An Instability of the Godunov Scheme

Alberto Bressan* Helge Kristian Jenssen† Paolo Baiti ‡

February 1, 2005

Abstract

We construct a solution to a 2×2 strictly hyperbolic system of conservation laws, showing that the Godunov scheme [12] can produce an arbitrarily large amount of oscillations. This happens when the speed of a shock is close to rational, inducing a resonance with the grid. Differently from the Glimm scheme or the vanishing viscosity method, for systems of conservation laws our counterexample indicates that no a priori BV bounds or L^1 stability estimates can in general be valid for finite difference schemes.

1 Introduction

Consider a strictly hyperbolic $m \times m$ system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0. \tag{1.1}$$

For initial data with small total variation, the existence of a unique entropy weak solution is well known [11], [7], [5]. A closely related question is the stability and convergence of various types of approximate solutions. For vanishing viscosity approximations

$$u_t + f(u)_x = \varepsilon\, u_{xx}\,, \tag{1.2}$$

*Department of Mathematics, Penn State University, UP, PA 16802, U.S.A.; E-mail: bressan@math.psu.edu

†Department of Mathematics, North Carolina State University, Raleigh, NC 27695, U.S.A.; E-mail: hkjensse@math.ncsu.edu. Research supported in part by the NSF under grant DMS-0206631

‡Dipartimento di Matematica e Informatica, Università di Udine, Via delle Scienze 206, Udine 33100, Italy; E-mail: baiti@dimi.uniud.it

uniform BV bounds, stability and convergence as $\varepsilon \to 0$ were recently established in [4]. Assuming that all the eigenvalues of the Jacobian matrix $Df(u)$ are strictly positive, similar results are also proved in [2] for solutions constructed by the semidiscrete (upwind) Godunov scheme

$$\frac{d}{dt}\, u_j(t) + \frac{1}{\Delta x}\Big[f\big(u_j(t)\big) - f\big(u_{j-1}(t)\big)\Big] \;=\; 0\,, \qquad u_j(t) = u(t,\ j\,\Delta x)\,. \quad (1.3)$$

Recently, Bianchini has succeeded in proving the same type of BV bounds for the Jin-Xin relaxation model [3].

In the present paper we study the case of fully discrete schemes, where the derivatives with respect to both time and space are replaced by finite differences. In sharp contrast with the previous situations, we show that *the total variation of an approximate solution constructed by the Godunov scheme can become arbitrarily large.* This typically happens when the solution contains a shock along a curve $x = \gamma(t)$, whose speed remains for a long time very close (but not exactly equal) to a given rational, for example

$$\dot\gamma(t) \approx \frac{1}{2}\,\frac{\Delta x}{\Delta t}\,.$$

In this case, a resonance occurs and a substantial amount of downstream oscillations is observed in the numerically computed solution. For general $m \times m$ systems, our counterexample indicates that a priori BV bounds and uniform stability estimates cannot hold for solutions generated by finite difference schemes. It leaves open the possibility that these difference schemes still converge to the unique entropy weak solution. In any case, a rigorous proof of this convergence cannot rely on the same arguments as in [11], based on uniform BV bounds and Helly's compactness theorem.

We recall that, for the 2×2 system of isentropic gas dynamics, the convergence of Lax-Friedrichs approximations was proved in [9], within the framework of compensated compactness. Further results were obtained for *straight line systems*, where all the Rankine-Hugoniot curves are straight lines. For these systems, uniform BV bounds, stability and convergence of Godunov and Lax-Friedrichs approximations were established in [17], [6] and [21]. The analysis relies on the fact that, due to the very particular geometry, the interaction of waves of the same family does not generate additional oscillations.

It is interesting to understand why the arguments in [4] or [2] break down, when applied to fully discrete schemes. A key ingredient in the analysis of vanishing viscosity approximations is the local decomposition of a viscous solution in terms of traveling waves. To achieve a good control the oscillations produced by interactions of waves of a same family, it is essential that the center manifold of traveling profiles has a certain degree of smoothness. This is precisely what fails in the case of fully discrete schemes. As remarked by Serre [20], for general hyperbolic systems the discrete shock profiles cannot depend continuously on the speed σ, in the BV norm. In the related paper [1] we constructed an explicit example showing how this happens.

Our basic example is provided by a 2×2 system in triangular form

$$\begin{aligned} u_t + f(u)_x &= 0, &(1.4)\\ v_t + \lambda v_x + g(u)_x &= 0. &(1.5)\end{aligned}$$

This system is strictly hyperbolic provided that $f'(u) > \lambda$ for all u. Choosing mesh sizes $\Delta t = \Delta x = 1$, the Godunov (upwind) scheme takes the form

$$\begin{aligned} u_j^{n+1} &= u_j^n - \big[f(u_j^n) - f(u_{j-1}^n)\big], &(1.6)\\ v_j^{n+1} &= \lambda v_{j-1}^n + (1-\lambda) v_j^n - \big[g(u_j^n) - g(u_{j-1}^n)\big]. &(1.7)\end{aligned}$$

We shall assume that

$$0 < \lambda < f'(u) < 1 \qquad \text{for all} \quad u \in \mathbb{R} \tag{1.8}$$

so that the usual linearized stability conditions are satisfied. Thanks to the triangular form of the system (1.4)-(1.5), the exact solution of a Cauchy problem can be computed explicitly. Indeed, one first solves the scalar equation for u, say by the method of characteristics. Then the function $g(u)_x$ is plugged as a source term into the second equation, which is linear in v.

More specifically, we shall consider a solution of (1.4)-(1.5) where the first component u contains a single shock, located along the curve $x = \gamma(t)$. One can arrange things so that the speed of the shock varies slowly in time, remaining close to some rational number, say $\dot\gamma(t) \approx 1/2$. The second component v will then satisfy a linear transport equation with a source located along a smooth curve, and a priori bounds on its total variation can be easily given.

For the corresponding approximate solution generated by the Godunov scheme (1.6)-(1.7), however, things turn out to be quite different. Because of the discretization, in the equation for the second component the source is not located along a smooth curve, but sampled at grid points. Since the speed of the shock is close to rational, this source "resonates" with the grid, producing an arbitrarily large amount of downstream oscillations as time progresses.

The plan of the paper is as follows. Section 2 contains a preliminary analysis of the heat equation with a moving source. We show that if the point sources are located at grid points and have an average speed close to rational, then the solution will contain downstream oscillations. By studying the strength and the location of these tail oscillations one gets a basic understanding of what happens for solutions generated by discrete schemes.

In Sections 3 - 5 we carry out a detailed construction of a Godunov approximate solution for the system (1.4)-(1.5), showing that the total variation can become arbitrarily large. Choosing the flux

$$f(u) = \ln\big((1-\mu) + \mu e^u\big)\,, \tag{1.9}$$

one can perform a nonlinear transformation introduced by Lax [14], [15] and explicitly compute the discrete Godunov solution u_j^n, in terms of binomial coefficients. In Section 6 we summarize the main features of our example, and discuss its significance toward a rigorous theory of discrete schemes for hyperbolic conservation laws. For readers' convenience, some results on the approximation of the binomial distribution in terms of heat kernels are recalled in an appendix.

2 The heat equation with a moving source

As a motivation for the following analysis, consider a solution of the finite difference scheme

$$u_j^{n+1} = u_j^n - \big[f(u_j^n) - f(u_{j-1}^n)\big] \tag{2.1}$$

in the form of a discrete shock with speed σ, say

$$u_j^n = \phi(j - \sigma n)\,.$$

For a scalar conservation law, the existence of discrete traveling profiles was proved in [13]. We assume that the flux f is strictly convex and that the shock profile satisfies

$$u^- \;=\; \lim_{s\to-\infty}\phi(s) \;>\; \lim_{s\to+\infty}\phi(s) \;=\; u^+\,.$$

Inserting this solution as a source in the second equation (1.7), we obtain

$$\begin{aligned} v_j^{n+1} &= \lambda v_{j-1}^n + (1-\lambda)v_j^n - \big[g(u_j^n) - g(u_{j-1}^n)\big] \\ &= \lambda v_{j-1}^n + (1-\lambda)v_j^n - \Big[g\big(\phi(j-\sigma n)\big) - g\big(\phi(j-1-\sigma n)\big)\Big]. \end{aligned} \tag{2.2}$$

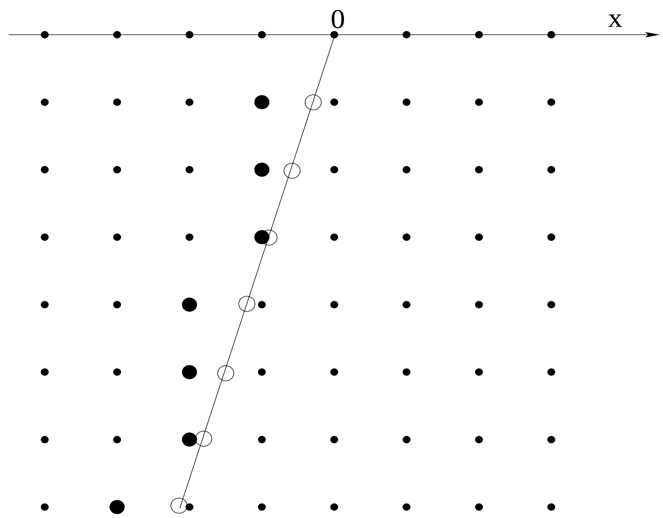

Figure 1: Discrete sources at integer points.

We can choose a function g such that

$$
\begin{array}{rcl}
g'(u) & = & 0 \quad \text{if } \left|u - \frac{u^+ + u^-}{2}\right| \geq \epsilon \\
g'(u) & > & 0 \quad \text{if } \left|u - \frac{u^+ + u^-}{2}\right| < \epsilon,
\end{array}
$$

for some $\epsilon > 0$ small. With this choice, the source terms on the right hand side of (2.2) will vanish outside a thin strip centered around the shock. Indeed, if the shock is located along the line $x = \sigma t$, a fairly good approximation is

$$
g(u_j^n) - g(u_{j-1}^n) \approx \begin{cases} g(u^+) - g(u^-) & \text{if } j - 1 = [\![\sigma n]\!] , \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}
$$

Here $[\![s]\!]$ denotes the largest integer $\leq s$. To understand the oscillations in (2.2) determined by the presence of these moving sources, we first study some model problems related to the standard heat equation. The approximation of a finite difference scheme by means of a second order diffusion equation is indeed a standard tool of analysis (see [16], p.117). In this section we review some calculations that highlight the mechanism responsible for generating variation in solutions to fully discrete schemes. For a detailed treatment we refer to the paper [1]. Rigorous estimates will then be worked out in Sections 3 - 5.

First, consider the easier case of the heat equation with point sources located on a discrete set of points $P_n = (n, \sigma n)$, with n integer (the white circles in Figure 1)

$$v_t - v_{xx} = \delta_{n,\sigma n}\,. \tag{2.4}$$

We assume that $\sigma > 0$ and consider a solution of (2.4) defined for $t \in]-\infty, 0[$. Its values at time $t = 0$ are now computed as

$$v(0, y) = \Phi(y) := \sum_{n\geq 1} G(n,\, y + \sigma n)\,. \tag{2.5}$$

Here

$$G(t, x) := \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}$$

is the standard heat kernel. To understand how the oscillations of $\Phi(y)$ decay as $y \to -\infty$, we express the sum in (2.5) as an integral

$$\Phi(y) := \sum_{n\geq 1} G(n,\, y + \sigma n) = \int_0^\infty G(t,\, y + \sigma t)\left(1 + h_1'(t)\right) dt \tag{2.6}$$

where

$$h_1(t) := [\![t]\!] - t + 1/2\,. \tag{2.7}$$

By induction, we can find a sequence of periodic and uniformly bounded functions h_m such that

$$h_m(t) = h_m(t+1)\,, \qquad \int_0^1 h_m(t)\, dt = 0\,, \qquad \frac{d}{dt} h_m(t) = h_{m-1}(t)\,.$$

Integrating by parts we obtain

$$\begin{aligned} \Phi(y) &= \int_0^\infty G(t,\, y + \sigma t)\left(1 + \frac{d^m}{dt^m} h_m(t)\right) dt \\ &= \frac{1}{\sigma} + (-1)^m \int_0^\infty \frac{d^m}{dt^m} G(t,\, y + \sigma t)\, h_m(t)\, dt\,. \end{aligned}$$

The identities

$$G(t, x) = t^{-1/2} G(1,\; x/\sqrt{t})\,, \qquad\qquad G_t = G_{xx}\,,$$

imply

$$\begin{aligned} \frac{\partial^m}{\partial x^m} G(t, x) &= t^{-(m+1)/2} \cdot \frac{\partial^m G(1,\; x/\sqrt{t})}{\partial x^m}, \\ \frac{\partial^m G(t, x)}{\partial t^m} &= t^{-(2m+1)/2} \cdot \frac{\partial^m}{\partial t^m} G(1,\; x/\sqrt{t})\,. \end{aligned}$$

In addition we observe that,

$$\sup_{\left|t+\frac{y}{\sigma}\right| < |y|^{\epsilon+1/2}} \left|\frac{d^m}{dt^m} G(t,\, y+\sigma t)\right| \;=\; O(1)\cdot e^{c_\epsilon y} \qquad\qquad \text{as } \; y\to -\infty\,,$$

for some constant $c_\varepsilon > 0$. Letting $y \to -\infty$, for every $m \geq 1$ the above estimates imply

$$\left|\Phi(y) - \frac{1}{\sigma}\right| \leq \int_0^\infty \left|\frac{d^m}{dt^m} G(t,\, y+\sigma t)\right| dt \;=\; O(1)\cdot y^{-m/2}. \tag{2.8}$$

Similarly,

$$\left|\Phi'(y)\right| \leq \int_0^\infty \left|\frac{d^{m+1}}{dt^{m+1}} G(t,\, y+\sigma t)\right| dt \;=\; O(1)\cdot y^{-(m+1)/2}. \tag{2.9}$$

Taking $m = 2$ in (2.9) one obtains the integrability of Φ', hence a bound on the total variation of Φ.

Next we outline the case when the point sources are located not at the points $P_n = (n, \sigma n)$, but at the points with integer coordinates $Q_n := \big(n,\, [\![\sigma n]\!]\big)$ (the black circles in Figure 1),

$$v_t - v_{xx} = \delta_{n, [\![\sigma n]\!]}\,.$$

Again we consider a solution defined for $t \in\,]-\infty,\, 0]$ and a direct computation yields

$$v(0,\, y-1) = \Psi(y) := \sum_{n\geq 1} G\big(n,\, y + [\![\sigma n]\!]\big).$$

Because of (2.8), to determine the asymptotic behavior as $y \to -\infty$, it suffices to estimate the difference

$$K(y) \;:=\; \Psi(y) - \Phi(y) \;=\; -\sum_{n\geq 1}\Big[G\big(n,\, y+\sigma n\big) - G\big(n,\, y + [\![\sigma n]\!]\big)\Big]\,.$$

It is here that, if the speed σ is close to a rational, a resonance is observed. To see a simple case, let $\sigma = 1+\varepsilon$, with $\varepsilon > 0$ small. Then we can approximate

$$\begin{aligned} K(y) \;&\approx\; -\sum_{n\geq 1} G_x(n,\, y+\sigma n)\big(\sigma n - [\![\sigma n]\!]\big) \\ &\approx\; -\int_0^\infty G_x(t,\, y+\sigma t)\,\big(\varepsilon t - [\![\varepsilon t]\!]\big)\, dt\,. \end{aligned} \tag{2.10}$$

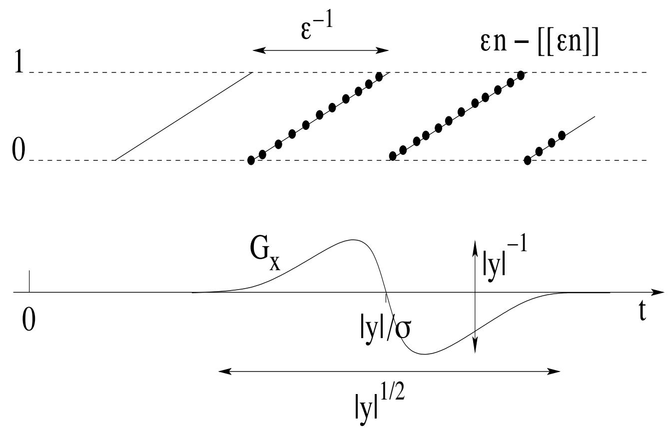

Figure 2: Interaction of G_x and fractional parts.

The functions appearing in the above integration are shown in Figure 2. We recall that

$$\int_0^\infty G_x(t,\, y+\sigma t)\, dt \;=\; -\int_0^\infty \frac{y+\sigma t}{4t\sqrt{\pi t}}\, \exp\left\{-\frac{(y+\sigma t)^2}{4t}\right\} dt \;=\; 0$$

for every $y<0$. Set $y_\varepsilon := -\varepsilon^{-2}$. When y ranges within the interval

$$I_\varepsilon := [y_\varepsilon,\ y_\varepsilon/2] \;=\; [-\varepsilon^{-2},\ -\varepsilon^{-2}/2]\,,$$

the integral in (2.10) can be of the same order of magnitude as

$$\int_0^\infty \big|G_x(t,\, y_\varepsilon+\sigma t)\big|\, dt \;\geq\; c_0\, y_\varepsilon^{-1/2} \;=\; c_0\,\varepsilon\,.$$

Moreover, each time that y increases by an amount $\Delta y = \varepsilon^{-1}$, the phase of the fractional part $[\![\varepsilon y]\!] - \varepsilon y$ goes through a full cycle, hence the map

$$y \mapsto \int_0^\infty G_x(t,\, y+\sigma t)\,\big(\varepsilon t - [\![\varepsilon t]\!]\big)\, dt$$

oscillates by an amount $\geq c_1\varepsilon$. In all, we have approximately $1/2\varepsilon$ cycles within the interval I_ε. Hence the total variation of the discrete profile $\Psi = \Psi^{(1+\varepsilon)}$ can be estimated as

$$\text{Tot. Var.}\,\left\{\Psi^{(1+\varepsilon)}\,;\quad I_\varepsilon\right\} \geq c_2 \tag{2.11}$$

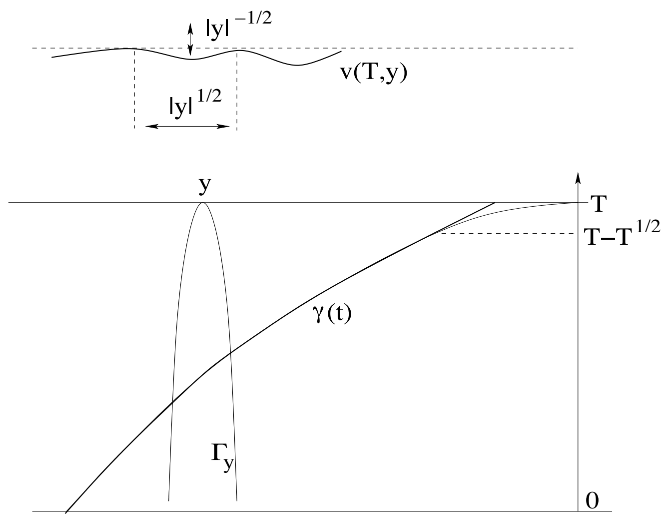

Figure 3: Oscillations produced by sources along the curve $\gamma(t)$.

for some constant $c_2 > 0$ independent of ε. By (2.11) it is clear that, as $\varepsilon \to 0+$, the functions $\Psi^{(1+\varepsilon)}$ do not form a Cauchy sequence and cannot converge in the space BV. The details of the preceding argument are given (for the Lax-Friedrichs scheme) in [1].

Finally we consider the case where the source travels with a carefully chosen variable speed. In the previous computation the source travelled with a constant speed $\sigma = 1+\varepsilon$, and a significant amount of oscillation was observed at a distance $|y_\varepsilon| = \varepsilon^{-2}$ downstream from the shock. To construct a solution whose oscillation becomes arbitrarily large, the idea is to choose a large time interval $[0, T]$ and generate a source with non-constant speed, say located along a curve $x = \gamma(t)$ with

$$\gamma(t) = (t-T) - 2\sqrt{T-t}\,, \qquad \dot{\gamma}(t) = 1 + \frac{1}{\sqrt{T-t}} \tag{2.12}$$

for $0 \leq t \leq T - T^{1/2}$, (see Figure 3). Consider the profile at the terminal time T of a solution of

$$v_t + v_{xx} = \delta_{n,[\![\gamma(n)]\!]} \qquad\qquad v(0,x) = 0, \tag{2.13}$$

defined for $t \in [0, T]$.

For a given point y the value $v(T, y)$ is essentially determined by the sources located inside the parabolic region

$$\Gamma_y := \Big\{(t,x) \ : \ x \in \big[y-(T-t)^{\epsilon+1/2},\ y+(T-t)^{\epsilon+1/2}\big]\Big\}.$$

By construction, if $-T/2 < y < -T^{1/2}$, this region contains a portion of the curve γ traveling with speed

$$\dot\gamma(t) = 1 + \frac{1}{\sqrt{T-t}} \approx 1 + \frac{1}{\sqrt{|y|}}\,.$$

This is precisely the resonant speed that can produce downstream oscillations at a distance $\approx y$ from the shock. As in (2.11), one has

$$\text{Tot. Var.}\ \big\{v(T,\cdot)\ ;\ \ [y,\,y/2]\big\} \geq c_2 \qquad\qquad -T/2 < y < -T^{1/2}\,. \tag{2.14}$$

Now consider the points

$$y_j \doteq -2^j\,T^{1/2} \qquad\qquad j = 0,1,\ldots,N-1\,,$$

where N is the largest integer $\leq \frac{1}{2}\log_2 T$. Notice that this choice of N guarantees that $-T/2 < y_N < \ldots < y_1 < y_0 = -T^{1/2}$. Observe that the choice of the variable speed $\dot\gamma$ at (2.12) is precisely what is needed in order to produce a uniformly positive variation on each of the intervals $[y_j,\,y_{j-1}]$. From (2.14) it thus follows

$$\text{Tot. Var.}\ \big\{v(T,\cdot)\big\} \geq \sum_{j=1}^{N} \text{Tot. Var.}\ \big\{v(T,\cdot)\,;\ [y_j,\,y_j/2]\big\} \geq c_2\cdot\left(\frac{1}{2}\log_2 T - 2\right)\,. \tag{2.15}$$

Choosing T large, we expect to find numerical solutions generated by the Godunov scheme whose total variation grows by an arbitrarily large amount. In the following sections we show that this is indeed the case. By providing rigorous estimates on all the approximations performed in the above formal analysis, we will prove that the discrete Godunov solution satisfies the same type of estimate as in (2.15).

3 Special solutions

3.1 Inviscid solution

It will be convenient to work with the particular scalar conservation law

$$u_t + \big[\ln(\nu + \mu e^u)\big]_x = 0\,, \tag{3.1}$$

where $\mu \in]\,1/2, 1\,[$ and $\nu := 1 - \mu$. This equation will serve as the first equation in the system (1.4)-(1.5). Notice that in this case the flux $f(u) = \ln(\nu + \mu e^u)$ is increasing and convex. Indeed,

$$f'(u) = \frac{\mu e^u}{\nu + \mu e^u}\,, \qquad f''(u) = \frac{\mu\nu e^u}{(\nu + \mu e^u)^2}\,,$$

so that

$$f'(u) \in]\,0, 1\,[\,, \qquad\qquad f''(u) > 0.$$

Note that

$$f(0) = 0, \qquad\qquad f'(0) = \mu > 1/2.$$

Our first goal is to construct an exact solution of the inviscid equation (3.1) containing exactly one shock, located along a curve $\gamma(t)$ which travels with a speed close to a rational number $\lambda_0 > \lambda$, where λ is as in (1.5). The role of this curve will be that its spatial translates are close to the level curves of a certain (everywhere defined) solution of the upwind scheme (1.6). This particular solution of the scheme will then be a good approximation of the exact solution of (3.1) which we construct here.

We now fix the speed λ in (1.5) to be 1/2. (This choice is not essential but it simplifies the calculations below which involve approximating binomial distributions with the heat kernel.) The shock solution of (1.4) will connect a variable state $u_-(t)$ to a fixed right state $u_+ < u_-(t)$, and we want this u-shock to travel faster than the advection speed $\lambda = 1/2$ of the second equation. We begin by fixing the right state $u_+ := 0$, and our solution will be constant equal to this value to the right of γ, i.e.

$$u(t, x) \equiv 0 \qquad\qquad \text{for } x > \gamma(t).$$

Next, in analogy with the analysis in Section 2, we insist that the shock curve $\gamma(t)$ should have a speed as in (2.12). For notational convenience we will make the explicit construction for negative times. We thus fix a large, positive (integer) time T, and define the curve $\gamma(t)$ by setting

$$\gamma(t) = \lambda_0 t - 2\sqrt{-t}, \qquad \text{for } t \in [-2T, -\sqrt{T}\,[\,. \tag{3.2}$$

We require γ to be smooth on $[-2T, 0]$ and such that

$$\dot\gamma(t) \equiv \dot\gamma(-\sqrt{T}) = \lambda_0 + \frac{1}{T^{1/4}} \qquad \text{for } t \in [-\sqrt{T}, 0\,]\,,$$

see Figure 3. Here $\lambda_0 \in]\,\mu, 1\,[\cap \mathbb{Q}$ denotes a fixed, rational speed. At every time $t \in [\,-2T, 0\,]$ the state $u_-(t)$ immediately to the left of the shock must then satisfy the Rankine-Hugoniot equation

$$\frac{\ln(\nu + \mu e^{u_-(t)})}{u_-(t)} = \dot{\gamma}(t). \tag{3.3}$$

To establish the existence and properties of the function $t \mapsto u_-(t)$, we let u_0 denote the left state corresponding to constant speed λ_0, i.e.

$$\frac{\ln(\nu + \mu e^{u_0})}{u_0} = \lambda_0\,,$$

and then consider the equation

$$\frac{\ln(\nu + \mu e^{u})}{u} = \sigma\,. \tag{3.4}$$

A straightforward argument shows that (3.4) has exactly one solution $u = \varphi(\sigma) \in]\,0, \infty\,[$ for each $\sigma \in]\,\mu, 1\,[$, and that φ depends smoothly on σ. Therefore, for t negative and sufficiently large, from (3.3) and (3.2) we recover a smooth function $t \mapsto u_-(t)$, where

$$\begin{aligned} u_-(t) &= \varphi\left(\lambda_0 + \frac{1}{\sqrt{-t}}\right) \\ &= \kappa_0 + \frac{\kappa_1}{(-t)^{1/2}} + \frac{\kappa_2}{(-t)} + \frac{O(1)}{(-t)^{3/2}}\,, \end{aligned} \tag{3.5}$$

for $t < 0$. We have here assumed the Taylor expansion

$$\varphi(\lambda_0 + \delta) = \kappa_0 + \kappa_1\delta + \kappa_2\delta^2 + O(1)\delta^3.$$

3.2 Discrete Cole-Hopf transformation

As observed by Lax [14], [15], for the particular flux function f in (3.1), one can perform a "discrete Cole-Hopf" transformation which linearizes the scheme (1.6). This enables us to write down explicit solutions of the Godunov scheme in the form of nonlinear superpositions of discrete traveling waves. More precisely, consider the Godunov scheme in this case, i.e.

$$u_j^{n+1} = u_j^n - \Big[\ln\left(\nu + \mu e^{u_j^n}\right) - \ln\left(\nu + \mu e^{u_{j-1}^n}\right)\Big], \tag{3.6}$$

where we recall that $\mu > 1/2$ and that $\nu = 1 - \mu$. We then have the following result due to Lax.

Lemma 3.1. *If $z_j^n > 0$ is a solution of*

$$z_j^{n+1} = \mu z_{j-1}^n + \nu z_j^n \,, \tag{3.7}$$

then a solution of (3.6) *is provided by*

$$u_j^n = \ln\left(\frac{z_{j-1}^n}{z_j^n}\right) .$$

Proof. A direct computation yields

$$\begin{aligned} u_j^{n+1} &= \ln\left(\frac{z_{j-1}^{n+1}}{z_j^{n+1}}\right) = \ln\left(\frac{\mu z_{j-2}^n + \nu z_{j-1}^n}{\mu z_{j-1}^n + \nu z_j^n}\right) \\ &= \ln\left(\frac{z_{j-1}^n}{z_j^n}\right) + \ln\left(\frac{\nu + \mu\frac{z_{j-2}^n}{z_{j-1}^n}}{\nu + \mu\frac{z_{j-1}^n}{z_j^n}}\right) \\ &= u_j^n + \ln\left(\nu + \mu e^{u_{j-1}^n}\right) - \ln\left(\nu + \mu e^{u_j^n}\right) . \end{aligned}$$

□

In this section we will give discrete traveling waves for the the linearized scheme (3.7) and use these to construct discrete traveling waves for (3.6). In the next section we will then use these as building blocks to write down an explicit solution of (3.6) which is a good approximation to the exact inviscid solution described in Section 3.1. Letting both variables in (3.7) range over $\mathbb{R}$ we get

$$z(t+1, x) = \mu z(t, x-1) + \nu z(t, x). \tag{3.8}$$

Special solutions of (3.8) are easy to find. For every $b > 0$ define $\sigma(b)$ as the Rankine-Hugoniot speed of the shock connecting the left state b to the right state $u_+ = 0$, i.e.

$$\sigma(b) := \frac{\ln(\nu + \mu e^b)}{b} . \tag{3.9}$$

Notice that $\sigma = \sigma(b)$ then satisfies the equation

$$\nu + \mu e^b = e^{b\sigma(b)}, \tag{3.10}$$

and it follows that for any $x_1 \in \mathbb{R}$ the function

$$z(t, x) = e^{-b[x - x_1 - \sigma(b)t]}$$

is a solution of the linearized equation (3.8). More generally, by linearity, any finite linear combination

$$z(t,x) = \sum_{j=0}^{N} e^{-b_j[x-x_j-\sigma(b_j)t]},$$

or any integral combination

$$z(t,x) = \int e^{-\zeta\left(x-x(\zeta)-\sigma(\zeta)t\right)}\, d\zeta, \tag{3.11}$$

are also solutions of (3.8). In turn, the formula

$$u(t,x) = \ln\left(\frac{z(t,\, x-1)}{z(t,x)}\right) \tag{3.12}$$

allows us to recover an explicit solution of the nonlinear difference equation

$$u(t+1,\, x) = u(t,x) - \Big[\ln\left(\nu + \mu e^{u(t,x)}\right) - \ln\left(\nu + \mu e^{u(t,x-1)}\right)\Big]. \tag{3.13}$$

As an example let $b_1 < b_2$, $x_1, x_2 \in \mathbb{R}$ and consider the solution

$$z(t,x) := e^{-b_1[x-x_1-\sigma(b_1)t]} + e^{-b_2[x-x_2-\sigma(b_2)t]}$$

of (3.8). A simple calculation shows that the corresponding function u in (3.12) represents a traveling wave solution of (3.13) with speed

$$\sigma^* = \frac{\ln(1+e^{b_1}) - \ln(1+e^{b_2})}{b_1 - b_2} = \frac{b_1\sigma(b_1) - b_2\sigma(b_2)}{b_1 - b_2},$$

and connecting $u(-\infty) = b_2$ to $u(+\infty) = b_1$. To see where the center of the wave is located, we observe that at the point

$$\bar{x} = \frac{x_1 b_1 - x_2 b_2}{b_1 - b_2} = x_2 + \frac{b_1}{b_1 - b_2}(x_1 - x_2)$$

one has

$$u(0,\bar{x}) = \ln\left(\frac{e^{b_1}e^{-b_1(\bar{x}-x_1)} + e^{b_2}e^{-b_2(\bar{x}-x_2)}}{e^{-b_1(\bar{x}-x_1)} + e^{-b_2(\bar{x}-x_2)}}\right) = \ln\left(\frac{e^{b_1}+e^{b_2}}{2}\right) \approx \frac{b_1+b_2}{2}\,.$$

We can thus think of the point $x^*(t) := \bar{x} + \sigma^* t$ as the center of the discrete traveling profile at time t.

4 Construction of a special discrete approximation

We proceed to use the information from Section 3.2 to construct an exact solution of the upwind scheme (3.6) that is a good approximation of the particular solution of (3.1) with a single shock along the curve γ. This exact solution will be obtained as a solution $u^d(x,t)$ of the difference equation (3.13) as in (3.12), where $z^d(t,x)$ is a special solution of the linearized difference equation (3.8). (Superscript d is used (only) here to emphasize that the solutions satisfy the everywhere defined difference equations (3.13) and (3.8).) We point out that the relevant property of the "good" approximation u^d, which will be needed in the final computations, is that its level curves should travel with speeds sufficiently close to the level curves of the exact inviscid solution of (3.1) constructed in Section 3.1.

To construct such a solution of (3.13) we consider the curve $\gamma(t)$ given by (3.2) for large negative times. As above we set $u_-(t) := \varphi(\dot\gamma(t))$, such that $\sigma(u_-(\tau)) = \dot\gamma(\tau)$, and define

$$z(t,x) := 1 + \int_{-2T}^{0} \exp\Big\{ - u_-(\tau)\,\big[x - \gamma(\tau) - \dot\gamma(\tau)(t-\tau)\big]\Big\} \cdot \psi(\tau)\,d\tau\,, \tag{4.1}$$

for $t \in \mathbb{R}$. By (3.11) this defines a smooth solution of the difference equation (3.8). The corresponding solution of (3.13) will be denoted by $u(t,x)$ (see (4.14) below). For reasons that will be clear from the following computations we choose

$$\psi(\tau) := (-\tau)^{-3/4}, \qquad \tau < 0. \tag{4.2}$$

To avoid several minus signs we let $s = -t$, $\eta = -x$, and set

$$\begin{aligned} Z(s,\eta) &:= z(t,x) \qquad\qquad (4.3)\\ &= 1 + \int_0^{2T} \exp\Big\{u_-(-\tau)\big[\eta + \gamma(-\tau) - \dot\gamma(-\tau)(s-\tau)\big]\Big\}\psi(-\tau)\,d\tau, \end{aligned}$$

where we have made the change of variables $\tau \mapsto -\tau$. We define

$$\Gamma(\tau) \;:=\; -\gamma(-\tau) = \lambda_0\tau + 2\sqrt{\tau}, \tag{4.4}$$

$$W(\tau) \;:=\; u_-(-\tau) = \varphi\left(\lambda_0 + \frac{1}{\sqrt{\tau}}\right), \tag{4.5}$$

$$g(\tau) \;:=\; \psi(-\tau) = \tau^{-3/4}, \tag{4.6}$$

for $\tau > 0$. Letting $y := \eta - \Gamma(s)$ and making the change of variables $\xi = \tau/s$, we obtain

$$
\begin{aligned}
&Z(s, \Gamma(s) + y) \\
&\quad = 1 + \int_0^{2T} \exp\Big\{ W(\tau) \Big[y + \Gamma(s) - \Gamma(\tau) - \dot{\Gamma}(\tau)(s - \tau) \Big] \Big\}\, g(\tau)\, d\tau \\
&\quad = 1 + \int_0^{2T} \exp\left\{ W(\tau) \left[y + \sqrt{s} \left(2 - \sqrt{\frac{\tau}{s}} - \sqrt{\frac{s}{\tau}} \right) \right] \right\} g(\tau)\, d\tau \\
&\quad = 1 + s^{1/4} \int_0^{2T/s} \exp\left\{ W(s\xi) \Big[y + \sqrt{s} \Big(2 - \sqrt{\xi} - \frac{1}{\sqrt{\xi}} \Big) \Big] \right\} g(\xi) d\xi\, . \qquad (4.7)
\end{aligned}
$$

We will consider times t in $[-T, 0]$, i.e. $s \in [0, T]$, such that the upper limit of integration in (4.7) is bounded away from $\xi = 1$. The factor $s^{1/4}$ will be important in the following computations and this is what dictates the particular choice in (4.2). We proceed to analyze in detail the level curves of $z(t, x)$ and $u(t, x)$ by considering the level curves of $Z(s, \eta)$. For $s \gg 1$, the main contribution to the integral in (4.7)

is provided within the region where $\xi \approx 1$. A careful analysis is required in order to obtain the exact behavior.

4.1 Level curves of $u(t, x)$

As a first step we show that the time derivatives of the level curves of $Z(s, \eta)$ behave (almost) like $\dot{\Gamma}(s)$. Before starting to estimate the various terms that are needed, we make a simplifying observation. For $\varepsilon > 0$, $\varepsilon \ll 1$, we let $I(s)$ denote the interval

$$
I(s) = [1 - s^{-\alpha}, 1 + s^{-\alpha}], \qquad\qquad \alpha = 1/4 - \varepsilon.
$$

Observation 1. *In what follows all ξ-integrals over $[0, 2T/s] \setminus I(s)$, as well as all boundary terms (obtained by differentiation of such integrals), are exponentially small. Namely, they are all of order $O(e^{-Cs^{\varepsilon}})$ for some C, $\varepsilon > 0$.*

Next we define the functions

$$\begin{aligned} F(\xi, s, y) &:= W(s\xi)\left[y + \sqrt{s}\left(2 - \sqrt{\xi} - \frac{1}{\sqrt{\xi}}\right)\right], & (4.8) \\ Q(s, y) &:= \left(\kappa_0 + \frac{\kappa_1}{\sqrt{s}}\right) y, & (4.9) \\ H(s, \xi) &:= -\frac{\kappa_0\sqrt{s}}{4}(\xi - 1)^2. & (4.10) \end{aligned}$$

Thus

$$Z(s, \Gamma(s) + y) = 1 + s^{1/4}\int_0^{2T/s} \exp\big[F(\xi, s, y)\big] g(\xi)\, d\xi.$$

Lemma 4.1. *For $y = O(1)$ and $s \gg 1$ we have*

$$\frac{d}{ds} Z(s, \Gamma(s) + y) = \frac{O(1)}{s^{3/2-\varepsilon}}, \tag{4.11}$$

Proof. We start by stating a detailed expansion of the function $F - Q - H$ where F, Q and H were defined by (4.8), (4.9) and (4.10):

$$\begin{aligned} F(\xi, s, y) - Q(s, y) - H(s, \xi) &= A + B(\xi - 1) + C(\xi - 1)^2 + D(\xi - 1)^3 \\ &\quad + E(\xi - 1)^4 + O(1)(\xi - 1)^5, \end{aligned}$$

where the coefficients A-E are given as

$$\begin{aligned} A(y, s) &= y\left(\frac{\kappa_2}{s} + \frac{O(1)}{s^{3/2}}\right), \\ B(y, s) &= -y\left(\frac{\kappa_1}{2\sqrt{s}} + \frac{\kappa_2}{s} + \frac{O(1)}{s^{3/2}}\right), \\ C(y, s) &= -\frac{\kappa_1}{4} + \frac{3\kappa_1 y - 2\kappa_2}{8\sqrt{s}} + \frac{\kappa_2 y + O(1)}{s} + \frac{O(1)y}{s^{3/2}}, \\ D(y, s) &= \frac{\kappa_0\sqrt{s}}{4} + \frac{3\kappa_1}{8} + \frac{8\kappa_2 - 5y\kappa_1}{16\sqrt{s}} + \frac{O(1) - \kappa_2 y}{s} + \frac{O(1)y}{s^{3/2}}, \\ E(y, s) &= O(1)\sqrt{s} + \frac{O(1)y}{\sqrt{s}}. \end{aligned}$$

The proof is a direct calculation which is omitted. The estimate (4.11) is now obtained by exploiting the fact that, when differentiating with respect to time, we can subtract off the leading order contribution, i.e. the term

$$\frac{d}{ds}\left\{s^{1/4}\int_{I(s)} e^{H(s,\xi)}\, d\xi\right\},$$

since this is exponentially small. We have

$$Z(s,\Gamma(s)+y) = 1 + e^{Q(s,y)} s^{1/4} \int_{I(s)} \exp[F(\xi,s,y) - Q(s,y)] g(\xi)\, d\xi.$$

such that, up to exponentially small terms,

$$\begin{aligned}
\frac{d}{ds} Z(s,\Gamma(s)+y) &= Q_s e^Q \cdot s^{1/4} \int_{I(s)} \exp[F-Q] g(\xi)\, d\xi \\
&\quad + e^Q \frac{\partial}{\partial s} \left\{ s^{1/4} \int_{I(s)} \exp[F-Q] g(\xi) - e^H \, d\xi \right\} \\
&= \frac{O(1)}{s^{3/2-\varepsilon}} + \frac{O(1)}{s} \cdot s^{1/4} \int_{I(s)} \big(\exp[F-Q-H] g(\xi) - 1 \big) e^H \, d\xi \\
&\quad + O(1) s^{1/4} \int_{I(s)} \Big\{ \big(\exp[F-Q-H] g(\xi) - 1 \big) e^H \Big\}_s \, d\xi, \qquad (4.12)
\end{aligned}$$

Using the expansion above and $g(\xi) = 1 - 3(\xi-1)/4 + O(1)(\xi-1)^2$, we get that (for $y = O(1)$ and $s \gg 1$)

$$\exp[F-Q-H] g(\xi) - 1 = \frac{O(1)}{s^{1/2-\varepsilon}} + \{\text{odd function of } (\xi-1)\}. \qquad (4.13)$$

For the last term in (4.12) we apply the expansion above again to get that

$$\begin{aligned}
&\Big\{ \big(\exp[F-Q-H] g(\xi) - 1 \big) e^H \Big\}_s \\
&= \Big[\big(\exp[F-Q-H] g(\xi) - 1 \big) H_s + \exp[F-Q-H] g(\xi) (F-Q-H)_s \Big] e^H \\
&= \left[\{\text{odd function of } (\xi-1)\} + \frac{O(1)}{s^{3/2-\varepsilon}} \right] e^H.
\end{aligned}$$

Here we have used the expansion of $g(\xi)$, (4.13), that

$$H_s = -\frac{\kappa_0 (\xi-1)^2}{8\sqrt{s}} = \frac{O(1)}{s^{1-\varepsilon}} \qquad \text{on } I(s),$$

and finally that

$$(F-Q-H)_s = \frac{\kappa_0}{2\sqrt{s}} (\xi-1)^3 + \frac{O(1)}{s^{3/2-\varepsilon}},$$

which follows from the expansion above. As

$$s^{1/4}\int_{I(s)} e^{H(s,\xi)}\,d\xi = O(1),$$

we therefore have that the two last terms in (4.12) are both $O(1)/s^{3/2-\varepsilon}$. The result follows. $\square$

We now define

$$U(s,\eta) := u(-s,-\eta) := \ln\left(\frac{z(-s,-\eta-1)}{z(-s,-\eta)}\right) = \ln\left(\frac{Z(s,\eta+1)}{Z(s,\eta)}\right). \tag{4.14}$$

A similar calculation as above shows that

$$Z(s,\Gamma(s)+y) = 1 + c_0 e^{Q(s,y)} + \frac{O(1)}{s^{1/2-\varepsilon}},$$

where

$$c_0 = t^{1/4}\int_{-\infty}^{\infty} \exp[H(t,\xi)]\,d\xi = 2\sqrt{\frac{\pi}{\kappa_0}}\,.$$

It follows that

$$U(s,\Gamma(s)) = \ln\left(\frac{1+c_0e^{\kappa_0}}{1+c_0}\right) + \frac{O(1)}{s^{1/2-\varepsilon}} =: a_0 + \frac{O(1)}{s^{1/2-\varepsilon}}.$$

For a sufficiently close to a_0, and well within $]\,U(-\infty),U(+\infty)\,[$, we denote the a-level curve of $U(s,\eta)$ by $X(s;a)$, i.e.

$$U(s,X(s;a)) \equiv a, \tag{4.15}$$

such that

$$U_s(s,X(s;a)) = -X_s(s;a)U_\eta(s,X(s;a)). \tag{4.16}$$

A computation shows that

$$Z_\eta(s,\Gamma(s)+y) = \kappa_0 c_0 e^{Q(s,y)} + \frac{O(1)}{s^{1/2-\varepsilon}},$$

whence

$$U_\eta(s,\Gamma(s)+y) = \frac{\kappa_0 c_0 e^{\kappa_0 y}(e^{\kappa_0}-1)}{(1+\kappa_0 c_0 e^{\kappa_0 y})(1+c_0e^{\kappa_0 y})} + \frac{O(1)}{s^{1/2-\varepsilon}},$$

which is nonvanishing and $O(1)$ for $y = O(1)$. For a fixed s and with $a = U(s, \Gamma(s) + y)$ it follows that

$$\begin{aligned} \frac{d}{ds} U(s, \Gamma(s) + y) &= U_\eta(s, \Gamma(s) + y)\big[X_s(s; a) - \dot{\Gamma}(s)\big] \\ &= O(1)\big[X_s(s; a) - \dot{\Gamma}(s)\big], \end{aligned} \tag{4.17}$$

for $y = O(1)$. On the other hand, using (4.14) and Lemma 4.1, we get

$$\frac{d}{ds} U(s, \Gamma(s) + y) = \frac{\frac{d}{ds} Z(s, \Gamma(s) + y + 1)}{Z(s, \Gamma(s) + y + 1)} - \frac{\frac{d}{ds} Z(s, \Gamma(s) + y)}{Z(s, \Gamma(s) + y)} = \frac{O(1)}{s^{3/2-\varepsilon}},$$

such that

$$X_s(s; a) - \dot{\Gamma}(s) = \frac{O(1)}{s^{3/2-\varepsilon}}. \tag{4.18}$$

Transforming back to t and x coordinates, it follows that the a-level curve $x(t; a)$ of $u(t, x)$ satisfies

$$x_t(t; a) = \dot{\gamma}(t) + \frac{O(1)}{(-t)^{3/2-\varepsilon}}, \tag{4.19}$$

for a sufficiently close to a_0. We conclude that there exists a smooth function $c(a)$ defined on an interval about a_0 such that

$$x(t; a) = c(a) + \gamma(t) + \frac{O(1)}{(-t)^{1/2-\varepsilon}}, \tag{4.20}$$

for $t \in [-2T, 0]$.

5 Estimates on the total variation

We now fix the solution $u(t, x) := U(-t, -x)$ defined in (4.14) and set

$$u_j^n := u(n, j).$$

By Lemma 3.1 these values provide a solution to the upwind scheme (1.6) for the particular flux function $f(u) = \ln(\nu + \mu e^u)$. Recalling that we consider negative times we let v_j^n denote the corresponding solution of (1.7) obtained by prescribing vanishing v-data at time $-2T$, and our goal is to show that the total variation of $V(j) := v_{j+1}^1$ over the set $\{j \in \mathbb{Z} \,|\, -T \leq j \leq 0\}$ is of order $\ln T$. Here T is as in Section 4. This will be accomplished in a series of lemmas which reduce the problem to an estimate involving the heat kernel and the curve $\gamma(t)$. In the course of doing this we will specify the precise assumptions on the function $g(u)$ in (1.7).

Recalling that we have fixed $\lambda = 1/2$ in (1.7) we get that the discrete Green kernel K_k^n for (the linear part of) (1.7) is given by

$$K_k^n := \frac{1}{2^n}\binom{n}{k}.$$

With vanishing data at time $-2T$ we get that the solution at time step $n = 1$ is given as

$$V(j) := v_{j+1}^1 = \sum_{n=0}^{2T}\sum_{k\in\mathbb{Z}} \psi_k^{-n} K_{j-k-1}^n, \tag{5.1}$$

where we have set

$$\psi_k^n := -\big[g(u_k^n) - g(u_{k-1}^n)\big] = -\int_{u(n,k-1)}^{u(n,k)} g'(\xi)\, d\xi.$$

We will make use of the following representation formula.

Lemma 5.1. *Let $x(t;\xi)$ denote the ξ-level curve of the particular solution $u(t,x)$ specified above, and assume that the function $g(u)$ in (1.7) has compact support. We then have the representation*

$$V(j) = \int_0^\infty g'(\xi)\Big[\sum_{n=0}^{2T} K_{j-[\![x(-n;\xi)]\!]}^n\Big]\, d\xi. \tag{5.2}$$

Proof. Using that the constructed solution $u(t,x)$ is monotone decreasing at each time we get that,

$$\begin{aligned}
V(j) &= \sum_{n=0}^{2T}\sum_{k\in\mathbb{Z}} \psi_k^{-n} K_{j-k+1}^n \\
&= -\sum_{n=0}^{2T}\sum_{k\in\mathbb{Z}} \left\{\int_{u(-n,k-1)}^{u(-n,k)} g'(\xi)\, d\xi\right\} K_{j-(k-1)}^n \\
&= -\sum_{n=0}^{2T}\sum_{k\in\mathbb{Z}} \int_{u(-n,k-1)}^{u(-n,k)} g'(\xi) K_{j-[\![x(-n;\xi)]\!]}^n\, d\xi \\
&= -\sum_{n=0}^{2T} \int_{u(-n,-\infty)}^{u(-n,+\infty)} g'(\xi) K_{j-[\![x(-n;\xi)]\!]}^n\, d\xi \\
&= \int_0^\infty g'(\xi)\Big[\sum_{n=0}^{2T} K_{j-[\![x(-n;\xi)]\!]}^n\Big]\, d\xi
\end{aligned}$$

$\square$

In the following computations we will use that the level curves $x(t;\xi)$ of $u(t,x)$ are well approximated by translates of $\gamma(t)$, which is true for large (negative) values of t by (4.20). Now, for T sufficiently large, the values $V(j)$ for $j \leq -\sqrt{T}$ are only influenced exponentially little by values of $x(t;\xi)$ and $u(t,x)$ for $t \in [-\sqrt{T}, 0]$. Furthermore, for $j \in [-T, -\sqrt{T}]$, the n-summations in (5.1) and (5.2) are unaffected, up to errors which are summable in j, by restricting n to the interval

$$I(j) := \Big[\frac{|j|}{\beta} - |j|^{1/2+\varepsilon}, \frac{|j|}{\beta} + |j|^{1/2+\varepsilon}\Big],$$

where $0 < \varepsilon \ll 1$ and $\beta = \lambda_0 - 1/2 > 0$. This fact will be used repeatedly in the rest of this section. From now on all values j will be in $[-T, -\sqrt{T}]$, i.e. far downstream.

For two functions $A(j)$ and $B(j)$ defined for integers j we write $A \approx B$ to indicate that the difference of the functions satisfy

$$\sum_{j \leq -\sqrt{T}} |A(j) - B(j)| < \infty.$$

With g as in Lemma 5.1 we thus have

$$V(j) \approx \int_0^\infty g'(\xi) \Big[\sum_{n \in I(j)} K^n_{j - [\![x(-n;\xi)]\!]} \Big]\, d\xi.$$

Since $n \in I(j)$ implies $|2(j - [\![x(-n;\xi)]\!]) - n| \leq O(1) n^{1/2+\varepsilon}$ we can use

Lemma 7.1 in the appendix to conclude that

$$
\begin{aligned}
V(j)-V(j-1) \approx \int_0^\infty g'(\xi)\Big[\sum_{n\in I(j)} K^n_{j-[\![x(-n;\xi)]\!]} - K^n_{j-[\![x(-n;\xi)]\!]-1}\Big]\,d\xi \\
= 4\int_0^\infty g'(\xi)\Big[\sum_{n\in I(j)} G_x\big(\tfrac{n}{2}, 2(j-[\![x(-n;\xi)]\!])-n\big)\Big]\,d\xi \\
-4\int_0^\infty g'(\xi)\Big[\sum_{n\in I(j)} G_{xx}(\tfrac{n}{2}, 2(j-[\![x(-n;\xi)]\!])-n)\Big]\,d\xi \\
+\int_0^\infty g'(\xi)\sum_{n\in I(j)} O(n^{-2+\varepsilon})\,d\xi \\
\approx 4\int_0^\infty g'(\xi)\Big[\sum_{n\in I(j)} G_x\big(\tfrac{n}{2}, 2(j-[\![x(-n;\xi)]\!])-n\big)\Big]\,d\xi \\
-4\int_0^\infty g'(\xi)\Big[\sum_{n\in I(j)} G_{xx}\big(\tfrac{n}{2}, 2(j-[\![x(-n;\xi)]\!])-n\big)\Big]\,d\xi \\
=: \mathcal{A}(j)-\mathcal{B}(j), \qquad (5.3)
\end{aligned}
$$

where we have used that $|I(j)| = O(1)|j|^{1/2+\varepsilon}$ and that $n = O(|j|)$ for $n \in I(j)$. We will estimate the variation of $V(j)$ on $[-T, -\sqrt{T}\,]$ by considering $\mathcal{A}(j)$ and $\mathcal{B}(j)$ separately.

5.1 Variation of $\mathcal{B}(j)$

We start by defining the functions

$$
\begin{aligned}
H(y;\xi) &:= \sum_{n\in I(j)} G_{xx}\big(\tfrac{n}{2}, 2(y-[\![x(-n;\xi)]\!])-n\big), &(5.4)\\
h(y;\xi) &:= \int_0^\infty G_{xx}\big(\tfrac{t}{2}, 2(y-x(-t;\xi))-t\big)\,dt. &(5.5)
\end{aligned}
$$

Lemma 5.2. *We have*

$$H(j;\xi) \approx h(j;\xi) \qquad \textit{uniformly for } \xi \textit{ in compacts.} \qquad (5.6)$$

Proof. We recall the notation h_1 for the sawtooth function introduced in

Section 2. Using the decay properties of the heat kernel, we have

$$
\begin{aligned}
\big|H(j;\xi)-h(j;\xi)\big| &= \Big|\sum_{n\in I(j)} G_{xx}\big(\tfrac{n}{2},2(j-[\![x(-n;\xi)]\!])-n\big) \\
&\qquad\qquad -\int_0^\infty G_{xx}\big(\tfrac{t}{2},2(j-x(-t;\xi))-t\big)\,dt\Big| \\
&= \Big|\sum_{n\in I(j)} \Big[G_{xx}\big(\tfrac{n}{2},2(j-x(-n;\xi))-n\big)+O(1)G_{xxx}\big(\tfrac{n}{2},\tilde{y}(j,n;\xi)\big)\Big] \\
&\qquad\qquad -\int_0^\infty G_{xx}\big(\tfrac{t}{2},2(j-x(-t;\xi))-t\big)\,dt\Big| \\
&\approx \Big|\int_0^\infty G_{xx}\big(\tfrac{t}{2},2(j-x(-t;\xi))-t\big)\big(1+h_1'(t)\big)\,dt \\
&\qquad\qquad -\int_0^\infty G_{xx}\big(\tfrac{t}{2},2(j-x(-t;\xi))-t\big)\,dt\Big| \\
&= \Big|\int_0^\infty \Big[\tfrac{1}{2}G_{xxt}\big(\tfrac{t}{2},2(j-x(-t;\xi))-t\big) \\
&\qquad\qquad +2[x_t(-t;\xi)-\tfrac{1}{2}]G_{xxx}\big(\tfrac{t}{2},2(j-x(-t;\xi))-t\big)\Big]h_1(t)\,dt\Big|,
\end{aligned}
$$

where $\tilde{y}(j,n;\xi)$ is between $2(j-x(n;\xi))-n$ and $2(j-[\![x(n;\xi)]\!])-n$. Using the decay properties of the heat kernel, that $x_t=O(1)$, and the fact that $t=O(1)|j|$ for $t\in I(j)$, we conclude that

$$
\big|H(j;\xi)-h(j;\xi)\big| \approx O(1)\int_{I(j)} \frac{1}{t^{5/2}}+\frac{1}{t^2}\,dt = \frac{O(1)}{|j|^{3/2-\varepsilon}}.
$$

□

Note that the $O(1)$ in the last term here is uniform with respect to ξ as ξ varies over a compact set. Since g is assumed to have compact support we conclude that the second term in (5.3) satisfies

$$
\int_0^\infty g'(\xi)\Big[\sum_{n\in I(j)} G_{xx}\big(\tfrac{n}{2},2(j-[\![x(n;\xi)]\!])-n\big)\Big]\,d\xi \approx \int_0^\infty g'(\xi)h(j;\xi)\,d\xi.
$$

Using (4.19) we now calculate

$$
\begin{aligned}
h(j;\xi) &= \int_0^\infty G_{xx}\big(\tfrac{t}{2}, 2(j-x(-t;\xi))-t\big)\,dt \\
&= \int_0^\infty G_{xx}\big(\tfrac{t}{2}, 2(j-x(-t;\xi))-t\big)[2x_t(-t;\xi)-1]\,dt \\
&\quad +2\int_0^\infty G_{xx}\big(\tfrac{t}{2}, 2(j-x(-t;\xi))-t\big)[1-x_t(-t;\xi)]\,dt \\
&= \int_0^\infty G_{xx}\big(\tfrac{t}{2}, 2(j-x(-t;\xi))-t\big)[2x_t(-t;\xi)-1] \\
&\qquad\qquad\qquad +\frac{1}{2}G_{xt}\big(\tfrac{t}{2}, 2(j-x(-t;\xi))-t\big)\,dt \\
&\quad +2\int_0^\infty G_{xx}\big(\tfrac{t}{2}, 2(j-x(-t;\xi))-t\big)\Big((1-\lambda_0)-\frac{1}{\sqrt{t}}+\frac{O(1)}{t^{3/2}}\Big)\,dt \\
&\quad -\frac{1}{2}\int_0^\infty G_{xxx}\big(\tfrac{t}{2}, 2(j-x(-t;\xi))-t\big)\,dt,
\end{aligned}
$$

where we have added and subtracted $G_{xt}=G_{xxx}$. The first integral on the right vanishes so that rearranging gives,

$$
\begin{aligned}
(2\lambda_0-1)h(j;\xi) &\approx -2\int_0^\infty G_{xx}\big(\tfrac{t}{2}, 2(j-x(-t;\xi))-t\big)\frac{dt}{\sqrt{t}}+\int_{I(j)}\frac{O(1)}{|j|^2}\,dt \\
&= \int_{I(j)}\frac{O(1)}{|j|^2}\,dt \;=\; \frac{O(1)}{|j|^{3/2-\varepsilon}}. \qquad (5.7)
\end{aligned}
$$

Hence $h(j;\xi)$ is summable in j, uniformly for ξ in compacts. It follows from Lemma 5.2 that the same is true for $H(j;\xi)$, and we conclude that the function $\mathcal{B}(j)$ is summable. Hence it contributes only a finite amount to the total variation of $V(j)$ independently of T.

5.2 Variation of $\mathcal{A}(j)$

It remains to show that the variation of $\mathcal{A}(j)$ in (5.3) grows indefinitely with T. This requires a more detailed analysis and we start by defining the functions

$$
\begin{aligned}
L(y;\xi) &:= \sum_{n\geq 0} G_x\big(\tfrac{n}{2}, 2(y-x(-n;\xi))-n\big), \qquad (5.8) \\
l(y;\xi) &:= \int_0^\infty G_x\big(\tfrac{t}{2}, 2(y-x(-t;\xi))-t\big)\,dt. \qquad (5.9)
\end{aligned}
$$

Note that the arguments in neither of these functions involve integer parts.

Lemma 5.3. *We have*

$$L(j;\xi) \approx l(j;\xi) \qquad \textit{uniformly for } \xi \textit{ in compacts.} \tag{5.10}$$

Proof. As above we use the function h_1 introduced in Section 2 and the decay properties of the heat kernel:

$$\begin{aligned}
\big|L(j;\xi) - l(j;\xi)\big| &= \Big| \int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\big(1 + h_1'(t)\big)\, dt \\
&\qquad - \int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\, dt\Big| \\
= \ & \Big| \int_0^\infty \Big[\tfrac{1}{2} G_{xt}\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big) \\
&\qquad + [2x_t(-t;\xi) - 1] G_{xx}\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\Big] h_1(t)\, dt\Big| \\
\leq \ & \frac{O(1)}{|j|^{3/2-\varepsilon}} + \Big| \int_0^\infty [2x_t(-t;\xi) - 1] G_{xx}\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big) h_1(t)\, dt\Big|.
\end{aligned}$$

To estimate the last integral we apply (4.19) and the function h_2 introduced in Section 2, to get

$$\begin{aligned}
&\int_0^\infty [2x_t(-t;\xi) - 1] G_{xx}\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big) h_1(t)\, dt \\
&= \int_0^\infty \left(2\lambda_0 - 1 + \frac{O(1)}{\sqrt{t}}\right) G_{xx}\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big) h_1(t)\, dt \\
&\approx (2\lambda_0 - 1) \int_0^\infty G_{xx}\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big) h_2'(t)\, dt + \frac{O(1)}{|j|^{3/2-\varepsilon}} \\
&= O(1) \int_0^\infty G_{xxt}\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big) h_2(t)\, dt \\
&\quad + O(1) \int_0^\infty G_{xxx}\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big) h_2(t)\, dt + \frac{O(1)}{|j|^{3/2-\varepsilon}} \\
&= \frac{O(1)}{|j|^{3/2-\varepsilon}},
\end{aligned}$$

and the conclusion follows. $\square$

Lemma 5.4. $l(j;\xi) \approx 0$ *uniformly for ξ in compacts.*

Proof. Using (4.19) we have

$$
\begin{aligned}
l(j;\xi) &= \int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\, dt \\
&= \int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)[2x_t(-t;\xi) - 1]\, dt \\
&\quad +2\int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)[1 - x_t(-t;\xi)]\, dt \\
&= \int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)[2x_t(-t;\xi) - 1] \\
&\qquad\qquad\qquad + \tfrac{1}{2}G_t\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\, dt \\
&\quad +2\int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\Big((1-\lambda_0) - \frac{1}{\sqrt{t}} + \frac{O(1)}{t^{3/2}}\Big)\, dt \\
&\quad -\frac{1}{2}h(j;\xi),
\end{aligned}
$$

where we have added and subtracted $G_t = G_{xx}$ and used (5.5). The first integral on the right-hand side here vanishes. Recalling the definition of $l(j;\xi)$, rearranging, and using the decay properties of the heat kernel together with what we have already proved about $h(j;\xi)$, we get

$$
l(j;\xi) \approx \frac{2}{1-2\lambda_0}\int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\frac{dt}{\sqrt{t}}\,.
$$

To estimate this last integral we repeat the same procedure:

$$
\begin{aligned}
&\int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\frac{dt}{\sqrt{t}} \\
&\quad = \int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\frac{[2x_t(-t;\xi) - 1]}{\sqrt{t}}\,dt \\
&\qquad +2\int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\frac{[1 - x_t(-t;\xi)]}{\sqrt{t}}\,dt \\
&\quad = \int_0^\infty \Big\{[2x_t(-t;\xi) - 1]G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big) \\
&\qquad\qquad\qquad +\tfrac{1}{2}G_t\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\Big\}\frac{dt}{\sqrt{t}} \\
&\qquad +2(1-\lambda_0)\int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\frac{dt}{\sqrt{t}} \\
&\qquad +O(1)\int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\frac{dt}{t} \\
&\qquad -\frac{1}{2}\int_0^\infty G_{xx}\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\frac{dt}{\sqrt{t}}.
\end{aligned}
$$

Integration by parts and decay of $G(t)$ shows that the first integral here is $O(1)/|j|^{3/2-\varepsilon}$. Rearranging and using the decay properties of the heat kernel thus gives

$$
\int_0^\infty G_x\big(\tfrac{t}{2}, 2(j - x(-t;\xi)) - t\big)\frac{dt}{\sqrt{t}} = \frac{O(1)}{|j|^{3/2-\varepsilon}}.
$$

We conclude that

$$
l(j;\xi) \approx \frac{O(1)}{|j|^{3/2-\varepsilon}},
$$

which completes the proof. □

From Lemma 5.3 and Lemma 5.4 it follows that $L(j;\xi)$ is summable with respect to j, uniformly for ξ in compacts. We conclude that the function

$$
M(j;\xi) := \sum_{n\in I(j)} G_x\big(\tfrac{n}{2}, 2(j - [\![x(-n;\xi)]\!]) - n\big) \tag{5.11}
$$

appearing in the first term in (5.3), may be expanded as

$$\begin{aligned} M(j;\xi) &\approx \sum_{n\geq 0} G_x\big(\tfrac{n}{2}, 2(j - [\![x(-n;\xi)]\!]) - n\big) \; - \; L(j;\xi) \\ &\approx \sum_{n\geq 0} G_{xx}\big(\tfrac{n}{2}, 2(j - x(-n;\xi)) - n\big) (\!(x(-n;\xi))\!), \end{aligned}$$

where $(\!(a)\!) := a - [\![a]\!]$ denotes fractional part. Assume now that λ_0 is a rational number,

$$\lambda_0 = \frac{p}{q},$$

say, and write $n = mq + i$ for $1 \leq i \leq q$, to get that

$$\begin{aligned} M(j;\xi) \;\approx\; & \sum_{m\geq 0} \sum_{i=1}^{q} G_{xx}\big(\tfrac{mq+i}{2}, 2(j - x(-mq-i;\xi)) - mq - i\big) \\ & \cdot (\!(x(-mq-i;\xi))\!), \end{aligned}$$

Taylor expanding G_{xx} about the points

$$\big(\tfrac{mq}{2}, 2(j - x(-mq;\xi)) - mq\big),$$

and using the decay properties of the heat kernel, yields

$$M(j;\xi) \approx \sum_{m\geq 0} G_{xx}\big(\tfrac{mq}{2}, 2(j - x(-mq;\xi)) - mq\big) \left[\sum_{i=1}^{q} (\!(x(-mq-i;\xi))\!)\right].$$

Recalling from (4.20) that the level curves $x(t;\xi)$ are given as

$$x(-t;\xi) = c(\xi) - \lambda_0 t + 2\sqrt{t} + \frac{O(1)}{t^{1/2-\varepsilon}}, \qquad \text{for } t > 0,$$

we obtain

$$\begin{aligned} M(j;\xi) &\approx \sum_{m\geq 0} G_{xx}\big(\tfrac{mq}{2}, 2(j - x(-mq;\xi)) - mq\big) \left[\sum_{i=1}^{q} (\!(x(-mq-i;\xi))\!)\right] \\ &= \sum_{m\geq 0} G_{xx}\big(\tfrac{mq}{2}, 2(j - x(-mq;\xi)) - mq\big) \\ &\qquad \cdot \left[\sum_{i=1}^{q} (\!(c(\xi) - \tfrac{ip}{q} + 2\sqrt{mq+i} + \tfrac{O(1)}{\sqrt{mq}})\!)\right]. \end{aligned} \tag{5.12}$$

Lemma 5.5. *We have*

$$M(j;\xi) \approx \sum_{m\geq 0} G_{xx}\big(\tfrac{mq}{2}, 2(j - x(-mq;\xi)) - mq\big) \left[\sum_{i=1}^{q} \big(\!\big(c(\xi) - \tfrac{ip}{q} + 2\sqrt{mq}\big)\!\big)\right]$$

uniformly for ξ in compacts.

Proof. By (5.12) it suffices to show that

$$\sum_j \Big| \sum_{m\geq 0} G_{xx}\big(\tfrac{mq}{2}, 2(j - x(-mq;\xi)) - mq\big) \\ \cdot \left[\sum_{i=1}^{q} \big(\!\big(c(\xi) - \tfrac{ip}{q} + 2\sqrt{mq} + \delta(m,i)\big)\!\big) - \big(\!\big(c(\xi) - \tfrac{ip}{q} + 2\sqrt{mq}\big)\!\big)\right] \Big| < \infty,$$

where the summations are over those m and i for which $mq + i \in I(j)$. We have also set

$$\delta(m,i) := 2\sqrt{mq+i} - 2\sqrt{mq} + \frac{O(1)}{\sqrt{mq}},$$

which is of magnitude $O(1)/\sqrt{|j|}$ for all m and i in question. Next, for fixed ξ and i, let

$$b := c(\xi) - \tfrac{ip}{q},$$

and define

$$\alpha(s) := b + 2\sqrt{ms}, \qquad \beta(s) := b + 2\sqrt{ms} + \delta(s,i).$$

Now, as s ranges over $I(j)$, both functions $(\!(\alpha(s))\!)$ and $(\!(\beta(s))\!)$ jump at most an $O(1)|j|^{\varepsilon}$ number of times. Also, if neither of these functions jump in an interval $(s-1, s+1)$, then they are $O(1)/\sqrt{|j|}$ close on that interval. It follows that the expression above is dominated by a sum of the form

$$\sum_j \left\{ \frac{O(1)|j|^{\varepsilon}}{|j|^{3/2}} + \frac{O(1)|j|^{1/2+\varepsilon}}{|j|^{3/2}\sqrt{|j|}} \right\},$$

which is finite. □

Making use of the identity

$$\sum_{i=1}^{q} \big(\!\big(a - \tfrac{ip}{q}\big)\!\big) = (\!(qa)\!) + \tfrac{q-1}{2}$$

and what we have already shown for $H(j;\xi)$, we infer that

$$\begin{aligned}
M(j;\xi) &\approx \sum_{m\geq 0} G_{xx}\big(\tfrac{mq}{2}, 2(j - x(-mq;\xi)) - mq\big) \\
&\qquad\qquad\qquad\qquad \cdot \Big[\big(\!\big(qc(\xi) + 2q\sqrt{mq}\big)\!\big) + \tfrac{q-1}{2}\Big] \\
&\approx \sum_{m\geq 0} G_{xx}\big(\tfrac{mq}{2}, 2(j - x(-mq;\xi)) - mq\big)\big(\!\big(qc(\xi) + 2q\sqrt{mq}\big)\!\big).
\end{aligned}$$

We next want to approximate $M(j;\xi)$ with an integral and we proceed as above for $H(j;\xi)$ and $h(j;\xi)$. We define

$$N(j;\xi) := \int_0^\infty G_{xx}\big(\tfrac{qs}{2}, 2(j - x(-qs;\xi)) - qs\big)\big(\!\big(qc(\xi) + 2q\sqrt{qs}\big)\!\big)\, ds.$$

Lemma 5.6. *$M(j;\xi) \approx N(j;\xi)$ uniformly for ξ in compacts.*

Proof. Using the function h_1 and decay properties of the heat kernel we have

$$\begin{aligned}
&M(j;\xi) - N(j;\xi) \\
&\approx \int_0^\infty G_{xx}\big(\tfrac{qs}{2}, 2(j - x(-qs;\xi)) - qs\big)\big(\!\big(qc(\xi) + 2q\sqrt{qs}\big)\!\big) h_1'(s)\, ds \\
&= -\int_0^\infty \frac{d}{ds}\Big[G_{xx}\big(\tfrac{qs}{2}, 2(j - x(-qs;\xi)) - qs\big)\big(\!\big(qc(\xi) + 2q\sqrt{qs}\big)\!\big)\Big] h_1(s)\, ds \\
&= -q\int_0^\infty \Big[G_{xxx}\big(\tfrac{qs}{2}, 2(j - x(-qs;\xi)) - qs\big)[2x_t(-qs;\xi) - 1] \\
&\qquad\qquad + \tfrac{1}{2}G_{xxt}\big(\tfrac{qs}{2}, 2(j - x(-qs;\xi)) - qs\big)\Big]\big(\!\big(qc(\xi) + 2q\sqrt{qs}\big)\!\big) h_1(s)\, ds \\
&\quad - \int_0^\infty G_{xx}\big(\tfrac{qs}{2}, 2(j - x(-qs;\xi)) - qs\big)\left\{\frac{d}{ds}\big(\!\big(qc(\xi) + 2q\sqrt{qs}\big)\!\big)\right\} h_1(s)\, ds \\
&\approx -\int_0^\infty G_{xx}\big(\tfrac{qs}{2}, 2(j - x(-qs;\xi)) - qs\big)\Big\{\tfrac{q^{3/2}}{\sqrt{s}} - \sum_{k\geq 0}\delta_{s_k}(s)\Big\} h_1(s)\, ds \\
&\approx \sum_{\{k\geq 0 \,|\, s_k \in I(j)\}} G_{xx}\big(\tfrac{qs_k}{2}, 2(j - x(-qs_k;\xi)) - qs_k\big) h_1(s_k),
\end{aligned}$$

where the s_k denote the s-values for which $qc(\xi) + 2q\sqrt{qs}$ is an integer. Clearly $s_k = O(k^2)$, such that the last sum contains only $O(1)|j|^{1/4+\varepsilon}$ terms. Since each term in this sum is $O(|j|^{-3/2})$ the conclusion follows. ∎

Making a change of variables ($\tau = qs$) and restricting the integration to $I(j)$ we get that

$$M(j;\xi) \approx \frac{1}{q}\int_{I(j)} G_{xx}\big(\tfrac{\tau}{2}, 2(j - x(-\tau;\xi)) - \tau\big)\big(\!\big(qc(\xi) + 2q\sqrt{\tau}\big)\!\big)\, d\tau. \quad (5.13)$$

Recalling that the level curve $x(-\tau;\xi)$ is given as

$$x(-\tau;\xi) = c(\xi) - \lambda_0\tau + 2\sqrt{\tau} + \frac{O(1)}{\sqrt{\tau}}, \quad (5.14)$$

we make a further change of variables $\tau \mapsto \sigma$ where

$$\tau = \frac{|j|}{\beta} + \frac{\sqrt{|j|}}{\beta^{3/2}}\left(\frac{\sigma}{\sqrt{2}} - 2\right).$$

A straightforward calculation then yields

$$G_{xx}\big(\tfrac{\tau}{2}, 2(j - x(-\tau;\xi)) - \tau\big) = \frac{C}{|j|^{3/2}}\big(2\sigma^2 - 1\big)e^{-\sigma^2} + \frac{O(1)}{|j|^{2-\delta}},$$

which holds for $\tau \in I(j)$, or, equivalently, for $|\sigma - 2\sqrt{2}| \leq O(1)|j|^{\varepsilon}$. Here, and below, C denotes various explicit numerical constants. Substituting into (5.13) we conclude that $M(j;\xi)$ satisfies

$$\begin{aligned} M(j;\xi) \;&\approx\; \frac{C}{|j|}\int_{-\infty}^{\infty}(2\sigma^2 - 1)e^{-\sigma^2} \\ &\qquad\qquad \cdot \left(\!\!\left(qc(\xi) + 2q\sqrt{\tfrac{|j|}{\beta}} + \tfrac{q(\sigma - \sqrt{8})}{\sqrt{2}\beta} + \tfrac{O(1)}{|j|^{1/2-\varepsilon}}\right)\!\!\right) d\sigma \\ &\approx\; \frac{C}{|j|}\int_{-\infty}^{\infty}(2\sigma^2 - 1)e^{-\sigma^2}\left(\!\!\left(qc(\xi) + 2q\sqrt{\tfrac{|j|}{\beta}} + \tfrac{q(\sigma - \sqrt{8})}{\sqrt{2}\beta}\right)\!\!\right) d\sigma. \end{aligned}$$

We can now finally proceed to estimate the variation of $V(j)$. We recall that the term $\mathcal{B}(j)$ in (5.3) is summable in j. Using the expression above for $M(j;\xi)$ we can estimate the variation of the term $\mathcal{A}(j)$ in (5.3). Setting

$$z_j := 2q\sqrt{\tfrac{|j|}{\beta}} \quad (5.15)$$

we thus get that

$$\begin{aligned} V(j) - V(j-1) &\approx 4\int_0^{\infty} g'(\xi)M(j;\xi)\, d\xi \\ &= \frac{C}{|j|}\int_{-\infty}^{\infty}(2\sigma^2 - 1)e^{-\sigma^2}\left\{\int_0^{\infty} g'(\xi)\left(\!\!\left(qc(\xi) + z_j + \tfrac{q(\sigma-\sqrt{8})}{\sqrt{2}\beta}\right)\!\!\right) d\xi\right\} d\sigma \\ &= \frac{C}{|j|}\int_{-\infty}^{\infty}(2\sigma^2 - 1)e^{-\sigma^2}\pi(\sigma;z_j)\, d\sigma\,. \end{aligned}$$

Here

$$\pi(\sigma; z) := \int_0^\infty g'(\xi) \left(\!\left(qc(\xi) + z + \tfrac{q(\sigma-\sqrt{8})}{\sqrt{2}\beta}\right)\!\right) d\xi\,, \tag{5.16}$$

with $c(\xi)$ as in (5.14). Defining the function

$$\Pi(z) := \int_{-\infty}^{\infty} (2\sigma^2 - 1)e^{-\sigma^2} \pi(\sigma; z)\, d\sigma\,, \tag{5.17}$$

we observe that $\Pi(z)$ is periodic with period 1 and (since $z_{-j} = z_j$) that

$$\underset{-T\le j\le -\sqrt{T}}{\text{Tot.Var.}}\; V(j) \;\ge\; \sum_{\sqrt{T}\le j\le T} \frac{C|\Pi(z_j)|}{j} \;-\; C_0, \tag{5.18}$$

for some finite constant C_0 independent of T.

In order to complete the argument showing that V has large total variation we need to introduce the following condition.

Assumption (A): *The smooth function g has compact support and is such that the 1-periodic function $\Pi(z)$ at (5.17) does not vanish identically.*

It is not difficult to see that this condition is satisfied for a large class of functions g. For example, if g' is formally replaced by a Dirac delta-function δ_{u^*} concentrated at a point u^* close to a_0, then we can directly compute

$$\Pi(z) = \int_{-\infty}^{\infty} (2\sigma^2 - 1)e^{-\sigma^2} \left(\!\left(qc(u^*) + z + \tfrac{q(\sigma-\sqrt{8})}{\sqrt{2}\beta}\right)\!\right) d\sigma \;\not\equiv 0.$$

By continuity, any function g whose derivative approximates δ_{u^*} will still satisfy the Assumption (A).

We now fix a smooth function g which satisfies (A), and observe that the resulting function Π is continuous. Therefore, for a sufficiently small number ε, $0 < \varepsilon < \sup_z |\Pi(z)|$, there exist numbers $a < b$ in $(0, 1)$ such that

$$|\Pi(z)| \ge \varepsilon \qquad \text{for all } z \in (n + a, n + b) \;=: J_n,$$

for all $n \ge 0$. Recalling the definition (5.15) of z_j we define, for each $n \in \mathbb{N}$, the integers

$$j(n) := \quad \text{smallest integer } j \text{ for which } z_j \in J_n \;\;,$$

and

$$k(n) := \quad \text{largest integer } j \text{ for which } z_j \in J_n \;\;.$$

That is,

$$j(n) = \left\lceil \frac{\beta(n+a)^2}{4q^2} \right\rceil \qquad \text{and} \qquad k(n) = \left\lfloor \frac{\beta(n+b)^2}{4q^2} \right\rfloor .$$

We conclude from (5.18) that

$$\begin{aligned}
\sum_{\sqrt{T} \le j \le T} \frac{|\Pi(z_j)|}{j} \quad &\ge \quad \sum_{\{n \mid J_n \subset [\sqrt{T}, T]\}} \sum_{\{j \mid z_j \in J_n\}} \frac{|\Pi(z_j)|}{j} \\
&\ge \quad \sum_{\{n \mid J_n \subset [\sqrt{T}, T]\}} \frac{\varepsilon(k(n) - j(n))}{k(n)} \\
&\ge \quad \sum_{\{n \mid J_n \subset [\sqrt{T}, T]\}} \frac{C}{n} \quad \ge \quad C \ln T.
\end{aligned}$$

This completes the proof and shows that there is no *a priori* bound available on the total variation of an approximate solution computed using the upwind scheme.

6 Concluding remarks

The analysis shows that it is possible to prescribe data for a strictly hyperbolic system of conservation laws in such a way that the total variation of the solution generated by the Godunov scheme increases by an arbitrarily large amount, after a large number of time steps. We observe that the total change in the speed of the curve γ decreases to zero as $T \to \infty$. Therefore, as initial data for the u-component, we can take an arbitrarily small perturbation of a single shock.

We summarize here the main features of our example:

1. Our 2×2 system is strictly hyperbolic. One characteristic field is linearly degenerate, the other is genuinely nonlinear. The characteristic speeds are both strictly contained inside the interval $[0, 1]$, so that the usual linearized stability conditions hold.

2. We can choose a sequence of initial data of the form

$$u_\nu(0, x) = \begin{cases} u^- + \phi_\nu(x) & \text{if} \quad x < 0\,, \\ \quad u^+ & \text{if} \quad x > 0\,, \end{cases} \qquad\qquad v_\nu(0, x) = 0\,,$$

where the functions ϕ_ν are smooth and satisfy

$$\text{Tot. Var.}\ \{\phi_\nu\} \to 0\,, \qquad\qquad \|\phi_\nu\|_{\mathcal{C}^k} \to 0$$

for all k, as $\nu \to \infty$. For these initial data, the exact solutions have uniformly bounded total variation, but the corresponding Godunov approximations $(u_\nu,\, v_\nu)$ satisfy

$$\text{Tot. Var.}\ \big\{v_\nu(T_\nu, \cdot)\big\} \to \infty\,,$$

for some sequence of times $T_\nu \to \infty$.

3. Consider a second sequence of Godunov solutions

$$(\tilde u_\nu\,,\ \tilde v_\nu)(t,x) = (u_\nu\,,\ v_\nu)(t\,,\ x-1)\,.$$

By the previous estimates on the total variation, it trivially follows that

$$\lim_{\nu\to\infty} \frac{\big\|v_\nu(T_\nu) - \tilde v_\nu(T_\nu)\big\|_{\mathbf{L}^1}}{\big\|u_\nu(0) - \tilde u_\nu(0)\big\|_{\mathbf{L}^1}} = \infty\,.$$

In other words, the Godunov approximations are also unstable in the $\mathbf{L}^1$ norm, w.r.t. perturbations of the initial data. This is in sharp contrast with the stability of Glimm or front tracking approximations [18], [7].

Our counterexample has many special features: it can be solved in triangular form, the first equation contains a very particular flux function, and the second equation is linear. All these additional features allow us to perform explicit calculations and derive rigorous estimates, but none of them seems to be essential for the validity of the result. Indeed, we expect that the instability highlighted in this paper will be a common feature of all discrete schemes for systems of conservation laws, under generic conditions on the flux functions. In this direction, we observe that in our example the oscillations in the v-component are spread out on a very large spatial interval. Requiring that the first field be genuinely nonlinear (instead of linearly degenerate) would not achieve any appreciable decay in these oscillations.

We observe also that our results do *not* show that the upwind scheme fails to converge. In light of the convergence results in [9] (obtained through compensated compactness arguments) we expect that one can in fact establish strong convergence also for the systems considered above. In this connection it is worth noticing that triangular 2×2 systems of the form (1.4)-(1.5) are endowed with strictly convex entropies.

A somewhat subtle point in the preceding analysis is the issue of small data. As presented above, our examples do not immediately fit the setting of Glimm's theorem. We started by giving a particular shock-wave solution of the first equation (1.4), and *then* chose a flux $g(u)$ for the second equation. However it is now possible to go back and see that if we first fix a smooth function $g(u)$, then we can give shock-wave solutions of (1.4) of arbitrarily small amplitude and with the properties: (i) its level curves travels with speed $\approx$ [rational]$+1/\sqrt{-t}$, and (ii) the assumption (A) is satisfied. One can then redo the previous analysis for the resulting system/data pair. Since the data for the second equation are identically zero, this provides an example within the setting of Glimm's theorem for small BV data.

7 Appendix

Approximation of binomial coefficients For completeness we include the argument for the approximation of the discrete Green's function in terms of the heat kernel. Following the notation in Feller's book [10],

$$a_k(\nu) := \left(\frac{1}{2}\right)^{2\nu} \binom{2\nu}{\nu+k},$$

we have

$$\begin{aligned} a_k(\nu) &= = a_0 \frac{\exp\left[\sum_{j=-k+1}^{0} \ln\left(1-\frac{j}{\nu}\right)\right]}{\exp\left[\sum_{j=1}^{k} \ln\left(1-\frac{j}{\nu}\right)\right]} \\ &= a_0 \exp\left[-\frac{2}{\nu}\left(\sum_{j=1}^{k-1} j\right) - \frac{2}{3\nu^2}\left(\sum_{j=1}^{k-1} j^3\right)\right. \\ &\qquad \left. -\frac{k}{\nu} + \frac{1}{2}\left(\frac{k}{\nu}\right)^2 - \frac{1}{3}\left(\frac{k}{\nu}\right)^3 + \frac{1}{4}\left(\frac{k}{\nu}\right)^4 + O\left(\frac{k^6}{\nu^5}\right)\right]. \end{aligned}$$

Thus,

$$a_k(\nu) = a_0 e^{-\frac{k^2}{\nu}} \exp\left[\frac{1}{2}\left(\frac{k}{\nu}\right)^2 - \frac{1}{3}\left(\frac{k}{\nu}\right)^3 - \frac{(k-1)^2k^2}{6\nu^3} + \frac{1}{4}\left(\frac{k}{\nu}\right)^4 + O\left(\frac{k^6}{\nu^5}\right)\right].$$

Defining

$$h := \sqrt{\frac{2}{\nu}}, \qquad \mathfrak{N}(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

and recalling Stirling's formula

$$n! = \sqrt{2\pi} n^{n+1/2} e^{-n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \frac{O(1)}{n^4}\right),$$

we get

$$a_k(\nu) = h\mathfrak{N}(kh)e^{\varepsilon_1 - \varepsilon_2}.$$

Here ε_1 is due to the expansion above and ε_2 is due to the approximation in Stirling's formula. We have

$$\varepsilon_1 = \varepsilon_1(k,\nu) := \frac{1}{2}\left(\frac{k}{\nu}\right)^2 - \frac{1}{3}\left(\frac{k}{\nu}\right)^3 - \frac{(k-1)^2k^2}{6\nu^3} + O(1)\left(\frac{k}{\nu}\right)^4,$$

while

$$\varepsilon_2 = \varepsilon_2(\nu) = -\frac{1}{8\nu} + \frac{1}{192\nu^3} + \frac{O(1)}{\nu^4}.$$

Discrete Green's function Applying the approximation above to the discrete Green's function

$$K_k^n := \left(\frac{1}{2}\right)^n \binom{n}{k},$$

we get for $n = 2m$, $k = 2l$ that

$$\begin{aligned} K_k^n = a_{k-m}(m) &= \sqrt{\frac{2}{m}}\mathfrak{N}\left(\sqrt{\tfrac{2}{m}}(k-m)\right)e^{\varepsilon_1-\varepsilon_2} \\ &= 2G\left(\frac{n}{2}, 2k-n\right)e^{\varepsilon_1-\varepsilon_2}, \end{aligned}$$

where $G(t,x) = e^{-x^2/4t}/2\sqrt{\pi t}$ denotes the heat kernel and

$$\begin{aligned} \varepsilon_1 &= \varepsilon_1(k-\tfrac{n}{2}, \tfrac{n}{2}) = \frac{1}{2}\left(\frac{2k-n}{n}\right)^2 - \frac{1}{3}\left(\frac{2k-n}{n}\right)^3 \\ &\qquad - \frac{(2k-n-2)^2(2k-n)^2}{12n^3} + O\left(\left(\frac{2k-n}{n}\right)^4\right), \end{aligned}$$

and

$$\varepsilon_2 = \varepsilon_2(\tfrac{n}{2}) = \frac{1}{4n} - \frac{1}{24n^3} + \frac{O(1)}{n^4}.$$

Lemma 7.1. *If* $|n-2k| \le O\left(n^{1/2+\delta}\right)$ $(0 < \delta \ll 1)$*, then*

$$K_k^n - K_{k-1}^n = 4\Big[G_x\left(\tfrac{n}{2}, 2k-n\right) - G_{xx}\left(\tfrac{n}{2}, 2k-n\right)\Big] + O\left(n^{-2+4\delta}\right).$$

Proof. Letting

$$\varepsilon_1 = \varepsilon_1(k - \tfrac{n}{2}, \tfrac{n}{2}), \qquad \varepsilon_2 = \varepsilon_2(\tfrac{n}{2}), \qquad \tilde{\varepsilon}_1 = \varepsilon_1(k - n/2 - 1, n/2),$$

we get that

$$\begin{aligned} K_k^n - K_{k-1}^n &= 2\Big[G\left(\tfrac{n}{2}, 2k-n\right) - G\left(\tfrac{n}{2}, 2k-n-2\right)\Big]e^{\varepsilon_1-\varepsilon_2} \\ &\quad +2G\left(\tfrac{n}{2}, 2k-n-2\right)\left(e^{\varepsilon_1-\varepsilon_2} - e^{\tilde{\varepsilon}_1-\varepsilon_2}\right) \\ &= 4\left[G_x\left(\tfrac{n}{2}, 2k-n\right) - G_{xx}\left(\tfrac{n}{2}, 2k-n\right) + \frac{O(1)}{n^2}\right]e^{\varepsilon_1-\varepsilon_2} \\ &\quad +2G\left(\tfrac{n}{2}, 2k-n-2\right)\left(e^{\varepsilon_1} - e^{\tilde{\varepsilon}_1}\right)e^{-\varepsilon_2}. \end{aligned}$$

Since $(2k-n)/n = O\left(n^{-1/2+\delta}\right)$ a calculation shows that

$$\varepsilon_1 - \varepsilon_2 = O\left(n^{-1+4\delta}\right),$$

and that

$$\varepsilon_1 - \tilde{\varepsilon}_1 = O\left(n^{-3/2+3\delta}\right).$$

Thus

$$\begin{aligned} &K_k^n - K_{n,k-1} \\ &= 4\left[G_x\left(\tfrac{n}{2}, 2k-n\right) - G_{xx}\left(\tfrac{n}{2}, 2k-n\right) + \frac{O(1)}{n^2}\right]\left(1 + O\left(n^{-1+4\delta}\right)\right) \\ &\quad +O(n^{-1/2})\left(O\left(n^{-3/2+3\delta}\right) + O\left(n^{-5/2}\right)\right) \\ &= 4\Big[G_x\left(\tfrac{n}{2}, 2k-n\right) - G_{xx}\left(\tfrac{n}{2}, 2k-n\right)\Big] + O(1)n^{-2+4\delta}. \end{aligned}$$

□

Acknowledgments A. Bressan and P. Baiti: Research supported in part by the European network HYKE, funded by the EC as contract HPRN-CT-2002-00282, and by the italian MIUR-COFIN-2002 research project 2002017219 "Equazioni Iperboliche e Paraboliche Nonlineari". H. K. Jenssen: Research supported in part by the NSF under grant DMS-0206631.

References

[1] P. Baiti, A. Bressan, H. K. Jenssen, Instability of Travelling Wave Profiles for the Lax-Friedrichs scheme, *Discrete Contin. Dynam. Systems*, to appear.

[2] S. Bianchini, BV solutions of the semidiscrete upwind scheme. *Arch. Rational Mech. Anal.* **167** (2003), 1–81.

[3] S. Bianchini, Hyperbolic limit of the Jin-Xin relaxation model, Comm. Pure Appl. Math. to appear.

[4] S. Bianchini, A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math.*, to appear.

[5] A. Bressan, "Hyperbolic systems of conservation laws. The one-dimensional Cauchy problem". Oxford University Press, 2000.

[6] A. Bressan, H.K. Jenssen, On the convergence of Godunov scheme for nonlinear hyperbolic systems. *Chinese Ann. Math. Ser. B* **21** (2000), 269–284.

[7] A. Bressan, T.P. Liu, T. Yang, L^1 stability estimates for $n \times n$ conservation laws. *Arch. Rational Mech. Anal.* **149** (1999), 1–22.

[8] A. Bressan, W. Shen, BV estimates for multicomponent chromatography with relaxation. *Discrete Contin. Dynam. Systems* **6** (2000), 21–38.

[9] X. Ding, G.Q. Chen, P. Luo, Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics. *Comm. Math. Phys.* **121** (1989), 63–84.

[10] W. Feller, "An introduction to probability theory and its applications", vol. 1, third edition, John Wiley & Sons, New York, 1968.

[11] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.* **18** (1965), 697–715.

[12] S. K. Godunov, A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics. (Russian) *Mat. Sb.* (N.S.) **47** (89) (1959), 271–306.

[13] G. Jennings, Discrete shocks. *Comm. Pure Appl. Math.* **27** (1974), 25–37.

[14] P. D. Lax, Weak solutions of nonlinear hyperbolic equations and their numerical computation, Comm. Pure Appl. Math. **7** (1954), 159–193.

[15] P. D. Lax, Hyperbolic systems of conservation laws II, Comm. Pure Appl. Math. **10** (1957), 537–566.

[16] R. J. LeVeque, "Numerical methods for conservation laws", Birkhäuser-Verlag, Basel, 1990.

[17] R. J. LeVeque, B. Temple, Stability of Godunov's method for a class of 2×2 systems of conservation laws. *Trans. Amer. Math. Soc.* **288** (1985), 115–123.

[18] T. P. Liu, T. Yang, Well-posedness theory for hyperbolic conservation laws. *Comm. Pure Appl. Math.* **52** (1999), 1553–1586.

[19] A. Majda, J. Ralston, Discrete shock profiles for systems of conservation laws. *Comm. Pure Appl. Math.* **32** (1979), 445–482.

[20] D. Serre, Remarks about the discrete profiles of shock waves. *Mat. Contemp.* **11** (1996), 153–170.

[21] T. Yang, H. Zhao, C. Zhu, BV estimates of Lax-Friedrichs' scheme for a class of nonlinear hyperbolic conservation laws. *Proc. Amer. Math. Soc.* **131** (2003), 1257–1266.