frac{1}{2}\int_{-\infty}^{\infty} \exp\left(-\left|q(\xi,t) - q(\eta,t)\right|\right) p(\eta,t)\,d\eta, \\ p_t(\xi,t) &= \frac{1}{2}p(\xi,t)\int_{-\infty}^{\infty} \operatorname{sgn}(\xi-\eta)\exp\left(-\left|q(\xi,t) - q(\eta,t)\right|\right) p(\eta,t)\,d\eta. \end{aligned} \tag{1.4}$$

In [7, 6], Camassa, Huang, and Lee discretize system (1.4) by considering a finite number n of "particles" whose positions and momenta are given by

$$q_i(t) = q(\xi_i,t) \quad p_i = p(\xi_i,t)$$

for some equidistributed ξ_i. By approximating the integrals in (1.4) by their Riemann sums, equation (1.4) reduces to the system of ordinary differential equations given by (1.3). For initial data such that $p_i > 0$, (1.3) has global solutions in time. In this case they show that the scheme is convergent in the following sense. Let p and q be solutions of (1.4) and $\{p_i(t)\}_{i=1}^n$ and $\{q_i(t)\}_{i=1}^n$ be solutions of (1.3) with initial conditions $p_i(0) = p(\xi_i,0)$ and $q_i(0) = q(\xi_i,0)$. Then, when the number of particles n increases, $\{p_i(t)\}_{i=1}^n$ and $\{q_i(t)\}_{i=1}^n$ converge uniformly for any time interval $[0,T]$ to $\{p(\xi_i,t)\}_{i=1}^n$ and $\{q(\xi_i,t)\}_{i=1}^n$ in some discrete l_1 norm.

The approach we adopt here is different, and we obtain a more general convergence result, see Theorem 3.1. However, the numerical method, which is based on solving (1.3), is the same. We consider distributional solutions of (1.2), and show first that multipeakons are indeed distributional solutions. Given general initial data for (1.2), we construct a sequence of multipeakons and prove that it converges to the exact solution of the equation when the number of peakons is increased appropriately. More precisely, we prove that, given $u_0 \in H^1(\mathbb{R})$ such that $u_0 - u_{0,xx}$ is a positive Radon measure, there exists a sequence of multipeakons that converges in $L^\infty_{\mathrm{loc}}(\mathbb{R}, H^1_{\mathrm{loc}}(\mathbb{R}))$ to the solution of the Camassa–Holm equation with initial data u_0. The proofs extend to the periodic case as well. Our proofs are constructive in the sense that we provide an explicit method, either by a collocation method or by a minimization technique (see Proposition 3.2 and Remark 3.4) to construct the multipeakon approximation. This gives a constructive proof of existence of solutions to the Camassa–Holm equation in the case where the initial data satisfy the condition mentioned above. Furthermore, this leads to a numerical method which, in contrast to the finite difference scheme presented in [13], does not contain any dissipation and preserves the $H^1(\mathbb{R})$ norm exactly. In the last section we illustrate the method on two numerical examples.

2. Global existence of multipeakon solutions

The Camassa–Holm equation may be rewritten as

$$m_t + um_x + 2mu_x = 0 \tag{2.1}$$

where the momentum m equals $u - u_{xx}$.

Definition 2.1. *We say that u in $L^1_{\mathrm{loc}}([0,T), H^1_{\mathrm{loc}})$ is a weak solution of the Camassa–Holm equation if it satisfies*

$$u_t - u_{xxt} + \frac{3}{2}(u^2)_x + \frac{1}{2}(u_x^2)_x - \frac{1}{2}(u^2)_{xxx} = 0 \tag{2.2}$$

in the sense of distributions.

When u is smooth, (2.1) and (2.2) are equivalent. Multipeakons are solutions of the form

$$u(x,t) = \sum_{i=1}^{n} p_i(t) e^{-|x-q_i(t)|}, \tag{2.3}$$

which are continuous and piecewise C^∞ functions in $H^1(\mathbb{R})$ for any given t. But since they have discontinuous first derivative, they cannot satisfy the Camassa–Holm equation in the classical sense. For functions with these properties the left-hand side of (2.2) is a distribution which consists of regular terms (piecewise C^∞ functions) and singular terms (Dirac functions or their derivatives at the points q_i) that we can compute explicitely. We only give the details of the computation of the last term, $(u^2)_{xxx}$, in (2.2), the other terms being obtained similarly. For each $i \in \{0, \dots, n+1\}$ we introduce the function

$$u_i(x,t) = \sum_{j=1}^{i} p_j(t) e^{-(x-q_j(t))} + \sum_{j=i+1}^{n} p_j(t) e^{(x-q_j(t))}$$

which is C^∞ in the space variable. Then (2.3) can be rewritten as

$$u(x,t) = \sum_{i=0}^{n} u_i(x,t)\chi_i(x)$$

where χ_i denotes the characteristic function of the interval $[q_i, q_{i+1})$ with the convention that $q_0 = -\infty$ and $q_{n+1} = \infty$. Since the χ_i have disjoint supports, we have

$$u^2 = \sum_{i=0}^{n} u_i^2 \chi_i \tag{2.4}$$

and, after differentiating (2.4),

$$\begin{aligned}
(u^2)_x &= \sum_{i=0}^{n} (u_i^2)_x \chi_i + \sum_{i=1}^{n} u_i^2(q_i)\delta_{q_i} - \sum_{i=0}^{n-1} u_i^2(q_{i+1})\delta_{q_{i+1}} \\
&= \sum_{i=0}^{n} (u_i^2)_x \chi_i + \sum_{i=1}^{n} \left(u_i^2(q_i) - u_{i-1}^2(q_i)\right)\delta_{q_i} \\
&= \sum_{i=0}^{n} (u_i^2)_x \chi_i + \sum_{i=1}^{n} \left[u^2\right]_{q_i} \delta_{q_i} \qquad (2.5)
\end{aligned}$$

where the bracket $[v]_{q_i}$ denotes the jump of v across q_i, that is, $[v]_{q_i} = v(q_i^+) - v(q_i^-)$. Since u is continuous, $\left[u^2\right]_{q_i} = 0$, and the last term in (2.5) vanishes. We differentiate (2.5) and get

$$\begin{aligned}
(u^2)_{xx} &= \sum_{i=0}^{n} (u_i^2)_{xx} \chi_i + \sum_{i=1}^{n} (u_i^2)_x(q_i)\delta_{q_i} - \sum_{i=0}^{n-1} (u_i^2)_x(q_{i+1})\delta_{q_{i+1}} \\
&= \sum_{i=0}^{n} (u_i^2)_{xx} \chi_i + \sum_{i=1}^{n} \left[(u^2)_x\right]_{q_i} \delta_{q_i}. \qquad (2.6)
\end{aligned}$$

On every interval (q_i, q_{i+1}), since $u = u_i$, u is differentiable and every derivative of u admits a limit when x tends to q_i from one side. It follows that the jump $\left[(u^2)_x\right]_{q_i}$ is a well-defined quantity and justifies its use in (2.6). Finally, after differentiating (2.6) once more, we get

$$(u^2)_{xxx} = \sum_{i=0}^{n} (u_i^2)_{xxx} \chi_i + \sum_{i=1}^{n} \left[(u^2)_{xx}\right]_{q_i} \delta_{q_i} + \sum_{i=1}^{n} \left[(u^2)_x\right]_{q_i} \delta'_{q_i}.$$

In a similar way we can compute the other terms in (2.2) and we end up with

$$
\begin{aligned}
u_t - u_{xxt} + \frac{3}{2}(u^2)_x + \frac{1}{2}(u_x^2)_x - \frac{1}{2}(u^2)_{xxx} \\
= \sum_{i=0}^{n} \left(u_{i,t} - u_{i,xxt} + \frac{3}{2}(u_i^2)_x + \frac{1}{2}(u_{i,x}^2)_x - \frac{1}{2}(u_i^2)_{xxx} \right) \chi_i \\
+ \sum_{i=1}^{n} \left(- [u_{xt}]_{q_i} + \frac{1}{2}\left[u_x^2\right]_{q_i} - \frac{1}{2}\left[(u^2)_{xx}\right]_{q_i} \right) \delta_{q_i} \\
+ \sum_{i=1}^{n} \left(- [u_t]_{q_i} - \frac{1}{2}\left[(u^2)_x\right]_{q_i} \right) \delta'_{q_i}.
\end{aligned} \tag{2.7}
$$

We already noted the equivalence between (2.2) and (2.1) when u is smooth. The same equivalence obviously holds for u_i and, after introducing m_i to denote $u_i - u_{i,xx}$, we have

$$
u_{i,t} - u_{i,xxt} + \frac{3}{2}(u_i^2)_x + \frac{1}{2}(u_{i,x}^2)_x - \frac{1}{2}(u_i^2)_{xxx} = m_{i,t} + u_i m_{i,x} + 2 m_i u_{i,x} = 0
$$

because, from the definition of u_i as a linear combination of e^{-x} and e^x, it is clear that $m_i = 0$. Thus, the first sum on the right-hand side of (2.7) vanishes. The values of the jumps in (2.7) can be computed from (2.3). We have

$$
[u_x]_{q_i} = -2p_i \tag{2.8}
$$

and, after some calculation,

$$
\begin{aligned}
&\left[(u^2)_{xx}\right]_{q_i} = 0, \quad [u_t]_{q_i} = 2p_i\dot{q}_i, \quad [u_{xt}]_{q_i} = -2\dot{p}_i, \\
&\left[u_x^2\right]_{q_i} = [u_x]_{q_i}\left(u_x(q_i^+) + u_x(q_i^-)\right) = 4p_i \sum_{j=1}^{n} p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i - q_j|}, \\
&\left[(u^2)_x\right]_{q_i} = 2u(q_i)\,[u_x]_{q_i} = -4p_i \sum_{j=1}^{n} p_j e^{-|q_i - q_j|}.
\end{aligned} \tag{2.9}
$$

Assume that the q_i are all distinct. Then (2.2) holds if and only if the coefficients multiplying δ_{q_i} and δ'_{q_i} in (2.7) all vanish. Hence, after using (2.9), (2.7) and (2.2), we end up with the system

$$
\begin{cases}
\dot{q}_i = \displaystyle\sum_{j=1}^{n} p_j e^{-|q_i - q_j|}, \\
\dot{p}_i = \displaystyle\sum_{j=1}^{n} p_i p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i - q_j|}
\end{cases} \tag{2.10}
$$

with the convention that $\operatorname{sgn}(x) = 0$ if $x = 0$. We summarize the discussion in the following lemma.

Lemma 2.2. *The function* (2.3) *is a weak solution of the Camassa–Holm equation if and only if* p_i*,* q_i *satisfy the system* (2.10) *of ordinary differential equations.*

The system (2.10) is Hamiltonian with Hamiltonian H given by

$$
H = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j e^{-|q_i - q_j|}.
$$

It means that (2.10) can be rewritten as

$$
\dot{q}_i = \frac{\partial H}{\partial p_i}, \ \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{2.11}
$$

From (2.3), the momentum is given by

$$m = 2\sum_{i=1}^{n} p_i \delta_{q_i}.$$

Hence,

$$\|u\|^2_{H^1(\mathbb{R})} = \langle u - u_{xx}, u\rangle_{H^{-1}} = \sum_{i=1}^{n} 2p_i u(q_i) = 2\sum_{i,j=1}^{n} p_i p_j e^{-|q_i-q_j|},$$

and the Hamiltonian H and the H^1 norm of u satisfy

$$H = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j e^{-|q_i-q_j|} = \frac{1}{4}\|u\|^2_{H^1(\mathbb{R})}. \tag{2.12}$$

Because of the sign function, the right-hand side in (2.10) is not Lipchitz, and we cannot apply Picard's theorem to get existence and uniqueness of solutions of (2.10). However, the Lipschitz condition would hold if we knew in advance that $q_i - q_j$ does not change sign. We are going to prove that the peaks do not cross and that the sign of $q_i - q_j$ is indeed preserved.

Let us first assume, without loss of generality, that the positions of the peaks at time $t = 0$, $\{q_i\}_{i=1}^n$, are distinct and ordered as follows

$$q_i(0) < q_j(0) \text{ for all } i < j. \tag{2.13}$$

We consider the system of ordinary differential equations

$$\begin{cases} \dot{q}_i = \displaystyle\sum_{j=1}^{n} p_j e^{-|q_i-q_j|}, \\ \dot{p}_i = \displaystyle\sum_{j=1}^{n} p_i p_j \operatorname{sgn}(i-j) e^{-|q_i-q_j|}. \end{cases} \tag{2.14}$$

This system is equivalent to (2.10) as long as the positions of the peaks q_i satisfy the ordering defined in (2.13). In contrast to (2.10), the system (2.14) fulfills the Lipchitz condition of Picard's theorem, and therefore there exists a unique maximal solution. If, in addition, the p_i are strictly positive initially then the solution exists for all time.

Lemma 2.3. *Let $\{p_i, q_i\}$ be the maximal solutions of* (2.14). *If we have*

$$q_i(t) < q_j(t) \quad \textit{for all } i < j, \tag{2.15}$$

$$p_i(t) > 0 \quad \textit{for all } i, \tag{2.16}$$

when $t = 0$, then $\{p_i(t), q_i(t)\}$ are globally defined on $[0,\infty)$ and inequalities (2.15) *and* (2.16) *remain true for all t.*

Proof. We call T the maximal time of existence. Let us assume that (2.15) and (2.16) do not hold for all $t \in [0, T)$. Then, since p_i and q_i are continuous, there exist t_0 in $[0, T)$ such that (2.15) and (2.16) hold in $[0, t_0)$ and either

$$q_i(t_0) = q_j(t_0) \text{ for some } i \text{ and } j \text{ with } i < j$$

or

$$p_i(t_0) = 0 \text{ for some } i.$$

In the first case when $q_i(t_0) = q_j(t_0) = \alpha$, we have that q_i and q_j are both solutions of the ordinary differential equation

$$\dot{q} = \sum_{k=1}^{n} p_k e^{-|q-q_k|}$$

with initial condition $q(t_0) = \alpha$. The function q plays the role of the unknown while p_k and q_k are given (they are the solutions of (2.14)). By Picard's theorem, we know that, given some initial condition, the solution is unique and therefore $q_i = q_j$ at least in a small interval centered around t_0. This contradicts the assumption that $q_i(t) < q_j(t)$ in $[0, t_0)$. In the second case when $p_i(t_0) = 0$, the function p_i is solution of

$$\dot{p} = p \sum_{j=1}^{n} p_j \operatorname{sgn}(i-j) e^{-|q_i - q_j|}$$

with initial condition $p(t_0) = 0$. Zero is an obvious solution and since the solution is unique, we must have $p_i = 0$ on $[0, T)$. This contradicts our assumption, and hence (2.15) and (2.16) hold for all time t in $[0, T)$. We denote by M the sum of all the p_i, i.e.,

$$M = \sum_{i=1}^{n} p_i. \tag{2.17}$$

M is preserved by solutions of (2.14). Indeed, we have

$$\frac{dM}{dt} = \sum_{i,j=1}^{n} p_i p_j \operatorname{sgn}(i-j) e^{-|q_i - q_j|} = 0. \tag{2.18}$$

We have proved that the p_i are positive for all t in $[0, T)$. Therefore we have

$$0 < p_i(t) < M,$$

for all i and all $t \in [0, T)$, which implies the p_i are bounded. It follows that $\dot{p}_i$ and $\dot{q}_i$ in (2.14) are bounded and the maximum solution is therefore defined for all time, i.e., $T = \infty$. □

Lemma 2.3 tells us that the ordering of the positions of the peaks is preserved, and in this case, as we already mentioned, (2.10) and (2.14) are equivalent. Thus we have established the following result.

Lemma 2.4. *If $q_i < q_j$ for $i < j$ and $p_i > 0$ at $t = 0$, then the system* (2.10) *has a unique, globally defined solution on* $[0, \infty)$.

Remark 2.5. A similar result is proved by other means in [6].

3. Convergence of multipeakon sequences

Multipeakon solutions can be used to prove the existence of solutions for the Camassa–Holm equation.

Theorem 3.1. *Given u_0 in $H^1(\mathbb{R})$ such that $m_0 = u_0 - u_{0,xx}$ is in $\mathcal{M}^+$, the space of positive finite Radon measures, there exists a sequence of multipeakons that converges in $L^\infty_{\mathrm{loc}}(\mathbb{R}, H^1_{\mathrm{loc}}(\mathbb{R}))$ to the unique solution of the Camassa–Holm equation with initial condition u_0.*

The proof of Theorem 3.1 is presented at the end of the section. The sequence of multipeakons mentioned in the theorem is denoted by $u^n(x,t) = \sum_{i=1}^{n} p_i^n(t) e^{-|x - q_i^n(t)|}$. We require that the initial conditions $u_0^n(x) = u^n(x, 0)$ satisfy the following properties

$$u_0^n \to u_0 \text{ in } H^1(\mathbb{R}), \tag{3.1a}$$

$$u_0^n \text{ is uniformly bounded in } L^1(\mathbb{R}), \tag{3.1b}$$

$$p_i^n \geq 0 \text{ for all } i \text{ and } n. \tag{3.1c}$$

In the next proposition we give a constructive proof that such sequences exist. The sequence u_0^n is defined by collocation: It coincides with the given initial function u_0 at a given number of points.

Proposition 3.2. *Given $u_0 \in H^1(\mathbb{R})$ such that $u_0 - u_{0,xx} \in \mathcal{M}^+$. For each n, let $q_{i,n} = i/n$. There exists a unique $(p_{i,n})_{i=-n^2}^{n^2}$ such that $u_0^n(x) = \sum_{i=-n^2}^{n^2} p_{i,n} e^{-|x - q_{i,n}|}$ coincides with u_0 at the $q_{i,n}$, that is,*

$$u_0^n(q_{i,n}) = u_0(q_{i,n}) \tag{3.2}$$

for all $i \in \{-n^2, \dots, n^2\}$. The initial multipeakon sequence u_0^n satisfies condition (3.1).

Proof. In order to simplify the notation, we write u and u^n instead of u_0 and u_0^n. First we show that (3.2) defines a unique $p_{i,n}$. The equation (3.2) is equivalent to the following system

$$A\overline{p} = \overline{u} \tag{3.3}$$

where $\overline{p}$ and $\overline{u}$ are vectors of $\mathbb{R}^{2n^2+1}$ given by $(p_{i,n})_{i=-n^2}^{n^2}$ and $(u(q_{i,n}))_{i=-n^2}^{n^2}$, respectively, and A equals the matrix

$$A = (A_{i,j})_{i,j=-n^2}^{n^2}, \quad A_{i,j} = e^{-|q_{i,n}-q_{j,n}|}.$$

The method is well-posed if A is invertible. In fact, A is symmetric and positive definite. Symmetry is obvious. To prove the positivity of A, we associate to any vector $\overline{r}$ in $\mathbb{R}^{2n^2+1}$ the function v in $H^1(\mathbb{R})$ given by $v(x) = \sum_{i=-n^2}^{n^2} r_{i,n}e^{-|x-q_{i,n}|}$. The H^1 norm of v has already been calculated, see (2.12), and we have

$$\overline{r}^t A\overline{r} = \frac{1}{2}\left\|v\right\|_{H^1(\mathbb{R})}^2 \geq 0. \tag{3.4}$$

Hence, A is positive. Let us prove that A is invertible. Assume $A\overline{r} = 0$. From (3.4), we have $v = 0$. Thus, since $v - v_{xx} = 2\sum_{i=-n^2}^{n^2} r_{i,n}\delta_{q_{i,n}}$, we have

$$2\sum_{i=-n^2}^{n^2} r_{i,n}\delta_{q_{i,n}} = 0. \tag{3.5}$$

Since the $q_{i,n}$ are all distinct, it follows that $\overline{r} = 0$. Hence, A is invertible, and thus there exists a unique $\overline{p}$ solving (3.3) for any given $\overline{u}$.

Let us prove (3.1a). Let f and v^n denote $u - u_{xx}$ and $u - u^n$, respectively. We want to prove that v^n tends to zero in $H^1(\mathbb{R})$. Note that $v^n - v^n_{xx} = f - 2\sum_{i=-n^2}^{n^2} p_{i,n}\delta_{q_{i,n}}$ is a Radon measure and we have

$$\left\|v^n\right\|_{H^1(\mathbb{R})}^2 = \langle v^n - v^n_{xx}, v^n\rangle = \langle f, v^n\rangle - 2p_{i,n}v^n(q_{i,n})$$

where the bracket $\langle \mu, g\rangle$ denotes the integration of g with respect to the Radon measure μ. By assumption (3.2), we have $v(q_{i,n}) = 0$, and it follows that

$$\left\|v^n\right\|_{H^1(\mathbb{R})}^2 = \langle f, v^n\rangle. \tag{3.6}$$

We consider a partition of unity of $\mathbb{R}$ that we denote $\{\phi_{i,n}\}_{i=-\infty}^{\infty}$ and which corresponds to the decomposition $\mathbb{R} = \cup_{i=-\infty}^{\infty}\left(\frac{i-1}{n}, \frac{i+1}{n}\right)$. The functions $\phi_{i,n}$ satisfy $0 \leq \phi_{i,n} \leq 1$, $\sum_{i=-\infty}^{\infty}\phi_{i,n} = 1$ and $\operatorname{supp}\phi_{i,n} \subset (\frac{i-1}{n}, \frac{i+1}{n})$. Then we have

$$\langle f, v^n\rangle = \langle f, \psi_n v^n\rangle + \sum_{i=-n^2}^{n^2}\langle f, \phi_i v^n\rangle \tag{3.7}$$

where $\psi_n = 1 - \sum_{i=-n^2}^{n^2}\phi_i$. We estimate separately the two terms on the right-hand side of (3.7). Since the support of ϕ_i is contained in $(q_{i-1,n}, q_{i+1,n})$, we have

$$\phi_i(x)v^n(x) \leq \sup_{x\in(q_{i-1,n},q_{i+1,n})}\left|v^n(x)\right|\phi_i(x). \tag{3.8}$$

Since

$$v^n(x) = v^n(q_{i,n}) + \int_{q_{i,n}}^{x} v^n_x(t)\,dt$$

and $v^n(q_{i,n}) = 0$, we have

$$\begin{aligned}\sup_{x\in(q_{i-1,n},q_{i+1,n})}\left|v^n(x)\right| &\leq \int_{q_{i-1,n}}^{q_{i+1,n}}\left|v^n_x(t)\right|\,dt \\ &\leq \sqrt{\frac{2}{n}}\left\|v^n\right\|_{H^1(\mathbb{R})} \quad \text{(Cauchy–Schwarz)}.\end{aligned} \tag{3.9}$$

The positivity of f directly implies that f is monotone: If $u \leq v$, then $\langle f, u\rangle \leq \langle f, v\rangle$. Hence, from (3.8), (3.9) and the monotonicity of f, we get

$$\sum_{i=-n^2}^{n^2} \langle f, \phi_i v^n\rangle \leq \sum_{i=-n^2}^{n^2} \sqrt{\frac{2}{n}} \left\| v^n \right\|_{H^1(\mathbb{R})} \langle f, \phi_i\rangle \leq \sqrt{\frac{2}{n}} \left\| v^n \right\|_{H^1(\mathbb{R})} \left\| f \right\|_{\mathcal{M}}. \tag{3.10}$$

Since $H^1(\mathbb{R})$ is continuously embedded in $L^\infty(\mathbb{R})$, we have, for some constant C independent of n,

$$\psi_n(x) v^n(x) \leq \left\| v^n \right\|_{L^\infty} \psi_n(x) \leq C \left\| v^n \right\|_{H^1(\mathbb{R})} \psi_n(x)$$

and, after using the monotonicity of f,

$$\langle f, \psi_n v^n\rangle \leq C \left\| v^n \right\|_{H^1(\mathbb{R})} \langle f, \psi_n\rangle \tag{3.11}$$

Gathering (3.6), (3.7), (3.10) and (3.11), we get

$$\left\| v^n \right\|_{H^1(\mathbb{R})}^2 \leq \sqrt{\frac{2}{n}} \left\| v^n \right\|_{H^1(\mathbb{R})} \left\| f \right\|_{\mathcal{M}} + C \left\| v^n \right\|_{H^1(\mathbb{R})} \langle f, \psi_n\rangle$$

which, after dividing both terms by $\left\| v^n \right\|_{H^1(\mathbb{R})}$,

$$\left\| v^n \right\|_{H^1(\mathbb{R})} \leq \sqrt{\frac{2}{n}} \left\| f \right\|_{\mathcal{M}} + C \langle f, \psi_n\rangle. \tag{3.12}$$

It remains to prove that $\langle f, \psi_n\rangle$ tends to zero. The space of Radon measures and C_0^*, the dual of C_0, where C_0 denotes the closure of C_c in $L^\infty(\mathbb{R})$, are isometrically isomorphic (see, e.g., [11, Chapter 7]), and we have

$$\left\| f \right\|_{\mathcal{M}} = \sup_{\substack{\varphi \in C_c \\ \left\| \varphi \right\|_{L^\infty} \leq 1}} \langle f, \varphi\rangle.$$

Therefore, for all $\varepsilon > 0$, there exists $\tilde{\varphi} \in C_c$ with $\left\| \tilde{\varphi} \right\|_{L^\infty} \leq 1$ and such that

$$\left\| f \right\|_{\mathcal{M}} \leq \langle f, \tilde{\varphi}\rangle + \varepsilon.$$

For n big enough, the supports of ψ_n and $\tilde{\varphi}$ do not intersect and therefore we have $\left\| \psi_n + \tilde{\varphi} \right\|_{L^\infty} \leq 1$. Hence,

$$\begin{aligned}
\langle f, \psi_n + \tilde{\varphi}\rangle &\leq \left\| f \right\|_{\mathcal{M}} \left\| \psi_n + \tilde{\varphi} \right\|_{L^\infty} \\
&\leq \left\| f \right\|_{\mathcal{M}} \\
&\leq \langle f, \tilde{\varphi}\rangle + \varepsilon
\end{aligned}$$

which implies

$$\langle f, \psi_n\rangle \leq \varepsilon,$$

and this proves that $\langle f, \psi_n\rangle \to 0$. Then, by (3.12), we get that v^n tends to zero, and (3.1a) is proved.

Let us prove (3.1b), namely that $p_{i,n} \geq 0$ for all $-n^2 \leq i \leq n^2$. Let f again denote $u - u_{xx}$. By assumption, f is positive. In a first step, we assume that f belongs to $C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. We will remove this smoothness assumption afterwards. A notable property of u^n is that it is always bounded by u, i.e.,

$$u^n(x) \leq u(x) \text{ for all } x. \tag{3.13}$$

We see this as follows. Let $v = u - u^n$. Since we have $u^n - u^n_{xx} = 0$ everywhere except at the $q_{i,n}$ and (3.2) holds, v satisfies, for every $i \in \{-n^2, \dots, n^2 - 1\}$, the Dirichlet problem

$$\begin{aligned}
&v - v_{xx} = f \quad \text{on } (q_{i,n}, q_{i+1,n}), \\
&v(q_{i,n}) = v(q_{i+1,n}) = 0.
\end{aligned} \tag{3.14}$$

The Green's function $G(x, \xi)$ is defined as the solution of (3.14) with $f = \delta(x - \xi)$. We can compute G (see, for example, [1]), and we get

$$G(x, \xi) = \frac{1}{\sinh(q_{i+1,n} - q_{i,n})} \begin{cases} \sinh(x - q_{i,n}) \sinh(q_{i+1,n} - \xi) & \text{for } q_{i,n} \leq x \leq \xi, \\ \sinh(\xi - q_{i,n}) \sinh(q_{i+1,n} - x) & \text{for } \xi < x \leq q_{i+1,n}. \end{cases} \tag{3.15}$$

The general solution of (3.14) is then given by

$$v(x) = \int_{q_{i,n}}^{q_{i+1,n}} G(x,\xi) f(\xi)\,d\xi. \tag{3.16}$$

Since $G(x,\xi)$ is positive, it follows from (3.16) that $v \geq 0$ on every interval $[q_{i,n}, q_{i+1,n}]$. On the intervals $(-\infty, q_{-n^2,n}]$ and $[q_{n^2,n}, \infty)$, v solves a Dirichlet problem similar to (3.14) and the Green's functions are obtained from (3.15) by letting $q_{-n^2-1,n}$ tend to $-\infty$ and $q_{n^2+1,n}$ to $+\infty$, respectively. The Green's functions are still positive and that implies, as before, that $v \geq 0$ on $(-\infty, q_{-n^2,n}] \cup [q_{n^2,n}, \infty)$. This concludes the proof of (3.13). From (2.8), we have

$$\begin{aligned} p_{i,n} &= -\frac{1}{2}\left[u^n_x\right]_{q_{i,n}} \\ &= -\frac{1}{2}\lim_{h\downarrow 0}\left[\frac{u^n(q_{i,n}+h) - u^n(q_{i,n})}{h} - \frac{u^n(q_{i,n}) - u^n(q_{i,n}-h)}{h}\right] \end{aligned}$$

and, after using (3.13) and (3.2),

$$\begin{aligned} p_{i,n} &\geq -\frac{1}{2}\lim_{h\downarrow 0}\left[\frac{u(q_{i,n}+h) - u(q_{i,n})}{h} - \frac{u(q_{i,n}) - u(q_{i,n}-h)}{h}\right] \\ &\geq -\frac{1}{2}\left[u_x\right]_{q_{i,n}}. \end{aligned}$$

Since f is smooth, u is smooth and therefore $\left[u_x\right]_{q_{i,n}} = 0$. Hence,

$$p_{i,n} \geq 0. \tag{3.17}$$

We want to prove (3.17) without any extra smoothness assumption on f. Let ρ be a positive, C^∞ and even function which satisfies $\int_{-\infty}^{\infty} \rho(x)\,dx = 1$. We denote by ρ_ε the mollifier $\rho_\varepsilon = \frac{1}{\varepsilon}\rho(x/\varepsilon)$. Let $f_\varepsilon = \rho_\varepsilon * f$ and $u_\varepsilon = \rho_\varepsilon * u$. The mollified function u_ε tends to u in $H^1(\mathbb{R})$ and therefore in $L^\infty(\mathbb{R})$. Hence, for all i in $\{-n^2, \dots, n^2\}$, $u_\varepsilon(q_{i,n})$ tends to $u(q_{i,n})$ or, using the previous notations,

$$\overline{u}_\varepsilon \to \overline{u}. \tag{3.18}$$

We can construct multipeakons u^n_ε from the regularized function u_ε whose coefficients $p_{\varepsilon,i,n}$ are determined by

$$A\overline{p}_\varepsilon = \overline{u}_\varepsilon, \quad \overline{p}_\varepsilon = (p_{\varepsilon,i,n})_{i=-n^2}^{n^2}. \tag{3.19}$$

Since f is positive, f_ε is positive and, since it also belongs to $C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, we have already established, see (3.17), that $\overline{p}_\varepsilon \geq 0$. Thus, by (3.18),

$$\overline{p} = A\overline{u} = \lim_{\varepsilon\to 0} A\overline{u}_\varepsilon = \lim_{\varepsilon\to 0} \overline{p}_\varepsilon, \tag{3.20}$$

implying that $\overline{p}$ is positive and (3.1b) is proved.

It remains to prove (3.1c), namely that u^n is bounded in $L^1(\mathbb{R})$. The regularized f_ε of f belongs to $C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and is positive. Hence, (3.13) holds when u^n and u are replaced by u^n_ε and u_ε:

$$u^n_\varepsilon \leq u_\varepsilon. \tag{3.21}$$

From (3.20), we have $\overline{p}_\varepsilon \to \overline{p}$ when $\varepsilon \to 0$. Then by looking at the defintions of u^n_ε and u^n it is clear that u^n_ε tends to u^n in $L^\infty(\mathbb{R})$. We have already seen that u_ε tends to u in $L^\infty(\mathbb{R})$. Hence, after letting ε tend to zero in (3.21), we get that (3.13) holds for all f without any further smoothness assumption. Morevover, u is positive since the positivity of $\overline{p}$ implies the positivity of u^n. From (3.13), we get

$$\int_{-\infty}^{\infty} u^n(x)\,dx \leq \int_{-\infty}^{\infty} u(x)\,dx. \tag{3.22}$$

If u belongs to $L^1(\mathbb{R})$, then a bound on $\|u^n\|_{L^1}$ follows directly from (3.22). Again, we consider the regularized f_ε of f. Since u_ε satisfies $u_\varepsilon - u_{\varepsilon,xx} = f_\varepsilon$, it is known that u_ε can be expressed as

$$u_\varepsilon(x) = \int_{-\infty}^{\infty} e^{-|x-y|} f_\varepsilon(y)\,dy.$$

Hence,

$$\begin{aligned}\int_{-\infty}^{\infty} u_\varepsilon\,dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-|x-y|} f_\varepsilon(y)\,dydx\\ &= 2\left\|f_\varepsilon\right\|_{L^1} \quad \text{(after applying Fubini's theorem)}\\ &= 2\left\|f_\varepsilon\right\|_{\mathcal{M}}.\end{aligned}$$

Since u_ε is positive and converges to u in $L^\infty(\mathbb{R})$, by Fatou's lemma, we get

$$\begin{aligned}\int_{-\infty}^{\infty} u(x)\,dx &\leq \liminf \int_{-\infty}^{\infty} u_\varepsilon(x)\,dx\\ &\leq 2\liminf \left\|f_\varepsilon\right\|_{\mathcal{M}}.\end{aligned} \tag{3.23}$$

Let us estimate $\left\|f_\varepsilon\right\|_{\mathcal{M}}$. For any continuous function ϕ with compact support, we have

$$\langle f_\varepsilon,\phi\rangle = \langle \rho_\varepsilon * f,\phi\rangle = \langle f,\phi*\rho_\varepsilon\rangle. \tag{3.24}$$

Note that the last equality in (3.24) holds because of the parity of ρ_ε (see, e.g., [11, Chapter 9] for general formulas on convolutions of distributions). Hence,

$$\begin{aligned}\left|\langle f_\varepsilon,\phi\rangle\right| &\leq \left\|f\right\|_{\mathcal{M}}\left\|\phi*\rho_\varepsilon\right\|_{L^\infty}\\ &\leq \left\|f\right\|_{\mathcal{M}}\left\|\phi\right\|_{L^\infty}\left\|\rho_\varepsilon\right\|_{L^1} \quad \text{(Young's inequality)}\end{aligned}$$

and, since $\left\|\rho_\varepsilon\right\|_{L^1} = 1$, it implies

$$\left\|f_\varepsilon\right\|_{\mathcal{M}} \leq \left\|f\right\|_{\mathcal{M}}.$$

Inequality (3.23) now gives

$$\int_{-\infty}^{\infty} u(x)\,dx \leq 2\left\|f\right\|_{\mathcal{M}}$$

which implies that u belongs to $L^1(\mathbb{R})$. From (3.22), we get that $\left\|u^n\right\|_{L^1}$ is bounded. This concludes the proof of the proposition. □

Remark 3.3. The initial multipeakon sequence $u_0^n(x) = \sum_{i=-n^2}^{n^2} p_{i,n}e^{-|x-q_{i,n}|}$ defined by setting

$$p_{i,n} = \frac{1}{2}\langle m_0,\phi_{i,n}\rangle \quad \text{for } i\in\{-n^2,\dots,n^2\}$$

where $\{\phi_i\}_{i=-\infty}^{\infty}$ denotes the partition of unity used in (3.7), also satisfies the condition (3.1). The proof of that result is much shorter than the proof of Proposition 3.2. However, the method is not directly applicable numerically (we would have to construct the ϕ_i and compute $2n^2+1$ integrals), which makes Proposition 3.2 more interesting.

Remark 3.4. Another natural way to construct a sequence of multipeakons from the set of points $q_{i,n}$, is to choose $\overline{p}$ so that it minimizes $\|u_0 - u_0^n\|_{H^1(\mathbb{R})}$, that is,

$$\overline{p} = \operatorname*{Argmin}_{p_{i,n}}\left\|u_0 - \sum_{i=-n^2}^{n^2} p_{i,n}e^{-|x-q_{i,n}|}\right\|_{H^1(\mathbb{R})}. \tag{3.25}$$

It turns out that the sequence that this minimization method produces and the one of Proposition 3.2 are the same. One can prove this as follows. We have

$$\begin{aligned}\|u_0-u_0^n\|_{H^1(\mathbb{R})}^2 &= \|u_0\|_{H^1(\mathbb{R})}^2 - 2\langle u_0^n,u_0\rangle_{H^1} + \|u_0^n\|_{H^1(\mathbb{R})}^2\\ &= \|u_0\|_{H^1(\mathbb{R})}^2 - 2\left\langle u_0^n - u_{0,xx}^n,u_0\right\rangle_{\mathcal{M}} + 2\overline{p}^tA\overline{p}\end{aligned}$$

and, since

$$\left\langle u_0^n - u_{0,xx}^n,u_0\right\rangle_{\mathcal{M}} = \left\langle \sum_{i=-n^2}^{n^2} 2p_{i,n}\delta_{q_{i,n}},u_0\right\rangle = 2\sum_{i=-n^2}^{n^2} p_{i,n}u_0(q_{i,n}) = 2\overline{p}^t\overline{u},$$

we get

$$\|u_0-u_0^n\|_{H^1(\mathbb{R})}^2 = \|u_0\|_{H^1(\mathbb{R})}^2 - 4\overline{p}^t\overline{u} + 2\overline{p}^tA\overline{p}. \tag{3.26}$$

By differentiating (3.26) with respect to $\overline{p}$, we can easily check that the minimizer $\overline{p}$ of (3.26) satisfies (3.3). In addition, since A is positive definite, $\overline{p}$ is the unique strict minimizer of (3.26).

The estimates contained in the following lemma will be needed to derive the existence of a converging subsequence.

Lemma 3.5. *Let $u^n(x,t)$ be a sequence of multipeakons with initial data satisfying* (3.1). *The following properties hold:*
(i) u^n *is uniformly bounded in* $H^1(\mathbb{R})$,
(ii) u^n_x *is uniformly bounded in* $L^\infty(\mathbb{R})$,
(iii) u^n_x *has a uniformly bounded total variation,*
(iv) u^n_t *is uniformly bounded in* $L^2(\mathbb{R})$.

Proof. From Lemma 2.4 we know, using assumption (3.1c), that the system (2.10) has a unique global solution, and hence we have a globally defined sequence of multipeakons denoted $u^n(x,t)$. In order to simplify the notation, we drop the superscript n on p^n_i and q^n_i and write

$$u^n = \sum_{i=1}^{n} p_i e^{-|x-q_i|}.$$

Property (i) is obvious because of (3.1a) and the fact that the H^1 norm is automatically preserved due to the Hamiltonian structure of (2.10). We have

$$u^n_x(t,x) = \sum_{i=1}^{n} -p_i(t)\operatorname{sgn}(x-q_i(t))e^{-|x-q_i(t)|} \quad \text{a.e.}$$

Hence,

$$\begin{aligned}
|u^n_x(x,t)| &\le \sum_{i=1}^{n} p_i(t) e^{-|x-q_i(t)|} && (p_i \ge 0)\\
&\le \|u^n\|_{L^\infty}\\
&\le C\,\|u^n\|_{H^1(\mathbb{R})} && (H^1(\mathbb{R}) \text{ is continuously embedded in } L^\infty(\mathbb{R}))
\end{aligned}$$

and (ii) follows from (i). The total variation of u^n_x equals $\|u^n_{xx}\|_{\mathcal{M}}$ (see, e.g., [10, Chapter 6]). We have

$$\|u^n_{xx}\|_{\mathcal{M}} \le \|u^n\|_{\mathcal{M}} + \|u^n - u^n_{xx}\|_{\mathcal{M}}. \tag{3.27}$$

Since $u^n \in L^1(\mathbb{R})$, we have $\|u^n\|_{\mathcal{M}} = \|u^n\|_{L^1}$ and

$$\|u^n\|_{L^1} = \sum_{i=1}^{n} p_i(t)\int_{-\infty}^{\infty} e^{-|x-q_i(t)|}\,dx = 2\sum_{i=1}^{n} p_i(t) = 2\sum_{i=1}^{n} p_i(0) = \|u^n_0\|_{L^1} \tag{3.28}$$

because $\sum_{i=1}^{n} p_i(t)$ is a constant of motion (see (2.17) and (2.18)). Since the p_i are positive, the fact that $m^n = u^n - u^n_{xx} = \sum_{i=1}^{n} 2p_i\delta_{q_i}$ and (3.28) imply that

$$\|u^n - u^n_{xx}\|_{\mathcal{M}} = 2\sum_{i=1}^{n} p_i = \|u^n_0\|_{L^1}. \tag{3.29}$$

Hence, from (3.27),

$$\|u^n_{xx}\|_{\mathcal{M}} \le 2\,\|u^n_0\|_{L^1}$$

and (iii) follows from (3.1b) .

The derivative u^n_t is given by

$$u^n_t = \sum_{i=1}^{n} \left(\dot{p}_i e^{-|x-q_i|} + p_i\dot{q}_i \operatorname{sgn}(x-q_i)e^{-|x-q_i|}\right),$$

or, after using (2.10),

$$u^n_t = \sum_{i,j=1}^{n} p_i p_j e^{-|x-q_i|} e^{-|q_i-q_j|}\left(\operatorname{sgn}(q_i-q_j) + \operatorname{sgn}(x-q_i)\right).$$

Hence, since the p_i are all positive,

$$\begin{aligned}\|u_t^n\|_{L^2} &\le 2\sum_{i,j=1}^{n} p_i p_j e^{-|q_i-q_j|}\left\|e^{-|x-q_i|}\right\|_{L^2}\\ &\le 2\sum_{i,j=1}^{n} p_i p_j e^{-|q_i-q_j|}\\ &\le \|u^n\|_{H^1(\mathbb{R})}^2\end{aligned}$$

and assertion (iv) follows from (i). □

To prove the existence of a converging subsequence of u^n in $C([0,T],H^1_{\mathrm{loc}}(\mathbb{R}))$ we recall the following compactness theorem adapted from Simon [18, Corollary 4].

Theorem 3.6 (Simon)**.** *Let X, B, Y be three continuously embedded Banach spaces*

$$X \subset B \subset Y$$

with the first inclusion, $X \subset B$, compact. We consider a set $\mathcal{F}$ of continuous functions mapping $[0,T]$ into X. If $\mathcal{F}$ is bounded in $L^\infty([0,T],X)$ and $\frac{\partial \mathcal{F}}{\partial t} = \left\{\frac{\partial f}{\partial t} \mid f\in\mathcal{F}\right\}$ is bounded in $L^r([0,T],Y)$ where $r>1$, then $\mathcal{F}$ is relatively compact in $C([0,T],B)$.

Proof of Theorem 3.1. Given initial data $u_0 \in H^1(\mathbb{R})$ with $u_0 - u_{0,xx} \in \mathcal{M}^+$ we know from Proposition 3.2 that there exists a sequence u_0^n satisfying condition (3.1). Furthermore, by using Lemma 2.4, we infer that there exists a sequence of multipeakons $u^n(x,t)$ such that $u^n|_{t=0} = u_0^n$. The sequence then possesses the properties stated in Lemma 3.5.

To apply Theorem 3.6, we have to determine the Banach spaces with the required properties. Let K be a compact subset of $\mathbb{R}$. We define $X = X(K)$ as the set of functions of $H^1(K)$ which have derivatives of bounded variation, that is,

$$X(K) = \left\{v \in H^1(K) \mid v_x \in \mathrm{BV}(K)\right\}$$

endowed with the norm

$$\|v\|_{X(K)} = \|v\|_{H^1(K)} + \|v_x\|_{BV(K)} = \|v\|_{H^1(K)} + \|v_x\|_{L^\infty(K)} + \mathrm{TV}_K(v_x).$$

It follows that $X(K)$ is a Banach space. Let us prove that the injection $X(K) \subset H^1(K)$ is compact. We consider a sequence v_n which is bounded in $X(K)$. By the Rellich–Kondrachov theorem, since $\|v_n\|_{H^1(K)}$ is bounded, there exists a subsequence (that we still denote v_n) which converges to some v in $L^2(\mathbb{R})$. Since $\mathrm{TV}_K(v_{n,x})$ is bounded, Helly's theorem allow us to extract another subsequence such that

$$v_{n,x} \to w \ \text{ a.e. in } K \tag{3.30}$$

for some $w \in L^\infty(K)$. We have $\|v_{n,x}\|_{L^\infty(K)}$ bounded. From (3.30) we get, by Lebesgue's dominated convergence theorem, that $v_{n,x} \to w$ in $L^2(K)$. Using the distributional definition of a derivative, it is not hard to check that w must coincide with v_x. Therefore v_n converges to v in $H^1(K)$ and $X(K)$ is compactly embedded in $H^1(K)$.

The estimates we have derived previously imply that u^n and u_t^n are uniformly bounded in $L^\infty([0,T],X(K))$ and $L^\infty([0,T],L^2(K))$, respectively. Since $X(K) \subset H^1(K) \subset L^2(K)$ with the first inclusion compact, Simon's theorem gives us the existence of a subsequence of u^n that converges to some $u \in H^1(K)$ in $C([0,T],H^1(K))$. We consider a sequence of compact sets K_m such that $\mathbb{R} = \cup_{m\in\mathbb{N}} K_m$ and a sequence of time T_m such that $\lim_{m\to\infty} T_m = \infty$. By a diagonal argument, we can find a subsequence (that we still denote u^n) that converges to some $u \in C([0,T_m],H^1(K_m))$ in $L^\infty([0,T_m],H^1(K_m))$ for all m. Therefore u belongs to $C(\mathbb{R},H^1_{\mathrm{loc}}(\mathbb{R}))$ and u^n converges to u in $L^\infty_{\mathrm{loc}}(\mathbb{R},H^1_{\mathrm{loc}}(\mathbb{R}))$.

It remains to prove that u is solution of the Camassa–Holm equation. This simply comes from the fact that the u^n are all weak solutions of (2.2), and since they converge to u in $L^\infty_{\mathrm{loc}}(\mathbb{R},H^1_{\mathrm{loc}}(\mathbb{R}))$, u is a weak solution of (2.2). The solutions of the Camassa–Holm equation for the class of initial data we are considering in the theorem are unique, see [8]. It implies that not only a subsequence, but the whole sequence of multipeakons converges to the solution. ☐

4. NUMERICAL RESULTS

Multipeakons can be used in a numerical scheme to solve the Camassa–Holm equation with initial data satisfying $u_0 - u_{0,xx} \in \mathcal{M}^+$. The scheme consists of solving the system of ordinary differential equations (2.10) where the initial conditions are computed as in Proposition 3.2.

In the numerical experiments that follow, we solve (2.10) by using the explicit Runge–Kutta solver `ode45` for ordinary differential equation from MATLAB. In the case where u_0 is sufficiently smooth, an initial multipeakon sequence can be obtained without having to solve (3.2). This is the aim of the following proposition.

Proposition 4.1. *Let u_0 be such that $u_0 - u_{0,xx}$ is a positive function in $H^1(\mathbb{R}) \cap L^1(\mathbb{R})$. We set*

$$\begin{aligned} q_{i,n} &= \frac{i}{n}, \\ p_{i,n} &= \frac{1}{2n}[u_0 - u_{0,xx}](q_{i,n}) = \frac{1}{2n} m_0(q_{i,n}). \end{aligned} \tag{4.1}$$

Then the sequence $u_0^n = \sum_{i=-n^2}^{n^2} p_{i,n} e^{-|x-q_{i,n}|}$ of multipeakons satisfies the conditions given in (3.1).

Proof. Condition (3.1c) follows directly from the definition of $p_{i,n}$ and the positivite of m_0. Let us prove (3.1a), i.e., that $u_0^n \to u_0$ in $H^1(\mathbb{R})$. It is enough to show that m_0^n tends to m_0 in H^{-1} because the mapping $v \mapsto v - v_{xx}$ is an homeomorphism from H^1 to H^{-1} (see [2, chapter 8]). For any function ϕ in $H^1(\mathbb{R})$, we have to prove that

$$\langle m_0^n, \phi \rangle = \sum_{i=-n^2}^{n^2} 2p_{i,n}\phi(q_{i,n}) = \frac{1}{n} \sum_{i=-n^2}^{n^2} m_0(q_{i,n})\phi(q_{i,n})$$

converges to

$$\langle m_0, \phi \rangle = \int_{\mathbb{R}} m_0(x)\phi(x)\,dx.$$

If ϕ is continuous with compact support, the above convergence simply follows from the fact that for continuous functions, the Riemann sums converge to the integral. To prove that $\langle m_0^n, \phi\rangle \to \langle m_0, \phi\rangle$ for any $\phi \in H^1(\mathbb{R})$, it is then enough to show that $\|m_0^n\|_{H^{-1}}$ is uniformly bounded. In fact, m_0^n is uniformly bounded in $\mathcal{M}$ and

$$\|m_0^n\|_{\mathcal{M}} = \frac{1}{n} \sum_{i=-n^2}^{n^2} m_0(q_{i,n}) \to \|m_0\|_{L^1}. \tag{4.2}$$

Let us prove (4.2). We have

$$\int_{\mathbb{R}} m_0(x)\,dx = \int_{-\infty}^{-n} m_0(x)\,dx + \int_{-n}^{n+\frac{1}{n}} m_0(x)\,dx + \int_{n+\frac{1}{n}}^{\infty} m_0(x)\,dx.$$

The first and the last integral tend to zero because m_0 belongs to $L^1(\mathbb{R})$. Then we have

$$\begin{aligned} \left| \int_{-n}^{n+\frac{1}{n}} m_0(x)\,dx - \frac{1}{n}\sum_{i=-n^2}^{n^2} m_0(q_{i,n}) \right| &\le \sum_{i=-n^2}^{n^2} \int_{q_{i,n}}^{q_{i+1,n}} |m_0(x) - m_0(q_{i,n})|\,dx \\ &\le \sum_{i=-n^2}^{n^2} \int_{q_{i,n}}^{q_{i+1,n}} \int_{q_{i,n}}^{x} |m_0'(\xi)|\,d\xi dx. \end{aligned}$$

We change the order of integration, introduce $\chi_{i,n}$ to denote the characteristic function of the interval $(q_{i,n}, q_{i+1,n})$, and get

$$\begin{aligned}
\left|\int_{-n}^{n+\frac{1}{n}} m_0(x)\,dx - \frac{1}{n}\sum_{i=-n^2}^{n^2} m_0(q_{i,n})\right| &\leq \sum_{i=-n^2}^{n^2}\int_{q_{i,n}}^{q_{i+1,n}}\int_{\xi}^{q_{i+1,n}} \left|m_0'(\xi)\right|\,dxd\xi\\
&= \int_{-\infty}^{\infty} |m_0'(\xi)| \sum_{i=-n^2}^{n^2}\chi_{i,n}(\xi)(q_{i+1,n}-\xi)\,d\xi\\
&\leq \|m_0'\|_{L^2}\Big[\int_{-\infty}^{\infty}\Big(\sum_{i=-n^2}^{n^2}\chi_{i,n}(\xi)(q_{i+1,n}-\xi)\Big)^2\,d\xi\Big]^{1/2}\\
&\leq \|m_0\|_{H^1(\mathbb{R})}\Big[\int_{-\infty}^{\infty}\sum_{i=-n^2}^{n^2}\chi_{i,n}(\xi)(q_{i+1,n}-\xi)^2\,d\xi\Big]^{1/2}\\
&\leq \|m_0\|_{H^1(\mathbb{R})}\Big[\sum_{i=-n^2}^{n^2}\int_{q_{i,n}}^{q_{i+1,n}}(q_{i+1,n}-\xi)^2\,d\xi\Big]^{1/2}\\
&\leq \|m_0\|_{H^1(\mathbb{R})}\Big[\sum_{i=-n^2}^{n^2}\frac{1}{3n^3}\Big]^{1/2}\\
&\leq \frac{1}{\sqrt{n}}\|m_0\|_{H^1(\mathbb{R})}
\end{aligned}$$

which tends to zero. This concludes the proof of (3.1a) and condition (3.1b) follows from (4.2) since we have, see (3.29),

$$\|u_0^n\|_{L^1} = 2\sum_{i=1}^{n} p_i = \|m_0^n\|_{\mathcal{M}}.$$

□

We tested our algorithm with smooth traveling waves. Smooth traveling waves are solutions of the form

$$u(x,t) = f(x-ct)$$

where f is solution of the second-order ordinary differential equation

$$f_{xx} = f - \frac{\alpha}{(f-c)^2}. \tag{4.3}$$

In order to give rise to a smooth traveling wave, the constants c and α cannot be chosen arbitrarily, see [15]. Here we consider periodic smooth traveling waves. The approach, based on functions in $H^1(\mathbb{R})$, which was developed in the previous sections, can be adapted to handle solutions with periodic boundary conditions. We then have to consider periodic multipeakons which are solutions of the form

$$u(x,t) = \sum_{i=1}^{n} p_i(t) G(x, q_i(t)) \tag{4.4}$$

where G is given by

$$G(x,y) = \frac{\cosh(d(x,y)-\frac{a}{2})}{\sinh\frac{a}{2}}.$$

In the expression above, a is the period and $d(x,y) = \min\left(|x-y|, a-|x-y|\right)$ is the distance in the interval $[0,a]$, identifying the end points 0 and a of the interval. The function $G(x,y)$ can be interpreted as the periodized version of $e^{-|x-y|}$ as we have $G(x,y) = \sum_{k=-\infty}^{\infty} e^{-|x-y+ka|}$. The coefficients p_i and q_i satisfy equation (2.11) when H is replaced by the Hamiltonian

$$H_{\mathrm{per}} = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j G(q_i, q_j).$$

For periodic functions, we have $H_{\mathrm{per}} = 4\,\|u\|_{H^1((0,a))}$ for u given by (4.4). It is not hard to prove that, with the necessary amendments, Theorem 3.1 and Proposition 3.2 hold also for periodic functions in $H^1([0,a])$.

A high precision solution of equation (4.3) is used as a reference solution for the smooth traveling wave. We take $\alpha = c = 3$. With initial condition $f(0) = 1, f_x(0) = 0$, it gives rise to a smooth traveling wave of period $a \approx 6.4723$. In our multipeakon scheme, we approximate initial data by using (4.1) because the initial data is smooth. In Figure 1, we show the result of such approximation in the case of 10 multipeakons.

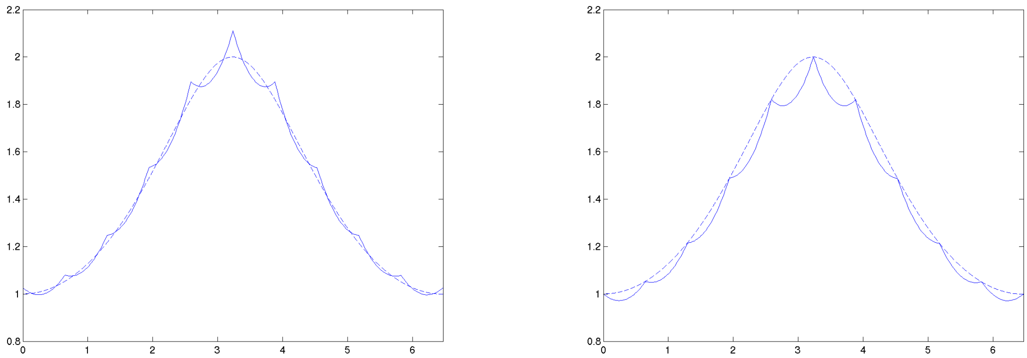

FIGURE 1. Approximation of a smooth traveling wave (dashed curve) by (4.4) with $n = 10$. On the left, the coefficients p_i are computed by using the method of Proposition 4.1 designed for smooth functions. On the right, they are computed by using the collocation method of Proposition 3.2.

Number of peakons	5	10	20	40
$\|u - u_{\mathrm{exact}}\|_{H^1(\mathbb{R})}$ at $t = 0$	1.48	0.76	0.38	0.20
Ratio		1.95	2	1.9
$\|u - u_{\mathrm{exact}}\|_{H^1(\mathbb{R})}$ at $t = 2$	1.31	0.68	0.34	0.17
Ratio		1.93	2	2

TABLE 1. Convergence rate in the case of a smooth traveling wave.

In Table 1 we give the error in the H^1 norm between the computed and the exact solutions at time $t = 0$ and $t = 2$ (at $t = 2$, the wave has approximately traveled over a distance equal to one period). We can see that the computed solution converges to the exact solution at a linear rate. It is to be noted that the error does not grow in time and is apparently only due to the error which is made in approximating the initial data.

Our next example deals with a initial data function u_0 which has discontinuous derivative. We take

$$u_0(x) = \frac{10}{(3 + |x|)^2}.$$

The function u_0 satisfies $u_0 - u_{0,xx} \geq 0$ and it is plotted in Figure 2. In our multipeakon scheme, we use Proposition 3.2 to set the initial sequence of multipeakons.

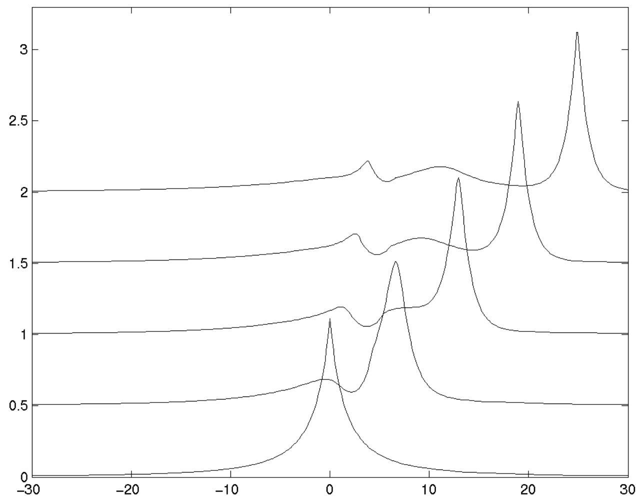

FIGURE 2. Solution with initial data $u_0(x) = 10(3 + |x|)^{-2}$ at $t = 0, 5, 10, 15, 20$ (from the bottom to the top).

In Figure 2 the solution is computed with very high resolution ($n = 1000$ peakons spread over the interval $[-30, 30]$) and in Table 4, the error is evaluated by taking this numerical solution as an approximation of the exact solution (except a time $t = 0$ where we can use u_0).

Number of peakons	61	127	251	501	1001
$\|u - u_{\text{exact}}\|_{H^1(\mathbb{R})}$ at $t = 0$	0.27	0.14	0.079	0.053	0.045
Ratio	–	1.93	1.77	1.49	1.18
$\|u - u_{\text{exact}}\|_{H^1(\mathbb{R})}$ at $t = 10$	0.58	0.18	0.074	0.028	–
Ratio	–	3.22	2.43	1.95	–

TABLE 2. Convergence rate for an initial data given by $u_0(x) = 10(3 + |x|)^{-2}$.

The convergence rate at time $t = 0$ is not linear, as in the previous case. This is due to the fact that we only took peakons on the interval $[-30, 30]$. We have considered the error in $H^1([-30, 30])$, and in that case the convergence is linear. As in the case of smooth traveling waves, the error does not grow in time showing the robustness of the algorithm.

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