Global solutions of the Hunter-Saxton equation

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Abstract. *We construct a continuous semigroup of weak, dissipative solutions to a nonlinear partial differential equations modeling nematic liquid crystals. A new distance functional, determined by a problem of optimal transportation, yields sharp estimates on the continuity of solutions with respect to the initial data.*

1 - Introduction

In this paper we investigate the Cauchy problem

$$u_t + \left(\frac{u^2}{2}\right)_x = \frac{1}{4}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) u_x^2\, dx\,, \qquad\qquad u(0,x) = \bar u(x)\,. \tag{1.1}$$

Formally differentiating the above equation with respect to the spatial variable x, we obtain

$$(u_t + uu_x)_x = \frac{1}{2}\, u_x^2, \tag{1.2}$$

whereas yet another differentiation leads to

$$u_{txx} + 2u_x u_{xx} + uu_{xxx} = 0. \tag{1.3}$$

Either of the forms (1.2) and (1.3) of the equation in (1.1) is known as the Hunter-Saxton equation. In this paper we analyse various concepts of solutions for the above equations, and construct a semigroup of globally defined solutions. Moreover, we introduce a new distance functional, related to a problem of optimal transportation, which monitors the continuous dependence of solutions on the initial data.

Physical significance

The Hunter-Saxton equation describes the propagation of waves in a massive director field of a nematic liquid cristal [HS], with the orientation of the molecules described by the field of unit vectors $\mathbf{n}(t,x) = \big(\cos u(t,x), \sin u(t,x)\big)$, x being the space variable in a reference frame moving with the linearized wave velocity, and t being a slow time variable. The liquid crystal state is a distinct phase of matter observed between the solid and liquid states. More specifically, liquids are isotropic (that is, with no directional order) and without a positional order of their molecules, whereas the molecules in solids are constrained to point only in certain directions and to be only in certain positions with respect to each other. The liquid crystal phase exists between the solid and the liquid phase - the molecules in a liquid crystal do not exhibit any positional order, but they do possess a certain degree of orientational order. Not all substances can have a liquid crystal phase e.g. water molecules melt directly from solid crystalline ice to liquid water. Liquid crystals are fluids made up of long rigid molecules, with an average orientation that specifies the local direction of the medium. Their orientation is described macroscopically by a field of unit vectors $\mathbf{n}(t,\mathbf{x})$). There are many types of liquid crystals, depending upon the amount of order in the material. Nematic liquid crystals are invariant under the transformation $\mathbf{n} \mapsto -\mathbf{n}$, in which case $\mathbf{n}$ is called a director field, so that the rodlike molecules have no positional order but tend to point in the same direction (along the director). The director field does not remain the same but generally fluctuates. Obtaining the equation governing the director field represents the crucial point for the modeling of nematic liquid crystals since it is advantageous to study the dynamics of director field instead of studying the dynamics of all the molecules. The fluctuations of the director field are mainly due to the thermodynamical force caused by elastic deformations in the form of twisting, bending, and splay (the latter being a fan-shaped spreading out from the original direction, bending being a change of direction, while twisting corresponds to a rotation of the direction in planes orthogonal to the axis of rotation). Consider director fields that lie on a circle and depend on a single spatial variable x so that twisting is not allowed. To describe the dynamics of the director field independently of the coupling with the fluid flow, let $u(t,x)$ be the perturbation about a constant value. The asymptotic equation for weakly nonlinear unidirectional waves is precisely equation (1.2), obtained as the Euler-Lagrange equation of the variational principle

$$\delta \int_{t_1}^{t_2} \int_{\mathbb{R}} (u_t u_x + u u_x^2)\, dx dt = 0,$$

for the internal stored energy of deformation of the director field if dissipative effects are neglected (corresponding to the case when inertia effects dominate viscosity) - see [HS] for the details of the derivation. Unlike other studies, in the Hunter-Saxton model the kinetic energy of the director field is not neglected. In the asymptotic regime in which (1.2) is derived, the nondimensionalized kinetic energy density is u_x^2 so that the condition

$$\int_{\mathbb{R}} u_x^2(t,x)\, dx < \infty \tag{1.4}$$

has to hold at any fixed time t for a physically meaningful solution to the Hunter-Saxton equation.

Equation (1.1) is also relevant in other physical situations, e.g. it is a high-frequency limit of the Camassa-Holm equation [DP], a nonlinear shallow water equation [CH, J] modeling solitons [CH] as well as breaking waves [CE].

Geometric interpretation

An interesting aspect of the Hunter-Saxton equation (see [KM]) is the fact that, for spatially periodic functions, it describes geodesic flow on the homogeneous space $\mathrm{Diff}(\mathcal{S})/\mathrm{Rot}(\mathcal{S})$ of the infinite-dimensional Lie group $\mathrm{Diff}(\mathcal{S})$ of smooth orientation-preserving diffeomorphisms of the unit circle $\mathcal{S}$ modulo the rotations $\mathrm{Rot}(\mathcal{S})$, with respect to the right-invariant homogeneous metric $\langle f,\, g\rangle = \int_{\mathcal{S}} f_x g_x \, dx$. The geometric interpretation of the Hunter-Saxton equation establishes a natural connection with the Camassa-Holm equation, which describes geodesic flow on $\mathrm{Diff}(\mathcal{S})$ with respect to the right-invariant metric $\langle f,\, g\rangle = \int_{\mathcal{S}} (fg + f_x g_x) \, dx$, see [K, CK]. A similar geometric interpretation of (1.1) on the diffeomorphism group of the line holds also for smooth initial data $\bar{u}$ in certain weighted function spaces but the involved technicalities are more intricated (see [C] for the case of the Camassa-Holm equation).

Integrable structure

The Hunter-Saxton equation has a an integrable structure. The equation has a reduction (see [BSS, HZ1]) to a finite dimensional completely integrable Hamiltonian system whose phase space consists of piecewise linear solutions of the form

$$u(t,x) = \sum_{i=1}^{n} \alpha_i(t)\, |x - x_i(t)|, \tag{1.5}$$

with the constraint

$$\sum_{i=1}^{n} \alpha_i(t) = 0, \tag{1.6}$$

the Hamiltonian being

$$H(x,\alpha) = \frac{1}{2} \sum_{i,j=1}^{n} \alpha_i \alpha_j\, |x_i - x_j|.$$

Due to their lack of regularity, functions of the form (1.5) are not classical solutions of (1.2). Below we will discuss in what sense they are weak solutions of the Hunter-Saxton equation. Let us point out that the constraint (1.6) is the necessary and sufficient condition to ensure that the distributional derivative $x \mapsto u_x(t,x)$ of a function of the form (1.5) belongs to the space $L^2(I\!R)$. Thus (1.4) holds.

In the family of smooth functions $u : I\!R \mapsto I\!R$ all of whose derivatives $\partial_x^n u$ decay rapidly as $x \to \pm\infty$, the Hunter-Saxton equation is bi-Hamiltonian [HZ1]. If D^{-1} is the skew-adjoint anti-derivative operator given by

$$(D^{-1}f)(x) = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty} \right) f(x)\, dx\,, \qquad f \in \mathcal{D}(I\!R),$$

the first Hamiltonian form for the Hunter-Saxton equation is

$$u_t = J_1 \frac{\delta \mathcal{H}_1}{\delta u}, \qquad J_1 = u_x D^{-2} - D^{-2} u_x, \qquad \mathcal{H}_1 = \frac{1}{2} \int_{I\!R} u_x^2 \, dx,$$

whereas the second, compatible, Hamiltonian structure is

$$u_t = J_2 \frac{\delta \mathcal{H}_2}{\delta u}, \qquad J_2 = D^{-1}, \qquad \mathcal{H}_2 = \frac{1}{2} \int_{I\!R} u u_x^2 \, dx,$$

Moreover, the Hunter-Saxton equation is formally integrable e.g. it has an associated Lax pair (see [BSS]). However, the complete integrability of the equation has been established only in the previously mentioned case when it reduces to a finite dimensional dynamical system.

The notion of solution

Physically relevant solutions of the Hunter-Saxton equation need to be of finite kinetic energy so that (1.4) must hold. This leads naturally to functions $u(t,x)$ with distributional derivative $u_x(t,\cdot)$ square integrable at every instant t. Note that the integrability assumption $u_x(t,\cdot) \in L^2(I\!R)$ already imposes a certain degree of regularity on the function u. This suggests that it might be possible to incorporate a reasonably high degree of regularity in the concept of weak solution to the Hunter-Saxton equation. Let us first consider the concept of weak solutions introduced by Hunter and Zheng [HZ2].

Definition 1.1 *A function $u(t,x)$ defined on $[0,T] \times I\!R$ is a solution of the equation (1.2) if*

(i) $u \in C([0,T] \times I\!R; I\!R)$ and $u(0,x) = \bar{u}(x)$ pointwise on $I\!R$;

(ii) For each $t \in [0,T]$, the map $x \mapsto u(t,x)$ is absolutely continuous with $u_x(t,\cdot) \in L^2(I\!R)$. Moreover, the map $t \mapsto u_x(t,\cdot)$ belongs to the space $L^\infty([0,T]; L^2(I\!R))$ and is locally Lipschitz continuous on $[0,T]$ with values in $H^{-1}_{loc}(I\!R)$;

(iii) Equation (1.2) holds in the sense of distributions.

Here and below, by a mapping f that is locally Lipschitz or locally bounded on $[0,T]$ with values in $H^{-1}_{loc}(I\!R)$ we understand the following: for every $n \geq 1$ there is a constant $K_n \geq 0$ such that

$$\sup_{\{\psi \in \mathcal{D}(-n,n):\ \|\psi\|_{H^1(I\!R)} \leq 1\}} \Big| \langle f(t) - f(s), \psi \rangle \Big| \leq K_n\, |t-s|, \qquad t,s \in [0,T],$$

respectively

$$\sup_{\{\psi \in \mathcal{D}(-n,n):\ \|\psi\|_{H^1(I\!R)} \leq 1\}} \Big| \langle f(t), \psi \rangle \Big| \leq K_n, \qquad t \in [0,T].$$

Here $\mathcal{D}(a,b)$ is the family of smooth functions $f : I\!R \to I\!R$ with compact support within $(a,b) \subset I\!R$.

To a function $u : [0,T] \times I\!R \to I\!R$ with the above properties associate the function $F : [0,T] \times I\!R \to I\!R$ defined by

$$F(t,x) = \frac{1}{4} \left(\int_{-\infty}^{x} - \int_{x}^{\infty} \right) u_x^2 \, dx. \tag{1.7}$$

Then $F \in L^\infty_{loc}([0,T] \times I\!R; I\!R) \subset L^2_{loc}([0,T] \times I\!R; I\!R)$. Moreover, $F_x = \frac{1}{2}\, u_x^2$ so that equation (1.2) becomes

$$(u_t + u u_x - F)_x = 0 \tag{1.8}$$

in the sense of distributions. Note that $uu_x \in L^2_{loc}([0,T] \times I\!R; I\!R)$. From (1.8) we infer the existence of a distribution $h(t)$ so that $u_t + uu_x - F = h(t) \otimes 1(x)$, where $1(x)$ stands for the constant function with value 1 on $I\!R$. If $H(t)$ is a primitive of the distribution $h(t)$, we deduce that the distribution $U = u - H(t) \otimes 1(x)$ satisfies $U_t = u_t - h(t) \otimes 1(x) = F - uu_x \in L^2_{loc}([0,T] \times I\!R; I\!R)$ and $U_x = u_x \in L^2_{loc}([0,T] \times I\!R; I\!R)$. Therefore $U \in H^1_{loc}([0,T] \times I\!R)$. Moreover, since $U_t = F - uu_x \in L^\infty_{loc}([0,T]; H^{-1}_{loc}(I\!R))$ ensures that U is locally Lipschitz as a function from $[0,T]$ to $H^{-1}_{loc}(I\!R)$ and so is also u, we deduce that $h(t) \otimes 1(x) = u - U$ shares this property too. But then $h : [0,T] \to I\!R$ has to be Lipschitz continuous. We infer that $u = U + H(t) \otimes 1(x)$ belongs to the space $H^1_{loc}([0,T] \times I\!R)$. Since the requirement (iii) in Definition 1.1 ensures that the identity

$$\int_0^T \int_{I\!R} \Big(\phi_{xt} u + \frac{1}{2}\, \phi_{xx} u^2 - \frac{1}{2}\, \phi u_x^2 \Big)\, dxdt = 0$$

holds for every smooth function $\phi : (0,T) \times I\!R \to I\!R$ with compact support in $(0,T) \times I\!R$, we see that the notion of weak solution in the sense of Definition 1.1 is stronger than the concept of weak solution introduced by Hunter and Saxton [HS]. Another useful conclusion that can be drawn from the previous considerations is that for a function u with regularity properties specified in (i)-(ii) of Definition 1.1, the requirement (iii) from Definition 1.1 is equivalent to asking that the equation

$$u_t + uu_x = F + h(t) \circ 1(x) \tag{1.9}$$

holds in distribution sense for some Lipschitz continuous function $h : [0,T] \to I\!R$. Any such function h is admissible. Among all these possibilities the most natural one corresponds to the special choice $h \equiv 0$. This leads us to the form (1.1) of the Hunter-Saxton equation.

In the following, we say that a map $t \mapsto u(t,\cdot)$ from $[0,T]$ into $\mathbf{L}^p_{loc}(I\!R)$ is *absolutely continuous* if, for every bounded interval $[a,b]$, the restriction of u to $[a,b]$ is absolutely continuous as a map with values in $\mathbf{L}^p\big([a,b]\big)$. We can thus adopt the following notion of a weak solution.

Definition 1.2 *A function $u(t,x)$ defined on $[0,T] \times I\!R$ is a solution of the equation (1.2) if*
(i) $u \in C([0,T] \times I\!R; I\!R)$ and $u(0,x) = \bar{u}(x)$ pointwise on $I\!R$;
(ii) For each $t \in [0,T]$, the map $x \mapsto u(t,x)$ is absolutely continuous with $u_x(t,\cdot) \in L^2(I\!R)$. Moreover, the map $t \mapsto u_x(t,\cdot)$ belongs to the space $L^\infty([0,T]; L^2(I\!R))$;
(iii) The map $t \mapsto u(t,\cdot) \in L^2_{loc}(I\!R)$ is absolutely continuous and satisfies the equation (1.1) for a.e. $t \in [0,T]$.

The concept of solution introduced in Definition 1.2 is stronger than that corresponding to Definition 1.1. Indeed, for a function u satisfying all the requirements of Definition 1.2 we infer by (1.1) that $u_{tx} \in L^\infty_{loc}([0,T]; H^{-1}_{loc}(I\!R))$ since $u_t = -uu_x + F$ and $uu_x,\ F \in L^\infty_{loc}([0,T]; L^2_{loc}(I\!R))$. This yields that the map $t \mapsto u_x(t,\cdot)$ is locally Lipschitz continuous on $[0,T]$ with values in $H^{-1}_{loc}(I\!R)$. We thus recover the apparently missing part from the requirement (ii) in Definition 1.1.

We remark that, even with this stronger definition, solution are far from unique. For example, consider the initial data

$$\bar{u}(x) = 0. \tag{1.10}$$

There are now two ways to prolong the solution for times $t > 0$. On one hand, we can define

$$u(t,x) = 0 \qquad\qquad x \in I\!R\,, \quad t \geq 0\,. \tag{1.11}$$

On the other hand, the function

$$u(t,x) \doteq \begin{cases} -2t & \text{if} \quad x \leq -t^2 \\[2ex] \frac{2x}{t} & \text{if} \quad |x| < t^2 \\[2ex] 2t & \text{if} \quad x \geq t^2 \end{cases} \qquad \text{for} \quad t \geq 0 \tag{1.12}$$

provides yet another solution. To distinguish between these two solutions, we need to consider the evolution equation satisfied by the "energy density" u_x^2, namely

$$(u_x^2)_t + (uu_x^2)_x = 0\,. \tag{1.13}$$

For smooth solution, the conservation law (1.13) is satisfied pointwise. Notice that the solution defined by (1.10), (1.12) satisfies the additional conservation law (1.13) in distributional sense, i.e.

$$\int\!\!\int_{I\!R_+ \times I\!R} \left\{ u_x^2 \varphi_t + uu_x^2\, \varphi_x \right\} dxdt = 0 \tag{1.14}$$

for every test function $\varphi \in \mathcal{C}_c^1(I\!R_+ \times I\!R)$ whose compact support is contained in the half plane where $t > 0$. On the contrary, the solution defined by (1.10)-(1.11) dissipates energy. More precisely, for every $t_2 \geq t_1 \geq 0$ we have

$$\int_{I\!R} u_x^2(t_2, x)\, \varphi(t_2, x)\, dx - \int_{I\!R} u_x^2(t_2, x)\, \varphi(t_2, x)\, dx \leq \int_{t_1}^{t_2} \int_{I\!R} \left\{ u_x^2\, \varphi_t + u u_x^2 \varphi_x \right\} dx dt\,, \tag{1.15}$$

for every test function $\varphi \in \mathcal{C}_c^1(I\!R_+ \times I\!R)$. In the sequel, we say that a solution is *dissipative* if the inequality (1.15) holds for every $t_2 > t_1 > 0$, $\varphi \in \mathcal{C}_c^1(I\!R_+ \times I\!R)$. Notice that the solution (1.10), (1.12) does not satisfy (1.15) when $t_1 = 2$, $t_2 > 0$.

At this point in the discussion it is worthwhile to point out that the most important feature in the definition of weak solutions is the requirement (1.4). A continuous function $u : [0, T] \times I\!R \to I\!R$ with square integrable distributional derivative $u_x(t, \cdot)$ belonging to the space $L^\infty([0, T]; L^2(I\!R))$ is not necessarily bounded, nor does it have a pre-determined asymptotic behaviour at infinity, as one can see from the example

$$u(t, x) = \begin{cases} |x|^{\frac{1}{5}} \sin(|x|^{\frac{1}{5}}) & \text{if} \quad t \geq 0,\ |x| \geq 1\,, \\ |x|^{\frac{2}{3}} \sin(|x|^{\frac{2}{3}}) & \text{if} \quad t \geq 0,\ |x| \leq 1\,. \end{cases}$$

Nevertheless, the possibility that some additional structural information about the behaviour of such functions at infinity might be inferred from some invariance properties of the Hunter-Saxton equation should be ruled out. To do this, consider solutions of the type (1.5) with the constraint (1.6). This type of solutions enter into the framework of Definition 1.2 and for any $N(t) > \max\{|x_1(t)|, ..., |x_n(t)|\}$ we have

$$u_t(t, x) = F(x) \qquad \text{a.e. on} \qquad |x| \geq N(t),$$

so that for all $j \geq N(t)$,

$$u_t(t, j) - u_t(-j) = F(j) - F(-j) = \frac{1}{2} \int_{-j}^{j} u_x^2(t, x)\, dx = \frac{1}{2} \int_{I\!R} u_x^2(t, x)\, dx. \tag{1.16}$$

But the quantity $I = \dfrac{1}{2} \displaystyle\int_{I\!R} u_x^2(t, x)\, dx$ is an invariant (time-independent) cf. [HZ1, BSS]. Moreover, the special form of the solutions guarantees that at every fixed $t \geq 0$,

$$u_\infty(t) \doteq \lim_{x \to \infty} u(t, x) = \sum_{i=1}^{n} \alpha_i(t)\, x_i(t) = - \lim_{x \to -\infty} u(t, x).$$

and $u_\infty(t) = u(t, j) = -u(t, -j)$ for all $j \geq N(t)$. Thus (1.16) yields

$$u_\infty(t) = u_\infty(0) + \frac{1}{4} I t, \qquad t \geq 0. \tag{1.17}$$

Unless $I = 0$, in which case u is constant, we see from (1.17) that the asymptotic behaviour of the solutions changes with time. On the basis of this set of examples we conclude that the asymptotic behaviour of the solutions at infinity should not be prescribed *a priori*. However, the previous set of examples indicates that a possible restriction would be to require $u \in L^\infty([0, T] \times I\!R)$ if $\bar{u} \in L^\infty(I\!R)$. In this case the space of functions introduced in Definition 1.2 (that is, bounded functions with all the properties specified in Definition 1.2 except the condition that u satisfies equation (1.1) in $L^2[-n,n]$ for every $n \geq 1$) is a Banach space when endowed with the norm

$$\|u\|_T = \sup_{(t,x)\in[0,T]\times\mathbb{R}} \{|u(t,x)|\} + \operatorname{ess-sup}_{t\in[0,T]} \int_{\mathbb{R}} u_x^2(t,x)\,dx. \tag{1.18}$$

It is also worth noticing that a function entering the framework of Definition 1.2 has further reqularity properties that are not explicitely stated. For example, we have the Hölder continuity property

$$|u(t,x) - u(t,y)| \leq K(t)\,\sqrt{|x-y|}, \qquad t \geq 0,\ x,y \in \mathbb{R},$$

with $K(t) = \|u_x(t,\cdot)\|_{L^2(\mathbb{R})}$, since

$$|u(t,x)-u(t,y)|^2 = \left|\int_x^y u_x(t,\zeta)\,d\zeta\right|^2 \leq |x-y|\cdot\left|\int_x^y u_x^2(t,\zeta)\,d\zeta\right| \leq |x-y|\int_{\mathbb{R}} u_x^2(t,\zeta)\,d\zeta.$$

2 - Global existence of dissipative solutions

For twice continuously differentiable solutions, the derivative $v \doteq u_x$ of the solution u to (1.1) satisfies the equations

$$v_t + uv_x \;=\; -\frac{v^2}{2}\,, \tag{2.1}$$

$$(v^2)_t + (uv^2)_x = 0\,. \tag{2.2}$$

Define the characteristic $t \mapsto \xi(t,y)$ as the solution to the O.D.E.

$$\frac{\partial}{\partial t}\,\xi(t,y) = u\big(t,\,\xi(t,y)\big)\,, \qquad\qquad \xi(0,y) = y\,. \tag{2.3}$$

From (1.2) it follows that the evolution of the gradient u_x along each characteristic is described by

$$\frac{d}{dt}\,u_x\big(t,\xi(t,y)\big) = -\frac{1}{2}u_x^2\big(t,\xi(t,y)\big)\,. \tag{2.4}$$

Observe that the solution of the O.D.E.

$$\dot z = -z^2/2\,, \qquad\qquad z(0) = z_0$$

is given by

$$z(t) = \frac{2z_0}{2+tz_0} \tag{2.5}$$

If $z_0 \geq 0$ this solution is defined for all $t \geq 0$, whereas if $z_0 < 0$, this solution approaches $-\infty$ at the blow-up time

$$T(z_0) = -2/z_0 \tag{2.6}$$

Note that if $\bar u(x) \not\equiv 0$ then there is some $x_0 \in \mathbb{R}$ with $\bar u(x_0) < 0$ so that the characteristic cruve $t \mapsto \xi(t,\bar u(x_0))$ will blowup in finite time. Nevertheless, if $\liminf_{x\in\mathbb{R}}\{\bar u_x(x)\} > -\infty$, then $T_0 > 0$, where

$$T_0 = \inf_{\{x\in\mathbb{R}:\ \bar u_x(x)<0\}} \left\{\frac{-2}{\bar u_x(x)}\right\} \geq 0, \tag{2.7}$$

and on the time interval $[0,T_0)$ the method of characteristics can be used to construct the unique solution of (1.1). Let us describe the construction in detail. From (2.3) we get

$$\frac{\partial}{\partial t}\,\xi_x = u_x(t,\xi)\;\cdot\;\xi_x = \frac{2\,\bar{u}_x}{2+t\,\bar{u}_x}\;\cdot\;\xi_x \tag{2.8}$$

since

$$u_x(t,\xi(t,y)) = \frac{2\,\bar{u}_x(y)}{2+t\,\bar{u}_x(y)} \tag{2.9}$$

in view of (2.4) and the solution formula (2.5). The unique solution of the linear O.D.E. (2.7) with initial data $\xi_x(0,y)=1$ is given by

$$\xi_x(t,y) = \Big(1+\frac{t}{2}\,\bar{u}_x(y)\Big)^2. \tag{2.10}$$

Since $1+\frac{t}{2}\,\bar{u}_x(y) > 0$ for $t\in[0,T_0)$, relation (2.10) shows that for each $t\in[0,T_0)$ the map $y\mapsto\xi(t,y)$ is an absolutely continuous increasing diffeomorphism of the line. Define the absolutely continuous function φ by

$$\varphi(y) = \frac{1}{4}\int_{I\!R}\mathrm{sign}(y-x)\,\bar{u}_x^2(x)\,dx \tag{2.11}$$

so that $\varphi_x(y)=\frac{1}{2}\,\bar{u}_x^2(y)$. Note that by (2.10),

$$\xi_{tx} = \bar{u}_x + t\,\frac{\bar{u}_x^2}{2}. \tag{2.12}$$

Since $\xi_t(0,y)=0$ as $\xi(0,y)=y$, integration of (2.12) with respect to the spatial variable x yields

$$\xi_t(t,y) = \bar{u}(y) + \frac{t}{4}\int_{I\!R}\mathrm{sign}(y-x)\,\bar{u}_x^2(x)\,dx \tag{2.13}$$

and thus

$$\xi(t,y) = y + \int_0^t \xi_t(s,y)\,ds = y + t\,\bar{u}(y) + \frac{t^2}{4}\int_{I\!R}\mathrm{sign}(y-x)\,\bar{u}_x^2(x)\,dx. \tag{2.14}$$

The value of the solution u along the characteristic curve $t\mapsto\xi(t,y)$ is

$$u(t,\xi(t,y)) = \bar{u}(y) + \frac{t}{2}\int_{I\!R}\mathrm{sign}(y-x)\,\bar{u}_x^2(x)\,dx. \tag{2.15}$$

This relation is obtained by combining (2.14) with (2.3). The increasing diffeomorphism of the line $y\mapsto\xi(t,y)$ given by (2.14) and formula (2.15) yield the unique solution of the Hunter-Saxton equation on the time interval $[0,T_0)$. The above approach works as long as $2+t\,\bar{u}_x(x)>0$ but breaks down at $T=T_0$ with T_0 given by (2.7). The reason for the breakdown is that

$$\liminf_{t\uparrow T_0}\,\{\inf_{x\in I\!R}\,u_x(t,x)\} = -\infty \tag{2.16}$$

in view of (2.9) and the definition (2.7) of T_0. Note that at $t=T_0$ we have might have $\xi_x(t,x)=0$ for all $x\in(a,b)\subset I\!R$ so that the map $y\mapsto\xi(t,y)$ is not any more an increasing diffeomorphism of the line. Nevertheless, the previous considerations suggest the following approach in the general case when $\bar{u}_x\in L^2(I\!R)$, covering situations when possibly $T_0=0$ as it is the case for e.g. $\bar{u}(x)=$

$x^{\frac{2}{3}}(1-x)^{\frac{2}{3}}\,\chi_{[0,1]}$. Here χ_A stands for the characteristic function of the set A, defined by $\chi_A(x)=1$ if $x\in A$ and $\chi_A(x)=0$ if $x\notin A$.

Let $\bar u\in C(I\!R)$ be such that its distributional derivative $\bar u_x$ is square integrable. Define $\varphi:I\!R_+\times I\!R\to I\!R$ by

$$\varphi_y(t,y)=\frac{1}{2}\,\bar u_x^2(y)\,\chi_{[\bar u_x>-2/t]}(y) \tag{2.17}$$

so that

$$\varphi(t,y)=\frac{1}{4}\int_{[\bar u_x>-2/t]}\operatorname{sign}(y-x)\,\bar u_x^2(x)\,dx,\qquad t>0, \tag{2.18}$$

with the understanding that

$$\varphi(0,y)=\frac{1}{4}\int_{I\!R}\operatorname{sign}(y-x)\,\bar u_x^2(x)\,dx.$$

In other words, if $u(t,\xi(t,x_0)$ blows up before $t_0>0$, then the point x_0 is not included in the domain of the integral defining $\varphi(t_0,\cdot)$, because

$$T\big(\bar u_x(x)\big)>t\qquad\text{if and only if}\qquad \bar u_x(x)>-2/t\,,$$

according to (2.4) and (2.6). Observe that (2.18) and Young's inequality yield

$$\|\varphi(t,\cdot)\|_{L^\infty(I\!R)}\le\frac{1}{4}\int_{I\!R}\bar u_x^2(x)\,dx,\qquad t\ge 0. \tag{2.19}$$

In the (t,x)-plane, the characteristic curve starting at y is obtained as

$$\xi(t,y)=y+t\bar u(y)+\int_0^t(t-s)\,\varphi(s,y)\,ds\,. \tag{2.20}$$

The value of the solution u along this curve is

$$u\big(t,\,\xi(t,y)\big)=\bar u(y)+\int_0^t\varphi(s,y)\,ds\,. \tag{2.21}$$

Observe that for all $t\ge 0$ and $y\in I\!R$,

$$\xi_t(t,y)=u(t,\xi(t,y)) \tag{2.22}$$

in view of (2.20)-(2.21).

Theorem 1. *Given any absolutely continuous function $\bar u:I\!R\to I\!R$ with derivative $\bar u_x\in L^2(I\!R)$, the formulas (2.18)-(2.20) provide a dissipative solution to (1.1), defined for all times $t\ge 0$.*

Proof. We proceed in several steps. First of all, for any fixed $t\ge 0$, the map $y\mapsto\xi(t,y)$ is absolutely continuous since $\varphi_y(t,\cdot)\in L^2(I\!R)$. We claim that for any fixed $t\ge 0$ the map $y\mapsto\xi(t,y)$ is nondecreasing on $I\!R$ with $\lim_{y\to\pm\infty}\xi(t,y)=\pm\infty$.

Indeed, if $\bar u_x(y)>-\frac{2}{t}$ then $\bar u_x(y)>-\frac{2}{s}$ for all $s\in[0,t]$ so that $\varphi_y(s,y)=\frac{1}{2}\,\bar u_x^2(y)$ for $s\in[0,t]$ by (2.17). Since

$$\xi_y(t,y)=1+t\bar u_x(y)+\int_0^t(t-s)\,\varphi_y(s,y)\,ds\,. \tag{2.23}$$

we find that in this case

$$\xi_y(t,y) = 1 + t\bar{u}_x(y) + \frac{t^2}{4}\,\bar{u}_x^2(y) = \frac{1}{4}\Big(2 + t\,\bar{u}_x(y)\Big)^2. \tag{2.24}$$

In the remaining cases we have that $\bar{u}_x(y) = -\frac{2}{t_0} \le -\frac{2}{t}$ for some $t_0 \in (0,t]$. Therefore (2.17) yields $\varphi_y(s,y) = \frac{1}{2}\,\bar{u}_x^2(y)$ for $s \in [0,t_0)$, while $\varphi_y(s,y) = 0$ for $s \in (t_0,t]$. From (2.23) we infer that

$$\xi_y(t,y) = 1 - \frac{2t}{t_0} + \frac{2t - t_0}{t_0} = 0. \tag{2.25}$$

The relations (2.24)-(2.25) confirm the monotonicity of the map $y \mapsto \xi(t,y)$. Since $\xi(0,x) = x$, it remains to prove that $\lim_{y\to\pm\infty}\xi(t,y) = \pm\infty$ for any $t > 0$. Fix $t > 0$. Since $\bar{u}_x \in L^2(I\!R)$, the Lebesgue measure $l(t)$ of the set $\{y \in I\!R:\ \bar{u}_x(y) \le -\frac{1}{t}\}$ is finite. On the complement $C(t)$ of this set we obviously have $\bar{u}_x(y) > -\frac{1}{t}$ and thus $\xi_y(t,y) \ge \frac{1}{4}$ by taking into account (2.24). Therefore, given $x_2 > x_1$, we infer that

$$\xi(t,x_2) - \xi(t,x_1) = \int_{x_1}^{x_2} \xi_y(t,y)\,dy \ge \int_{[x_1,x_2]\,\cap\, C(t)} \xi_y(t,y)\,dy \ge \int_{[x_1,x_2]\,\cap\, C(t)} \frac{1}{4}\,dy \ge \frac{x_2 - x_1 - l(t)}{4}.$$

This proves the claim about the limiting behaviour of $\xi(t,\cdot)$ at $\pm\infty$. While for times t up to the blow-up time T_0, given by (2.7), the map $y \mapsto \xi(t,y)$ is an absolutely continuous diffeomorphism of the real line, for $t \ge T_0$ this map is nondecreasing and onto but is not necessarily a bijection. Nevertheless, we would like to define the solution u by the formula (2.21) for all $t \ge 0$.

To show that u is well-defined via (2.21), due to the monotone and surjective character of the map $y \mapsto \xi(t,y)$, it is sufficient to show that if $\xi(t,y_1) = \xi(t,y_2)$ for some $y_2 > y_1$, then the values of u given by (2.21) are also equal. Indeed, we must have that $\xi(t,y) = \xi(t,y_1)$ for all $y \in [y_1,y_2]$ and a glance at (2.24)-(2.25) confirms that $\bar{u}_x(y) \le -\frac{2}{t}$ for $y \in [y_1,y_2]$. This means that for every fixed $y \in (y_1,y_2)$ we have $\bar{u}_x(y) = -\frac{2}{t_0(y)}$ for some $t_0(y) \in [0,t]$. Consequently $\varphi_y(s,y) = \frac{1}{2}\,\bar{u}_x^2(y)\,\chi_{[0,t_0(y)]}(s)$ for $s \in [0,t]$ and differentiation of the right-hand side of (2.21) yields

$$\partial_y\Big(\bar{u}(y) + \int_0^t \varphi(s,y)\,ds\Big) = \bar{u}_x(y) + \int_0^{t_0(y)} \frac{1}{2}\,\bar{u}_x^2(y)\,ds = -\frac{2}{t_0(y)} + \frac{t_0(y)}{2}\,\frac{4}{t_0^2(y)} = 0$$

for $y \in (y_1,y_2)$. In particular, the values of the right-hand side of (2.21) are equal when evaluated at (t,y_1) and at (t,y_2). This proves that u is well-defined.

The next step is to prove that for every $t \ge 0$, the map $y \mapsto u(t,y)$ is continuous on $I\!R$ with distributional derivative in $L^2(I\!R)$. Given $t \ge 0$ and $y_0 \in I\!R$, let $I_0 = \{x \in I\!R:\ \xi(t,x) = y_0\}$. The previously established properties of the map $x \mapsto \xi(t,x)$ ensure that $I_0 = [a,b]$ for some $a \le b$. For any sequence $y_n \to y_0$, choose $x_n \in I\!R$ with $\xi(t,x_n) = y_n$. If we show that $\min\{|x_n - a|,\,|x_n - b|\} \to 0$ as $n \to \infty$, by the continuous dependence on the y-variable of the right-hand side of (2.21), we infer that

$$u(t,y_n) = u(t,\xi(t,n)) \to u(t,\xi(t,a)) = u(t,\xi(t,b)) = u(t,y_0)$$

since $\xi(t,x_n) \to \xi(t,a) = \xi(t,b) = y_0$. Thus $y \mapsto u(t,y)$ would be continuous at y_0. If it would be possible that $\min\{|x_{n_k} - a|,\,|x_{n_k} - b|\} \ge \varepsilon > 0$ for a sequence $n_k \to \infty$, then

$$|y_{n_k} - y_0| = |\xi(t,x_{n_k}) - \xi(t,a)| = |\xi(t,x_{n_k}) - \xi(t,b)| \ge \min\{y_0 - \xi(t,a-\varepsilon),\,\xi(t,b+\varepsilon)\} > 0$$

must hold by the definition of $[a,b]$ and the monotonicity property of the function $x \mapsto \xi(t,x)$. But this is a contradiction since $y_n \to y_0$ as $n \to \infty$. We therefore proved the continuity of the map

$y \mapsto u(t,y)$ for every fixed $y \in I\!R$. Actually, a glance at the previous considerations confirms the continuity of the map $u : I\!R_+ \times I\!R \to I\!R$ since $\xi : I\!R_+ \times I\!R \to I\!R$ is continuous in view of (2.14). To show that for each $t \geq 0$ the distributional derivative $u_x(t,\cdot)$ belongs to $L^2(I\!R)$, due to the absolute continuity of the nondecreasing surjective map $\xi(t,\cdot) : I\!R \to I\!R$, we first show that at every point $y = \xi(t,x)$ where $\xi_x(t,x) > 0$ exists, $u_x(t,y) \in I\!R$ exists. Indeed, at such a point y the right-hand side of (2.21), formally equal to $u_x(t,\xi(t,x)) \cdot \xi_x(t,x)$, is differentiable with derivative

$$\bar{u}_x(x) + \int_0^t \varphi_y(t,x)\,ds = \bar{u}_x(x) + \frac{t}{2}\,\bar{u}_x^2(x),$$

since in view of (2.24)-(2.25) we must have $\bar{u}_x(x) > -\frac{2}{t}$ and $\varphi_y(s,x) = \frac{1}{2}\,\bar{u}_x^2(x)$ for all $s \in [0,t]$. Since $\xi_y(t,x) = 1 + t\,\bar{u}_x(x) + \frac{t^2}{4}\,\bar{u}_x^2(x)$, we infer that $u_x(t,y)$ exists, being given by the formula

$$u_x(t,y) = \frac{\bar{u}_x(x) + \frac{t}{2}\,\bar{u}_x^2(x)}{1 + t\bar{u}_x(x) + \frac{t^2}{4}\,\bar{u}_x^2(x)} = \frac{\bar{u}_x(x)}{1 + \frac{t}{2}\,\bar{u}_x(x)}, \tag{2.26}$$

where $y = \xi(t,x)$. From (2.26) we deduce that for any interval $[x_1,x_2]$ where $\xi_x(t,x) > 0$ a.e., we have

$$\begin{aligned}\int_{\xi(t,x_1)}^{\xi(t,x_2)} u_x^2(t,y)\,dy &= \int_{\xi(t,x_1)}^{\xi(t,x_2)} u_x^2(t,\xi(t,x)) \cdot \xi_x(t,x)\,dx \\ &= \int_{y_1}^{y_2} \frac{\bar{u}_x^2(x)}{\Big(1 + \frac{t}{2}\,\bar{u}_x(x)\Big)^2} \Big(1 + t\,\bar{u}_x(x) + \frac{t^2}{4}\,\bar{u}_x^2(x)\Big)\,dx = \int_{y_1}^{y_2} \bar{u}_x^2(x)\,dx\end{aligned}$$

if $y_1 = \xi(t,x_1)$, $y_2 = \xi(t,x_2)$ and if we take into account (2.24). Summing up over such intervals, we obtain that

$$\int_{I\!R} u_x^2(t,x)\,dx = \int_{\{\bar{u}_x(x) > -\frac{2}{t}\}} \bar{u}_x^2(x)\,dx. \tag{2.27}$$

In particular, the map $t \mapsto \|u_x(t,\cdot)\|_{L^2(I\!R)}$ is nondecreasing on $I\!R_+$. Moreover, we also have that

$$\int_{-\infty}^{\xi(t,y)} u_x^2(t,x)\,dx = \int_{\{x \in (-\infty,y]:\ \bar{u}_x(x) > -\frac{2}{t}\}} \bar{u}_x^2(x)\,dx$$

and

$$\int_{\xi(t,y)}^{\infty} u_x^2(t,x)\,dx = \int_{\{x \in [y,\infty):\ \bar{u}_x(x) > -\frac{2}{t}\}} \bar{u}_x^2(x)\,dx.$$

A comparison with (2.18) yields

$$\varphi(t,y) = \frac{1}{4}\int_{I\!R} \text{sign}\,\Big(\xi(t,y) - x\Big)\,u_x^2(t,x)\,dx. \tag{2.28}$$

Furthermore, if $\xi_x(t,x) > 0$ exists, relation (2.20) ensures for $y = \xi(t,x)$ the existence of $u_t(t,y)$, given by the formula

$$u_t(t,y) = -\,u_x(t,y)\,u(t,y) + \varphi(t,x)$$

obtained by differentiation and taking into acount (2.22). In combination with (2.28), this yields

$$\Big(u_t + uu_x\Big)(t,\xi(t,x)) = \frac{1}{4}\int_{I\!R} \mathrm{sign}\Big(\xi(t,x)-\zeta\Big)\, u_x^2(t,\zeta)\,d\zeta,$$

which is precisely (1.1) evaluated at $(t,\xi(t,x))$. In view of the previously established properties of the map $x \mapsto \xi(t,x)$ we deduce that the constructed function u satisfies also the condition (iii) of Definition 2.1. Since the other properties required by Definition 2.1 were proved above, we conclude that u qualifies as a solution to (1.1) in the sense of Definition 2.1. This completes the proof of Theorem 1. $\diamondsuit$

3 - A distance functional

If $\bar u : I\!R \to I\!R$ is also bounded, in addition to being continuous and with distributional derivative $\bar u_x \in L^2(I\!R)$, then the global solution $u(t,\cdot)$ constructed in Theorem 1 will be bounded at every fixed time $t \geq 0$. More precisely, in view of (2.19) and (2.21) we have that

$$\sup_{t\geq 0,\, x\in I\!R} \big|u(t,x)\big| \leq \sup_{x\in I\!R} \big|\bar u(x)\big| + \frac{t}{4}\int_{I\!R} \bar u_x^2(x)\,dx\,.$$

Thus, if $\bar u : I\!R \to I\!R$ is a bounded continuous function with distributional derivative $\bar u_x \in L^2(I\!R)$, then at each fixed time $t \geq 0$, the solution $u(t,\cdot)$ to (1.1), constructed in Theorem 1, belongs to the Banach space $\mathcal{X}$ of bounded continuous functions $f : I\!R \to I\!R$ with distributional derivative $f_x \in L^2(I\!R)$, endowed with the norm

$$\|f\|_{\mathcal{X}} = \sup_{x\in I\!R}\{|f(x)|\} + \Big(\int_{I\!R} f_x^2(x)\,dx\Big)^{\frac{1}{2}}.$$

The Banach space $\mathcal{X}$ seems suitable for (1.1) - see also [BZZ] where a construction similar to the one performed in Theorem 1 is presented. However, the map $t \mapsto u(t,\cdot)$ is generally not continuous from $I\!R_+$ to $\mathcal{X}$. Indeed, if for some $\tau > 0$ we have that the set $\{x \in I\!R :\ \bar u_x(x) = -\frac{2}{\tau}\}$ is of positive Lebesgue measure, then a discontinuity occurs at time $t = \tau$ for the map $t \mapsto u(t,\cdot) \in \mathcal{X}$ since from (2.27) we infer that for $t < \tau$,

$$\int_{I\!R} u_x^2(t,x)\,dx - \int_{I\!R} u_x^2(\tau,x)\,dx \geq \int_{\{x\in x:\ \bar u_x(x) = -\frac{2}{\tau}\}} \bar u_x^2(x)\,dx > 0.$$

Our aim will be to construct a distance functional in the space of solutions to (1.1) with respect to which we will have both continuity with respect to time as well as continuity with respect to the initial data for the solutions to (1.1). More precisely, for non-smooth solutions the conservation law (2.2) is replaced by

$$(v^2)_t + (uv^2)_x = -\mu\,, \tag{3.1}$$

where μ is the positive measure on the t-x plane defined as

$$\mu(\Omega) = \int_{\Big\{\big(T(y)\,,\ \xi(T(y),y)\big)\,\in\,\Omega\Big\}} \bar u_x^2(y)\,dy$$

for every open set $\Omega \subset I\!\!R_+ \times I\!\!R$. Here $T(y)$ is the blow-up time along the characteristic curve starting at y, namely

$$T(y) \doteq \begin{cases} -2/\bar u_x(y) & \text{if} \quad \bar u_x(y) < 0\,, \\ \infty & \text{otherwise.} \end{cases}$$

For any $\bar u \in \mathcal{X}$, we can use the semigroup notation $S_t \bar u \doteq u(t,\cdot)$ to denote the solution of (1.1) constructed in Section 2. Indeed

$$S_0 \bar u = \bar u\,, \qquad S_{t+s}\bar u = S_t\big(S_s \bar u\big)\,. \tag{3.2}$$

To prove (3.2), we first show that

$$\xi_1(t+s,y) = \xi_2(t,\xi_1(s,y)), \qquad t,\, s \geq 0,\ y \in I\!\!R, \tag{3.3}$$

where ξ_2 is the characteristic built upon the initial data $y \mapsto u(s,\xi_1(s,y))$. To check (3.3), we view both expressions as functions of t. At $t=0$ they are both equal to $\xi_1(s,y)$. For $t > 0$, differentiation of (3.3) yields

$$\bar u(y) + \int_0^{t+s} \varphi_1(r,y)\,dr = u(s,\xi_1(s,y)) + \int_0^t \varphi_2(r,\xi_1(s,y))\,dr \tag{3.4}$$

in view of (2.20). We use (2.21) to express the right-hand side of (3.4) as

$$\bar u(y) + \int_0^s \varphi_1(r,y)\,dr + \int_0^t \varphi_2(r,\xi_1(s,y))\,dr.$$

Therefore, to get (3.4), which yields (3.3) by integration, it suffices to show that

$$\int_s^{t+s} \varphi_1(r,y)\,dr = \int_0^t \varphi_2(r,\xi_1(s,y))\,dr. \tag{3.5}$$

To prove (3.5), we note that by (2.18),

$$\int_0^t \varphi_2(r,\xi_1(s,y))\,dr = \frac{1}{4}\int_0^t \int_{\{x:\ u_x(s,x) > -\frac{2}{r}\}} \mathrm{sign}\Big(\xi_1(s,y) - x\Big)\, u_x^2(s,x)\,dx\,dr$$

$$= \frac{1}{4}\int_0^t \int_{\{x:\ u_x(s,\xi_1(s,x)) > -\frac{2}{r}\}} \mathrm{sign}\Big(\xi_1(s,y) - \xi_1(s,x)\Big)\, u_x^2\Big(s,\xi_1(s,x)\Big)\,\partial_x \xi_1(s,x)\,dx\,dr$$

if we change variables $x \mapsto \xi_1(s,x)$. Taking now (2.9)-(2.10) into account, we infer that

$$\begin{aligned} \int_0^t \varphi_2(r,\xi_1(s,y))\,dr &= \frac{1}{4}\int_0^t \int_{\{x:\ u_x(s,\xi_1(s,x)) > -\frac{2}{r}\}} \mathrm{sign}\Big(\xi_1(s,y) - \xi_1(s,x)\Big)\, \bar u_x^2(x)\,dx\,dr \\ &= \frac{1}{4}\int_0^t \int_{\{x:\ u_x(s,\xi_1(s,x)) > -\frac{2}{r}\}} \mathrm{sign}(y-x)\, \bar u_x^2(x)\,dx\,dr \end{aligned}$$

since the function $x \mapsto \xi_1(s,x)$ is nondecreasing. But

$$u_x(s,\xi_1(s,x)) = \frac{2\,\bar u_x(x)}{2 + s\,\bar u_x(x)} > -\frac{2}{r} \quad \text{if and only if} \quad \bar u_x(x) > -\frac{2}{s+r}$$

since the function $y \mapsto \dfrac{2y}{2+s\,y}$ is strictly increasing for $y > -\frac{2}{s}$, so that in the end we get

$$
\begin{aligned}
\int_0^t \varphi_2(r,\xi_1(s,y))\,dr &= \frac{1}{4}\int_0^t \int_{\{x:\ \bar u_x(x) > -\frac{2}{r+s}\}} \operatorname{sign}(y-x)\,\bar u_x^2(x)\,dx\,dr \\
&= \frac{1}{4}\int_s^{t+s} \int_{\{x:\ \bar u_x(x) > -\frac{2}{\tau}\}} \operatorname{sign}(y-x)\,\bar u_x^2(x)\,dx\,d\tau
\end{aligned}
\tag{3.6}
$$

where $\tau = r+s$. On the other hand, by (2.18),

$$
\int_s^{t+s} \varphi_1(r,y)\,dr = \frac{1}{4}\int_s^{t+s} \int_{\{x:\ \bar u_x(x) > -\frac{2}{\tau}\}} \operatorname{sign}(y-x)\,\bar u_x^2(x)\,dx\,d\tau
$$

so that (3.4) holds and (3.3) is proved. Knowing (3.3), to infer $S_{t+s}\bar u = S_t(S_s \bar u)$, it suffices to show that

$$
u(t+s,\xi_1(t+s,y)) = u(t,\xi_2(t,\xi_1(s,y))).
$$

But, by (2.21), the left-hand side is precisely

$$
\bar u(y) + \int_0^{t+s} \varphi(r,y)\,dr = \bar u(y) + \int_0^s \varphi_1(r,y)\,dr + \int_s^{t+s} \varphi_1(r,y)\,dr = u(s,\xi_1(s,y)) + \int_s^{t+s} \varphi_1(r,y)\,dr,
$$

which, taking into account (3.5), equals to

$$
u(s,\xi_1(s,y)) + \int_0^t \varphi_2(r,\xi_1(s,y))\,ds = u(t,\xi_2(t,\xi_1(s,y)))
$$

in view of (2.21). This completes the proof of (3.2).

Notice that in general the map $t \mapsto S_t \bar u$ is NOT continuous from $[0,\infty[$ into $\mathcal{X}$. It is thus interesting to identify some distance $J(u,v)$ which is well adapted to the evolution generated by (1.1). More precisely, given an arbitrary constant M, in this section we shall construct a functional $J(u,v)$ with the following property: For any initial data $\bar u, \bar v \in \mathcal{X}$ with

$$
\|\bar u_x\|_{\mathbf{L}^2} \le M\,, \qquad\qquad \|\bar v_x\|_{\mathbf{L}^2} \le M\,,
$$

the corresponding dissipative solutions u, v constructed in Theorem 1 satisfy

$$
J\big(u(t)\,,\ v(t)\big) \le e^{C_M\,t} J(\bar u,\ \bar v).
$$

To begin the construction, consider the metric space

$$
X \doteq \Big(\mathbb{R}^2 \times\,]-\pi/2\,,\ \pi/2]\Big) \cup \{\infty\}
\tag{3.7}
$$

with distance

$$
\begin{aligned}
d\Big((x,u,w),\ (\tilde x,\tilde u,\tilde w)\Big) &\doteq \min\Big\{|x-\tilde x| + |u-\tilde u| + \kappa_0\,|w-\tilde w|\,, \quad \kappa_0\,|\pi/2+w| + \kappa_0|\pi/2+\tilde w|\Big\}\,, \\
d\Big((x,u,w),\ \infty\Big) &= \kappa_0\,|\pi/2+w|\,.
\end{aligned}
\tag{3.8}
$$

Here κ_0 is a suitably large constant, whose precise value will be specified later. Notice that X is obtained from the metric space $I\!R^2 \times [-\pi/2\,,\ \pi/2]$ by identifying all points $(x,\,u,\,-\pi/2)$ into a single point, called "∞".

Let $M(X)$ be the space of all bounded Radon measures on X. To each function $u \in H^1_{loc}(I\!R)$ with $u_x \in L^2(I\!R)$ we now associate the measure $\mu^u \in \mathcal{M}(X)$ defined as

$$\mu^u\big(\{\infty\}\big) = 0\,, \qquad\qquad \mu^u(A) = \int_{\big\{x\in I\!R:\ (x,\,u(x),\,\arctan\ u_x(x)\,)\in A\big\}} u_x^2(x)\,dx \tag{3.9}$$

for every Borel set $A \subseteq I\!R^2\times\,]-\pi/2\,,\ \pi/2]\,$.

As distance between two functions $u, v \in \mathcal{X}$ we now introduce a kind of Kantorovich distance $J(u,v)$ related to an optimal transportation problem. Call $\mathcal{F}$ the family of all triples (ψ,ϕ_1,ϕ_2), where $\phi_1,\phi_2 : I\!R \mapsto [0,1]$ are simple Borel measurable maps (that is, their range is a finite number of points and the preimage of each such point is a Borel set) and $\psi : I\!R \mapsto I\!R$ is a nondecreasing absolute continuous surjective map. Assuming that

$$\phi_1(x)\,u_x^2(x) = \psi'(x)\cdot\phi_2\big(\psi(x)\big)\,v_x^2\big(\psi(x)\big) \qquad\qquad \text{for a.e. } x \in I\!R\,, \tag{3.10}$$

we define

$$\begin{aligned} J^{(\psi,\phi_1,\phi_2)}(u,v) \doteq &\int d\Big(\big(x,\,u(x),\,\arctan u_x(x)\big)\,,\ \big(\psi(x),\,v(\psi(x)),\,\arctan v_x(\psi(x))\big)\Big)\cdot\phi_1(x)\,u_x^2(x)\,dx \\ &+\int d\Big(\big(x,\,u(x),\,\arctan u_x(x)\big)\,,\ \infty\Big)\cdot\big(1-\phi_1(x)\big)\,u_x^2(x)\,dx \\ &+\int d\Big(\big(\psi(x),\,v(\psi(x)),\,\arctan v_x(\psi(x)\big)\,,\ \infty\Big)\cdot\big(1-\phi_2(\psi(x))\big)\,v_x^2(\psi(x))\,\psi'(x)\,dx\,. \end{aligned} \tag{3.11}$$

Observe that (ψ,ϕ_1,ϕ_2) can be regarded as a **transportation plan**, in order to transport the measure μ^u onto the measure μ^v. Since these two positive measures need not have the same total mass, we allow some of the mass to be transferred to the point ∞. More precisely, the mass transferred is $(1-\phi_1)\cdot\mu^u$ and $(1-\phi_2)\cdot\mu^v$. The last two integrals in (3.11) account for the additional cost of this transportation. Integrating (3.10) over the real line, one finds

$$\int_{I\!R}\phi_1(x)\,u_x^2(x)\,dx = \int_{I\!R}\phi_2(y)\,v_x^2(y)\,dy\,.$$

We can thus transport the measure $\phi_1\mu^u$ onto $\phi_2\,\mu^v$ by a map $\Psi : \big(x,\,u(x)\ \arctan u_x(x)\big) \mapsto \big(y,\,v(y),\,\arctan v_x(y)\big)$, with $y=\psi(x)$. The associated cost is given by the first integral in (3.11). In this case the measure $\phi_2\,\mu^v$ is obtained as the push-forward of the measure $\phi_1\mu^u$. We recall that the **push-forward** of a measure μ by a mapping Ψ is defined as $(\Psi\#\mu)(A) \doteq \mu(\Psi^{-1}(A))$ for every measurable set A. Here $\Psi^{-1}(A) \doteq \big\{z\ :\ \Psi(z)\in A\big\}$.

We now define our distance functional by optimizing over all transportation plans, namely

$$J(u,v) \doteq \inf_{(\psi,\phi_1,\phi_2)}\{J^{(\psi,\phi_1,\phi_2)}(u,v)\} \tag{3.12}$$

where the infimum is taken over all triples $(\psi,\phi_1,\phi_2)\in\mathcal{F}$ such that (3.10) holds.

To check that (3.12) actually defines a distance, let $u,\ v,\ w \in \mathcal{X}$ be given functions.

1. Let us show that $J(u,v) = J(v,u)$. In order to do this, it is enough to prove that for every triple $(\psi,\, \phi_1,\, \phi_2) \in \mathcal{F}$ satisfying (3.10) and every $\varepsilon > 0$, there is a triple $(\eta, \varphi_1, \varphi_2) \in \mathcal{F}$ satisfying (3.10) such that $\eta : I\!\!R \to I\!\!R$ is a strictly increasing absolutely continuous bijection and

$$\left| J^{(\eta,\varphi_1,\varphi_2)}(u,v) - J^{(\psi,\phi_1,\phi_2)}(u,v) \right| \le \varepsilon. \tag{3.13}$$

Indeed, given $(\psi, \phi_1, \phi_2) \in \mathcal{F}$ satisfying (3.10), define $\tilde{\psi} = \eta^{-1}$, $\tilde{\phi_1} = \varphi_2$, $\tilde{\phi_2} = \varphi_1$. The properties of η ensure the absolute continuity of $\tilde{\psi}$ (see [N]) so that we obtain $J^{(\tilde{\psi},\tilde{\phi_1},\tilde{\phi_2})}(v,u) = J^{(\eta,\varphi_1,\varphi_2)}(u,v)$ by performing the change of variables $x \mapsto \eta(x)$. Since $\varepsilon > 0$ was arbitrary, we infer that $J(v,u) \le J(u,v)$. Interchanging the roles of u and v we get $J(u,v) = J(v,u)$.

To prove (3.13), it is convenient to view $\psi : I\!\!R \to I\!\!R$ as a maximal monotone multifunction $\psi : I\!\!R \mapsto \mathcal{P}(I\!\!R)$ with domain and range $I\!\!R$. Here $\mathcal{P}(I\!\!R)$ is the family of all subsets of $I\!\!R$. The conditions for a multifunction $F : I\!\!R \mapsto \mathcal{P}(I\!\!R)$ to be maximal monotone with domain and range $I\!\!R$ may be explicited as follows [Z]:

- for every $x \in I\!\!R$, the set $F(x) \subset I\!\!R$ is nonempty (i.e. the domain of F is $I\!\!R$);

- for every $y \in I\!\!R$ there is at least some $x \in I\!\!R$ with $y \in F(x)$, expressing the fact that the range of F is $I\!\!R$;

- there are no couples (x_1, y_1) and (x_2, y_2) with $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ such that $x_1 < x_2$ and $y_2 < y_1$, meaning that F is monotone);

- if we associate to F its graph $\{(x,y) \in I\!\!R^2 :\ y \in F(x)\}$, then this graph has no proper extension satisfying the first three properties (condition defining the maximal monotonicity property).

We recall some important features presented by such maps [AA, Z]:

- the set $F(x)$ is an interval of the form $[a_x, b_x]$ with $a_x \le b_x$ for all $x \in I\!\!R$ and $a_x = b_x$ for all $x \in I\!\!R$, except perhaps an at most countable set (so F is singlevalued with the exception of at most countably many points);

- F is a.e. differentiable, that is, for almost all $x_0 \in I\!\!R$ there exists $F'(x_0) \in I\!\!R$ such that

$$\lim_{x \to x_0,\, y \in F(x)} \frac{y - F(x_0) - (x - x_0)\, F'(x_0)}{x - x_0} = 0;$$

- we can define the inverse $F^{-1} : I\!\!R \to \mathcal{P}(I\!\!R)$ of F by asking $y \in F^{-1}(x)$ if and only if $x \in F(x)$ and F^{-1} is again a maximal monotone multifunction with domain and range $I\!\!R$.

Since the multifunction ψ^{-1} is maximal montone, let $\{y_n\}$ be the (at most countable) set of points where it is multivalued, that is, $\psi^{-1}(y) = [a_n, b_n]$ with $b_n > a_n$. Then $\psi(x) = y_n$ for $x \in [a_n, b_n]$ and, ψ being absolutely continuous, $\psi_x > 0$ a.e. on $I\!\!R - \bigcup_n [a_n, b_n]$ since ψ is strictly increasing on this set. Given $\gamma > 0$, the absolute continuity of $\psi : I\!\!R \to I\!\!R$ allows us to choose some $\delta > 0$ such that the total variation of ψ over the union of disjoint closed intervals with the sum of their lengths less than δ is less than γ cf. [BGH]. On each interval $[a_n - \frac{\delta}{2^n}, b_n + \frac{\delta}{2^n}]$ we replace ψ with the linear function η which takes the values $\psi(a_n - \frac{\delta}{2^n})$, respectively $\psi(b_n + \frac{\delta}{2^n})$ at the endpoints. By the way a_n and b_n were defined, we know that $\psi(b_n + \frac{\delta}{2^n}) > \psi(a_n - \frac{\delta}{2^n})$ so that $\eta'(x)$ is a positive constant on $[a_n - \frac{\delta}{2^n}, b_n + \frac{\delta}{2^n}]$ with

$$\sum_n \int_{a_n - \frac{\delta}{2^n}}^{b_n + \frac{\delta}{2^n}} \eta'(x)\, dx \le \sum_n \Big(\psi(b_n + \frac{\delta}{2^n}) - \psi(a_n - \frac{\delta}{2^n}) \Big) \le \gamma.$$

Setting $\eta(x) = \psi(x)$ for $x \notin [a_n - \frac{\delta}{2^n}, b_n + \frac{\delta}{2^n}]$, we obtain a strictly increasing absolutely continuous bijection $\eta : I\!\!R \to I\!\!R$. Let us now show that the triple $(\eta, \varphi_1, \varphi_2) \in \mathcal{F}$ satisfies both (3.10) and

(3.13), where φ_1, φ_2 are defined by setting $\varphi_1(x) = 0$ for $x \in [a_n - \frac{\delta}{2^n}, b_n + \frac{\delta}{2^n}]$ and $\varphi_1(x) = \phi_1(x)$ for $x \notin [a_n - \frac{\delta}{2^n}, b_n + \frac{\delta}{2^n}]$, while $\varphi_2(\eta(x)) = \phi_2(\psi(x)))$ for $x \notin [a_n - \frac{\delta}{2^n}, b_n + \frac{\delta}{2^n}]$ and $\varphi_2(\eta(x)) = 0$ for $x \in [a_n - \frac{\delta}{2^n}, b_n + \frac{\delta}{2^n}]$. On the complement of the set $\bigcup_n [a_n - \frac{\delta}{2^n}, b_n + \frac{\delta}{2^n}]$ relation (3.10) clearly holds a.e. for $(\eta, \varphi_1, \varphi_2)$, being unmodified from (3.10) for (ψ, ϕ_1, ϕ_2). If $x \in [a_n - \frac{\delta}{2^n}, b_n + \frac{\delta}{2^n}]$, then (3.10) for $(\eta, \varphi_1, \varphi_2)$ holds again since both sides are zero as $\varphi_1(x) = \varphi_2(\eta(x)) = 0$ in this case. Finally, to check (3.13), notice that if we denote

$$E_\delta = \bigcup_n \Big\{ [a_n - \frac{\delta}{2^n}, a_n] \cup [b_n, b_n + \frac{\delta}{2^n}] \Big\}, \qquad A = \bigcup_n [a_n, b_n],$$

then

$$\Big| J^{(\eta,\varphi_1,\varphi_2)}(u,v) - J^{(\psi,\phi_1,\phi_2)}(u,v) \Big| \le 2\kappa_0 \pi \int_{E_\delta} u_x^2 \, dx + 2\kappa_0 \pi \int_{E_\delta \cup A} v_x^2(\eta(x))\, \eta'(x)\, dx. \tag{3.14}$$

Indeed, the distance d is less that $2\kappa_0 \pi$ and the integrands in $J^{(\eta,\varphi_1,\varphi_2)}(u,v)$ and $J^{(\psi,\phi_1,\phi_2)}(u,v)$ agree on the complement of the set $\bigcup_n [a_n - \frac{\delta}{2^n}, b_n + \frac{\delta}{2^n}]$ by definition. Also, for a.e. $x \in [a_n, b_n]$ we have $\phi_1(x)\, u_x^2 = 0$ by (3.10) as $\psi'(x) = 0$, and $\varphi_1(x) = 0$ by its definition. We obtain (3.14). Since the absolutely continuous map η maps $E_\delta \cup A$ into a set of Lebesgue measure less than γ, and u_x^2, $v_x^2 \in L^1(I\!\!R)$, from (3.14) we infer (3.13) by choosing $\delta > 0$ and $\gamma > 0$ small enough. This completes the argumentation needed to show that $J(u,v) = J(v,u)$.

2. Choosing $\psi(x) = x$, $\phi_1(x) = \phi_2(x) = 1$, we immediately see that $J(u,u) = 0$. Moreover, we have $J(u,v) > 0$ if $u \neq v$. To check this, note that $J(u,v) = 0$ implies that there is a sequence $(\psi^n, \phi_1^n, \phi_2^n)$ along which $J^{(\psi^n,\phi_1^n,\phi_2^n)}(u,v) \to 0$. The second term in (3.11) yields

$$\Big(\frac{\pi}{2} + \arctan u_x(x)\Big)\Big(1 - \phi_1^n(x)\Big)\, u_x^2(x) \to 0 \quad \text{in} \quad L^1(I\!\!R),$$

so that along a subsequence $(1 - \phi_1^{n_k})\, u_x^2 \to 0$ a.e. on $I\!\!R$ since $u_x > -\infty$ a.e. On the set $S = \{x \in I\!\!R: \; u_x(x) \neq 0\}$ we therefore have $\phi_1^{n_k} \to 1$ a.e. Moreover, the first term in (3.11) forces

$$\phi_1^{n_k}(x) u_x^2(x) \cdot \min \Big\{ |x - \psi^{n_k}(x)| + |u(x) - v(\psi^{n_k}(x))| + \kappa_0 |\arctan u_x(x) - \arctan v_x(\psi^{n_k}(x))|,$$
$$\kappa_0 \Big[\frac{\pi}{2} + \arctan u_x(x) + \frac{\pi}{2} + \arctan v_x(\psi^{n_k}(x))\Big] \Big\} \to 0 \quad \text{in} \quad L^1(I\!\!R). \tag{3.15}$$

Since $u_x > -\infty$ a.e. ensures

$$\frac{\pi}{2} + \arctan u_x(x) + \frac{\pi}{2} + \arctan v_x(\psi^{n_k}(x)) \ge \frac{\pi}{2} + \arctan u_x(x) > 0 \quad \text{a.e. on} \quad I\!\!R,$$

we infer from (3.15), by passing to another subsequence, that

$$|x - \psi^{n_k}(x)| + |u(x) - v(\psi^{n_k}(x))| \to 0 \quad \text{a.e. on} \quad S.$$

In view of the continuity of v, $\psi^{n_k}(x) \to x$ a.e. on S guarantees $v(\psi^{n_k}(x)) \to v(x)$ a.e. on S so that $u = v$ a.e. on S since also $v(\psi^{n_k}(x)) \to u(x)$ a.e. on S. Repeating this argument with the roles of u and v reversed, we find that $u = v$ a.e. on the set $\{x \in I\!\!R: \; v_x \neq 0\}$. Combining this with the previous conclusion, we have $u = v$ a.e. on the complement of the set $\{x \in I\!\!R: u_x = v_x = 0\}$. Since u_x, $v_x \in L^2(I\!\!R)$, this is possible only if $u = v$ on $I\!\!R$. Thus $J(u,v) = 0$ if and only if $u = v$.

3. Finally, to prove the triangle inequality, it suffices to show that for every choice of $(\psi^\flat, \phi_1^\flat, \phi_2^\flat)$ satisfying (3.10), and of $(\psi^\sharp, \phi_1^\sharp, \phi_2^\sharp)$ satisfying (3.10) for (v, w), the triplet (ψ, ϕ_1, ϕ_2) defined by

$$\psi(x) = \psi^\sharp(\psi^\flat(x)), \qquad \phi_1(x) = \phi_1^\flat(x) \cdot \phi_1^\sharp(\psi^\flat(x)), \quad \phi_2(y) = \phi_2^\sharp(y) \cdot \phi_2^\flat(\psi^\flat(x)),$$

satisfies (3.10) for (u, w) and

$$J^{(\psi,\phi_1,\phi_2)}(u, w) \leq J^{(\psi^\flat,\phi_1^\flat,\phi_2^\flat)}(u, v) + J^{(\psi^\sharp,\phi_1^\sharp,\phi_2^\sharp)}(v, w). \tag{3.16}$$

Notice that composing the relation (3.10) for (v, w) a.e. to the right with $\psi^\flat$, and multiplying the outcome by $\phi_2^\flat \circ \psi^\flat \cdot (\psi^\flat)'$, we infer that (3.10) holds a.e. on $I\!R$ for (u, w) with our choice of (ψ, ϕ_1, ϕ_2) and we can now concentrate on proving (3.16).

To simplify matters, we introduce the following notation

$$P_1 = (x,\, u,\, \arctan u_x), \quad P_2 = (\psi^\flat,\, v \circ \psi^\flat,\, \arctan v_x \circ \psi^\flat), \quad P_3 = (\psi,\, w \circ \psi,\, \arctan w_x \circ \psi),$$

$$m_1 = u_x^2, \qquad m_2 = v_x^2 \circ \psi^\flat \cdot (\psi^\flat)', \qquad m_3 = w_x^2 \circ \psi \cdot \psi'.$$

The relations of type (3.10) yield then that a.e. on $I\!R$,

$$\phi_1^\flat \cdot m_1 = \phi_2^\flat \circ \psi^\flat \cdot m_2, \qquad \phi_1^\sharp \circ \psi^\flat \cdot m_2 = \phi_2^\sharp \circ \psi \cdot m_3, \qquad \phi_1 \cdot m_1 = \phi_2 \circ \psi \cdot m_3. \tag{3.17}$$

Also,

$$J^{(\psi,\phi_1,\phi_2)}(u, w) = \int_{I\!R} \Big\{ d(P_1, P_3) \cdot \phi_1 m_1 + d(P_1, \infty) \cdot (1 - \phi_1) m_1 + d(P_3, \infty) \cdot (1 - \phi_2 \circ \psi) m_3 \Big\}\, dx,$$

$$J^{(\psi^\flat,\phi_1^\flat,\phi_2^\flat)}(u, v) = \int_{I\!R} \Big\{ d(P_1, P_2) \cdot \phi_1^\flat m_1 + d(P_1, \infty) \cdot (1 - \phi_1^\flat) m_1 + d(P_2, \infty) \cdot (1 - \phi_2^\flat \circ \psi^\flat) m_2 \Big\}\, dx,$$

$$J^{(\psi^\sharp,\phi_1^\sharp,\phi_2^\sharp)}(v, w) = \int_{I\!R} \Big\{ d(P_2, P_3) \cdot \phi_1^\sharp \circ \psi^b m_1 + d(P_2, \infty) \cdot (1 - \phi_1^\sharp \circ \psi^\flat) m_2 + d(P_3, \infty) \cdot (1 - \phi_2^\sharp \circ \psi) m_3 \Big\}\, dx,$$

the last relation being obtained after the change of variables $x \mapsto \psi^\flat(x)$ in the integral. We will prove (3.16) by deriving an appropriate inequality valid a.e. pointwise between the integrands in the previous expressions. Since

$$(1 - \phi_2^\flat \circ \psi^\flat)(1 - \phi_1^\sharp \circ \psi^\flat) \geq 0,$$

we have

$$1 - \phi_2^\flat \circ \psi^\flat + 1 - \phi_1^\sharp \circ \psi^\flat \geq \phi_2^\flat \circ \psi^\flat (1 - \phi_1^\sharp \circ \psi^\flat) + \phi_1^\sharp \circ \psi^\flat (1 - \phi_2^\flat \circ \psi^\flat).$$

Multiplication of both sides by $d(P_2, \infty) \cdot m_2$ leads to

$$\begin{aligned} d(P_2, \infty) \cdot (1 - \phi_2^\flat \circ \psi^\flat) m_2 + d(P_2, \infty) \cdot (1 - \phi_1^\sharp \circ \psi^\flat) m_2 &\geq d(P_2, \infty) \cdot \phi_1^\flat (1 - \phi_1^\sharp \circ \psi^\flat) m_1 \\ &+ d(P_2, \infty) \cdot \phi_1^\sharp \circ \psi^\flat (1 - \phi_2^\flat \circ \psi^\flat) m_2 \end{aligned} \tag{3.18}$$

in view of (3.17). Multiply now the inequalities

$$d(P_1, P_2) - d(P_1, \infty) + d(P_2, \infty) \geq 0, \qquad d(P_2, P_3) - d(P_3, \infty) + d(P_2, \infty) \geq 0,$$

by $\phi_1^\flat(1-\phi_1^\sharp\circ\psi^\flat)m_1$, respectively $\phi_1^\sharp\circ\psi^\flat(1-\phi_2^\flat\circ\psi^\flat)m_2$, and add them up. The outcome yields in combination with (3.18) that

$$d(P_1,P_2)\cdot\phi_1^\flat(1-\phi_1^\sharp\circ\psi^\flat)m_1-d(P_1,\infty)\cdot\phi_1^\flat(1-\phi_1^\sharp\circ\psi^\flat)m_1+d(P_2,P_3)\cdot\phi_1^\sharp\circ\psi^\flat(1-\phi_2^\flat\circ\psi^\flat)m_2$$

$$+d(P_2,\infty)\cdot(1-\phi_2^\flat\circ\psi^\flat)m_2+d(P_2,\infty)\cdot(1-\phi_1^\sharp\circ\psi^\flat)m_2\geq d(P_3,\infty)\cdot\phi_1^\sharp\circ\psi^\flat(1-\phi_2^\flat\circ\psi^\flat)m_2.$$

Adding to both sides the quantity

$$d(P_1,\infty)\cdot m_1+d(P_3,\infty)\cdot m_3+d(P_1,P_2)\cdot\phi_1^\flat\cdot\phi_1^\sharp\circ\psi^\flat\cdot m_1-d(P_1,\infty)\cdot\phi_1^\flat\cdot\phi_1^\sharp\circ\psi^\flat\cdot m_1$$

$$+d(P_2,P_3)\cdot\phi_1^\sharp\circ\psi^\flat\cdot\phi_2^\flat\circ\psi^\flat\cdot m_2-d(P_3,\infty)\cdot\phi_2^\sharp\circ\psi\cdot m_3$$

we deduce by (3.17) that the integrand of $J^{(\psi^\flat,\phi_1^\flat,\phi_2^\flat)}(u,v)+J^{(\psi^\sharp,\phi_1^\sharp,\phi_2^\sharp)}(v,w)$, equal a.e. precisely to the left-hand side of the new inequality, is a.e. pointwise larger than

$$d(P_3,\infty)\cdot\phi_1^\sharp\circ\psi^\flat(1-\phi_2^\flat\circ\psi^\flat)m_2+d(P_1,\infty)\cdot m_1+d(P_3,\infty)\cdot m_3+d(P_1,P_2)\cdot\phi_1^\flat\cdot\phi_1^\sharp\circ\psi^\flat\cdot m_1$$

$$-d(P_1,\infty)\cdot\phi_1^\flat\cdot\phi_1^\sharp\circ\psi^\flat\cdot m_1+d(P_2,P_3)\cdot\phi_1^\sharp\circ\psi^\flat\cdot\phi_2^\flat\circ\psi^\flat\cdot m_2-d(P_3,\infty)\cdot\phi_2^\sharp\circ\psi\cdot m_3.$$

Taking into account (3.17) and the definition $\phi_1=\phi_1^\flat\cdot\phi_1^\sharp\circ\psi^\flat$, we see that the above expression equals

$$d(P_1,\infty)\cdot(1-\phi_1)m_1+d(P_3,\infty)\cdot(1-\phi_2\circ\psi)m_3+\Big(d(P_1,P_2)+d(P_2,P_3)\cdot\phi_1 m_1\Big)$$

$$\geq d(P_1,\infty)\cdot(1-\phi_1)m_1+d(P_3,\infty)\cdot(1-\phi_2\circ\psi)m_3+d(P_1,P_3)\cdot\phi_1 m_1.$$

The lower estimate is a.e. precisely the integrand in $J^{(\psi,\phi_1,\phi_2)}(u,w)$ and (3.16) holds. The proof that J satisfies the triangle inequality is therefore completed.

In the remainder of this section we examine how the distance $J(\cdot,\cdot)$ behaves in connection with solutions of the equation (1.1).

Continuity w.r.t. time. Let $t\mapsto u(t)$ be the solution of (1.1) constructed in Section 2. For any fixed $t>0$, we define a transportation plan of $\mu^{\bar u}$ to $\mu^{u(t)}$ by setting

$$\psi(x)\doteq\xi(t,x)\,,\qquad \phi_1(x)\doteq\begin{cases}1 & \text{if}\quad T(x)>t\,,\\ 0 & \text{if}\quad T(x)\leq t\,,\end{cases}\qquad \phi_2(x)\equiv 1\,. \tag{3.19}$$

Relation (3.6) follows from (2.9)-(2.10) on $\{T(x)>t\}$ and from (2.25) on $\{T(x)\leq t\}$. The cost of this plan is estimated by

$$\begin{aligned}J^{(\psi,\phi_1,\phi_2)}\big(\bar u\,,\ u(t)\big)\leq \int_{\{T(x)>t\}}&\Big\{\big|x-\xi(t,x)\big|+\big|\bar u(x)-u(t,\xi(t,x))\big|\\ &+\kappa_0\big|\arctan\bar u_x(x)-\arctan u_x(t,\xi(t,x))\big|\Big\}\,\bar u_x^2(x)\,dx\\ +\int_{\{T(x)\leq t\}}&\big|\pi/2+\arctan\bar u_x(x)\big|\,\bar u_x^2(x)\,dx\,.\end{aligned}\tag{3.20}$$

By (2.4) we have that a.e.

$$\left|\frac{d}{dt}\arctan u_x\big(t,\xi(t,x)\big)\right| = \left|\frac{\frac{d}{dt}u_x\big(t,\xi(t,x)\big)}{1+u_x^2\big(t,\xi(t,x)\big)}\right| \le \frac{1}{2}. \tag{3.21}$$

An integration on $[0,t]$ yields

$$\big|\arctan \bar{u}_x(x) - \arctan u_x\big(t,\xi(t,x)\big)\big| \le \frac{t}{2}, \qquad t \ge 0. \tag{3.22}$$

On the other hand, using (2.20), we get

$$|x-\xi(t,x)| \le t\,|\bar{u}(x)| + \int_0^t (t-s)\,|\varphi(s,x)|\,ds \le t\,|\bar{u}(x)| + \frac{t^2}{8}\int_{I\!R}\bar{u}_x^2(x)\,dx, \qquad t\ge 0,\ x\in I\!R, \tag{3.23}$$

if we take into account (2.19). From (2.21) and (2.19), we also infer

$$\big|\bar{u}(x) - u\big(t,\xi(t,x)\big)\big| \le \int_0^t |\varphi(s,y)|\,dy \le \frac{t}{4}\int_{I\!R}\bar{u}_x^2(x)\,dx, \qquad t\ge 0,\ x\in I\!R. \tag{3.24}$$

To estimate the last term in (3.20), notice that

$$\{x\in I\!R:\ T(x)\le t\} = \{x\in I\!R:\ -\frac{2}{\bar{u}_x(x)} \le t\} = \{x\in I\!R:\ \bar{u}_x(x)\le -\frac{2}{t}\}, \qquad t>0. \tag{3.25}$$

Furthermore, since $\lim\limits_{x\to-\infty} x(\frac{\pi}{2}+\arctan x) = -1$, there is a constant $c>0$ such that

$$0 \le \frac{\pi}{2} + \arctan y \le \frac{c}{|y|}, \qquad y\le -1,$$

whereas

$$\left|\frac{\pi}{2}+\arctan y\right|\,y^2 \le \pi \qquad \text{if} \quad -1\le y\le 0,$$

so that

$$\left|\frac{\pi}{2}+\arctan \bar{u}_x(x)\right|\,\bar{u}_x^2(x) \le \pi + c\,|\bar{u}_x(x)| \qquad \text{if} \quad \bar{u}_x(x)\le -\frac{2}{t}. \tag{3.26}$$

On the other hand, if $\bar{u}_x(x)\le -\frac{2}{t}$, then $t^2\bar{u}_x^2(x)\ge 4$ so that

$$\int_{\{T(x)\le t\}} 1\,dx \le \frac{t^2}{4}\int_{\{T(x)\le t\}}\bar{u}_x^2(x)\,dx. \tag{3.27}$$

From (3.25)-(3.27) we infer that

$$\int_{\{T(x)\le t\}}\left|\frac{\pi}{2}+\arctan\bar{u}_x(x)\right|\,\bar{u}_x^2(x)\,dx \le \frac{\pi t^2}{4}\,\|\bar{u}_x\|_{L^2}^2 + c\int_{\{T(x)\le t\}}|\bar{u}_x(x)|\,dx$$

$$\le \frac{\pi t^2}{4}\,\|\bar{u}_x\|_{L^2}^2 + c\Big(\int_{\{T(x)\le t\}}1\,dx\Big)^{\frac12}\Big(\int_{\{T(x)\le t\}}\bar{u}_x^2(x)\,dx\Big)^{\frac12} \le \frac{\pi t^2}{4}\,\|\bar{u}_x\|_{L^2}^2 + \frac{ct}{2}\,\|\bar{u}_x\|_{L^2}^2.$$

By (3.20), (3.22)-(3.25) and the previous inequality we conclude

$$J^{(\psi,\phi_1,\phi_2)}\big(\bar u\,,\ u(t)\big)\le\Big(\frac{\pi t}{4}+\frac{c+\kappa_0}{2}+\|\bar u\|_{L^\infty}+\frac{t+2}{8}\,\|\bar u_x\|^2_{L^2}\Big)\,t\,\|\bar u_x\|^2_{L^2},\qquad t\ge 0. \tag{3.28}$$

It is now clear that each semigroup trajectory $t\mapsto S_t\bar u$ is Lipschitz continuous as a map from $[0,\infty[$ into the metric space X equipped with our distance functional J. The Lipschitz constant remains uniformly bounded as $\bar u$ ranges over bounded subsets of X.

Continuity w.r.t. the initial data. We now consider two distinct solutions and study how the distance $J\big(u(t)\,,\ \tilde u(t)\big)$ varies in time. Recall that the solution $u=u(t,x)$ is computed by (2.20)–(2.22), also in the case where the gradient blows up. The same formula of course holds for $\tilde u$. Let $(\psi_0,\phi_{1,0},\phi_{2,0})$ be an optimal transportation plan of the measure $\mu^{u(0)}$ to the measure $\mu^{\tilde u(0)}$. In view of the approximation property established in (3.13), we can restrict our attention to the case when ψ_0 is strictly increasing on $I\!R$. For any $t>0$, we define a transportation plan $(\psi^t,\phi_1^t,\phi_2^t)$ of the measure $\mu^{u(t)}$ to $\mu^{\tilde u(t)}$ as follows:

$$\psi^t\big(\xi(t,y)\big)\doteq\tilde\xi(t,\ \tilde y)\quad\text{for}\quad\tilde y=\psi_0(y),$$

$$\phi_1^t\big(\xi(t,y)\big)\doteq\begin{cases}\phi_{1,0}(y) & \text{if}\quad T(y)>t\quad\text{and}\quad\widetilde T(\tilde y)>t\quad\text{for }\tilde y=\psi_0(y)\,,\\ 0 & \text{otherwise},\end{cases}$$

$$\phi_2^t\big(\tilde\xi(t,\ \tilde y)\big)\doteq\begin{cases}\phi_{2,0}(\tilde y) & \text{if}\quad T(y)>t\quad\text{and}\quad\widetilde T(\tilde y)>t\quad\text{for }y=\psi_0^{-1}(\tilde y)\,,\\ 0 & \text{otherwise}.\end{cases}$$

If initially the point y is mapped to $\tilde y=\psi_0(y)$, then at any later time $t>0$ the point $\xi(t,y)$ along the u-characteristic starting from y is sent to the point $\tilde\xi(t,\ \tilde y)$ along the $\tilde u$-characteristic starting from $\tilde y=\psi_0(y)$. We thus transport the mass from the point $\Big(\xi(t,y)\,,\ u\big(t,\xi(t,y)\big)\,,\ \arctan u_x\big(t,\xi(t,y)\big)\Big)$ to the corresponding point $\Big(\tilde\xi(t,\tilde y)\,,\ \tilde u\big(t,\tilde\xi(t,\tilde y)\big)\,,\ \arctan\tilde u_x\big(t,\tilde\xi(t,\tilde y)\big)\Big)$ with $\tilde y=\psi_0(y)$, except in the case where blow up has occurred within time t along one (or both) characteristics $\xi(\cdot,y)$, $\tilde\xi(\cdot,\tilde y)$. In this later case, the mass is transported to the point ∞.

To check (3.10), it suffices to show that a.e.

$$\phi_1^t(\xi(t,y))\cdot u_x^2(t,\xi(t,y))\cdot\xi_x(t,y)=\phi_2^t\Big(\psi^t(\xi(t,y))\Big)\cdot(\psi^t)'(\xi(t,y))\cdot\xi_x(t,y)\cdot u_x^2\Big(t,\psi^t(\xi(t,y))\Big).$$

Since the relations $\tilde y=\psi_0(y)$, $\psi^t(\xi(t,y))=\tilde\xi(t,\ \tilde y)$, and $(\psi^t)'(\xi(t,y))\cdot\xi_x(t,y)=\tilde\xi_x(t,\ \psi_0(y))\cdot\psi_0'(y)$ all hold a.e., the desired identity holds a.e. on the complement of the set $\{y:\ \tilde y=\psi_0(y),\ T(y)>t,\ \tilde T(\tilde y)>t\}$ where both sides equal zero since $\phi_1^t(\xi(t,y))=\phi_2^t(\tilde\xi(t,\tilde y))=0$. The identity holds also a.e. on the set $\{y:\ \tilde y=\psi_0(y),\ T(y)>t,\ \tilde T(\tilde y)>t\}$ since there, in view of (2.9)-(2.10), it practically amounts to relation (3.10) for $(\phi_{1,0},\ \phi_{2,0},\ \psi_0)$.

In the following, our main goal is to provide an estimate on the time derivative of the function

$$J(t)=J^{(\psi^t,\phi_1^t,\phi_2^t)}\big(u(t),\tilde u(t)\big).$$

Throughout the remainder of this section, by $\{\tilde T(\tilde y)\gtrless t\}$ we understand the set of all $y\in I\!R$ such that $\psi_0(y)=\{\tilde y\}$ and $\tilde T(\tilde y)\gtrless t$. Since $u_x^2(t,\xi(t,y))\cdot\xi_x(t,y)=\bar u_x^2(y)$ on $\{T(y)>t\}$ by (2.9)-(2.10) and $\tilde u_x^2(t,\,\tilde\xi(t,\tilde y))\cdot\tilde\xi_x(t,\tilde y)=\tilde u_x^2(0,\tilde y)$ on $\{\tilde T(\tilde y)>t\}$, while $\psi^t(\xi(t,y))=\tilde\xi(t,\tilde y)$ for $\psi_0(y)=\{\tilde y\}$, performing the change of variables $y\mapsto\xi(t,y)$, we see that

$$
\begin{aligned}
J(t)=\int_{\{T(y)>t,\,\tilde T(\tilde y)>t\}} &\min\Big\{|\xi(t,y)-\tilde\xi(t,\tilde y)|+|u(t,\xi(t,y))-\tilde u(t,\tilde\xi(t,\tilde y))|\\
&\quad+\kappa_0|\arctan u_x(t,\xi(t,y))-\arctan\tilde u_x(t,\tilde\xi(t,\tilde y))|,\\
&\quad\kappa_0\Big(\pi+\arctan u_x(t,\xi(t,y))+\arctan\tilde u_x(t,\tilde\xi(t,\tilde y))\Big)\Big\}\,\phi_{1,0}(y)\,\bar u_x^2(y)\,dy\\
&+\kappa_0\int_{\{T(y)>t,\,\tilde T(\tilde y)>t\}}\Big(\frac{\pi}{2}+\arctan u_x(t,\xi(t,y))\Big)\Big(1-\phi_{1,0}(y)\Big)\,\bar u_x^2(y)\,dy\\
&+\kappa_0\int_{\{T(y)\le t\ \text{or}\ \tilde T(\tilde y)\le t\}}\Big(\frac{\pi}{2}+\arctan u_x(t,\xi(t,y))\Big)\,\bar u_x^2(y)\,dy\\
&+\kappa_0\int_{\{T(y)>t,\,\tilde T(\tilde y)>t\}}\Big(\frac{\pi}{2}+\arctan\tilde u_x(t,\tilde\xi(t,\,\tilde y))\Big)\Big(1-\phi_{2,0}(\tilde y)\Big)\,\tilde u_x^2(0,\tilde y)\,\psi_0'(y)\,dy\\
&+\kappa_0\int_{\{T(y)\le t\ \text{or}\ \tilde T(\tilde y)\le t\}}\Big(\frac{\pi}{2}+\arctan\tilde u_x(t,\,\tilde\xi(t,\,\tilde y))\Big)\,\tilde u_x^2(0,\tilde y)\,\psi_0'(y)\,dy.
\end{aligned}
$$

To simplify notation, let

$$
S(t)=\{T(y)>t,\,\tilde T(\tilde y)>t\},\qquad S^c(t)=I\!R-S(t),\tag{3.29}
$$

$$
\begin{aligned}
E(t,y)=\min\Big\{&|u(t,\xi(t,y))-\tilde u(t,\tilde\xi(t,\tilde y))|+\kappa_0|\arctan u_x(t,\xi(t,y))-\arctan\tilde u_x(t,\tilde\xi(t,\tilde y))|\\
&+|\xi(t,y)-\tilde\xi(t,\tilde y)|,\quad\kappa_0\Big(\pi+\arctan u_x(t,\xi(t,y))+\arctan\tilde u_x(t,\tilde\xi(t,\tilde y))\Big)\Big\}.
\end{aligned}\tag{3.30}
$$

Since for $h\ge 0$ we have

$$
S(t+h)\subset S(t),\qquad S^c(t)\subset S^c(t+h),\tag{3.31}
$$

we deduce that

$$
\begin{aligned}
J(t+h)-J(t)=&\int_{S(t)}\Big(E(t+h,y)-E(t,y)\Big)\,\phi_{1,0}(y)\,\bar u_x^2(y)\,dy\\
&-\int_{S(t)\setminus S(t+h)}E(t+h,y)\,\phi_{1,0}(y)\,\bar u_x^2(y)\,dy\\
&+\kappa_0\int_{S(t)}\Big(\arctan u_x(t+h,\xi(t+h,y))-\arctan u_x(t,\xi(t,y))\Big)\Big(1-\phi_{1,0}(y)\Big)\,\bar u_x^2(y)\,dy\\
&+\kappa_0\int_{S(t)}\Big(\arctan\tilde u_x(t+h,\tilde\xi(t+h,\tilde y))-\arctan\tilde u_x(t,\tilde\xi(t,\tilde y))\Big)\Big(1-\phi_{2,0}(\tilde y)\Big)\,\tilde u_x^2(0,\tilde y)\,\psi_0'(y)\,dy\\
&-\kappa_0\int_{S(t)\setminus S(t+h)}\Big(\frac{\pi}{2}+\arctan u_x(t+h,\xi(t+h,y))\Big)\Big(1-\phi_{1,0}(y)\Big)\,\bar u_x^2(y)\,dy\\
&-\kappa_0\int_{S(t)\setminus S(t+h)}\Big(\frac{\pi}{2}+\arctan\tilde u_x(t+h,\tilde\xi(t+h,\tilde y))\Big)\Big(1-\phi_{2,0}(\tilde y)\Big)\,\tilde u_x^2(0,\tilde y)\,\psi_0'(y)\,dy\\
&+\kappa_0\int_{S^c(t)}\Big(\arctan u_x(t+h,\xi(t+h,y))-\arctan u_x(t,\xi(t,y))\Big)\,\bar u_x^2(y)\,dy
\end{aligned}
$$

$$+\kappa_0\int_{S^c(t)}\Big(\arctan\tilde u_x(t+h,\tilde\xi(t+h,\tilde y))-\arctan\tilde u_x(t,\tilde\xi(t,\tilde y))\Big)\,\tilde u_x^2(0,\tilde y)\,\psi_0'(y)\,dy$$

$$+\kappa_0\int_{S^c(t+h)\setminus S^c(t)}\Big(\frac{\pi}{2}+\arctan u_x(t+h,\xi(t+h,y))\Big)\,\bar u_x^2(y)\,dy$$

$$+\kappa_0\int_{S^c(t+h)\setminus S^c(t)}\Big(\frac{\pi}{2}+\arctan\tilde u_x(t+h,\tilde\xi(t+h,\tilde y))\Big)\,\tilde u_x^2(0,\tilde y)\,\psi_0'(y)\,dy. \tag{3.32}$$

Noticing that $S(t)\setminus S(t+h)=S(t)\cap S^c(t+h)=S^c(t+h)\setminus S^c(t)$, we see that the combination of the fifth and ninth terms above, with that of the sixth and tenth, added to the second term, amount to

$$\kappa_0\int_{S(t)\setminus S(t+h)}\Big(\frac{\pi}{2}+\arctan u_x(t+h,\xi(t+h,y))\Big)\,\phi_{1,0}(y)\,\bar u_x^2(y)\,dy$$

$$+\kappa_0\int_{S(t)\setminus S(t+h)}\Big(\frac{\pi}{2}+\arctan\tilde u_x(t+h,\tilde\xi(t+h,\tilde y))\Big)\,\phi_{2,0}(\tilde y)\,\tilde u_x^2(0,\tilde y)\,\psi_0'(y)\,dy$$

$$-\int_{S(t)\setminus S(t+h)}E(t+h,y)\,\phi_{1,0}(y)\,\bar u_x^2(y)\,dy$$

$$=\kappa_0\int_{S(t)\setminus S(t+h)}\Big(\pi+\arctan u_x(t+h,\xi(t+h,y))+\arctan\tilde u_x(t+h,\tilde\xi(t+h,\tilde y))\Big)\,\phi_{1,0}(y)\,\bar u_x^2(y)\,dy$$

$$-\int_{S(t)\setminus S(t+h)}E(t+h,y)\,\phi_{1,0}(y)\,\bar u_x^2(y)\,dy \tag{3.33}$$

by (3.10) for $(\phi_{1,0},\,\phi_{2,0},\,\psi_0)$. In view of (3.29)-(3.30), on $S(t)\setminus S(t+h)$ we have that

$$E(t+h,y)=\pi+\arctan u_x(t+h,\xi(t+h,y))+\arctan\tilde u_x(t+h,\tilde\xi(t+h,\tilde y))$$

since at least one of the expressions $u_x(t+h,\xi(t+h,y))$ and $\tilde u_x(t+h,\tilde\xi(t+h,\tilde y))$ is precisely $-\infty$ on this set. Thus the whole expression (3.33) is identically zero. Therefore (3.32) yields

$$J(t+h)-J(t)=\int_{S(t)}\Big(E(t+h,y)-E(t,y)\Big)\,\phi_{1,0}(y)\,\bar u_x^2(y)\,dy$$

$$+\kappa_0\int_{S(t)}\Big(\arctan u_x(t+h,\xi(t+h,y))-\arctan u_x(t,\xi(t,y))\Big)\Big(1-\phi_{1,0}(y)\Big)\,\bar u_x^2(y)\,dy$$

$$+\kappa_0\int_{S(t)}\Big(\arctan\tilde u_x(t+h,\tilde\xi(t+h,\tilde y))-\arctan\tilde u_x(t,\tilde\xi(t,\tilde y))\Big)\Big(1-\phi_{2,0}(\tilde y)\Big)\,\tilde u_x^2(0,\tilde y)\,\psi_0'(y)\,dy$$

$$+\kappa_0\int_{S^c(t)}\Big(\arctan u_x(t+h,\xi(t+h,y))-\arctan u_x(t,\xi(t,y))\Big)\,\bar u_x^2(y)\,dy$$

$$+\kappa_0\int_{S^c(t)}\Big(\arctan\tilde u_x(t+h,\tilde\xi(t+h,\tilde y))-\arctan\tilde u_x(t,\tilde\xi(t,\tilde y))\Big)\,\tilde u_x^2(0,\tilde y)\,\psi_0'(y)\,dy. \tag{3.34}$$

In view of (3.29), we have $S^c(t)=\{T(y)\le t\}\cup\{\tilde T(\tilde y)\le t\}$. On the set $\{T(y)\le t\}$ we have $\arctan u_x(t+h,\xi(t+h,y))=\arctan u_x(t,\xi(t,y))=-\infty$ so that in the fourth term in (3.34) only the integral over $\{T(y) > t,\, \tilde{T}(\tilde{y}) \leq t\}$ might have a nonzero contribution. Thus the second and fourth terms in (3.34) combine to

$$\kappa_0 \int_{\{T(y)>t\}} \Big(\arctan u_x(t+h, \xi(t+h,y)) - \arctan u_x(t, \xi(t,y))\Big) \Big(1 - \phi_{1,0}(y)\Big)\, \bar{u}_x^2(y)\, dy$$

$$+\kappa_0 \int_{\{\tilde{T}(\tilde{y}) \leq t < T(y)\}} \Big(\arctan u_x(t+h, \xi(t+h,y)) - \arctan u_x(t, \xi(t,y))\Big)\, \phi_{1,0}(y)\, \bar{u}_x^2(y)\, dy. \quad (3.35)$$

Similarly, the third and fifth terms combine to

$$\kappa_0 \int_{\{\tilde{T}(\tilde{y})>t\}} \Big(\arctan \tilde{u}_x(t+h, \xi(t+h,\tilde{y})) - \arctan \tilde{u}_x(t, \xi(t,\tilde{y}))\Big) \Big(1 - \phi_{2,0}(\tilde{y})\Big)\, \tilde{u}_x^2(0,\tilde{y})\, \psi_0'(y)\, dy$$

$$+\kappa_0 \int_{\{T(y) \leq t < \tilde{T}(\tilde{y})\}} \Big(\arctan \tilde{u}_x(t+h, \xi(t+h,\tilde{y})) - \arctan \tilde{u}_x(t, \xi(t,\tilde{y}))\Big)\, \phi_{2,0}(\tilde{y})\, \tilde{u}_x^2(0,\tilde{y})\, \psi_0'(y)\, dy. \quad (3.36)$$

To transform suitably the first term in (3.34), let us denote by $E^1(t,y)$ the first expression in the minimum (3.30), and by $E^2(t,y)$ the second. If $E(t,y) = E^2(t,y)$, then

$$E(t+h,y) - E(t,y) \leq \kappa_0 \Big(\arctan u_x(t+h, \xi(t+h,y)) - \arctan u_x(t, \xi(t,y))$$

$$+ \arctan \tilde{u}_x(t+h, \tilde{\xi}(t+h,\tilde{y})) - \arctan \tilde{u}_x(t, \tilde{\xi}(t,\tilde{y}))\Big), \quad (3.37)$$

since $E(t+h,y) \leq E^2(t+h,y)$. On the other hand, if $E(t,y) = E^1(t,y)$, then the triangle inequality and the relation $E(t+h,y) \leq E^1(t+h,y)$ ensure that

$$E(t+h,y) - E(t,y) \leq \Big|\xi(t+h,y) - \xi(t,y) + \tilde{\xi}(t+h,\tilde{y}) - \tilde{\xi}(t,\tilde{y})\Big|$$

$$+\Big|u(t+h, \xi(t+h,y)) - \tilde{u}(t+h, \tilde{\xi}(t+h,\tilde{y})) - u(t, \xi(t,y)) + \tilde{u}(t, \xi(t,\tilde{y}))\Big| \quad (3.38)$$

$$+\kappa_0 \Big|\arctan u_x(t+h, \xi(t+h,y)) - \arctan u_x(t, \xi(t,y)) + \arctan \tilde{u}_x(t+h, \tilde{\xi}(t+h,\tilde{y})) - \arctan \tilde{u}_x(t, \tilde{\xi}(t,\tilde{y}))\Big|.$$

Letting $h \downarrow 0$ in (3.34), and taking into account (3.38) and the considerations preceding it, we deduce that

$$\limsup_{h\downarrow 0} \frac{J(t+h) - J(t)}{h} \leq \kappa_0 J_0(t) + \int_{S(t)} \phi_{1,0}(y)\, \bar{u}_x^2(y) \left|\frac{d}{dt}\, \xi(t,y) - \frac{d}{dt}\, \tilde{\xi}(t, \psi_0(y))\right| dy$$

$$+ \int_{S(t)} \phi_{1,0}(y)\, \bar{u}_x^2(y) \left|\frac{d}{dt}\, u(t, \xi(t,y)) - \frac{d}{dt}\, \tilde{u}(\tilde{\xi}(t, \psi_0(y)))\right| dy$$

$$+\kappa_0 \int_{S(t)} \phi_{1,0}(y)\, \bar{u}_x^2(y) \left|\frac{d}{dt}\ \arctan\ u_x(t, \xi(t,y)) - \frac{d}{dt}\ \arctan\ \tilde{u}_x(\tilde{\xi}(t, \psi_0(y)))\right| dy \quad (3.39)$$

where

$$J_0(t) = \int_{[T(y)>t]} \Big(1 - \phi_{1,0}(y)\Big) \left[\frac{d}{dt}\ \arctan\ u_x(t, \xi(t,y))\right] \bar{u}_x^2(y)\, dy$$

$$+\int_{[\tilde T(\tilde y)>t]} \Big(1-\phi_{2,0}(\tilde y)\Big) \Big[\frac{d}{dt} \text{ arctan } \tilde u_x(t,\xi(t,\tilde y))\Big] \, \tilde u_x^2(0,\tilde y)\, d\tilde y$$

$$+\int_{[\tilde T(\tilde y)\le t<T(y)]} \phi_{1,0}(y) \Big[\frac{d}{dt} \text{ arctan } u_x(t,\xi(t,y))\Big] \, \bar u_x^2(y)\, dy$$

$$+\int_{[T(y)\le t<\tilde T(\tilde y)]} \phi_{2,0}(\tilde y) \Big[\frac{d}{dt} \text{ arctan } \tilde u_x(t,\xi(t,\tilde y))\Big] \, \tilde u_x^2(0,\tilde y)\, d\tilde y \le 0,$$

the last inequality being true by (2.4).

Before proceeding with the further analysis of (3.39), we establish a few *a priori* bounds. From (2.22) we get

$$\Big|\frac{d}{dt}\,\xi(t,y) - \frac{d}{dt}\,\tilde\xi(t,\tilde y)\Big| = \Big|u(t,\xi(t,y)) - \tilde u(t,\tilde\xi(t,\tilde y))\Big|. \tag{3.40}$$

Also, note that if $v=\arctan z(t)$ and $\dot z = -\dfrac{z^2}{2}$, then

$$\dot v = -\frac{z^2}{2+2z^2} = -\frac{1}{2}\,\sin^2\, v.$$

Since $|\sin^2\alpha - \sin^2\beta| \le |\alpha-\beta|$ by the mean-value theorem as $|(\sin^2 z)'| = 2|\sin z\,\cos z| = |\sin(2z)| \le 1$, we infer three useful facts if we set $z=u_x(t,\xi(t,x))$. First of all,

$$\frac{d}{dt}\,\arctan u_x(t,\xi(t,y)),\ \ \frac{d}{dt}\,\arctan \tilde u_x(t,\tilde u(t,\tilde\xi(t,\tilde y)) \le 0. \tag{3.41}$$

Secondly,

$$\Big|\frac{d}{dt}\,\text{ arctan } u(t,\xi(t,y)) - \frac{d}{dt}\,\text{ arctan } \tilde u(t,\tilde\xi(t,\tilde y))\Big| \le \frac{1}{2}\,\Big|\arctan u_x(t,\xi(t,y)) - \arctan \tilde u_x(t,\tilde\xi(t,\tilde y))\Big|. \tag{3.42}$$

Furthermore, if $\arctan z \le -\dfrac{\pi}{4}$, then $\sin(\arctan z) \in [-1,-\dfrac{1}{\sqrt2}]$, so that

$$\frac{d}{dt}\arctan u_x(t,\xi(t,y)) \le -\frac{1}{4} \quad \text{if} \quad \arctan u_x(t,\xi(t,y)) \le -\frac{\pi}{4}. \tag{3.43}$$

On the other hand, using first (2.21) and then (2.18), we have

$$\Big|\frac{d}{dt}\,u(t,\xi(t,y_0)) - \frac{d}{dt}\,\tilde u(t,\tilde\xi(t,\tilde y_0))\Big| = \Big|\varphi(t,y_0) - \tilde\varphi(t,\tilde y_0))\Big|$$

$$= \frac{1}{4}\,\Big|\int_{\{T(y)>t\}} \text{sign}(y_0-y)\,\bar u_x^2(y)\,dy - \int_{\{\tilde T(\tilde y)>t\}} \text{sign}(\tilde y_0-\tilde y)\,\tilde u_x^2(0,\tilde y)\,d\tilde y\Big|$$

$$= \frac{1}{4}\,\Big|\int_{-\infty}^{y_0} \bar u_x^2(y)\chi_{[T(y)>t]}dy - \int_{-\infty}^{\tilde y_0} \tilde u_x^2(0,\tilde y)\,\chi_{[\tilde T(\tilde y)>t]}d\tilde y$$

$$-\int_{y_0}^{\infty} \bar u_x^2(y)\chi_{[T(y)>t]}dy - \int_{\tilde y_0}^{\infty} \tilde u_x^2(0,\tilde y)\,\chi_{[\tilde T(\tilde y)>t]}d\tilde y\Big|$$

$$= \frac{1}{4}\,\Big|\int_{-\infty}^{y_0} \bar u_x^2(y)\chi_{[T(y)>t]}dy - \int_{-\infty}^{y_0} \tilde u_x^2(0,\psi_0(y))\,\chi_{[\tilde T(\tilde y)>t]}\,\psi_0'(y)\,dy$$

$$- \int_{y_0}^{\infty} \bar{u}_x^2(y) \chi_{[T(y)>t]} dy - \int_{y_0}^{\infty} \tilde{u}_x^2(0, \psi_0(y)) \, \chi_{[\tilde{T}(\tilde{y})>t]} \, \psi_0'(y) \, dy \Big|$$

$$= \frac{1}{4} \Big| \int_{-\infty}^{y_0} \Big(1 - \phi_{1,0}(y) + \phi_{1,0}(y) \Big) \bar{u}_x^2(y) \chi_{[T(y)>t]} dy$$

$$- \int_{-\infty}^{y_0} \Big(1 - \phi_{2,0}(\psi_0(y)) + \phi_{2,0}(\psi_0(y)) \Big) \tilde{u}_x^2(0, \psi_0(y)) \, \chi_{[\tilde{T}(\tilde{y})>t]} \, \psi_0'(y) \, dy$$

$$- \int_{y_0}^{\infty} \Big(1 - \phi_{1,0}(y) + \phi_{1,0}(y) \Big) \bar{u}_x^2(y) \chi_{[T(y)>t]} dy$$

$$- \int_{y_0}^{\infty} \Big(1 - \phi_{2,0}(\psi_0(y)) + \phi_{2,0}(\psi_0(y)) \Big) \tilde{u}_x^2(0, \psi_0(y)) \, \chi_{[\tilde{T}(\tilde{y})>t]} \, \psi_0'(y) \, dy \Big|$$

after performing in the next to the last step in two of the integrals the change of variables $\tilde{y} = \psi_0(y)$. Since (3.10) for $(\psi_0, \phi_{1,0}, \phi_{2,0})$ ensures that $\phi_{1,0}(y) \, \bar{u}_x^2(y) = \phi_{2,0}(\psi_0(y)) \, \tilde{u}_x^2(0, \psi_0(y)) \, \psi_0'(y)$ a.e. on $S(t)$, we deduce that

$$\Big| \frac{d}{dt} \, u(t, \xi(t, y_0)) - \frac{d}{dt} \, \tilde{u}(t, \tilde{\xi}(t, \tilde{y}_0)) \Big|$$

$$\le \frac{1}{4} \int_{[T(y)>t]} \Big(1 - \phi_{1,0}(y) \Big) \, \bar{u}_x^2(y) \, dy + \frac{1}{4} \int_{[\tilde{T}(\tilde{y})>t]} \Big(1 - \phi_{2,0}(\tilde{y}) \Big) \tilde{u}_x^2(0, \tilde{y}) \, d\tilde{y}$$

$$+ \frac{1}{4} \int_{[\tilde{T}(\tilde{y}) \le t < T(y)]} \phi_{1,0}(y) \, \bar{u}_x^2(y) \, dy + \frac{1}{4} \int_{[T(y) \le t < \tilde{T}(\tilde{y})]} \phi_{2,0}(\tilde{y}) \, \tilde{u}_x^2(0, \tilde{y}) \, d\tilde{y}. \tag{3.44}$$

Let us now introduce the following sets

$$S_1 = \{ y \in I\!R : \; \arctan \, u_x(t, \xi(t, y)) \le -\frac{\pi}{4} \quad \text{and} \quad \arctan \, \tilde{u}_x(t, \tilde{\xi}(t, \tilde{y})) \le -\frac{\pi}{4} \},$$

$$S_2 = \{ y \in I\!R : \; \arctan \, u_x(t, \xi(t, y)) > -\frac{\pi}{4} \quad \text{and} \quad \arctan \, \tilde{u}_x(t, \tilde{\xi}(t, \tilde{y})) > -\frac{\pi}{4} \},$$

$$S_3 = \{ y \in I\!R : \; \arctan \, u_x(t, \xi(t, y)) > -\frac{\pi}{4} \ge \arctan \, \tilde{u}_x(t, \tilde{\xi}(t, \tilde{y})) \},$$

$$S_4 = \{ y \in I\!R : \; \arctan \, \tilde{u}_x(t, \tilde{\xi}(t, \tilde{y})) > -\frac{\pi}{4} \ge \arctan \, u_x(t, \xi(t, y)) \}.$$

The integral on the right-hand side of (3.44) over S_1 is, in view of (3.43), bounded from above by $|J_0(t)| = - \, J_0(t)$. The integral over S_2 is, in view of the formula for $J(t)$ preceding relation (2.29), bounded from above by $\dfrac{J(t)}{\pi \kappa_0}$. To evaluate the contribution over the integral over S_3, notice that the same formula for $J(t)$ yields

$$J(t) \ge \kappa_0 \int_{S(t) \cap S_3} \Big(\arctan u_x(t, \xi(t, y)) - \arctan \tilde{u}_x(t, \tilde{\xi}(t, \tilde{y})) \Big) \, \phi_{1,0}(y) \, \bar{u}_x^2(y) \, dy$$

$$+ \kappa_0 \int_{S(t) \cap S_3} \Big(\frac{\pi}{2} + \arctan u_x(t, \xi(t, y)) \Big) \Big(1 - \phi_{1,0}(y) \Big) \, \bar{u}_x^2(y) \, dy$$

$$+ \kappa_0 \int_{S^c(t) \cap S_3} \Big(\frac{\pi}{2} + \arctan u_x(t, \xi(t, y)) \Big) \, \bar{u}_x^2(y) \, dy$$

$$+ \kappa_0 \int_{S(t) \cap S_3} \Big(\frac{\pi}{2} + \arctan \tilde{u}_x(t, \tilde{\xi}(t, \, \tilde{y})) \Big) \Big(1 - \phi_{2,0}(\tilde{y}) \Big) \, \tilde{u}_x^2(0, \tilde{y}) \, \psi_0'(y) \, dy$$

$$+ \kappa_0 \int_{S^c(t) \cap S_3} \Big(\frac{\pi}{2} + \arctan \tilde{u}_x(t, \, \tilde{\xi}(t, \, \tilde{y})) \Big) \, \tilde{u}_x^2(0, \tilde{y}) \, \psi_0'(y) \, dy.$$

Using (3.10), the sum of the first term and the fourth term is larger than

$$\kappa_0 \int_{S(t)\,\cap\, S_3} \Big(\frac{\pi}{2} + \arctan u_x(t,\xi(t,y))\Big)\, \tilde{u}_x^2(0,\tilde{y})\, \psi_0'(y)\, dy$$

$$\geq \frac{\kappa_0 \pi}{4} \int_{S(t)\,\cap\, S_3} \tilde{u}_x^2(0,\tilde{y})\, \psi_0'(y)\, dy = \frac{\kappa_0 \pi}{4} \int_{S(t)\,\cap\, S_3} \tilde{u}_x^2(0,\tilde{y})\, d\tilde{y}$$

$$\geq \frac{\kappa_0 \pi}{4} \int_{S(t)\,\cap\, S_3} \Big(1-\phi_{2,0}(\tilde{y})\Big)\, \tilde{u}_x^2(0,\tilde{y})\, d\tilde{y} = \frac{\kappa_0 \pi}{4} \int_{[\tilde{T}(\tilde{y})>t]\,\cap\, S_3} \Big(1-\phi_{2,0}(\tilde{y})\Big)\, \tilde{u}_x^2(0,\tilde{y})\, d\tilde{y},$$

with the last equality enforced by $S_3 \subset [T(y) > t]$. The second term is bounded from below by

$$\frac{\kappa_0 \pi}{4} \int_{S(t)\,\cap\, S_3} \Big(1-\phi_{1,0}(y)\Big)\, \bar{u}_x^2(y)\, dy = \frac{\kappa_0 \pi}{4} \int_{[T(y)>t]\,\cap\, S_3} \Big(1-\phi_{1,0}(y)\Big)\, \bar{u}_x^2(y)\, dy$$

$$-\frac{\kappa_0 \pi}{4} \int_{[T(y)>t\geq \tilde{T}(\tilde{y})]\,\cap\, S_3} \Big(1-\phi_{1,0}(y)\Big)\, \bar{u}_x^2(y)\, dy = \frac{\kappa_0 \pi}{4} \int_{[T(y)>t]\,\cap\, S_3} \Big(1-\phi_{1,0}(y)\Big)\, \bar{u}_x^2(y)\, dy$$

$$-\frac{\kappa_0 \pi}{4} \int_{[T(y)>t\geq \tilde{T}(\tilde{y})]\,\cap\, S_3} \bar{u}_x^2(y)\, dy + \frac{\kappa_0 \pi}{4} \int_{[T(y)>t\geq \tilde{T}(\tilde{y})]\,\cap\, S_3} \phi_{1,0}(y)\, \bar{u}_x^2(y)\, dy$$

if we recall (3.10). The third term is bounded from below by

$$\frac{\kappa_0 \pi}{4} \int_{S^c(t)\,\cap\, S_3} \bar{u}_x^2(y)\, dy \geq \frac{\kappa_0 \pi}{4} \int_{[T(y)>t\geq \tilde{T}(\tilde{y})]\,\cap\, S_3} \bar{u}_x^2(y)\, dy.$$

Summing up, we get

$$J(t) \geq \frac{\kappa\pi}{4} \Big\{ \int_{[T(y)>t]\,\cap S_3} \Big(1-\phi_{1,0}(y)\Big)\, \bar{u}_x^2(y)\, dy + \int_{[\tilde{T}(\tilde{y})>t]\,\cap S_3} \Big(1-\phi_{2,0}(\tilde{y})\Big) \tilde{u}_x^2(0,\tilde{y})\, d\tilde{y}$$

$$+ \int_{[\tilde{T}(\tilde{y})\leq t<T(y)]\,\cap S_3} \phi_{1,0}(y)\, \bar{u}_x^2(y)\, dy + \int_{[T(y)\leq t<\tilde{T}(\tilde{y})]\,\cap S_3} \phi_{2,0}(\tilde{y})\, \tilde{u}_x^2(0,\tilde{y})\, d\tilde{y} \Big\},$$

since the last term on the right is zero as $S_3 \subset [T(y) > t]$. A similar relation holds with S_4 instead of S_3. Consequently, putting together all this information about the various inequalities valid on the disjoint sets S_1, S_2, S_3, and S_4, we conclude by (3.44) that

$$\Big| \frac{d}{dt}\, u(t,\xi(t,y_0)) - \frac{d}{dt}\, \tilde{u}(t,\tilde{\xi}(t,\tilde{y}_0)) \Big| \leq -J_0(t) + \frac{3J(t)}{\kappa_0 \pi}. \tag{3.45}$$

To obtain now a suitable estimate on

$$\limsup_{h\downarrow 0} \frac{J(t+h)-J(t)}{h} = \limsup_{h\downarrow 0} \int_{S(t)} \frac{E(t+h,y)-E(t,y)}{h}\, \phi_{1,0}(y)\, \bar{u}_x^2(y)\, dy + \kappa_0\, J_0(t) \tag{3.46}$$

we distinguish two cases. If $E(t, y)$ is the second component $E^2(t, y)$ of the minimum in (3.30), then by (3.37) and (3.41) we can estimate the contribution of the first term in (3.46) by zero from above. On the other hand, if the minimum is $E^1(t, y)$, then the first integral term in (3.46) is not larger (pointwise) than the nonnegative expression

$$\Big(E(t,y) + \frac{3J(t)}{\kappa_0 \pi} - J_0(t)\Big)\, \phi_{1,0}(y)\, \bar{u}_x^2(y)$$

in view of the estimates (3.40), (3.42), and (3.45). We conclude that

$$\limsup_{h\downarrow 0} \frac{J(t+h)-J(t)}{h} \leq J(t) + \Big(\frac{3J(t)}{\kappa_0 \pi} - J_0(t)\Big)\, \|\bar u_x^2\|_{L^1(I\!R)} + \kappa_0\, J_0(t).$$

Since $J_0(t) \leq 0$, choosing the constant $\kappa_0 \doteq \|\bar u_x^2\|_{\mathbf{L}^1(I\!R)}$ we now have

$$\frac{d}{dt} J^{(\psi^t,\phi_1^t,\phi_2^t)}\big(u(t),\, v(t)\big) \leq 2\, J^{(\psi^t,\phi_1^t,\phi_2^t)}\big(u(t),\, v(t)\big)\,.$$

Optimizing over all triples $(\psi^0, \phi_1^0, \phi_2^0)$ we conclude

$$J\big(u(t),\, v(t)\big) \leq J\big(u(0),\, v(0)\big)\, e^{2t}, \qquad t \geq 0. \tag{3.47}$$

Summing up the considerations made above, we proved the following result.

Theorem 2. *The trajectories $t \mapsto u(t)$ of (1.1) constructed in Theorem 1 are locally Lipschitz continuous as maps from $[0,\infty)$ into the metric space $\mathcal{X}$ equipped with the distance functional J. Moreover, the distance between two trajectories is also locally Lipschitz continuous as a map from $[0,\infty)$ into $\mathcal{X}$.*

4 - Concluding remarks

The following example shows that, in some sense, our distance functional J in (3.11) is "sharp". Indeed, the convergence of the initial data in $\mathbf{L}^\infty(I\!R) \cap \mathbf{L}^1(I\!R)$ together with the weak convergence of the derivatives $\bar u_x$ and $\bar u_x^2$ in $\mathbf{L}^2(I\!R)$ does not guarantee the convergence of the corresponding solutions at later times $t > 0$.

Example 1. Consider the functions $f, g : [0,1] \mapsto [0,1]$ defined as

$$f(x) \doteq \begin{cases} 1-2x & \text{if} \quad x \in [0,\ 1/2], \\ 0 & \text{if} \quad x \in [1/2\,,\ 1], \end{cases} \qquad\qquad g(x) \doteq \begin{cases} 1-3x & \text{if} \quad x \in [0,\ 1/6], \\ 1/2 & \text{if} \quad x \in [1/6\,,\ 1/2], \\ 1-x & \text{if} \quad x \in [1/2\,,\ 1]. \end{cases}$$

Observe that

$$\int_0^1 f'(x)\, dx = \int_0^1 g'(x)\, dx = -1\,, \qquad\qquad \int_0^1 \big[f'(x)\big]^2\, dx = \int_0^1 \big[g'(x)\big]^2\, dx = 2\,.$$

Next, consider the function

$$h(x) \doteq \begin{cases} 1-|x| & \text{if} \quad |x| \leq 1, \\ 0 & \text{if} \quad |x| \geq 1, \end{cases}$$

and define the sequences of initial values

$$\bar u_n(x) = \begin{cases} h(x) & \text{if} \quad x \notin [0,1], \\ h(i/n) + \frac{1}{n} f(nx - i + 1) & \text{if} \quad x \in \big[\frac{i-1}{n}\,,\ \frac{i}{n}\big] \end{cases} \qquad\qquad i = 1, \ldots, n,$$

$$\bar v_n(x) = \begin{cases} h(x) & \text{if} \quad x \notin [0,1], \\ h(i/n) + \frac{1}{n} g(nx - i + 1) & \text{if} \quad x \in \left[\frac{i-1}{n}\,,\ \frac{i}{n}\right] \qquad i = 1,\ldots,n. \end{cases}$$

Letting $n \to \infty$ we now have the strong convergence $\left\|\bar u_n - \bar v_n\right\|_{\mathbf{L}^\infty(I\!R)} \to 0$. Moreover, by construction it is easy to see that at each point $x \in [0,1]$,

$$\lim_{n\to\infty} \int_0^x \Big((\bar u_n)_x(y) - (\bar v_n)_x(y) \Big)\, dy = \lim_{n\to\infty} \int_0^x \Big((\bar u_n)_x^2(y) - (\bar v_n)_x^2(y) \Big)\, dy = 0$$

so that in $\mathbf{L}^2[0,1]$ one has the weak convergence

$$(\bar u_n)_x - (\bar v_n)_x \rightharpoonup 0\,, \qquad\qquad (\bar u_n)_x^2 - (\bar v_n)_x^2 \rightharpoonup 0\,,$$

since both sequences are bounded in $\mathbf{L}^2[0,1]$ and the previous observation identifies the zero function as the only possible weak limit. However

$$u(t) \doteq \lim_{n\to\infty} u_n(t) \ \neq \ \lim_{n\to\infty} v_n(t) \doteq v(t)$$

for every $t \in (2/3, 1)$, where $T = 2/3$ is the time at which the gradients of the functions v_n blow up. The last assertion follows at once from (2.27). $\diamondsuit$

We also would like to highlight the importance of requiring that the transport map ψ in (3.10) be monotone nondecreasing. If in (3.11) we were to take the minimization over all maps ψ, not necessarily monotone, we would obtain the classical Kantorovich-Rubinstein distance between measures, which generates the weak topology on the space of bounded, positive measures [V]. By restricting ourselves to monotone nondecreasing maps ψ, the corresponding distance functional generates a much stronger topology.

Example 2. Consider the sequence of Lipschitz functions

$$u^m(x) \doteq \begin{cases} 0 & \text{if} \quad x \notin [0,1] \\ x - (i-1)/m & \text{if} \quad (i-1)/m \leq x \leq (2i-1)/2m \qquad i = 1,\ldots,m\,. \\ i/m - x & \text{if} \quad (2i-1)/2m \leq x \leq i/m \end{cases}$$

In this case, $u_x = \pm 1$ and $\arctan u_x = \pm\pi/4$. The corresponding measures μ^{u^m} defined at (3.9) converge weakly to the measure μ on $I\!R^2 \times [-\pi/2\,,\ \pi/2]$ defined as

$$\mu(A) \doteq \frac{1}{2}\,\text{meas}\Big\{x \in [0,1]\,;\quad (x,\,0,\,\pi/4) \in A\Big\} + \frac{1}{2}\,\text{meas}\Big\{x \in [0,1]\,;\quad (x,\,0,\,-\pi/4) \in A\Big\}\,.$$

In particular, these measures form a Cauchy sequence in the Kantorovich metric. However, these same functions u^m do not form a Cauchy sequence w.r.t. the distance J. Indeed, let $m < n$. Notice that in our case $\kappa_0 \doteq \|\bar u_x^2\|_{\mathbf{L}^1(I\!R)} = 1$. Consider the open intervals

$$I_i^{m+} = \left]\frac{i-1}{m}\,,\frac{2i-1}{2m}\right[\,, \qquad\qquad I_i^{m-} = \left]\frac{2i-1}{2m}\,,\frac{i}{m}\right[\,,$$

where u_x^m takes the values $+1$ and -1, respectively. Define the intervals I_j^{n+}, I_j^{n-} similarly. Now consider any transportation plan (ψ, ϕ_1, ϕ_2) relating u^m to u^n via (3.10), with ψ non-decreasing.

For each $i = 1, \ldots, m$, call ν_i the number of distinct intervals I_j^{n+} which intersect the image $\psi(I_i^{m+})$. Since ψ is monotone, if $\nu_i \geq 2$, this implies that the image $\psi(I_i^{m+})$ entirely covers at least $\nu_i - 1$ distinct intervals I_j^{n-}. Because $u_x^m = 1$ on I_i^{m+} and $u_x^n = -1$ on each I_j^{n-}, on the union of these intervals I_j^{n-} we have $\arctan u_x^m(\psi(x)) = -\arctan u_x^n(x) = \frac{\pi}{4}$ so that the integrand contribution from the first two parts of (3.11) is pointwise larger than $\frac{\pi}{2}\,\phi_1(x) + \frac{\pi}{4}\,(1-\phi_1(x)) \geq \frac{\pi}{4}$, which therefore accounts for a cost $\geq \pi(\nu_i - 1)/8n$. Next, if $\nu_1 + \ldots + \nu_m = n^* < n$, there must be $n - n^*$ intervals $I_{j(1)}^{n+}, \ldots, I_{j(n^*-n)}^{n+}$ which do not intersect any of the sets $\psi(I_i^{m+})$, for $i = 1, \ldots, m$. These intervals must be contained in the image of some I_i^{m-}, or in the image of the set $\psi\big(I\!\!R\setminus[0,1]\big)$, where $u^m \equiv 0$. This accounts for a cost $\geq \pi(n-n^*)/4n$.

The above argument shows that, for any $m < n$, the cost of any transportation plan relating u^m to u^n is bounded below by

$$J^{(\psi,\phi_1,\phi_2)}(u^m, u^n) \geq \frac{\pi}{8n} \cdot \max\left\{ \sum_{i=1}^m (\nu_i - 1)\,,\ n - \sum_{i=1}^m \nu_i \right\} \geq \frac{\pi}{8n} \cdot \frac{n-m}{2}\,.$$

For any fixed m, the right hand side approaches $\pi/16$ as $n \to \infty$. Therefore, the above is not a Cauchy sequence, in our transportation metric. $\diamondsuit$

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