Wave decomposition in Riemann problems for heterogeneous media

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1 Wave decomposition for balance laws with spatially-varying flux functions

A lot of problems arising from continuum mechanics can be described by the non-autonomous system of conservation laws in the form

$$\begin{aligned} \mathbf{q}_t + [\mathbf{g}(\mathbf{q}, \mathbf{b}(x))]_x &= \mathbf{0},\ x \in \mathbb{R},\ t > 0, \\ \mathbf{q}(x,0) &= \mathbf{q}_0(x),\ x \in \mathbb{R}, \end{aligned} \tag{1}$$

where $\mathbf{g} = \mathbf{g}(\mathbf{q}, \mathbf{b}):\ \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$ is a sufficiently smooth flux function, $\mathbf{b} = \mathbf{b}(x):\ \mathbb{R} \to \mathbb{R}^m$ and $\mathbf{q}_0 = \mathbf{q}_0(x) : \mathbb{R} \to \mathbb{R}^m$. This system can be rewritten to

$$\begin{aligned} \mathbf{u}_t + [\mathbf{f}(\mathbf{u})]_x &= \mathbf{0},\ x \in \mathbb{R},\ t > 0, \\ \mathbf{u} = [\mathbf{q}^T, \mathbf{w}^T]^T,\ \mathbf{f} &= [\mathbf{g}^T(\mathbf{q}, \mathbf{w}), \mathbf{0}]^T, \\ [\mathbf{q}^T(x,0), \mathbf{w}^T(x,0)]^T &= [\mathbf{q}_0^T(x), \mathbf{b}^T(x)]^T,\ x \in \mathbb{R}. \end{aligned} \tag{2}$$

There are two basic approaches towards solving this augmented system. The first one is based on componentwise methods (for example, the semi-discrete central scheme where the degenerate conservation law $\mathbf{w}_t = 0$ is solved exactly). The second approach is based on upwind schemes. If we use a finite volume method based on Roe's linearization $\mathbf{f}(\mathbf{U}_{j+1}^n) - \mathbf{f}(\mathbf{U}_j^n) = \bar{\mathbf{A}}_{j+1/2}^n(\mathbf{U}_{j+1}^n - \mathbf{U}_j^n)$ where the approximation of $\mathbf{u}_j^n = \mathbf{u}(x_j, t_n)$ is denoted by $\mathbf{U}_j^n$, $\mathbf{w}_j = \mathbf{w}(x_j)$ and

$$\mathbf{f}(\mathbf{U}_{j+1}^n) - \mathbf{f}(\mathbf{U}_j^n) = \begin{bmatrix} \mathbf{g}(\mathbf{Q}_{j+1}^n, \mathbf{w}_{j+1}) - \mathbf{g}(\mathbf{Q}_j^n, \mathbf{w}_j) \\ \mathbf{0} \end{bmatrix}$$

we obtain $\bar{\mathbf{A}}_{j+1/2}^n$ (the approximation of the Jacobi matrix) in the following form

$$\bar{\mathbf{A}}_{j+1/2}^n = \begin{bmatrix} (\bar{\mathbf{A}}_{11})_{j+1/2}^n & (\bar{\mathbf{A}}_{12})_{j+1/2}^n \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{3}$$

where

$$[\bar{\mathbf{A}}_{11}(x_j)]_{j+1/2}^n(\mathbf{Q}_{j+1}^n - \mathbf{Q}_j^n) = \mathbf{g}(\mathbf{Q}_{j+1}^n, \mathbf{w}_j) - \mathbf{g}(\mathbf{Q}_j^n, \mathbf{w}_j),$$

$$(\bar{\mathbf{A}}_{11})_{j+1/2}^n = \frac{1}{2}[\bar{\mathbf{A}}_{11}(x_j)]_{j+1/2}^n + \frac{1}{2}[\bar{\mathbf{A}}_{11}(x_{j+1})]_{j+1/2}^n,$$

$$[\bar{\mathbf{A}}_{12}(\mathbf{Q}_j^n)]_{j+1/2}^n(\mathbf{w}_{j+1} - \mathbf{w}_j) = \mathbf{g}(\mathbf{Q}_j^n, \mathbf{w}_{j+1}) - \mathbf{g}(\mathbf{Q}_j^n, \mathbf{w}_j),$$

$$(\bar{\mathbf{A}}_{12})_{j+1/2}^n = \frac{1}{2}[\bar{\mathbf{A}}_{12}(\mathbf{Q}_j^n)]_{j+1/2}^n + \frac{1}{2}[\bar{\mathbf{A}}_{12}(\mathbf{Q}_{j+1}^n)]_{j+1/2}^n,$$

The decomposition (used in the upwind schemes) has the form

$$\sum_{p=1}^{m+k} |\bar{\lambda}^{n,p}_{j+1/2}|\, \bar{\Upsilon}^{n,p}_{j+1/2} \bar{\mathbf{r}}^{n,p}_{j+1/2} = \begin{bmatrix} \sum_{p=1}^{m} |\bar{\lambda}^{n,p}_{j+1/2}|\, \bar{\Upsilon}^{n,p}_{j+1/2} (\bar{\mathbf{r}}_{11})^{n,p}_{j+1/2} \\ \mathbf{0} \end{bmatrix} \tag{4}$$

where

$$\bar{\mathbf{\Upsilon}}^n_{j+1/2} = (\bar{\mathbf{R}}^n_{j+1/2})^{-1} (\mathbf{U}^n_{j+1} - \mathbf{U}^n_j), \tag{5}$$

$\bar{\mathbf{r}}^{n,p}_{j+1/2}$ are the eigenvectors and $\bar{\lambda}^{n,p}_{j+1/2}$ are the eigenvalues of the matrix $\bar{\mathbf{A}}^n_{j+1/2}$ (the eigenvectors are columns of $\bar{\mathbf{R}}^n_{j+1/2}$) and $(\bar{\mathbf{r}}_{11})^{n,p}_{j+1/2}$ are the eigenvectors of the matrix $[\bar{\mathbf{A}}_{11}]^n_{j+1/2}$.

2 Waves in heterogeneous media – flux-based wave decomposition

We consider the hyperbolic system of conservation laws in the form

$$\mathbf{q}_t + [\mathbf{g}(\mathbf{q}, \mathbf{B}(x))]_x = \mathbf{0} \tag{6}$$

where $\mathbf{q} = [-p/K(x), \varrho(x) v]^T$

$$\mathbf{g}(\mathbf{q}, \mathbf{B}(x)) = \mathbf{B}(x)\mathbf{q} = \begin{bmatrix} 0 & \frac{-1}{\varrho(x_j)} \\ -K(x_j) & 0 \end{bmatrix} \mathbf{q},$$

$p = p(x,t)$ is the pressure, $v = v(x,t)$ is the velocity, $\varrho = \varrho(x)$ is the density and $K = K(x)$ is the bulk modulus of compressibility. This system is equivalent to

$$\mathbf{h}_t + \mathbf{B}(x)\mathbf{h}_x = \mathbf{0} \tag{7}$$

where $\mathbf{h} = \mathbf{B}(x)\mathbf{q}$ (but the augmented formulation (2) based on conservative equation (6) is more convenient to define a generalized solution for the system with discontinuous coefficients). In ([1]) it is proposed approach which is suitable for the non-autonomous case $\mathbf{g} = \mathbf{g}(\mathbf{q}, x)$. There is used the decomposition (flux-based wave decomposition)

$$\mathbf{g}(\mathbf{Q}^n_{j+1}, x_{j+1}) - \mathbf{g}(\mathbf{Q}^n_j, x_j) = \sum_{p=1}^{m} \bar{\Xi}^{n,p}_{j+1/2} \bar{\mathbf{r}}^{n,p}_{j+1/2} = \sum_{p=1}^{m} \bar{\mathbf{z}}^{n,p}_{j+1/2} \tag{8}$$

where $\bar{\mathbf{\Xi}}^n_{j+1/2} = (\bar{\mathbf{R}}^n_{j+1/2})^{-1} [\mathbf{g}(\mathbf{Q}^n_{j+1}, x_{j+1}) - \mathbf{g}(\mathbf{Q}^n_j, x_j)]$. Herein, the matrix $\bar{\mathbf{R}}^n_{j+1/2}$ is composed of the eigenvectors of the matrix $\bar{\mathbf{B}}(x_j, x_{j+1}) \approx \mathbf{B}(x_{j+1/2})$ (i. e., these formulas are partially based on the decomposition of $\mathbf{B}(x)\mathbf{h}_x$). The wave-propagation algorithm is in the form

$$\mathbf{Q}^{n+1}_j = \mathbf{Q}^n_j - \frac{\tau}{h} [\mathcal{B}^+(\Delta \mathbf{Q}^n_{j-1/2}) + \mathcal{B}^-(\Delta \mathbf{Q}^n_{j+1/2})] \tag{9}$$

where

$$\mathcal{B}^+(\Delta \mathbf{Q}^n_{j-1/2}) = \sum_{\bar{\lambda}^{n,p}_{j-1/2} > 0} \bar{\mathbf{z}}^{n,p}_{j-1/2},$$

$$\mathcal{B}^-(\Delta \mathbf{Q}^n_{j+1/2}) = \sum_{\bar{\lambda}^{n,p}_{j+1/2} < 0} \bar{\mathbf{z}}^{n,p}_{j+1/2}.$$

3 Waves in heterogeneous media – wave decomposition based on augmented formulation

First, we consider the simple initial value problem

$$\begin{aligned} q_t + [b(x)q]_x &= 0, \quad x \in \mathbb{R}, t > 0, \quad b(x) > 0 \ \forall x \in \mathbb{R}, \\ q(x,0) &= q_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

We rewrite this non-autonomous problem to the autonomous one (we add an appropriate degenerate conservation law) and obtain

$$\begin{aligned} \mathbf{u}_t + [\mathbf{f}(\mathbf{u})]_x &= \mathbf{0}, \ x \in \mathbb{R}, \ t > 0, \ \mathbf{u} = [q, w]^T, \ \mathbf{f} = [wq, 0]^T, \\ [q(x,0), w(x,0)]^T &= [q_0(x), b(x)]^T, \ x \in \mathbb{R} \end{aligned} \tag{10}$$

Then we apply the approach used in Sec. 1. Roe's linearization has the form

$$\mathbf{f}(\mathbf{U}^n_{j+1}) - \mathbf{f}(\mathbf{U}^n_j) = \bar{\mathbf{A}}^n_{j+1/2}(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j), \ \bar{\mathbf{A}}^n_{j+1/2} = \begin{bmatrix} \frac{1}{2}(W^n_j + W^n_{j+1}) & \frac{1}{2}(Q^n_j + Q^n_{j+1}) \\ 0 & 0 \end{bmatrix} \tag{11}$$

We can formulate the first-order upwind scheme in the form

$$\begin{aligned} Q^{n+1}_j &= Q^n_j - \tfrac{\tau}{h}[G^n_{j+1/2} - G^n_{j-1/2}], \\ G^n_{j+1/2} &= \tfrac{1}{2}(W^n_j Q^n_j + W^n_{j+1} Q^n_{j+1}) - \tfrac{1}{2}\bar{\beta}^n_{j+1/2}(Q^n_{j+1} - Q^n_j), \\ \bar{\beta}^n_{j+1/2} &= \operatorname{sign}(W^n_j + W^n_{j+1})\bar{b}^n_{j+1/2}, \\ \bar{b}^n_{j+1/2} &= \tfrac{W^n_{j+1}Q^n_{j+1} - W^n_j Q^n_j}{Q^n_{j+1} - Q^n_j}, \ \bar{b}^n_{j+1/2} = 0 \text{ if } Q^n_{j+1} = Q^n_j. \end{aligned} \tag{12}$$

If we choose $\bar{\mathbf{B}}(x_j, x_{j+1}) = \frac{1}{2}(b(x_j) + b(x_{j+1}))$ we obtain the scheme which is equivalent to the method described in Sec. 2.

Furthermore, we will study more general initial value problem (the problem (6))in the form

$$\begin{aligned} \mathbf{q}_t + [\mathbf{B}(x)\mathbf{q}]_x &= 0, \ x \in \mathbb{R}, \ t > 0, \\ \mathbf{q}(x,0) &= \mathbf{q}_0(x), \ x \in \mathbb{R}, \end{aligned} \tag{13}$$

rewritten to the initial value problem

$$\begin{aligned} \mathbf{u}_t + [\mathbf{f}(\mathbf{u})]_x &= \mathbf{0}, \ x \in \mathbb{R}, \ t > 0, \\ \mathbf{u} = [\mathbf{q}^T, \mathbf{w}_1, \dots, \mathbf{w}_m]^T, \ \mathbf{f} &= [\mathbf{q}^T \mathbf{B}^T(x), \mathbf{0}]^T, \\ [\mathbf{q}(x,0), \mathbf{W}(x,0)] &= [\mathbf{q}_0(x), \mathbf{B}(x)], \ x \in \mathbb{R}. \end{aligned} \tag{14}$$

where $\mathbf{w}_i$ are the rows of the matrix $\mathbf{W}(x,t)$. We obtain the approximation of the Jacobi matrix (3) where $[\bar{\mathbf{A}}_{11}(x_j)]^n_{j+1/2} = \mathbf{B}(x_j)$, $(\bar{\mathbf{A}}_{11})^n_{j+1/2} = \frac{1}{2}[\bar{\mathbf{A}}_{11}(x_j)]^n_{j+1/2} + \frac{1}{2}[\bar{\mathbf{A}}_{11}(x_{j+1})]^n_{j+1/2}$,

$$\begin{aligned} (\bar{\mathbf{A}}_{12})^n_{j+1/2} &= \frac{1}{2}\begin{bmatrix} \left(\mathbf{Q}^n_j\right)^T & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \dots & \dots & \mathbf{0} & \left(\mathbf{Q}^n_j\right)^T \end{bmatrix} + \\ &+ \frac{1}{2}\begin{bmatrix} \left(\mathbf{Q}^n_{j+1}\right)^T & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \dots & \dots & \mathbf{0} & \left(\mathbf{Q}^n_{j+1}\right)^T \end{bmatrix} \end{aligned}$$

is $m \times m^2$ matrix and $\mathbf{Q}_j^n$ is the approximation of $\mathbf{q}_j^n = \mathbf{q}(x_j, t_n)$.

We formulate the scheme in the wave-propagation form (see [2]) based on the decomposition (4) and (5)

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \tfrac{\tau}{h}[\mathcal{A}^+(\Delta\mathbf{Q}_{j-1/2}^n) + \mathcal{A}^-(\Delta\mathbf{Q}_{j+1/2}^n)],$$

$$\mathcal{A}^+(\Delta\mathbf{Q}_{j-1/2}^n) = \sum_{\substack{p=1 \\ \bar{\lambda}_{j-1/2}^{n,p} > 0}}^{m} \varXi_{j-1/2}^{n,p} (\bar{\mathbf{r}}_{11})_{j-1/2}^{n,p}, \tag{15}$$

$$\mathcal{A}^-(\Delta\mathbf{Q}_{j+1/2}^n) = \sum_{\substack{p=1 \\ \bar{\lambda}_{j+1/2}^{n,p} < 0}}^{m} \varXi_{j+1/2}^{n,p} (\mathbf{r}_{11})_{j+1/2}^{n,p}.$$

where $\varXi_{j+1/2}^{n,p} = \varUpsilon_{j+1/2}^{n,p} \bar{\lambda}_{j+1/2}^{n,p}$. Because all the wave families in this case are linearly degenerate therefore any solution consists of contact discontinuities (i.e., we don't have use any entropy fix procedure). We obtain the algorithm that has a very similar structure to the scheme described in Sec. 2.

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