DISCRETE AND CONTINUOUS DYNAMICAL SYSTEMS
Volume **00**, Number **0**, Xxxx **XXXX**

Website: http://AIMsciences.org

pp. **000–000**

WELLPOSEDNESS FOR A PARABOLIC-ELLIPTIC SYSTEM

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Abstract. We show existence of a unique, regular global solution of the parabolic-elliptic system $u_t + f(t,x,u)_x + g(t,x,u) + P_x = (a(t,x)u_x)_x$ and $-P_{xx} + P = h(t,x,u,u_x) + k(t,x,u)$ with initial data $u|_{t=0} = u_0$. Here $\inf_{(t,x)} a(t,x) > 0$. Furthermore, we show that the solution is stable with respect to variation in the initial data u_0 and the functions f, g etc. Explicit stability estimates are provided. The regularized generalized Camassa–Holm equation is a special case of the model we discuss.

1. Introduction.

In this paper we study a system that constitutes a generalized and regularized Camassa–Holm equation. More specifically, we consider the system

$$\begin{aligned} u_t + \big(f(t,x,u)\big)_x + g(t,x,u) + P_x &= \big(a(t,x)u_x\big)_x, \\ -P_{xx} + P &= h(t,x,u,u_x) + k(t,x,u), \\ u|_{t=0} &= u_0, \end{aligned} \tag{1.1}$$

on the domain $(t,x) \in \Pi_T := [0,T] \times \mathbb{R}$. Consider the special case, where we in particular assume the inviscid case $a = 0$, given by

$$\begin{aligned} u_t + \frac{1}{2}(u^2)_x + P_x &= 0, \\ -P_{xx} + P &= u^2 + \frac{1}{2}(u_x)^2 + \gamma u, \\ u|_{t=0} &= u_0. \end{aligned} \tag{1.2}$$

This system formally reduces, by applying the operator $u \mapsto u - u_{xx}$ to the first equation in (1.2), to the Camassa–Holm equation

$$u_t - u_{txx} + \gamma u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

Due to the presence of the term $a(t,x)$ in (1.1), we see that it constitutes a *viscous* regularization of a spatially and temporally varying Camassa–Holm equation. In this paper we address the question of wellposedness of the system (1.1). In particular, we focus on stability of solutions with respect to variation not only in the initial data, but also variation with respect to the functions f, a, etc. Furthermore, we are interested in the vanishing viscosity limit of (1.1), i.e., when $a \to 0$, and this problem is discussed in a subsequent paper [4].

Formally, by applying the operator $(1 - \partial_x^2)^{-1}$ to the second equation, we see that the system (1.1) can be written as the integro-differential Cauchy problem

$$\begin{aligned} u_t &+ \big(f(t,x,u)\big)_x + g(t,x,u) \\ &+ \frac{1}{2}\frac{\partial}{\partial x}\int_{\mathbb{R}} e^{-|x-y|}\Big(h\big(t,y,u(t,y),u_x(t,y)\big) + k\big(t,y,u(t,y)\big)\Big)dy = \big(a(t,x)u_x\big)_x, \end{aligned}$$

2000 *Mathematics Subject Classification.* Primary: 35A05; Secondary: 35B30.

Key words and phrases. Parabolic-elliptic system, wellposedness, Camassa–Holm equation.

Partially supported by the BeMatA program of the Research Council of Norway and the European network HYKE, contract HPRN-CT-2002-00282. KHK has been supported in part by an Outstanding Young Investigators Award from the Research Council of Norway.

$$u|_{t=0} = u_0.$$

The Camassa–Holm equation [2] has received extensive attention the last decade. It can be used as a model of unidirectional gravitational shallow water waves on a flat bottom when γ is positive [8]. The velocity is given by u. The equation possesses intriguing properties; it has a bi-Hamiltonian structure, it is completely integrable, and it experiences wave breaking, that is, the breakdown of smooth solutions in finite time, for a large class of initial data. The Cauchy problem has been extensively studied. We refer to [5, 7, 14, 15] and references therein for more complete information. Furthermore, Dai [6] derived the equation

$$u_t - u_{txx} + 3uu_x = \gamma(2u_x u_{xx} + uu_{xxx})$$

as a model for small amplitude radial deformation waves in cylindrical compressible hyperelastic rods.

Another related example in the inviscid case is that of a simplified model for radiating gases given by the hyperbolic-elliptic system

$$u_t + \frac{1}{2}(u^2)_x = -q_x, \quad -q_{xx} + q = -u_x, \quad u|_{t=0} = u_0, \tag{1.3}$$

see [9, 11]. However, this system is not covered by our assumptions because of presence of the viscous term in (1.1) and the assumptions on the functions f, h.

In this paper we study the Cauchy problem for the viscous regularization. More precisely, we study the system (1.1) where a is bounded away from zero. We establish short-time existence of a unique smooth solution by showing a contraction mapping principle. An energy estimate makes it possible to extend the result to a global result in time. Stability is established by a homotopy argument, see also [1, 3], where one connects by a smooth path two distinct solutions u and v with different data and coefficients. Our main result reads: Let u and v be solutions of

$$\begin{aligned} u_t + \big(f_0(t,x,u)\big)_x + g_0(t,x,u) + P_x &= \big(a_0(t,x)u_x\big)_x, \\ -P_{xx} + P &= h_0(t,x,u,u_x) + k_0(t,x,u), \\ u|_{t=0} &= u_0, \\ v_t + \big(f_1(t,x,v)\big)_x + g_1(t,x,v) + Q_x &= \big(a_1(t,x)v_x\big)_x, \\ -Q_{xx} + Q &= h_1(t,x,v,v_x) + k_1(t,x,v), \\ v|_{t=0} &= v_0, \end{aligned} \tag{1.4}$$

respectively. In addition to certain regularity assumptions (essentially boundedness of various derivatives) on the functions, we assume that

$$\begin{aligned} h_i(t,x,u,q)q - \frac{1}{2}f_{i,uu}(t,x,u)q^3 &\le C_0(u^2+q^2), \ \ i=0,1, \\ |h_i(t,x,u,q)| &\le C_0(|u| + u^2 + q^2), \ \ i=0,1, \\ f_{0,x}(t,x,0) &= f_{1,x}(t,x,0), \\ g_0(t,x,0) &= g_1(t,x,0), \\ k_0(t,x,0) &= k_1(t,x,0), \\ h_{0,u}(t,x,0,q) &= h_{1,u}(t,x,0,q), \\ h_{0,q}(t,x,u,0) &= h_{1,q}(t,x,u,0). \end{aligned}$$

Then we show that

$$\begin{aligned} \|u(t,\cdot\,) - v(t,\cdot\,)\|_{H^1(\mathbb{R})} \le \|u_0 - v_0\|_{H^1(\mathbb{R})}\, e^{K_0 t} \\ + K_1 t\big[\, \|f_{1,ux} - f_{0,ux}\|_{L^\infty(\mathcal{I})} + \|f_{1,u} - f_{0,u}\|_{L^\infty(\mathcal{I})} \\ + \|g_{1,u} - g_{0,u}\|_{L^\infty(\mathcal{I})} \end{aligned} \tag{1.5}$$

$$
\begin{aligned}
&+\|a_{1,x}-a_{0,x}\|_{L^\infty(\Pi_T)}+\|a_1-a_0\|_{L^\infty(\Pi_T)}\\
&+\|h_{0,u}-h_{1,u}\|_{L^\infty(\mathcal{J})}+\|k_{0,u}-k_{1,u}\|_{L^\infty(\mathcal{I})}\\
&+\|h_{0,q}-h_{1,q}\|_{L^\infty(\mathcal{J})}\Big],
\end{aligned}
$$

where

$$
\mathcal{I}:=\Pi_T\times\Big[-\frac{C_2}{\sqrt{2}},\frac{C_2}{\sqrt{2}}\Big],\quad \mathcal{J}:=\mathcal{I}\times\Big[-\frac{C_2}{\sqrt{2}},\frac{C_2}{\sqrt{2}}\Big],
$$

and K_0, K_1 and C_2 are constants that may only depend on the time horizon T and on the viscous coefficient a.

2. **Existence and uniqueness.** We consider the parabolic-elliptic initial-value problem

$$
\begin{aligned}
u_t+\big(f(t,x,u)\big)_x+g(t,x,u)+P_x&=\big(a(t,x)u_x\big)_x,\\
-P_{xx}+P&=h(t,x,u,u_x)+k(t,x,u),\\
u|_{t=0}&=u_0,
\end{aligned}
\tag{2.1}
$$

for $(t,x)\in\Pi_T$, where $T>0$.

Remark 2.1. *Since $e^{-|x|}/2$ is the Green's function of the operator $u\mapsto -u_{xx}+u$, (2.1) is equivalent to the following integro-differential Cauchy problem*

$$
\begin{aligned}
&u_t+\big(f(t,x,u)\big)_x+g(t,x,u)+P_x=\big(a(t,x)u_x\big)_x,\\
&P(t,x)=\frac{1}{2}\int_{\mathbb{R}}e^{-|x-y|}\Big(h\big(t,y,u(t,y),u_x(t,y)\big)+k\big(t,y,u(t,y)\big)\Big)dy,\\
&u|_{t=0}=u_0.
\end{aligned}
\tag{2.2}
$$

Regarding the initial data $u_0\colon\mathbb{R}\to\mathbb{R}$ we assume

$$
u_0\in H^\ell(\mathbb{R})\qquad\text{for some }\ \ell\geq 2. \tag{2.3}
$$

In the following sections we shall assume:

(**H**.**1**) the coefficients a, f, g, h, and k are smooth;

(**H**.**2**) there exists a constant $C_0>0$ such that the following hold

$$
f(\,\cdot\,,\,\cdot\,,0)=0, \tag{2.4a}
$$

$$
\frac{1}{C_0}\leq a(\cdot,\cdot)\leq C_0, \tag{2.4b}
$$

$$
\Big\|\frac{\partial^i f}{\partial x^i}(\,\cdot\,,\,\cdot\,,u)\Big\|_{L^\infty}\leq C_0\,|u|,\quad 1\leq i\leq \ell+1, \tag{2.4c}
$$

$$
\Big\|\frac{\partial^j a}{\partial x^j}\Big\|_{L^\infty},\ \Big\|\frac{\partial^{i+j} f}{\partial x^i\partial u^j}\Big\|_{L^\infty}\leq C_0,\quad j\geq 1,\ 2\leq i+j\leq \ell+1, \tag{2.4d}
$$

$$
\|k_x(\,\cdot\,,\,\cdot\,,u)\|_{L^\infty},\ \Big\|\frac{\partial^i g}{\partial x^i}(\,\cdot\,,\,\cdot\,,u)\Big\|_{L^\infty}\leq C_0\,|u|,\quad 0\leq i\leq \ell, \tag{2.4e}
$$

$$
\Big\|\frac{\partial^{i+j} g}{\partial x^i\partial u^j}\Big\|_{L^\infty}\leq C_0,\quad j\geq 1,\ 1\leq i+j\leq \ell+1, \tag{2.4f}
$$

$$
u\longmapsto\Big\|\frac{\partial^{i+j} k}{\partial x^i\partial u^j}(\,\cdot\,,\,\cdot\,,u)\Big\|_{L^\infty}\ \text{ is in }\ L^\infty_{\mathrm{loc}}(\mathbb{R}),\quad 0\leq i+j\leq \ell+1, \tag{2.4g}
$$

$$
(u,q)\longmapsto\Big\|\frac{\partial^{i+j+p} h}{\partial x^i\partial u^j\partial q^p}(\,\cdot\,,\,\cdot\,,u,q)\Big\|_{L^\infty}\ \text{ is in }\ L^\infty_{\mathrm{loc}}(\mathbb{R}^2),\quad 0\leq i+j+p\leq \ell+1, \tag{2.4h}
$$

$$
h(t,x,u,q)q-\frac{1}{2}f_{uu}(t,x,u)q^3\leq C_0(u^2+q^2), \tag{2.4i}
$$

$$
\|h(\,\cdot\,,\,\cdot\,,u,q)\|_{L^\infty}\leq C_0(|u|+u^2+q^2), \tag{2.4j}
$$

for $(t,x)\in\Pi_T$ and $u,q\in\mathbb{R}$.

In particular, (2.4d) shows that f_{uu} is bounded, and hence f is at most quadratic in u.

Example 2.2. *Consider the viscosity approximation of the generalized Camassa–Holm equation*

$$\begin{aligned}\frac{\partial u_\varepsilon}{\partial t}+\gamma u_\varepsilon\frac{\partial u_\varepsilon}{\partial x}+\frac{\partial P_\varepsilon}{\partial x}&=\varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2},\\ -\frac{\partial^2 P_\varepsilon}{\partial x^2}+P_\varepsilon&=\frac{1}{2}g(u_\varepsilon)+\frac{\gamma}{2}\Big(\frac{\partial u_\varepsilon}{\partial x}\Big)^2-\frac{\gamma}{2}u_\varepsilon^2+\kappa u_\varepsilon,\\ u_\varepsilon|_{t=0}&=u_{\varepsilon,0},\end{aligned}$$

where $u_{\varepsilon,0}$ satisfies (2.3)*. With $f=\gamma u^2/2$, $h=\gamma q^2/2$, and $k=g(u)-\frac{1}{2}u^2+\kappa u$, we see that our assumptions are fulfilled, see* [4]*. With $\varepsilon=0$ the system formally reduces to, with $u=u_\varepsilon$,*

$$u_t-u_{txx}+\frac{1}{2}g(u)_x+\kappa u_x=\gamma(2u_xu_{xx}+uu_{xxx}),$$

which is denoted the hyperelastic-rod wave equation, see [6]*.*

The main result of this section is the following.

Theorem 2.3. *Let $T>0$. Assume* (**H.1**)*,* (**H.2**)*, and* (2.3)*. Then there exists a unique smooth solution $u\in C\big([0,T];H^\ell(\mathbb{R})\big)$ of the Cauchy problem* (2.1)*.*

The proof of this theorem is divided into a local (in time) existence result, which is discussed first, and an extension theorem.

2.1. **Local existence and uniqueness.** We begin by proving the following result.

Theorem 2.4 (Local existence)**.** *Assume* (**H.1**)*,* (**H.2**)*, and* (2.3)*. There exists a positive time T_0 such that* (2.1) *has a unique, local, smooth solution defined on* $[0,T_0]\times\mathbb{R}$.

Let $G=G(t,s,x,y)$ be the Green's function associated to the operator $u\mapsto u_t-\big(a(t,x)u_x\big)_x$, see [10, Chapter IV, Section 11]. Let $T_0>0$. Define the following quantities:

$$\begin{aligned}U(t,x)&:=\int_{\mathbb{R}}G(t,0,x,y)u_0(y)dy,\\ \Lambda_1(u)(t,x)&:=\int_0^t\int_{\mathbb{R}}G(t,s,x,y)\Big[\big(f(s,y,u(s,y))\big)_y+g\big(s,y,u(s,y)\big)\Big]dyds,\\ \Lambda_2(u)(t,x)&:=\frac{1}{2}\int_0^t\int_{\mathbb{R}\times\mathbb{R}}G(t,s,x,y)e^{-|y-\xi|}\Big[h\big(s,\xi,u(s,\xi),u_\xi(s,\xi)\big)\\ &\qquad\qquad+k\big(s,\xi,u(s,\xi)\big)\Big]d\xi dyds,\\ \Lambda(u)&:=U-\Lambda_1(u)-\Lambda_2(u),\end{aligned}$$

for each $(t,x)\in\Pi_{T_0}$, $u\in C([0,T_0];H^1(\mathbb{R}))$.

The following lemmas are needed.

Lemma 2.5. *Let $T_0>0$ and assume* (2.3)*,* (**H.1**)*,* (**H.2**)*. Then*

$$U\in C([0,T_0];H^1(\mathbb{R})),\qquad \frac{\partial U}{\partial t}\in C([0,T_0];H^{-1}(\mathbb{R})). \tag{2.5}$$

Moreover, U is the smooth solution of

$$\begin{cases}U_t=(a(t,x)U_x)_x, & t>0,\ x\in\mathbb{R},\\ U|_{t=0}=u_0.\end{cases} \tag{2.6}$$

Proof. Due to regularity of u_0, the fact that U solves (2.6) is consequence of [10, Chapter IV, Section 14]. We have to prove (2.5). From (2.6) and (2.4d), we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}(U^2+U_x^2)dx &= \int_{\mathbb{R}}\big(UU_t+U_xU_{tx}\big)dx\\ &= \int_{\mathbb{R}}\big(U(aU_x)_x+U_x(a_xU_x+aU_{xx})_x\big)dx\\ &= -\int_{\mathbb{R}}\big(aU_x^2+a_xU_xU_{xx}+aU_{xx}^2\big)dx\\ &= -\int_{\mathbb{R}}\big(aU_x^2-\frac{a_{xx}}{2}U_x^2+aU_{xx}^2\big)dx \le \frac{C_0}{2}\|U(t,\cdot)\|^2_{H^1(\mathbb{R})}.\end{aligned}$$

Clearly this implies that $U \in L^\infty([0,T_0];H^1(\mathbb{R}))$. Writing U as convolution of the Green's function G and of the initial condition u_0, the first part of (2.5) is consequence of [10, Chapter 13, (13.3)]. For the second part, we consider equation (2.6) and observe that $U_{xx} \in C([0,T_0];H^{-1}(\mathbb{R}))$. □

Lemma 2.6. *Let $T_0 > 0$ and assume* (2.3), (**H.1**), *and* (**H.2**). *Then*

$$\Lambda_1(u) \in C([0,T_0];H^1(\mathbb{R})), \qquad \frac{\partial\Lambda_1(u)}{\partial t} \in C([0,T_0];H^{-1}(\mathbb{R})), \tag{2.7}$$

for each $u \in C([0,T_0];H^1(\mathbb{R}))$, and

$$\|\Lambda_1(u_1)-\Lambda_1(u_2)\|_{L^\infty([0,T_0];H^1(\mathbb{R}))} \le C_1e^{C_1T_0}\sqrt{T_0}\|u_1-u_2\|^2_{L^\infty([0,T_0];H^1(\mathbb{R}))}, \tag{2.8}$$

for each u_1, $u_2 \in C([0,T_0];H^1(\mathbb{R}))$ and some constant $C_1 = C_1(C_0) > 0$. Moreover, $v = \Lambda_1(u)$ is the smooth solution of

$$\begin{cases} v_t = (a(t,x)v_x)_x + \big(f(t,x,u)\big)_x + g\big(t,x,u\big), & t>0,\ x\in\mathbb{R},\\ v|_{t=0} = 0, & x\in\mathbb{R}.\end{cases} \tag{2.9}$$

Proof. Due to regularity of f, g, u, the fact that $\Lambda_1(u)$ solves (2.9) is consequence of [10, Chapter IV, Section 14]. We have to prove (2.7). From (2.9) and (**H.2**) we get[1]

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}(v^2+v_x^2)dx = \int_{\mathbb{R}}\big(vv_t+v_xv_{tx}\big)dx\\ &= \int_{\mathbb{R}}\big(-av_x^2+\frac{a_{xx}}{2}v_x^2-av_{xx}^2-v_xf+vg-v_{xx}(f)_x-v_{xx}g\big)dx\\ &\le \int_{\mathbb{R}}\Big(\big(\frac{1}{C_0}-a\big)\big(v_x^2+v_{xx}^2\big)+\frac{a_{xx}}{2}v_x^2+\frac{C_0}{2}v^2+\frac{C_0}{2}(f)_x^2+\frac{2+C_0^2}{4C_0}g^2+\frac{C_0}{4}f^2\Big)dx\\ &\le \frac{C_0}{2}\|v(t,\cdot)\|^2_{H^1(\mathbb{R})}+c_1\|u(t,\cdot)\|^4_{H^1(\mathbb{R})},\end{aligned}$$

for some constant $c_1 = c_1(C_0) > 0$. Hence, using the Gronwall inequality

$$\|v\|^2_{L^\infty([0,T_0];H^1(\mathbb{R}))} \le e^{C_0T_0/2}T_0\|u\|^4_{L^\infty([0,T_0];H^1(\mathbb{R}))}. \tag{2.10}$$

Clearly this implies $v \in L^\infty([0,T_0];H^1(\mathbb{R}))$. Writing U as convolution of the Green's function G and of the source, the first part of (2.5) is consequence of [10, Chapter 13, (13.3)]. For the second part, we look at the equation (2.9) and observe that $v_{xx} \in C([0,T_0];H^{-1}(\mathbb{R}))$.

To prove (2.8), observe that $\Lambda_1(u_1)-\Lambda_1(u_2)$ is the smooth solution of

$$\begin{cases} v_t = (a(t,x)v_x)_x + \Big[f(t,x,u_1)-f(t,x,u_2)\Big]_x + \Big[g(t,x,u_1)-g(t,x,u_2)\Big],\\ v|_{t=0} = 0,\end{cases}$$

[1]Observe that, e.g., f_x and $(f)_x$ are distinct as $(f)_x = (f(t,x,u(x,t)))_x = f_x + f_uu_x$.

since the source is in $L^\infty([0,T_0];H^{-1}(\mathbb{R}))$, we can argue as for (2.10). $\square$

Lemma 2.7. *Let $T_0>0$ and assume* (2.3), (**H.1**), (**H.2**). *Then*

$$\Lambda_2(u)\in C([0,T_0];H^1(\mathbb{R})),\qquad \frac{\partial\Lambda_2(u)}{\partial t}\in C([0,T_0];H^{-1}(\mathbb{R})), \tag{2.11}$$

for each $u\in C([0,T_0];H^1(\mathbb{R}))$, and

$$\begin{aligned}&\|\Lambda_2(u_1)-\Lambda_2(u_2)\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}\\ &\qquad\le C_2\sqrt{T_0}\big(\|u_1-u_2\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}+\|u_1-u_2\|^2_{L^\infty([0,T_0];H^1(\mathbb{R}))}\big),\end{aligned} \tag{2.12}$$

for each u_1, $u_2\in C([0,T_0];H^1(\mathbb{R}))$ and some constant $C_2=C_2(C_0)>0$. Moreover, $w=\Lambda_2(u)$ is the smooth solution of

$$\begin{cases} w_t=(a(t,x)w_x)_x+P_x, & t>0,\ x\in\mathbb{R},\\ -P_{xx}+P=h(t,x,u,u_x)+k(t,x,u), & t>0,\ x\in\mathbb{R},\\ w|_{t=0}=0.\end{cases} \tag{2.13}$$

Proof. Due to regularity of h, k, u, the fact that $\Lambda_2(u)$ solves (2.13) is consequence of [10, Chapter IV, Section 14] and Remark 2.1. We have to prove (2.11). From (2.13) and (**H.2**), we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\frac12(w^2+w_x^2)dx&=\int_{\mathbb{R}}\big(ww_t+w_xw_{tx}\big)dx\\ &=\int_{\mathbb{R}}\big(-aw_x^2+\frac{a_{xx}}{2}w_x^2-aw_{xx}^2+wP_x+w_xP-w_xh-w_xk\big)dx\\ &\le\int_{\mathbb{R}}\Big(\big(\frac{1}{C_0}-a\big)w_x^2-aw_{xx}^2+\frac{a_{xx}}{2}w_x^2+\frac{C_0}{2}h^2+\frac{C_0}{2}k^2\Big)dx\\ &\le\frac{C_0}{2}\|w(t,\cdot)\|^2_{H^1(\mathbb{R})}+c_2\Big(\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}+\|u(t,\cdot)\|^4_{H^1(\mathbb{R})}\Big),\end{aligned}$$

for some constant $c_2=c_2(C_0)>0$. Clearly this implies $w\in L^\infty([0,T_0];H^1(\mathbb{R}))$. Writing U as convolution of the Green's function G and of the source, the first part of (2.5) is consequence of [10, Chapter 13, (13.3)]. For the second part, we have simply to consider equation (2.13) and note that $w_{xx}\in C([0,T_0];H^{-1}(\mathbb{R}))$.

To prove (2.12), observe that $\Lambda_1(u_1)-\Lambda_1(u_2)$ is the smooth solution of

$$\begin{cases} w_t=(a(t,x)w_x)_x+P_x,\\ -P_{xx}+P=\Big[h(t,x,u_1,u_{1,x})-h(t,x,u_2,u_{2,x}))\Big]+\Big[k(t,x,u_1)-k(t,x,u_2)\Big],\\ w|_{t=0}=0.\end{cases}$$

and use the previous argument. $\square$

Proof of Theorem 2.4. Choose a time

$$0\le T_0\le\min\{1,\,c_3,\,c_4\},$$

where

$$\begin{aligned}c_3&:=\frac{1}{16C_1^2(\|U\|^2_{L^\infty([0,1];H^1(\mathbb{R}))}+3)^2e^{2C_1}},\\ c_4&:=\frac{1}{8C_2^2(\|U\|_{L^\infty([0,1];H^1(\mathbb{R}))}+2\|U\|^2_{L^\infty([0,1];H^1(\mathbb{R}))}+3)^2},\end{aligned}$$

and consider the ball

$$\mathcal{B}:=\Big\{u\in C([0,T_0];H^1(\mathbb{R}))\mid\|u-U\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}\le1\Big\}.$$

Observe that

$$u\in\mathcal{B}\Longrightarrow\Lambda(u)\in\mathcal{B},$$

$$u,\, v\in\mathcal{B} \Longrightarrow \|\Lambda(u)-\Lambda(v)\|_{L^\infty([0,T_0];H^1(\mathbb{R}))} \le \frac{1}{\sqrt{2}}\|u-v\|_{L^\infty([0,T_0];H^1(\mathbb{R}))},$$

indeed, by (2.8) and (2.12),

$$\begin{aligned}
\|\Lambda(u)-U\|_{L^\infty([0,T_0];H^1(\mathbb{R}))} &\le \|\Lambda_1(u)\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}+\|\Lambda_2(u)\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}\\
&\le C_1\sqrt{T_0}e^{C_1T_0}\|u\|^2_{L^\infty([0,T_0];H^1(\mathbb{R}))}\\
&\quad + C_2\sqrt{T_0}\big(\|u\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}+\|u\|^2_{L^\infty([0,T_0];H^1(\mathbb{R}))}\big)\\
&\le 2C_1\sqrt{T_0}e^{C_1T_0}\big(\|u-U\|^2_{L^\infty([0,T_0];H^1(\mathbb{R}))}+\|U\|^2_{L^\infty([0,1];H^1(\mathbb{R}))}\big)\\
&\quad + C_2\sqrt{T_0}\big(\|u-U\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}+\|U\|_{L^\infty([0,1];H^1(\mathbb{R}))}\\
&\quad + 2\|u-U\|^2_{L^\infty([0,T_0];H^1(\mathbb{R}))}+2\|U\|^2_{L^\infty([0,1];H^1(\mathbb{R}))}\big)\\
&\le 2C_1e^{C_1T_0}\sqrt{T_0}\big(1+\|U\|^2_{L^\infty([0,1];H^1(\mathbb{R}))}\big)\\
&\quad + C_2\sqrt{T_0}\big(3+\|U\|_{L^\infty([0,1];H^1(\mathbb{R}))}+2\|U\|^2_{L^\infty([0,1];H^1(\mathbb{R}))}\big)\\
&\le \sqrt{T_0}\Big(\frac{1}{2\sqrt{2c_3}}+\frac{1}{2\sqrt{2c_4}}\Big)\le\frac{1}{\sqrt{2}},
\end{aligned}$$

$$\begin{aligned}
&\|\Lambda(u)-\Lambda(v)\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}\\
&\quad\le \|\Lambda_1(u)-\Lambda_1(v)\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}+\|\Lambda_2(u)-\Lambda_2(v)\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}\\
&\quad\le C_1\sqrt{T_0}e^{C_1T_0}\|u-v\|^2_{L^\infty([0,T_0];H^1(\mathbb{R}))}\\
&\qquad + C_2\sqrt{T_0}\big(\|u-v\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}+\|u-v\|^2_{L^\infty([0,T_0];H^1(\mathbb{R}))}\big)\\
&\quad\le \sqrt{T_0}\|u-v\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}\\
&\qquad\times\big[(C_1e^{C_1}+C_2)\big(1+\|u-U\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}+\|v-U\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}\big)\big]\\
&\quad\le \sqrt{T_0}3\big[C_1e^{C_1}+C_2\big]\|u-v\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}\\
&\quad\le \sqrt{T_0}\Big(\frac{1}{2\sqrt{2c_3}}+\frac{1}{2\sqrt{2c_4}}\Big)\|u-v\|_{L^\infty([0,T_0];H^1(\mathbb{R}))}\\
&\quad\le \frac{1}{\sqrt{2}}\|u-v\|_{L^\infty([0,T_0];H^1(\mathbb{R}))},
\end{aligned}$$

for each $u,\, v\in\mathcal{B}$. Hence, the operator Λ is a contraction on $\mathcal{B}$. Due to the contraction mapping principle, there exists a unique $u\in\mathcal{B}$ such that

$$u=\Lambda(u).$$

In particular,

$$u\in C([0,T_0];H^1(\mathbb{R})),$$

and from (2.5), (2.7), and (2.11), we get

$$\frac{\partial u}{\partial t}\in C([0,T_0];H^{-1}(\mathbb{R})),$$

namely u is the unique weak solution of (2.1) defined on $[0,T_0]\times\mathbb{R}$. The proof of the smoothness of this solutions follows by the classical regularity theory for parabolic problems (see, e.g., [13, Section 2.2]). Here, we simply sketch the steps of the argument. We begin by observing that $u_x\in L^2_{\text{loc}}([0,T_0]\times\mathbb{R})$, then we prove that

$$u\in L^2([0,T_0];H^{\ell+1}(\mathbb{R}))\cap C([0,T_0];H^{\ell}(\mathbb{R})),\qquad \frac{\partial^k u}{\partial t^k}\in L^2([0,T_0];H^{l_k}(\mathbb{R})),$$

for $k=0,\dots,\lfloor(\ell+1)/2\rfloor$, where $\lfloor x\rfloor$ denotes the largest integer not exceeding x, and where

$$l_k:=\begin{cases}\ell+1-2k, & \text{if } k=0,\dots,\lfloor \ell/2\rfloor,\\ 2, & \text{if } k=\lfloor(\ell+1)/2\rfloor.\end{cases}$$

To conclude we use Sobolev embedding. □

A trivial consequence of the previous theorem is the following corollary.

Corollary 2.8 (Uniqueness). *Assume that* (**H.1**), (**H.2**), *and* (2.3) *hold. The initial value problem* (2.1) *has at most one smooth solution defined in* $[0,T]\times\mathbb{R}$.

2.2. **Energy estimate.** We prove the following a priori estimates.

Theorem 2.9 (H^1-estimate). *Assume that* (**H.1**), (**H.2**), *and* (2.3) *hold. Let* u *be the unique, local, smooth solution to* (2.1). *Then*

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}\le\frac{1}{\sqrt{2}}\ \|u(t,\cdot)\|_{H^1(\mathbb{R})}\le\frac{1}{\sqrt{2}}e^{11C_0t/2}\,\|u_0\|_{H^1(\mathbb{R})}\tag{2.14}$$

for each $0\le t\le T_0$. *In particular,*

$$u\in C([0,T_0];H^1(\mathbb{R})).$$

Proof. Fix $0<t\le T_0$. Multiplying the first equation in (2.1) by u and integrating over $\mathbb{R}$, we get

$$\int_{\mathbb{R}}u_tu\,dx=\int_{\mathbb{R}}\big(a(t,x)u_x\big)_xu\,dx-\int_{\mathbb{R}}\Big(\big(f(t,x,u)\big)_xu+g(t,x,u)u+P_xu\Big)dx.\tag{2.15}$$

Integrating by parts and using (**H.1**), (**H.2**),

$$\begin{aligned}\int_{\mathbb{R}}u_tu\,dx&=\frac12\frac{d}{dt}\int_{\mathbb{R}}u^2\,dx=\frac12\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ \int_{\mathbb{R}}\big(a(t,x)u_x\big)_xu\,dx&=-\int_{\mathbb{R}}a(t,x)u_x^2\,dx\le0,\\ \int_{\mathbb{R}}g(t,x,u)u\,dx&\le C_0\int_{\mathbb{R}}u^2\,dx,\\ \int_{\mathbb{R}}\big(f(t,x,u)\big)_xu\,dx&=-\int_{\mathbb{R}}f(t,x,u)u_x\,dx\\ &=-\int_{\mathbb{R}}F_u(t,x,u)u_x\,dx\\ &=-\int_{\mathbb{R}}\big(F(t,x,u)\big)_x\,dx+\int_{\mathbb{R}}F_x(t,x,u)\,dx\\ &=\int_{\mathbb{R}}F_x(t,x,u)\,dx\\ &\le\frac{C_0}{2}\int_{\mathbb{R}}u^2\,dx,\end{aligned}$$

where

$$F(t,x,z):=\int_0^zf(t,x,\zeta)\,d\zeta,\qquad 0\le t\le T,\ x,z\in\mathbb{R}.$$

Hence, from (2.15), we get

$$\frac12\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}\le C_0\frac32\int_{\mathbb{R}}u^2\,dx-\int_{\mathbb{R}}P_xu\,dx.\tag{2.16}$$

Differentiating the first equation in (2.1) with respect to x, we have

$$\begin{aligned}u_{tx}&+\big(f_x(t,x,u)+f_u(t,x,u)u_x\big)_x+g_x(t,x,u)+g_u(t,x,u)u_x\\ &=\big(a_x(t,x)u_x+a(t,x)u_{xx}\big)_x-P_{xx}\\ &=\big(a_x(t,x)u_x+a(t,x)u_{xx}\big)_x-P+h(t,x,u,u_x)+k(t,x,u).\end{aligned}\tag{2.17}$$

Multiplying this equation by u_x and integrating on $\mathbb{R}$, we get

$$\int_{\mathbb{R}}u_{tx}u_x\,dx=-\int_{\mathbb{R}}\big(f_x(t,x,u)+f_u(t,x,u)u_x\big)_xu_x\,dx$$

$$
\begin{aligned}
&- \int_{\mathbb{R}} \big(g_x(t,x,u) + g_u(t,x,u)u_x\big)u_x\,dx \\
&+ \int_{\mathbb{R}} \big(a_x(t,x)u_x + a(t,x)u_{xx}\big)_x u_x\,dx \qquad (2.18)\\
&- \int_{\mathbb{R}} Pu_x\,dx + \int_{\mathbb{R}} h(t,x,u,u_x)u_x\,dx + \int_{\mathbb{R}} k(t,x,u)u_x\,dx.
\end{aligned}
$$

Again, integrating by parts and using (**H.1**) and (**H.2**), we find

$$
\begin{aligned}
&\int_{\mathbb{R}} u_{tx}u_x\,dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} u_x^2\,dx = \frac{1}{2}\frac{d}{dt}\|u_x(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&\int_{\mathbb{R}} \big(a_x(t,x)u_x + a(t,x)u_{xx}\big)_x u_x dx\\
&\qquad = -\frac{1}{2}\int_{\mathbb{R}} a_x(t,x)\big(u_x^2\big)_x\,dx - \int_{\mathbb{R}} a(t,x)u_{xx}^2\,dx\\
&\qquad \le \frac{1}{2}\int_{\mathbb{R}} a_{xx}(t,x)u_x^2\,dx \le \frac{C_0}{2}\int_{\mathbb{R}} u_x^2\,dx,\\
&-\int_{\mathbb{R}} \big(f_x(t,x,u) + f_u(t,x,u)u_x\big)_x u_x\,dx\\
&\qquad = -\int_{\mathbb{R}} \Big(f_{xx}(t,x,u)u_x + \frac{3}{2}f_{ux}(t,x,u)u_x^2 + \frac{1}{2}f_{uu}(t,x,u)u_x^3\Big)\,dx\\
&\qquad \le C_0\frac{5}{2}\int_{\mathbb{R}} u_x^2\,dx - \frac{1}{2}\int_{\mathbb{R}} f_{uu}(t,x,u)u_x^3\,dx,\\
&-\int_{\mathbb{R}} \big(g_x(t,x,u) + g_u(t,x,u)u_x\big)u_x dx \le C_0\int_{\mathbb{R}} |uu_x|\,dx + C_0\int_{\mathbb{R}} u_x^2\,dx\\
&\qquad \le C_0\frac{1}{2}\int_{\mathbb{R}} u^2 dx + C_0\frac{3}{2}\int_{\mathbb{R}} u_x^2 dx,\\
&\int_{\mathbb{R}} k(t,x,u)u_x\,dx = \int_{\mathbb{R}} \big(K(t,x,u)\big)_x\,dx - \int_{\mathbb{R}} K_x(t,x,u)\,dx\\
&\qquad = -\int_{\mathbb{R}} K_x(t,x,u)\,dx \le \frac{C_0}{2}\int_{\mathbb{R}} u^2\,dx,
\end{aligned}
$$

where

$$
K(t,x,z) := \int_0^z k(t,x,\zeta)d\zeta, \qquad 0 \le t \le T,\ x,z \in \mathbb{R}.
$$

Therefore, from (2.18) and (**H.2**), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u_x(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad \le C_0\int_{\mathbb{R}} u^2\,dx + C_0\frac{9}{2}\int_{\mathbb{R}} u_x^2\,dx + \int_{\mathbb{R}} \Big(hu_x - \frac{1}{2}f_{uu}u_x^3\Big)dx + \int_{\mathbb{R}} P_x u\,dx \qquad (2.19)\\
&\quad \le 2C_0\int_{\mathbb{R}} u^2\,dx + C_0\frac{11}{2}\int_{\mathbb{R}} u_x^2\,dx + \int_{\mathbb{R}} P_x u\,dx,
\end{aligned}
$$

using (2.4i). Summing (2.16) and (2.19),

$$
\frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|^2_{H^1(\mathbb{R})} \le C_0\frac{7}{2}\int_{\mathbb{R}} u^2\,dx + C_0\frac{11}{2}\int_{\mathbb{R}} u_x^2\,dx \le C_0\frac{11}{2}\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}. \qquad (2.20)
$$

The Gronwall inequality implies

$$
\|u(t,\cdot)\|^2_{H^1(\mathbb{R})} \le \|u_0\|^2_{H^1(\mathbb{R})} e^{11C_0 t}, \qquad 0 \le t \le T_0, \qquad (2.21)
$$

that proves the second inequality in (2.14). Finally, since (see [12, Theorem 8.5])

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}} \|u(t,\cdot)\|_{H^1(\mathbb{R})}, \qquad 0 \le t \le T_0,$$

also the first one is proved. □

Remark 2.10. *Estimate* (2.19) *requires the assumption* (2.4i)*. In the case of the Camassa–Holm equation* [4] *we can improve* (2.19) *and obtain the energy conservation. Observe that the system for radiating gases, see equation* (1.3)*, does not satisfy* (2.4i)*, and indeed in that model there is no energy conservation.*

2.3. **Global existence.** In this section we prove Theorem 2.3. The following lemma is needed.

Lemma 2.11 (H^2-estimate)**.** *Let $T_0 > 0$. Assume that* (**H.1**)*,* (**H.2**)*, and* (2.3) *hold. Let u be the unique, local, smooth solution to* (2.1)*. Then, there exists a positive constants such that*

$$\|u_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} \le \|u_{0,xx}\|^2_{L^2(\mathbb{R})} e^{At} + \frac{B}{A}\big(e^{At} - 1\big), \tag{2.22}$$

for each $0 \le t \le T_0$, where

$$A := 2\big[27C_0^3 \|u_0\|^2_{H^1(\mathbb{R})} e^{11C_0T} + 6C_0^3 + C_0\big],$$
$$B := 2\Big[39C_0^3 \|u_0\|^2_{H^1(\mathbb{R})} e^{11C_0T} + \Big(\frac{1}{2} + 6C_0\Big) \|u_0\|^4_{H^1(\mathbb{R})} e^{22C_0T}\Big].$$

In particular

$$u \in C([0,T_0]; H^2(\mathbb{R})).$$

Proof. Fix $0 < t \le T_0$. Differentiating (2.17) with respect to x,

$$\begin{aligned} u_{txx} = &-\big(f_{xx} + 2f_{ux}u_x + f_{uu}u_x^2 + f_u u_{xx}\big)_x \\ &- \big(g_x + g_u u_x\big)_x + \big(a_{xx}u_x + 2a_x u_{xx} + a u_{xxx}\big)_x \\ &- P_x + \big(h(t,x,u,u_x)\big)_x + \big(k(t,x,u)\big)_x. \end{aligned} \tag{2.23}$$

Multiplying (2.23) by u_{xx} and integrating on $\mathbb{R}$, we get

$$\begin{aligned} \int_{\mathbb{R}} u_{txx}u_{xx}\,dx = &-\int_{\mathbb{R}} \big(f_{xx} + 2f_{ux}u_x + f_{uu}u_x^2 + f_u u_{xx}\big)_x u_{xx}dx \\ &- \int_{\mathbb{R}} \big(g_x + g_u u_x\big)_x u_{xx}dx + \int_{\mathbb{R}} \big(a_{xx}u_x + 2a_x u_{xx} + a u_{xxx}\big)_x u_{xx}dx \\ &- \int_{\mathbb{R}} P_x u_{xx}dx + \int_{\mathbb{R}} \big(h(t,x,u,u_x)\big)_x u_{xx}dx + \int_{\mathbb{R}} \big(k(t,x,u)\big)_x u_{xx}dx. \end{aligned} \tag{2.24}$$

Observe that, by (2.14), (**H.1**), and (**H.2**), we find

$$\int_{\mathbb{R}} u_{txx}u_{xx}dx = \frac{1}{2}\frac{d}{dt}\|u_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})},$$

$$\begin{aligned} -\int_{\mathbb{R}} &\big(f_{xx} + 2f_{ux}u_x + f_{uu}u_x^2 + f_u u_{xx}\big)_x u_{xx}dx \\ &= \int_{\mathbb{R}} \big(f_{xx} + 2f_{ux}u_x + f_{uu}u_x^2 + f_u u_{xx}\big) u_{xxx}dx \\ &\le \frac{3C_0}{2}\int_{\mathbb{R}} \big(f_{xx} + 2f_{ux}u_x + f_{uu}u_x^2 + f_u u_{xx}\big)^2 dx + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\ &\le \frac{3C_0}{2}\int_{\mathbb{R}} \big(C_0|u| + 2C_0|u_x| + C_0u_x^2 + C_0u_{xx}\big)^2 dx + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \end{aligned}$$

$$
\begin{aligned}
&\le 6C_0 \int_{\mathbb{R}} \left(C_0^2|u|^2 + 4C_0^2|u_x|^2 + C_0^2 u_x^4 + C_0^2 u_{xx}^2\right)dx + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\
&\le 24C_0^3\|u(t,\cdot)\|^2_{H^1(\mathbb{R})} + 6C_0^3\|u_x(t,\cdot)\|^4_{L^4(\mathbb{R})} \\
&\qquad + 6C_0^3\|u_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\
&\le 24C_0^3\|u(t,\cdot)\|^2_{H^1(\mathbb{R})} + 18C_0^3\|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad + 6C_0^3\|u_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\
&\le 24C_0^3 e^{11C_0 t}\|u_0\|^2_{H^1(\mathbb{R})} \\
&\qquad + 6C_0^3\left(3e^{11C_0 t}\|u_0\|^2_{H^1(\mathbb{R})} + 1\right)\|u_{xx}(t,\cdot)\|^2_{L^2} + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx,
\end{aligned}
$$

$$
\begin{aligned}
-\int_{\mathbb{R}} &\left(g_x + g_u u_x\right)_x u_{xx} dx = \int_{\mathbb{R}} \left(g_x + g_u u_x\right) u_{xxx} dx \\
&\le \frac{3C_0}{2}\int_{\mathbb{R}} \left(g_x + g_u u_x\right)^2 dx + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\
&\le 3C_0 \int_{\mathbb{R}} \left(g_x^2 + g_u^2 u_x^2\right)dx + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\
&\le 3C_0^3 \int_{\mathbb{R}} \left(u^2 + u_x^2\right)dx + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\
&= 3C_0^3\|u(t,\cdot)\|^2_{H^1(\mathbb{R})} + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\
&\le 3C_0^3 e^{11C_0 t}\|u_0\|^2_{H^1(\mathbb{R})} + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx,
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{R}} &\left(a_{xx}u_x + 2a_x u_{xx} + a u_{xxx}\right)_x u_{xx} dx \\
&= -\int_{\mathbb{R}} a_{xx}u_x u_{xxx} dx - 2\int_{\mathbb{R}} a_x u_{xx} u_{xxx} dx - \int_{\mathbb{R}} a u_{xxx}^2 dx \\
&\le \frac{3C_0}{2}\int_{\mathbb{R}} a_{xx}^2 u_x^2 dx - \frac{5}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx - \int_{\mathbb{R}} a_x (u_{xx}^2)_x dx \\
&\le \frac{3C_0^3}{2}\|u_x(t,\cdot)\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} a_{xx} u_{xx}^2 dx - \frac{5}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\
&\le \frac{3C_0^3}{2} e^{11C_0 t}\|u_0\|^2_{H^1(\mathbb{R})} + C_0\|u_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} - \frac{5}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx,
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{R}} &\left(h(t,x,u,u_x)\right)_x u_{xx} dx = -\int_{\mathbb{R}} h u_{xxx} dx \\
&\le \frac{3C_0}{2}\int_{\mathbb{R}} h^2 dx + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\
&\le 6C_0^3 \int_{\mathbb{R}} (u^2 + u^4 + u_x^4) dx + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \\
&\le \left(6C_0^3 + \|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\right)\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad + 6C_0^3\|u_x(t,\cdot)\|^4_{L^4(\mathbb{R})} + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx \le \\
&\le \left(6C_0^3 + \|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\right)\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad + 18C_0^3\|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx
\end{aligned}
$$

$$\begin{aligned}&\leq \Big(6C_0^3 + \frac{1}{2}\left\|u_0\right\|^2_{H^1(\mathbb{R})} e^{11C_0t}\Big)\left\|u_0\right\|^2_{H^1(\mathbb{R})} e^{11C_0t}\\&\qquad + 9C_0^3\left\|u_0\right\|^2_{H^1(\mathbb{R})} e^{11C_0t}\left\|u_{xx}(t,\,\cdot\,)\right\|^2_{L^2(\mathbb{R})} + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx,\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}}\big(k(t,x,u)\big)_x u_{xx}dx &= -\int_{\mathbb{R}} k u_{xxx}dx\\&\leq \frac{3C_0}{2}\int_{\mathbb{R}} k^2 dx + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx\\&= \frac{3C_0^3}{2}\|u(t,\,\cdot\,)\|^2_{L^2(\mathbb{R})} + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx\\&\leq \frac{3C_0^3}{2}\left\|u_0\right\|^2_{H^1(\mathbb{R})} e^{11C_0t} + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx.\end{aligned}$$

Here we have used the inequality

$$\left\|u_x(t,\cdot)\right\|^2_{L^4(\mathbb{R})} \leq 3\left\|u(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\left\|u_{xx}(t,\cdot)\right\|_{L^2(\mathbb{R})} \tag{2.25}$$

which follows from

$$\begin{aligned}\int_{\mathbb{R}} u_x^4(t,x)dx &= \int_{\mathbb{R}}\big((uu_x^3)_x - 3uu_x^2u_{xx}\big)dx\\&= -3\int_{\mathbb{R}} uu_x^2u_{xx}\,dx\\&\leq 3\left\|u(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\left\|u_x^2(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|u_{xx}(t,\cdot)\right\|_{L^2(\mathbb{R})}\end{aligned}$$

using the generalized Hölder inequality.

Finally, we have to estimate the nonlocal term P. Observe that

$$-\int_{\mathbb{R}} P_x u_{xx}dx = \int_{\mathbb{R}} P u_{xxx}dx \leq \frac{3C_0}{2}\int_{\mathbb{R}} P^2 dx + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2 dx, \tag{2.26}$$

moreover, using Remark 2.1 and (**H.2**),

$$\begin{aligned}P(t,x) &= \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|}\Big(h\big(t,y,u(t,y),u_x(t,y)\big) + k\big(t,y,u(t,y)\big)\Big)dy\\&\leq C_0(P_1+P_2),\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}P_1(t,x) &:= \int_{\mathbb{R}} e^{-|x-y|}|u(t,y)|dy,\\P_2(t,x) &:= \int_{\mathbb{R}} e^{-|x-y|}\big(u^2(t,y)+u_x^2(t,y)\big)dy.\end{aligned}$$

Since

$$\int_{\mathbb{R}} e^{-|x-y|}dy = 2,$$

using the Tonelli theorem and the Hölder inequality, we find

$$\begin{aligned}\int_{\mathbb{R}}\left|P_1(t,x)\right|^2 dx &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-|x-y|}\left|u(t,y)\right| dy\Big)^2 dx\\&\leq \int_{\mathbb{R}\times\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-|x-y|}dy\Big)e^{-|x-y|}\left|u(t,y)\right|^2 dxdy\\&= 2\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-|x-y|}dx\Big)\left|u(t,y)\right|^2 dy\\&= 4\left\|u(t,\,\cdot\,)\right\|^2_{L^2(\mathbb{R})} \leq 4\left\|u(t,\,\cdot\,)\right\|^2_{H^1(\mathbb{R})},\end{aligned} \tag{2.28}$$

$$\left|P_2(t,x)\right| = \int_{\mathbb{R}} e^{-|x-y|}\big(u^2(t,y)+u_x^2(t,y)\big)dy$$

$$\leq \int_{\mathbb{R}} \big(u^2(t,y) + u_x^2(t,y)\big)dx = \|u(t,\cdot)\|^2_{H^1(\mathbb{R})},$$

$$\int_{\mathbb{R}} |P_2(t,x)|\, dx = \int_{\mathbb{R}\times\mathbb{R}} e^{-|x-y|}\big(u^2(t,y) + u_x^2(t,y)\big)dxdy$$
$$= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} e^{-|x-y|}dx\Big)\big(u^2(t,y) + u_x^2(t,y)\big)dy = 2\|u(t,\cdot)\|^2_{H^1(\mathbb{R})},$$

hence

$$\int_{\mathbb{R}} |P_2(t,x)|^2\, dx \leq \|P_2(t,\cdot)\|_{L^\infty(\mathbb{R})}\|P_2(t,\cdot)\|_{L^1(\mathbb{R})} \leq 2\|u(t,\cdot)\|^4_{H^1(\mathbb{R})}. \tag{2.29}$$

Therefore, from (2.26), (2.27), (2.28), (2.29), and Theorem 2.9, we get

$$-\int_{\mathbb{R}} P_x u_{xx} dx \leq 6C_0^3\big(\|u(t,\cdot)\|^2_{H^1(\mathbb{R})} + \|u(t,\cdot)\|^4_{H^1(\mathbb{R})}\big) + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2\, dx$$
$$\leq 6C_0^3 e^{11C_0 t}\|u_0\|^2_{H^1(\mathbb{R})}\big(1 + e^{11C_0 t}\|u_0\|^2_{H^1(\mathbb{R})}\big) + \frac{1}{6C_0}\int_{\mathbb{R}} u_{xxx}^2\, dx.$$

Then, from (2.24),

$$\frac{d}{dt}\|u_{xx}(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq A\|u_{xx}(t,\cdot)\|^2_{H^1(\mathbb{R})} + B, \tag{2.30}$$

where A and B are defined in the statement. Then, using the Gronwall inequality we are done. □

Proof of Theorem 2.3. We know that the initial value problem (2.1) is locally well-posed in time (see Lemma 2.4); it remains to prove that the solutions are defined in the large. Due to Theorem 2.9 and Lemma 2.11, there exists a unique (global) solution that belongs to $C([0,T];H^2(\mathbb{R}))$, which is a subset of $C([0,T];W^{1,\infty}(\mathbb{R}))$ (see [12, Theorem 8.5]). We have to prove that

$$\|u(t,\cdot)\|_{H^\ell(\mathbb{R})} < +\infty, \qquad 0 \leq t \leq T, \tag{2.31}$$

and

$$u \in C^\infty(\langle 0,T\rangle \times \mathbb{R}). \tag{2.32}$$

We begin by proving (2.31). Since the argument is similar to the one of Lemma 2.11, we only sketch it. Let $i \in \{0,\dots,\ell\}$. Differentiating (2.17) i times with respect to x we find

$$\frac{\partial^{i+1}u}{\partial t\partial x^i} + \frac{\partial^{i+1}}{\partial x^{i+1}}\big(f(t,x,u)\big) + \frac{\partial^i}{\partial x^i}\big(g(t,x,u)\big) + \frac{\partial^{i+1}P}{\partial x^{i+1}} = \frac{\partial^{i+1}}{\partial x^{i+1}}\big(a(t,x)u_x\big). \tag{2.33}$$

Multiplying (2.33) by $\dfrac{\partial^i u}{\partial x^i}$, integrating over $\mathbb{R}$ and summing on i,

$$\sum_{i=0}^{\ell}\int_{\mathbb{R}} \frac{\partial^{i+1}u}{\partial t\partial x^i}\frac{\partial^i u}{\partial x^i}dx + \sum_{i=0}^{\ell}\int_{\mathbb{R}} \frac{\partial^{i+1}}{\partial x^{i+1}}\big(f(t,x,u)\big)\frac{\partial^i u}{\partial x^i}dx$$
$$+\sum_{i=0}^{\ell}\int_{\mathbb{R}} \frac{\partial^i}{\partial x^i}\big(g(t,x,u)\big)\frac{\partial^i u}{\partial x^i}dx + \sum_{i=0}^{\ell}\int_{\mathbb{R}} \frac{\partial^{i+1}P}{\partial x^{i+1}}\frac{\partial^i u}{\partial x^i}dx$$
$$= \sum_{i=0}^{\ell}\int_{\mathbb{R}} \frac{\partial^{i+1}}{\partial x^{i+1}}\big(a(t,x)u_x\big)\frac{\partial^i u}{\partial x^i}dx. \tag{2.34}$$

Observe that by integrating by parts,

$$\sum_{i=0}^{\ell}\int_{\mathbb{R}} \frac{\partial^{i+1}u}{\partial t\partial x^i}\frac{\partial^i u}{\partial x^i}dx = \frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|^2_{H^\ell(\mathbb{R})}. \tag{2.35}$$

From (2.4b) and (2.4d)

$$\begin{aligned}\sum_{i=0}^{\ell}\int_{\mathbb{R}}\frac{\partial^{i+1}}{\partial x^{i+1}}\big(a(t,x)u_x\big)\frac{\partial^i u}{\partial x^i}dx &= \sum_{i=0}^{\ell}\sum_{j=0}^{i+1}\int_{\mathbb{R}}\binom{i+1}{j}\frac{\partial^j a}{\partial x^j}\frac{\partial^{i-j+2}u}{\partial x^{i-j+2}}\frac{\partial^i u}{\partial x^i}dx \\ &= \sum_{i=0}^{\ell}\sum_{j=2}^{i+1}\int_{\mathbb{R}}\binom{i+1}{j}\frac{\partial^j a}{\partial x^j}\frac{\partial^{i-j+2}u}{\partial x^{i-j+2}}\frac{\partial^i u}{\partial x^i}dx \\ &\quad+\sum_{i=0}^{\ell}\int_{\mathbb{R}}a\frac{\partial^{i+2}u}{\partial x^{i+2}}\frac{\partial^i u}{\partial x^i}dx+\sum_{i=0}^{\ell}(i+1)\int_{\mathbb{R}}\frac{\partial a}{\partial x}\frac{\partial^{i+1}u}{\partial x^{i+1}}\frac{\partial^i u}{\partial x^i}dx \\ &= \sum_{i=0}^{\ell}\sum_{j=2}^{i+1}\int_{\mathbb{R}}\binom{i+1}{j}\frac{\partial^j a}{\partial x^j}\frac{\partial^{i-j+2}u}{\partial x^{i-j+2}}\frac{\partial^i u}{\partial x^i}dx \\ &\quad-\sum_{i=0}^{\ell}\int_{\mathbb{R}}a\Big(\frac{\partial^{i+1}u}{\partial x^{i+1}}\Big)^2dx+\sum_{i=0}^{\ell}i\int_{\mathbb{R}}\frac{\partial a}{\partial x}\frac{\partial^{i+1}u}{\partial x^{i+1}}\frac{\partial^i u}{\partial x^i}dx \\ &\le c_1\|u(t,\cdot)\|^2_{H^\ell(\mathbb{R})}-\frac{1}{C_0}\Big\|\frac{\partial u}{\partial x}(t,\cdot)\Big\|^2_{H^\ell(\mathbb{R})},\end{aligned}\tag{2.36}$$

where $c_1 = c_1(\ell, C_0) > 0$. From (2.4c) and (2.4d)

$$\begin{aligned}&\sum_{i=0}^{\ell}\int_{\mathbb{R}}\frac{\partial^{i+1}}{\partial x^{i+1}}\big(f(t,x,u)\big)\frac{\partial^i u}{\partial x^i}dx \\ &\quad=-\sum_{i=0}^{\ell}\int_{\mathbb{R}}\frac{\partial^i}{\partial x^i}\big(f(t,x,u)\big)\frac{\partial^{i+1}u}{\partial x^{i+1}}dx \\ &\quad\le\frac{C_0}{2}\sum_{i=0}^{\ell}\int_{\mathbb{R}}\Big(\frac{\partial^i}{\partial x^i}\big(f(t,x,u)\big)\Big)^2dx+\frac{1}{2C_0}\sum_{i=0}^{\ell}\int_{\mathbb{R}}\Big(\frac{\partial^{i+1}u}{\partial x^{i+1}}\Big)^2dx \\ &\quad\le c_2\|u(t,\cdot)\|^2_{H^\ell(\mathbb{R})}+\frac{1}{2C_0}\Big\|\frac{\partial u}{\partial x}(t,\cdot)\Big\|^2_{H^\ell(\mathbb{R})},\end{aligned}\tag{2.37}$$

where $c_2 = c_2(\ell, C_0, \|u_0\|_{H^2(\mathbb{R})}, T) > 0$. From (2.4e) and (2.4f),

$$\begin{aligned}&\sum_{i=0}^{\ell}\int_{\mathbb{R}}\frac{\partial^i}{\partial x^i}\big(g(t,x,u)\big)\frac{\partial^i u}{\partial x^i}dx \\ &\le\frac{1}{2}\sum_{i=0}^{\ell}\int_{\mathbb{R}}\Big(\frac{\partial^i}{\partial x^i}\big(g(t,x,u)\big)\Big)^2dx+\frac{1}{2}\sum_{i=0}^{\ell}\int_{\mathbb{R}}\Big(\frac{\partial^i u}{\partial x^i}\Big)^2dx\le c_3\|u(t,\cdot)\|^2_{H^\ell(\mathbb{R})},\end{aligned}\tag{2.38}$$

where $c_3 = c_3(\ell, C_0, \|u_0\|_{H^2(\mathbb{R})}, T) > 0$. From (2.4e), (2.4g), and (2.4h), since u, u_x are bounded,

$$\begin{aligned}&\sum_{i=0}^{\ell}\int_{\mathbb{R}}\frac{\partial^{i+1}P}{\partial x^{i+1}}\frac{\partial^i u}{\partial x^i}dx \\ &\quad=-\sum_{i=0}^{\ell}\int_{\mathbb{R}}\frac{\partial^i P}{\partial x^i}\frac{\partial^{i+1}u}{\partial x^{i+1}}dx \\ &\quad\le\frac{C_0}{2}\sum_{i=0}^{\ell}\int_{\mathbb{R}}\Big(\frac{\partial^i P}{\partial x^i}\Big)^2dx+\frac{1}{2C_0}\sum_{i=0}^{\ell}\int_{\mathbb{R}}\Big(\frac{\partial^{i+1}u}{\partial x^{i+1}}\Big)^2dx \\ &\quad\le c_4\|u(t,\cdot)\|^2_{H^\ell(\mathbb{R})}+\frac{1}{2C_0}\Big\|\frac{\partial u}{\partial x}(t,\cdot)\Big\|^2_{H^\ell(\mathbb{R})}+c_4,\end{aligned}\tag{2.39}$$

where $c_4 = c_4(\ell, C_0, \|u_0\|_{H^2(\mathbb{R})}, T) > 0$. Then, by (2.34), (2.35), (2.36), (2.37), (2.38) and (2.39),

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{H^\ell(\mathbb{R})} \le c_5\|u(t,\cdot)\|^2_{H^\ell(\mathbb{R})} + c_5,$$

where $c_5 = c_5(\ell, C_0, \|u_0\|_{H^2(\mathbb{R})}, T) > 0$, clearly this implies (2.31).

We conclude by proving (2.32). Let $0 < \tau < T$. Consider the new Cauchy problem

$$\begin{cases} \omega_t + \big(f(t,x,\omega)\big)_x + g(t,x,\omega) + \Omega_x = \big(a(t,x)\omega_x\big)_x, \\ -\Omega_{xx} + \Omega = h_0(t,x,\omega,\omega_x) + k_0(t,x,\omega), \\ \omega|_{t=0} = u(\tau,\cdot). \end{cases}$$

Since $u(\tau,\cdot) \in H^\ell(\mathbb{R})$ (see (2.31)), by Theorem 2.4, there exists a unique smooth solution $\omega = \omega(t,x)$ defined on the strip $[\tau, \tau+\varepsilon\rangle \times \mathbb{R}$, for some $\varepsilon > 0$. By Corollary 2.8, we have

$$u(t,x) = \omega(t-\tau, x), \qquad \tau \le t \le \tau+\varepsilon, \;\; x \in \mathbb{R},$$

and, due to the smoothness of ω, this implies (2.32). □

3. **Stability.** Let $u = u(t,x)$ be the solution of the Cauchy problem (see Theorem 2.3)

$$\begin{aligned} u_t + \big(f_0(t,x,u)\big)_x + g_0(t,x,u) + P_x &= \big(a_0(t,x)u_x\big)_x, \\ -P_{xx} + P &= h_0(t,x,u,u_x) + k_0(t,x,u), \\ u|_{t=0} &= u_0, \end{aligned} \tag{3.1}$$

for $(t,x) \in \Pi_T$, and $v = v(t,x)$ be the solution of

$$\begin{aligned} v_t + \big(f_1(t,x,v)\big)_x + g_1(t,x,v) + Q_x &= \big(a_1(t,x)v_x\big)_x, \\ -Q_{xx} + Q &= h_1(t,x,v,v_x) + k_1(t,x,v), \\ v|_{t=0} &= v_0, \end{aligned} \tag{3.2}$$

for $(t,x) \in \Pi_T$. Regarding the initial data u_0, $v_0 : \mathbb{R} \to \mathbb{R}$, we assume

$$u_0, v_0 \in H^\ell(\mathbb{R}) \qquad \text{for some } \ \ell \ge 2. \tag{3.3}$$

In the following sections we shall assume:

(**H.3**) the coefficients f_i, g_i, a_i, h_i and k_i are smooth;
(**H.4**) there exists a constant $C_0 > 0$ such that the following hold

$$\begin{aligned} f_i(\cdot,\cdot,0) &= 0, \\ \frac{1}{C_0} &\le a_i(\cdot,\cdot) \le C_0, \\ \left\|\frac{\partial^l f_i}{\partial x^l}(\cdot,\cdot,u)\right\|_{L^\infty} &\le C_0\,|u|, \quad 1 \le l \le \ell+1, \\ \|a_i\|_{C^\infty}, \left\|\frac{\partial^{l+j} f_i}{\partial x^l \partial u^j}\right\|_{L^\infty} &\le C_0, \quad j \ge 1,\, 2 \le l+j \le \ell+1, \\ \|k_{i,x}(\cdot,\cdot,u)\|_{L^\infty}, \left\|\frac{\partial^l g_i}{\partial x^l}(\cdot,\cdot u)\right\|_{L^\infty} &\le C_0\,|u|, \quad 0 \le l \le \ell, \\ \left\|\frac{\partial^{l+j} g_i}{\partial x^l \partial u^j}\right\|_{L^\infty} &\le C_0, \quad j \ge 1,\, 1 \le l+j \le \ell+1, \\ u \longmapsto \left\|\frac{\partial^{l+j} k_i}{\partial x^l \partial u^j}(\cdot,\cdot,u)\right\|_{L^\infty} &\text{ are in } L^\infty_{\mathrm{loc}}(\mathbb{R}), \quad 0 \le l+j \le \ell+1, \\ (u,q) \longmapsto \left\|\frac{\partial^{l+j+p} h_i}{\partial x^l \partial u^j \partial q^p}(\cdot,\cdot,u,q)\right\|_{L^\infty} &\text{ is in } L^\infty_{\mathrm{loc}}(\mathbb{R}^2), \quad 0 \le l+j+p \le \ell+1, \end{aligned}$$

$$h_i(t,x,u,q)q - \frac{1}{2} f_{i,uu}(t,x,u)q^3 \le C_0(u^2+q^2),$$
$$\|h_i(\,\cdot\,,\,\cdot\,,u,q)\|_{L^\infty} \le C_0(|u|+u^2+q^2),$$

for every for $i=0,1$, $(t,x)\in\Pi_T$, and $u,\ q\in\mathbb{R}$;

(**H.5**) the following identities hold

$$\begin{aligned} f_{0,x}(t,x,0) &= f_{1,x}(t,x,0),\\ g_0(t,x,0) &= g_1(t,x,0),\\ k_0(t,x,0) &= k_1(t,x,0),\\ h_{0,u}(t,x,0,q) &= h_{1,u}(t,x,0,q),\\ h_{0,q}(t,x,u,0) &= h_{1,q}(t,x,u,0), \end{aligned}$$

for every $(t,x)\in\Pi_T$ and $u,\ q\in\mathbb{R}$.

Clearly, these assumptions are satisfied in the case of the viscosity approximation of the generalized Camassa–Holm equation (see Example 2.2).

From Theorem 2.9 and Lemma 2.11 we know that

$$\|u(t,\cdot)\|_{H^2(\mathbb{R})},\|v(t,\cdot)\|_{H^2(\mathbb{R})} \le C_2, \qquad 0\le t\le T, \tag{3.4}$$

where

$$\begin{aligned} C_2 &:= \big(\|u_0\|_{H^1(\mathbb{R})}+\|v_0\|_{H^1(\mathbb{R})}\big)e^{11C_0T}\\ &\quad+\big(\|u_{0,xx}\|_{H^1(\mathbb{R})}+\|v_{0,xx}\|_{H^1(\mathbb{R})}\big)e^{11C_0T}e^{A_1T}+\frac{B_1}{A_1}(e^{A_1T}-1),\\ A_1 &:= 2\big[9c_1C_0^3\big(\|u_0\|^2_{H^1(\mathbb{R})}+\|v_0\|^2_{H^1(\mathbb{R})}\big)e^{11C_0T}+6C_0^3+C_0\big],\\ B_1 &:= 2\Big[39C_0^3\big(\|u_0\|^2_{H^1(\mathbb{R})}+\|v_0\|^2_{H^1(\mathbb{R})}\big)e^{11C_0T}\\ &\quad+\Big(\frac{1}{2}+6C_0\Big)\big(\|u_0\|^4_{H^1(\mathbb{R})}+\|v_0\|^4_{H^1(\mathbb{R})}\big)4e^{22C_0T}\Big]. \end{aligned}$$

The main result of this section is the following.

Theorem 3.1. *Assume* (**H.3**), (**H.4**), (**H.5**), *and* (3.3). *Then*

$$\begin{aligned} \|u(t,\cdot)-v(t,\cdot)\|_{H^1(\mathbb{R})} \le \|u_0-v_0\|_{H^1(\mathbb{R})}\,e^{K_0t}&\\ +K_1t\big[\,\|f_{1,ux}-f_{0,ux}\|_{L^\infty(\mathcal{I})}&+\|f_{1,u}-f_{0,u}\|_{L^\infty(\mathcal{I})}\\ +\|g_{1,u}-g_{0,u}\|_{L^\infty(\mathcal{I})}&\\ +\|a_{1,x}-a_{0,x}\|_{L^\infty(\Pi_T)}&+\|a_1-a_0\|_{L^\infty(I)}\\ +\|h_{0,u}-h_{1,u}\|_{L^\infty(\mathcal{J})}&+\|k_{0,u}-k_{1,u}\|_{L^\infty(\mathcal{I})}\\ +\|h_{0,q}-h_{1,q}\|_{L^\infty(\mathcal{J})}\,\big]&, \end{aligned} \tag{3.5}$$

for each $0\le t\le T$, *where*

$$\begin{aligned} \mathcal{I} &:= \Pi_T\times\Big[-\frac{C_2}{\sqrt{2}},\frac{C_2}{\sqrt{2}}\Big],\\ \mathcal{J} &:= \mathcal{I}\times\Big[-\frac{C_2}{\sqrt{2}},\frac{C_2}{\sqrt{2}}\Big],\\ K_0 &:= \kappa(C_0C_2+C_0^2C_2^2+C_0^3C_2^2+C_2+1),\\ K_1 &:= \Big(\sqrt{\frac{5C_0}{2}}+1\Big)e^{K_0T}, \end{aligned}$$

for some positive constant κ *independent of* T, C_0, u_0, *and* v_0.

3.1. **The homotopy argument.** Our approach, as in [3], is based on the following homotopy argument. Let $0 \le \theta \le 1$. The function ω_θ interpolates between the functions u and v. More precisely, denote by ω_θ the solution of the initial value problem (see Theorem 2.3)

$$\begin{aligned} \omega_{\theta,t} + \big(f_\theta(t,x,\omega_\theta)\big)_x + g_\theta(t,x,\omega_\theta) + \Omega_{\theta,x} &= \big(a_\theta(t,x)\omega_{\theta,x}\big)_x, \\ -\Omega_{\theta,xx} + \Omega_\theta &= h_\theta(t,x,\omega_\theta,\omega_{\theta,x}) + k_\theta(t,x,\omega_\theta), \\ \omega_\theta|_{t=0} &= \theta v_0 + (1-\theta)u_0, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} a_\theta &:= \theta a_1 + (1-\theta)a_0, \\ f_\theta &:= \theta f_1 + (1-\theta)f_0, \\ g_\theta &:= \theta g_1 + (1-\theta)g_0, \\ h_\theta &:= \theta h_1 + (1-\theta)h_0, \\ k_\theta &:= \theta k_1 + (1-\theta)k_0. \end{aligned}$$

Clearly

$$\omega_0 = u, \qquad \omega_1 = v.$$

Indeed

$$\theta \longmapsto \omega_\theta(t,x)$$

is a curve joining $u(t,x)$ and $v(t,x)$, and

$$\|u(t,\cdot) - v(t,\cdot)\|_{H^1(\mathbb{R})} \equiv \mathrm{dist}_{H^1(\mathbb{R})}\big(u(t,\cdot), v(t,\cdot)\big) \le \mathrm{length}_{H^1(\mathbb{R})}\big(\omega_\theta(t,\cdot)\big), \tag{3.7}$$

for each $t \ge 0$.

Lemma 3.2 (Smoothness of $\theta \mapsto \omega_\theta$)**.** *Assume* (**H**.**3**)*,* (**H**.**4**)*, and* (3.3)*. The curve*

$$\theta \in [0,1] \longmapsto \omega_\theta(t,\cdot) \in C^2(\mathbb{R})$$

is of class C^1. In particular, we infer

$$\mathrm{length}_{H^1(\mathbb{R})}\big(\omega_\theta(t,\cdot)\big) = \int_0^1 \left\| \frac{\partial \omega_\theta}{\partial \theta}(t,\cdot) \right\|_{H^1(\mathbb{R})} d\theta, \tag{3.8}$$

for each $0 \le t \le T$.

Proof. Consider the map

$$\begin{aligned} \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \colon \mathcal{D} &\longrightarrow C\big(\mathbb{R}, H^{\ell-2}(\mathbb{R})\big) \times H^{\ell-2}(\mathbb{R}), \\ \mathcal{F}_1(\theta,\omega,\Omega) &:= \omega_t + \big(f_\theta(t,x,\omega)\big)_x + g_\theta(t,x,\omega) + \Omega_x - \big(a_\theta(t,x)\omega_x\big)_x, \\ \mathcal{F}_2(\theta,\omega,\Omega) &:= -\Omega_{xx} + \Omega - h_\theta(t,x,\omega,\omega_x) - k_\theta(t,x,\omega), \end{aligned}$$

where

$$\mathcal{D} := \Big\{ (\theta,\omega,\Omega) \in [0,1] \times C\big([0,T]; H^\ell(\mathbb{R})\big)^2 \mid \omega|_{t=0} = \theta v_0 + (1-\theta)u_0 \Big\}.$$

From the definition of ω_θ and Ω_θ,

$$\mathcal{F}(\theta,\omega_\theta,\Omega_\theta) \equiv 0, \qquad 0 \le \theta \le 1. \tag{3.9}$$

Observe that $\mathcal{F}$ is of class C^1, and

$$\begin{aligned} \frac{\partial \mathcal{F}_1}{\partial \theta}(\theta,\omega,\Omega) &= \big(f_1(t,x,\omega) - f_0(t,x,\omega)\big)_x \\ &\quad + g_1(t,x,\omega) - g_0(t,x,\omega) - \Big(\big(a_1(t,x) - a_0(t,x)\big)\omega_x\Big)_x, \\ \frac{\partial \mathcal{F}_2}{\partial \theta}(\theta,\omega,\Omega) &= h_0(t,x,\omega,\omega_x) - h_1(t,x,\omega,\omega_x) \\ &\quad + k_0(t,x,\omega) - k_1(t,x,\omega), \end{aligned}$$

$$\frac{\partial \mathcal{F}_1}{\partial \Omega}(\theta,\omega,\Omega)\big[(\theta',z,Z)\big] = Z_x,$$
$$\frac{\partial \mathcal{F}_2}{\partial \Omega}(\theta,\omega,\Omega)\big[(\theta',z,Z)\big] = -Z_{xx} + Z.$$

Using

$$\frac{\partial \mathcal{F}_i}{\partial \omega}(\theta,\omega,\Omega)\big[(\theta',z,Z)\big] = \frac{\partial \mathcal{F}_i}{\partial \varepsilon}(\theta,\omega+\varepsilon z,\Omega)\bigg|_{\varepsilon=0}, \qquad i=1,2,$$

we find

$$\frac{\partial \mathcal{F}_1}{\partial \omega}(\theta,\omega,\Omega)\big[(\theta',z,Z)\big] = z_t + \big(f_{\theta,u}(t,x,\omega)z\big)_x + g_{\theta,u}(t,x,\omega)z - \big(a_\theta(t,x)z_x\big)_x,$$
$$\frac{\partial \mathcal{F}_2}{\partial \omega}(\theta,\omega,\Omega)\big[(\theta',z,Z)\big] = -h_{\theta,u}(t,x,\omega,\omega_x)z - h_{\theta,q}(t,x,\omega,\omega_x)z_x - k_{\theta,u}(t,x,\omega)z.$$

Observe that $(\theta',z,Z)\in\mathcal{D}$ satisfies the equation

$$\frac{\partial \mathcal{F}}{\partial(\omega,\Omega)}(\theta,\omega,\Omega)\big[(\theta',z,Z)\big] = (\zeta,\xi)$$

if and only if (θ',z,Z) is solution of the linear initial value problem

$$\begin{aligned}
z_t + \big(f_{\theta,u}(t,x,\omega)z\big)_x + g_{\theta,u}(t,x,\omega)z + Z_x &= \big(a_\theta(t,x)z_x\big)_x + \zeta(t,x),\\
-Z_{xx} + Z &= h_{\theta,u}(t,x,\omega,\omega_x)z + k_{\theta,u}(t,x,\omega)z\\
&\quad + h_{\theta,q}(t,x,\omega,\omega_x)z_x + \xi(t,x),\\
z|_{t=0} &= \theta' v_0 + (1-\theta')u_0.
\end{aligned}$$

Since this problem admits a unique solution (see Theorem 2.3), $\frac{\partial \mathcal{F}}{\partial(\omega,\Omega)}(\theta,\omega,\Omega)$ is invertible. By the implicit function theorem, the curve $\theta \longmapsto \omega_\theta$ is of class C^1 and clearly (3.8) holds. This concludes the proof. $\square$

Differentiating the equations in (3.6) with respect to θ, we have

$$\begin{aligned}
&\frac{\partial^2 \omega_\theta}{\partial t \partial \theta} + \frac{\partial}{\partial x}\Big(\frac{\partial f_\theta}{\partial u}(t,x,\omega_\theta)\frac{\partial \omega_\theta}{\partial \theta}\Big) + \frac{\partial}{\partial x}\Big(f_1(t,x,\omega_\theta) - f_0(t,x,\omega_\theta)\Big)\\
&\quad + \frac{\partial g_\theta}{\partial u}(t,x,\omega_\theta)\frac{\partial \omega_\theta}{\partial \theta} + \big(g_1(t,x,\omega_\theta) - g_0(t,x,\omega_\theta)\big) + \frac{\partial^2 \Omega_\theta}{\partial x \partial \theta}\\
&\quad = \frac{\partial}{\partial x}\Big(a_\theta(t,x)\frac{\partial^2 \omega_\theta}{\partial x \partial \theta}\Big) + \frac{\partial}{\partial x}\Big(\big(a_1(t,x) - a_0(t,x)\big)\frac{\partial \omega_\theta}{\partial x}\Big),\\
&-\frac{\partial^3 \Omega_\theta}{\partial^2 x \partial \theta} + \frac{\partial \Omega_\theta}{\partial \theta} = \frac{\partial h_\theta}{\partial u}(t,x,\omega_\theta,\omega_{\theta,x})\frac{\partial \omega_\theta}{\partial \theta} + \frac{\partial h_\theta}{\partial q}(t,x,\omega_\theta,\omega_{\theta,x})\frac{\partial^2 \omega_\theta}{\partial x \partial \theta}\\
&\quad + \big(h_1(t,x,\omega_\theta,\omega_{\theta,x}) - h_0(t,x,\omega_\theta,\omega_{\theta,x})\big)\\
&\quad + \frac{\partial k_\theta}{\partial u}(t,x,\omega_\theta)\frac{\partial \omega_\theta}{\partial \theta} + \big(k_1(t,x,\omega_\theta) - k_0(t,x,\omega_\theta)\big).
\end{aligned}$$

Denoting

$$\begin{aligned}
z_\theta &:= \frac{\partial \omega_\theta}{\partial \theta},\\
Z_\theta &:= \frac{\partial \Omega_\theta}{\partial \theta},\\
\alpha(\theta,t,x) &:= \frac{\partial f_\theta}{\partial u}(t,x,\omega_\theta),\\
\beta(\theta,t,x) &:= \frac{\partial g_\theta}{\partial u}(t,x,\omega_\theta),\\
\gamma(\theta,t,x) &:= \frac{\partial}{\partial x}\Big(f_1(t,x,\omega_\theta) - f_0(t,x,\omega_\theta)\Big) + \big(g_1(t,x,\omega_\theta) - g_0(t,x,\omega_\theta)\big)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \frac{\partial}{\partial x}\Big(\big(a_1(t,x) - a_0(t,x)\big)\frac{\partial \omega_\theta}{\partial x}\Big),\\
&\delta(\theta,t,x) := a_\theta(t,x),\\
&\eta(\theta,t,x) := \big(h_0(t,x,\omega_\theta,\omega_{\theta,x}) - h_1(t,x,\omega_\theta,\omega_{\theta,x})\big) + \big(k_0(t,x,\omega_\theta) - k_1(t,x,\omega_\theta)\big),\\
&\mu(\theta,t,x) := \frac{\partial h_\theta}{\partial u}(t,x,\omega_\theta,\omega_{\theta,x}) + \frac{\partial k_\theta}{\partial u}(t,x,\omega_\theta),\\
&\nu(\theta,t,x) := \frac{\partial h_\theta}{\partial q}(t,x,\omega_\theta,\omega_{\theta,x}),
\end{aligned}
$$

for each $0 \le \theta \le 1$, we find

$$
\begin{aligned}
z_{\theta,t} + \big(\alpha z_\theta\big)_x + \beta z_\theta + \gamma + Z_{\theta,x} &= \big(\delta z_{\theta,x}\big)_x,\\
-Z_{\theta,xx} + Z_\theta &= \mu z_\theta + \nu z_{\theta,x} + \eta,
\end{aligned}
\tag{3.10}
$$

for $(t,x) \in \Pi_T$. Moreover, observe that

$$
z_\theta|_{t=0} = v_0 - u_0, \qquad 0 \le \theta \le 1. \tag{3.11}
$$

3.2. **Energy estimate.** In this section we prove Theorem 3.1.

Remark 3.3. *By* (**H**.**4**)*,* (**H**.**5**)*, and* (3.4)*, we infer*

$$
\begin{aligned}
\|\alpha\|_{L^\infty} &\le \|f_{0,u}\|_{L^\infty(\mathcal{I})} + \|f_{1,u}\|_{L^\infty(\mathcal{I})} \le C_3,\\
\|\alpha_x\|_{L^\infty} &\le \|f_{0,ux}\|_{L^\infty(\mathcal{I})} + \|f_{1,ux}\|_{L^\infty(\mathcal{I})}\\
&\qquad + \big(\|f_{0,uu}\|_{L^\infty(\mathcal{I})} + \|f_{1,uu}\|_{L^\infty(\mathcal{I})}\big)\sqrt{2}C_2 \le C_3,\\
\|\beta\|_{L^\infty} &\le \|g_{0,u}\|_{L^\infty(\mathcal{I})} + \|g_{1,u}\|_{L^\infty(\mathcal{I})} \le C_3,\\
\|\delta_{xx}\|_{L^\infty} &\le \|a_{0,xx}\|_{L^\infty(\Pi_T)} + \|a_{1,xx}\|_{L^\infty(\Pi_T)} \le C_3,\\
\|\mu\|_{L^\infty} &\le \|h_{0,u}\|_{L^\infty(\mathcal{J})} + \|h_{1,u}\|_{L^\infty(\mathcal{J})} + \|k_{0,u}\|_{L^\infty(\mathcal{I})} + \|k_{1,u}\|_{L^\infty(\mathcal{I})} \le C_3,\\
\|\nu\|_{L^\infty} &\le \|h_{0,q}\|_{L^\infty(\mathcal{J})} + \|h_{1,q}\|_{L^\infty(\mathcal{J})} \le C_3,
\end{aligned}
\tag{3.12}
$$

where

$$
C_3 := c_2 C_0 C_2,
$$

for some positive constant c_2 independent on C_0, u_0, and v_0. Moreover,

$$
\frac{1}{C_0} \le \delta(\,\cdot\,,\,\cdot\,,\,\cdot\,) \le C_0,
$$

$$
\begin{aligned}
\|\gamma(\theta,t,\,\cdot\,)\|_{L^2(\mathbb{R})} &\le \|f_{1,ux} - f_{0,ux}\|_{L^\infty(\mathcal{I})}\|\omega_\theta(t,\,\cdot\,)\|_{L^2(\mathbb{R})}\\
&\quad + \|f_{1,u} - f_{0,u}\|_{L^\infty(\mathcal{I})}\|\omega_{\theta,x}(t,\,\cdot\,)\|_{L^2(\mathbb{R})}\\
&\quad + \|g_{1,u} - g_{0,u}\|_{L^\infty(\mathcal{I})}\|\omega_\theta(t,\,\cdot\,)\|_{L^2(\mathbb{R})}\\
&\quad + \|a_{1,x} - a_{0,x}\|_{L^\infty(\Pi_T)}\|\omega_{\theta,x}(t,\,\cdot\,)\|_{L^2(\mathbb{R})}\\
&\quad + \|a_1 - a_0\|_{L^\infty(\Pi_T)}\|\omega_{\theta,xx}(t,\,\cdot\,)\|_{L^2(\mathbb{R})}\\
&\le C_2\big[\|f_{1,ux} - f_{0,ux}\|_{L^\infty(\mathcal{I})} + \|f_{1,u} - f_{0,u}\|_{L^\infty(\mathcal{I})}\\
&\quad + \|g_{1,u} - g_{0,u}\|_{L^\infty(\mathcal{I})}\\
&\quad + \|a_{1,x} - a_{0,x}\|_{L^\infty(\Pi_T)} + \|a_1 - a_0\|_{L^\infty(\Pi_T)}\big],
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
\|\eta(\theta,t,\,\cdot\,)\|_{L^2(\mathbb{R})} &\le \|h_{0,u} - h_{1,u}\|_{L^\infty(\mathcal{J})}\|\omega_\theta(t,\,\cdot\,)\|_{L^2(\mathbb{R})}\\
&\quad + \|k_{0,u} - k_{1,u}\|_{L^\infty(\mathcal{I})}\|\omega_\theta(t,\,\cdot\,)\|_{L^2(\mathbb{R})}\\
&\quad + \|h_{0,q} - h_{1,q}\|_{L^\infty(\mathcal{J})}\|\omega_{\theta,x}(t,\,\cdot\,)\|_{L^2(\mathbb{R})}\\
&\le C_2\big[\|h_{0,u} - h_{1,u}\|_{L^\infty(\mathcal{J})}\\
&\quad + \|k_{0,u} - k_{1,u}\|_{L^\infty(\mathcal{I})} + \|h_{0,q} - h_{1,q}\|_{L^\infty(\mathcal{J})}\big],
\end{aligned}
\tag{3.14}
$$

for each $0 \le \theta \le 1$ and $0 \le t \le T$.

The following lemma is needed.

Lemma 3.4. *Assume* (**H.3**)*,* (**H.4**)*,* (**H.5**)*, and* (3.3)*. Then*

$$\begin{aligned}\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})} \le{}& \|z_\theta(0,\cdot)\|^2_{H^1(\mathbb{R})} e^{K_2 t} \\ &+ \frac{5C_0}{2}\int_0^t e^{K_2(t-\tau)}\|\gamma(\theta,\tau,\cdot)\|^2_{L^2(\mathbb{R})} d\tau \\ &+ \int_0^t e^{K_2(t-\tau)}\|\eta(\theta,\tau,\cdot)\|^2_{L^2(\mathbb{R})} d\tau,\end{aligned} \tag{3.15}$$

for each $0 \le t \le T$ *and* $0 \le \theta \le 1$*, where*

$$K_2 := 5C_3 + C_3^2 + 3C_0C_3^2 + C_2 + 2.$$

Proof. Let $0 < \theta < 1$ and $t > 0$. Multiplying the first equation of (3.10) by z_θ and integrating on $\mathbb{R}$,

$$\begin{aligned}\int_{\mathbb{R}} z_{\theta,t} z_\theta dx ={}& -\int_{\mathbb{R}} \big(\alpha z_\theta\big)_x z_\theta dx - \int_{\mathbb{R}} \beta z_\theta^2 dx \\ &- \int_{\mathbb{R}} \gamma z_\theta dx - \int_{\mathbb{R}} Z_{\theta,x} z_\theta dx + \int_{\mathbb{R}} \big(\delta z_{\theta,x}\big)_x z_\theta dx.\end{aligned} \tag{3.16}$$

Integrating by parts and using (**H.3**), (**H.4**), (**H.5**), (3.12),

$$\begin{aligned}-\int_{\mathbb{R}} \beta z_\theta^2 dx &\le \|\beta\|_{L^\infty}\int_{\mathbb{R}} z_\theta^2 dx \le C_3 \|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})}, \\ \int_{\mathbb{R}} \big(\delta z_{\theta,x}\big)_x z_\theta dx &= -\int_{\mathbb{R}} \delta z_{\theta,x}^2 dx \le 0, \\ -\int_{\mathbb{R}} \big(\alpha z_\theta\big)_x z_\theta dx &= \int_{\mathbb{R}} \alpha z_\theta z_{\theta,x} dx \\ &\le \|\alpha\|^2_{L^\infty}\frac{1}{2}\int_{\mathbb{R}} z_\theta^2 dx + \frac{1}{2}\int_{\mathbb{R}} z_{\theta,x}^2 dx \\ &\le \frac{C_3^2}{2}\|z_\theta(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|z_{\theta,x}(t,\cdot)\|^2_{L^2(\mathbb{R})} \le \frac{C_3^2}{2}\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})}, \\ -\int_{\mathbb{R}} \gamma z_\theta dx &\le \frac{1}{2}\int_{\mathbb{R}} \gamma^2 dx + \frac{1}{2}\int_{\mathbb{R}} z_\theta^2 dx \le \frac{1}{2}\int_{\mathbb{R}} \gamma^2 dx + \frac{1}{2}\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})}.\end{aligned} \tag{3.17}$$

Then from (3.16) and (3.17),

$$\frac{1}{2}\frac{d}{dt}\|z_\theta(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C_4 \|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})} - \int_{\mathbb{R}} Z_{\theta,x} z_\theta dx + \frac{1}{2}\|\gamma(\theta,t,\cdot)\|^2_{L^2(\mathbb{R})}, \tag{3.18}$$

where

$$C_4 := C_3 + \frac{C_3^2}{2} + \frac{1}{2}.$$

Differentiating the first equation in (3.10) with respect to x, we have

$$\begin{aligned}z_{\theta,tx} + \big(\alpha_x z_\theta + \alpha z_{\theta,x}\big)_x + (\beta z_\theta)_x + \gamma_x &= -Z_{\theta,xx} + \big(\delta_x z_{\theta,x} + \delta z_{\theta,xx}\big)_x \\ &= -Z_\theta + \mu z_\theta + \nu z_{\theta,x} + \eta + \big(\delta_x z_{\theta,x} + \delta z_{\theta,xx}\big)_x.\end{aligned} \tag{3.19}$$

Multiplying this equation by $z_{\theta,x}$ and integrating on $\mathbb{R}$, we get

$$\begin{aligned}\int_{\mathbb{R}} z_{\theta,tx} z_{\theta,x} dx ={}& -\int_{\mathbb{R}} \big(\alpha_x z_\theta + \alpha z_{\theta,x}\big)_x z_{\theta,x} dx \\ &- \int_{\mathbb{R}} \big(\beta z_\theta\big)_x z_{\theta,x} dx - \int_{\mathbb{R}} \gamma_x z_{\theta,x} dx \\ &- \int_{\mathbb{R}} Z_\theta z_{\theta,x} dx + \int_{\mathbb{R}} \mu z_\theta z_{\theta,x} dx + \int_{\mathbb{R}} \nu z_{\theta,x}^2 dx\end{aligned} \tag{3.20}$$

$$+\int_{\mathbb{R}} \eta z_{\theta,x} dx + \int_{\mathbb{R}} \big(\delta_x z_{\theta,x} + \delta z_{\theta,xx}\big)_x z_{\theta,x} dx.$$

Again, integrating by parts and using (**H.3**), (**H.4**), (**H.5**), (3.12),

$$\int_{\mathbb{R}} z_{\theta,tx} z_{\theta,x}\, dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} z_{\theta,x}^2\, dx = \frac{1}{2}\frac{d}{dt}\|z_{\theta,x}(t,\cdot)\|^2_{L^2(\mathbb{R})}, \tag{3.21}$$

$$\begin{aligned}
-\int_{\mathbb{R}} \big(\alpha_x z_\theta + \alpha z_{\theta,x}\big)_x z_{\theta,x}\, dx &= \int_{\mathbb{R}} \big(\alpha_x z_\theta + \alpha z_{\theta,x}\big) z_{\theta,xx}\, dx\\
&\le \frac{3C_0}{4}\int_{\mathbb{R}} \big(\alpha_x z_\theta + \alpha z_{\theta,x}\big)^2 dx + \frac{1}{3C_0}\int_{\mathbb{R}} z_{\theta,xx}^2 dx\\
&\le \frac{3C_0C_3^2}{4}\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})} + \frac{1}{3C_0}\int_{\mathbb{R}} z_{\theta,xx}^2 dx,\\
-\int_{\mathbb{R}} \big(\beta z_\theta\big)_x z_{\theta,x} dx &= \int_{\mathbb{R}} \beta z_\theta z_{\theta,xx} dx\\
&\le \frac{3C_0}{4}\int_{\mathbb{R}} \beta^2 z_\theta^2 dx + \frac{1}{3C_0}\int_{\mathbb{R}} z_{\theta,xx}^2 dx\\
&\le \frac{3C_0C_3^2}{4}\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})} + \frac{1}{3C_0}\int_{\mathbb{R}} z_{\theta,xx}^2 dx,\\
-\int_{\mathbb{R}} \gamma_x z_{\theta,x}\, dx &= \int_{\mathbb{R}} \gamma z_{\theta,xx} dx \le \frac{3C_0}{4}\int_{\mathbb{R}} \gamma^2 dx + \frac{1}{3C_0}\int_{\mathbb{R}} z_{\theta,xx}^2 dx,\\
\int_{\mathbb{R}} \mu z_\theta z_{\theta,x} dx &\le \|\mu\|_{L^\infty}\int_{\mathbb{R}} |z_\theta z_{\theta,x}|\, dx\\
&\le \frac{C_2}{2}\int_{\mathbb{R}} z_\theta^2 dx + \frac{C_2}{2}\int_{\mathbb{R}} z_{\theta,x}^2 dx = \frac{C_2}{2}\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})},\\
\int_{\mathbb{R}} \nu z_{\theta,x}^2 dx &\le \|\nu\|_{L^\infty}\int_{\mathbb{R}} z_{\theta,x}^2 dx \le C_3\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})},\\
\int_{\mathbb{R}} \big(\delta_x z_{\theta,x} + \delta z_{\theta,xx}\big)_x z_{\theta,x} dx &= -\int_{\mathbb{R}} \big(\delta_x z_{\theta,x} + \delta z_{\theta,xx}\big) z_{\theta,xx} dx\\
&= -\frac{1}{2}\int_{\mathbb{R}} \delta_x (z_{\theta,x}^2)_x dx - \int_{\mathbb{R}} \delta z_{\theta,xx}^2 dx\\
&\le \frac{1}{2}\int_{\mathbb{R}} \delta_{xx} z_{\theta,x}^2 dx - \frac{1}{C_0}\int_{\mathbb{R}} z_{\theta,xx}^2 dx\\
&\le \frac{1}{2}\|\delta_{xx}\|_{L^\infty}\int_{\mathbb{R}} z_{\theta,x}^2 dx - \frac{1}{C_0}\int_{\mathbb{R}} z_{\theta,xx}^2 dx\\
&\le \frac{C_3}{2}\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})} - \frac{1}{C_0}\int_{\mathbb{R}} z_{\theta,xx}^2 dx,\\
\int_{\mathbb{R}} \eta z_{\theta,x} dx &\le \frac{1}{2}\int_{\mathbb{R}} \eta^2 dx + \frac{1}{2}\int_{\mathbb{R}} z_{\theta,x}^2 dx\\
&\le \frac{1}{2}\int_{\mathbb{R}} \eta^2 dx + \frac{1}{2}\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})}.
\end{aligned}$$

Therefore, from (3.20) and (3.21),

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z_{\theta,x}(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C_5\|z_\theta(t,\cdot)\|^2_{H^1(\mathbb{R})} &+ \int_{\mathbb{R}} Z_{\theta,x} z_\theta\, dx\\
&+ \frac{3C_0}{4}\|\gamma(\theta,t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\eta(\theta,t,\cdot)\|^2_{L^2(\mathbb{R})},
\end{aligned} \tag{3.22}$$

where

$$C_5 := \frac{3C_0C_3^2}{2} + \frac{C_2}{2} + \frac{3C_3}{2} + \frac{1}{2}.$$

Summing (3.18) and (3.22),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|z_\theta(t,\,\cdot\,)\|^2_{H^1(\mathbb{R})} &\leq (C_4+C_5)\|z_{\theta,x}(t,\,\cdot\,)\|^2_{H^1(\mathbb{R})} \\ &\quad + \frac{5C_0}{4}\|\gamma(\theta,t,\,\cdot\,)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\eta(\theta,t,\,\cdot\,)\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{3.23}$$

The claim now follows from (3.23) and the Gronwall inequality. □

Proof of Theorem 3.1. The claim is direct consequence of (3.8), (3.11), (3.15), (3.13), and (3.14). □

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