# Two-dimensional regular shock reflection for the pressure gradient system of conservation laws

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**Abstract**

We establish the existence of a *global* solution to a regular reflection of a shock hitting a ramp for the pressure gradient system of equations. The set-up of the reflection is the same as that of Mach's experiment for the compressible Euler system, i.e., a straight shock hitting a ramp. We assume that the angle of the ramp is close to 90 degrees. The solution has a reflected bow shock wave, called the diffraction of the planar shock at the compressive corner, which is mathematically regarded as a free boundary in the self-similar variable plane. The pressure gradient system of equations is a subsystem, and an approximation, of the full Euler system, and we offer a couple of derivations.

**Keywords:** Free boundary, shock waves, oblique derivative, tangential oblique derivative, boundary value problem, diffraction, compressive corner, 2-D Riemann problem, regular reflection, Mach reflection, gas dynamics.

**AMS subject classification:** Primary: 35L65, 35J70, 35R35; Secondary: 35J65.

## 1 Introduction

We are interested in solving multi-dimensional systems of conservation laws. The primary system is the well-known Euler system for compressible gases ([22]). Much effort has been devoted to the system, see the survey papers [55][3][4][34][36], the conference proceedings [30] by Glimm and Majda, the monograph [29], and the recent progress Shuxing Chen [16][17][18], Chen, Xin, and Yin [19][20], Zhang [70], and Guiqiang Chen

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and Feldman [14][15]. However, the Euler system remains formidable because of its complexity. Simplified models that capture various isolated features of the Euler system may be proposed and studied to pave the road. Immediate models are the isentropic case, the irrotational (potential) flow equations, and the steady flows [54][65][66][67], some of these assumptions are already made in some of the aforementioned papers. A remarkable and distinct model is the unsteady transonic small disturbance system (UTSD) which was proposed and studied ([21][38] [54][10][11][12][8][9][39][60][63][26]) for transonic solutions and the transition from Mach reflection to regular reflection. Despite the conveniences these models bring, all these models have their difficulties. In recent years, another model called the pressure gradient system is proposed ([71][69]), which seems to exihibit new and complementary conveniences. See [13] for a similar system. In this paper we present further motivational work on the pressure gradient system, and in particular we present the global existence of a self-similar solution that is similar to the regular reflection seen in Mach's experiment for the full Euler system, see Section 4.

We spend Section 2 deriving the pressure gradient model and providing supportive materials. Section 3 covers the set-up of the shock reflection problem and basic classical treatment. The precise statement of the result is given in Section 4. Sections 5–9 are devoted to the proof. Section 10 is on fine properties of the velocity. And in Section 11 we offer some discussions. In the Appendix we provide theory that are somewhat implied by theorems in various papers but not yet clearly stated in writing.

# 2 The pressure gradient system

The *pressure gradient system* takes the form

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ E_t + (up)_x + (vp)_y = 0, \end{cases} \tag{1}$$

where $E = \frac{1}{2}(u^2 + v^2) + p$.

The pressure gradient system is a reduction of a subsystem appeared in the flux splitting scheme of Li and Cao [45] and Agarwal and Halt [1]. Its mathematical structure and numerical simulations were studied in Zhang, Li, and Zhang [69], where one sees striking similarity to the Euler system. The pressure gradient system was intuitively justified for its own physical validity when the velocity is small and the gas constant $\gamma$ is large in Zheng [71], where the existence of a subsonic solution was also established, which adds on further attractiveness of the model. Its regime of physical validity is clarified further in the formal presentation of an asymptotic derivation to be presented later in this section. See the books by Li et al [44] or Zheng [72] for more background information.

We recall that the full Euler system for an ideal fluid takes the form

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u + pI) = 0, \\ (\rho E)_t + \nabla \cdot (\rho E u + pu) = 0, \end{cases} \tag{2}$$

where

$$E := \frac{1}{2}|u|^2 + e,$$

where $e$ is the internal energy. For a polytropic gas, there holds

$$e = \frac{1}{\gamma - 1}\frac{p}{\rho},$$

where $\gamma > 1$ is a dimensionless gas constant (*the adiabatic exponent*). More precisely $\gamma = 1 + Rc_v^{-1}$ where $R$ is the constant in the equation of state for ideal gases and $c_v$ is the specific heat at constant volume ([22]).

## 2.1 The flux-splitting derivation

Separating the pressure from the inertia in the flux of the Euler equations

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uE + up \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vE + vp \end{pmatrix}_y = 0, \tag{3}$$

where

$$E = \frac{1}{2}(u^2 + v^2) + \frac{1}{\gamma - 1}\frac{p}{\rho},$$

we obtain two systems of equations

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 \\ \rho uv \\ \rho uE \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 \\ \rho vE \end{pmatrix}_y = 0 \tag{4}$$

and

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}_t + \begin{pmatrix} 0 \\ p \\ 0 \\ up \end{pmatrix}_x + \begin{pmatrix} 0 \\ 0 \\ p \\ vp \end{pmatrix}_y = 0\,. \tag{5}$$

Agarwal and Halt [1] have used this splitting (4)(5) to form a scheme in numerical computations for airfoil flows and observed a consistent improvement over other schemes (Roe, AUSM, CUSP, and Van Leer). System (4) is called the *zero pressure* system, or the *transport* or *convective* system. System (5) is called the variable-density *pressure gradient system*. This splitting corresponds to the separation of the two mechanisms– pressure difference and inertia– that are causes for fluid motion.

We focus on system (5). Let us do some simple reduction. From the first equation of system (5) we obtain

$$\rho_t = 0.$$

Thus $\rho$ is independent of time. For simplicity, let us assume $\rho = 1$. Then system (5) can be written as (1) with $E = (u^2 + v^2)/2 + p/(\gamma - 1)$. It can be seen easily that the transformation

$$\begin{cases} p = (\gamma - 1)P, \\ t = \frac{1}{\gamma-1}T, \end{cases} \tag{6}$$

will effectively rescale the gas constant $\gamma$ to 2. Thus system (1) with $E = (u^2+v^2)/2+p$ will be the primary system for us to study.

For smooth solutions or in regions where a solution is smooth, system (1) can be simplified to be

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ p_t + pu_x + pv_y = 0. \end{cases} \tag{7}$$

From system (7) we can find a decoupled equation

$$\left(\frac{p_t}{p}\right)_t = p_{xx} + p_{yy}. \tag{8}$$

## 2.2 The asymptotic derivation

Besides the previous somewhat brutal derivation, there is a soft derivation. Let us write the energy in the form

$$E = \frac{1}{2}|u|^2 + \varepsilon p/\rho$$

where

$$\varepsilon := \frac{1}{\gamma - 1}.$$

We propose to look for an asymptotic solution of the form

$$\begin{array}{rcll} \rho & = & \rho_0 + & \varepsilon\rho_1 + O(\varepsilon^2), \\ u & = & & \varepsilon u_1 + O(\varepsilon^2), \\ p & = & & \varepsilon p_1 + O(\varepsilon^2). \end{array} \tag{9}$$

This scaling corresponds to sound speeds of the order $O(1)$:

$$c = \sqrt{\gamma p/\rho} = O(\varepsilon^0).$$

So we scale space and time variables by the same factor (order $O(1)$) to study acoustic phenomena.

The leading order perturbation equation from conservation of mass is

$$(\rho_0)_t = 0,$$

so

$$\rho_0 = \rho_0(x).$$

The leading order equation from conservation of momentum $O(\varepsilon)$ and conservation of energy $O(\varepsilon^2)$ constitute the variable-density pressure gradient system

$$\begin{cases} (\rho_0 u_1)_t + \nabla p_1 = 0, \\ \left(\frac{1}{2}\rho_0 |u_1|^2 + p_1\right)_t + \nabla \cdot (p_1 u_1) = 0. \end{cases} \tag{10}$$

This is the same as in the flux-splitting derivation of the previous subsection.

John Hunter became interested in the system during the author's talk on its Riemann problem at an FRG workshop in Pittsburgh in 2003, and wanted to fix the handwaving-type argument for the physical validity of the system loosely presented in the talk (based on Zheng [71]). This formal derivation presented here is what he sent the author a few days after the meeting ([37]). We note that this asymptotic regime lacks strong physical sense because there is no such physical material for very large $\gamma$. There is no other nonphysical concerns though. For example, it is physical to have $p = \varepsilon p_1 + O(\varepsilon^2) \to 0$ since $p$ is an independent variable and one can adjust the temperature to achieve it.

## 2.3 Progress of research

Both Cauchy and Riemann problems for systems (5), (1), or (8) are open. The self-similar coordinates

$$\xi = \frac{x}{t}, \quad \eta = \frac{y}{t}$$

can reduce the Riemann problem by one dimension. However, all three systems (5), (1), and (8), and even their linearized versions are of mixed type in the self-similar coordinates. Peng Zhang, Jiequan Li, and Tong Zhang [69] have given a set of conjectures for solutions to the four-wave Riemann problem for these systems, see the book of Li, et. al. [44], or Section 9.3 of [72]. Zheng has established the existence of solutions in the elliptic region [71][72].

More precisely, equation (8) in the self-similar coordinates $(\xi, \eta)$ takes the form

$$\begin{aligned} (p-\xi^2)p_{\xi\xi} \quad & -2\xi\eta p_{\xi\eta} + (p-\eta^2)p_{\eta\eta} \\ & + \tfrac{1}{p}(\xi p_\xi + \eta p_\eta)^2 - 2(\xi p_\xi + \eta p_\eta) = 0. \end{aligned} \tag{11}$$

The eigenvalues of the coefficient matrix of the second order terms of (11) can be found to be $p$ and $p-\xi^2-\eta^2$. Zheng proved in [71] the existence of a weak solution for equation (11) in any open, bounded and convex region $\Omega \subset \mathbb{R}^2$ with smooth boundary and the degenerate boundary datum

$$p|_{\partial\Omega} = \xi^2 + \eta^2 \tag{12}$$

provided that the boundary of $\Omega$ does not contain the origin $(0,0)$.

Kyungwoo Song [62] has removed the restriction on the origin and the smoothness of the boundary. Kim and Song [43] have obtained regularity of the solution in the interior of the domain and continuity up to and including the boundary. Dai and Zhang [23] have obtained the interaction of two rarefaction waves adjacent to the vacuum. For numerical simulations, see [44].

## 2.4 Closeness to the Euler

How good does the pressure gradient system approximate the full Euler system? We will show solid evidence in a future paper [74]. For now we notice that the full Euler system can be rewritten in a form in which the pressure gradient system plays a dominating role.

First let us show the *maximum principle* for $p$ of the full Euler system, although it is known, see Serre [61]. In the self-similar plane and for smooth solutions, the system takes the form:

$$\begin{cases} \frac{1}{\rho}\partial_s \rho + u_\xi + v_\eta = 0, \\ \partial_s u + \frac{1}{\rho} p_\xi = 0, \\ \partial_s v + \frac{1}{\rho} p_\eta = 0, \\ \frac{1}{\gamma p}\partial_s p + u_\xi + v_\eta = 0 \end{cases} \tag{13}$$

where

$$\partial_s := (u-\xi)\partial_\xi + (v-\eta)\partial_\eta.$$

By differentiating the fourth equation in (13) in the flow stream direction $\partial_s$ and using the second and third equations, we obtain a second-order equation for $p$:

$$-\nabla\cdot(\frac{1}{\rho}\nabla p) + \partial_s \frac{\partial_s p}{\gamma p} - \left(\frac{\partial_s p}{\gamma p}\right)^2 - \frac{\partial_s p}{\gamma p} + 2(u_\xi v_\eta - u_\eta v_\xi) = 0. \tag{14}$$

The combination $u_\xi v_\eta - u_\eta v_\xi$ can be manipulated to depend on $(v_\eta, p_\xi, p_\eta)$ homogeneously with degree 2. In fact, using the last three equations in (13) we find that

$$u_\xi v_\eta - u_\eta v_\xi = \frac{v_\eta p_\eta}{(v-\eta)\rho} - \frac{v_\eta p_\xi}{(u-\xi)\rho} - \frac{p_\xi p_\eta}{(u-\xi)(v-\eta)\rho^2} + \frac{p_\eta p_s}{\gamma p\rho(v-\eta)}. \tag{15}$$

When $\nabla p$ vanishes, so does $u_\xi v_\eta - u_\eta v_\xi$. Therefore there holds the maximum principle for $p$ in the subsonic domains.

Furthermore, we show that there holds the *ellipticity principle*. Let

$$\varphi := c^2 - (u-\xi)^2 - (v-\eta)^2. \tag{16}$$

Assuming the variables are in $C^2$, we derive an equation for $\varphi$:

$$-c^2\Delta\varphi + \partial_s^2\varphi = O(\varepsilon) + (\frac{\partial_s\varphi}{c})^2 - \frac{4\varphi\partial_s\varphi}{c^2} - 3\partial_s\varphi + \frac{3\varphi^2 + (\varphi - c^2)^2}{c^2} + c^2. \tag{17}$$

Thus so long as $|O(\varepsilon)| < c^2$, there holds the *ellipticity principle*:

$$\min_{\Omega}\varphi \geq \min_{\partial\Omega}\varphi. \tag{18}$$

This means that our solution will be subsonic in a region if we use a barrier to force the domain subsonic.

Since the right-hand side of the $\varphi$ equation is so much positive, the ellipticity holds uniformly independent of $\varepsilon > 0$. That is, consider

$$F := \varphi - \beta\omega; \tag{19}$$

where $\omega > 0$ in $\Omega$ and

$$\omega = 0 \qquad \text{on } \partial\Omega. \tag{20}$$

Then there exists a small $\beta > 0$, independent of $\varepsilon > 0$, such that

$$F > 0 \qquad \text{in } \Omega, \tag{21}$$

thus

$$\varphi > \beta\omega > 0 \qquad \text{in } \Omega. \tag{22}$$

For ellipticity principle for the potential equation, see Elling and Liu [24].

## 2.5 Comparison of models

The Burgers' equation $u_t + (u^2/2)_x = \varepsilon u_{xx}$ has played an essential role in the theory of one-dimensional systems of conservation laws. No such a model has emerged for multi-dimensional systems. Current various models seem to have different physics captured. As the names suggest, the isentropic, the irrotational, and the steady flows are different. The UTSD model is regarded as describing the transition between Mach and regular reflections; i.e., it is best used locally at the triple point. The pressure gradient system can have global solutions that are similar to those of Euler system seen in numerical simulations and physical experiments, established in this paper. Thus this model possesses essential physics.

What separates this model from others is the series of features: It is a neat set of three evolutionary conservation laws, the pressure variable can decouple from the other two to form a single quasilinear equation whose quasilinearity is lower than that for the potential flow equation (or the potential formulation of the pressure gradient system). By quasilinearity I mean the order of the highest order of derivatives involved in the coefficients of the principal part of the second-order equation. In the pressure gradient system, the coefficients of the principal part of the second-order equation for the pressure do not depend on $\nabla p$, thus its quasilinearity is zero. In the potential flow equation (p.248 of [22]), however, the coefficients of the principal part of the second-order equation for the potential depend on $\nabla p$, thus its quasilinearity is one. This small difference makes the pressure gradient system more accessible since the state of the art in elliptic theory has been employed fully here. The success here, as intended as a model, may induce better utilization of the elliptic theory to handle the potential flow in the near future.

The model's simplicity has allowed us to establish the existence a global solution in this paper that resembles the regular reflection of a straight shock hitting a ramp for the adiabatic Euler system. This is the first of such a result.

So this model is simple enough for mathematical treatment and yet captures essential physics. In addition, its potential to fully approximate a major part of the full Euler system is promising, see a forthcoming paper [74].

# 3 Set-up of regular reflection

So we consider pressure gradient system (1). We consider self-similar solutions. In the self-similar plane $(\xi, \eta) = (x/t, y/t)$, the system of equations are

$$\begin{cases} -\xi u_\xi - \eta u_\eta + p_\xi &= 0, \\ -\xi v_\xi - \eta v_\eta + p_\eta &= 0, \\ -\xi E_\xi - \eta E_\eta + (pu)_\xi + (pv)_\eta &= 0. \end{cases} \tag{23}$$

Consider a flat shock hitting a wedge with half angle $\theta_w \in (0, \pi/2)$. The state ahead of the shock is $(p, u, v) = (p_0, 0, 0)$ for some $p_0 > 0$. The state behind the shock is $(p_1, u_1, 0)$ with $p_1 > p_0$. To connect the two states with a single forward shock, we need the relation

$$u_1 = \frac{p_1 - p_0}{\sqrt{\bar{p}_{10}}}, \qquad p_1 > p_0. \tag{24}$$

The overhead bar denotes the average: $\bar{p}_{10} = (p_1 + p_0)/2$. Assuming the shock reflection is a regular one, then it hits the ramp at the location

$$(\xi, \eta) = (\xi_{10}, \eta_{10}) := (\sqrt{\bar{p}_{10}},\ \tan\theta_w \sqrt{\bar{p}_{10}}). \tag{25}$$

See Figure 1 for the set-up.

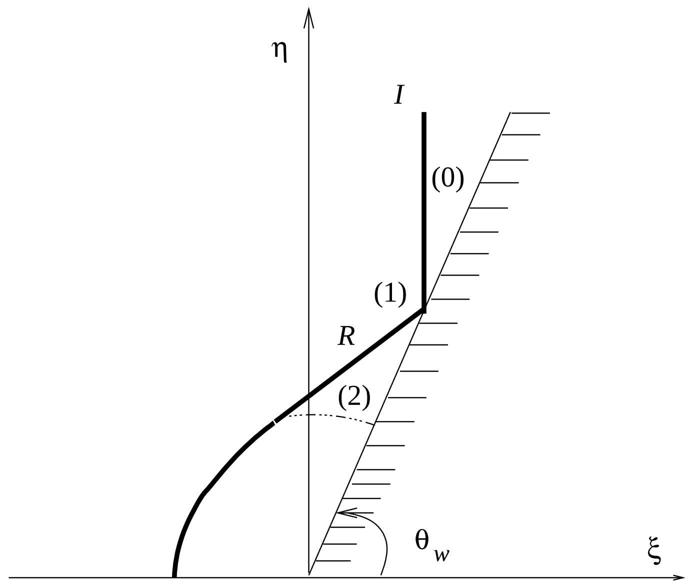


Figure 1. Regular reflection

*Two free parameters*: We see that system (1) is invariant under the translation $(u, v) \to (u - a, v - b)$, which we have utilized in assuming that the velocity is zero ahead of the incident shock. The system enjoys another invariance: $(p, u, v, x, y) \to (\alpha^2 p, \alpha u, \alpha v, \alpha x, \alpha y)$ which we can use to scale $p_0 = 1$. Thus there is only one free variable $p_1/p_0$ in describing the data. The entire experiment can thus be characterized by the two parameters $(p_1/p_0, \theta_w)$.

*Algebraic portion:* To find the reflected shock and the state between it and the ramp, denoted by state 2, we need the Rankine-Hugoniot relation in 2-D. Let $\eta = \eta(\xi)$ with slope $\sigma = \eta'(\xi)$ be a shock curve. Then,

$$\begin{aligned} &(\eta - \xi\sigma)[u] + \sigma[p] = 0 \\ &(\eta - \xi\sigma)[v] - [p] = 0 \\ &(\eta - \xi\sigma)[E] + \sigma[pu] - [pv] = 0. \end{aligned} \tag{26}$$

We can solve them to obtain the contact discontinuity $\sigma = \eta/\xi = [v]/[u], [p] = 0$ and the shocks

$$\begin{aligned} &\tfrac{d\eta}{d\xi} = \sigma_\pm := \tfrac{\xi\eta \pm \sqrt{\bar{p}(\xi^2+\eta^2-\bar{p})}}{\xi^2-\bar{p}}, \\ &[p] = \xi[u] + \eta[v], \\ &[p]^2 = \bar{p}([u]^2 + [v]^2). \end{aligned} \tag{27}$$

We use the minus branch for the reflected shock. A useful and equivalent form for the

Rankine-Hugoniot relation is

$$\frac{d\eta}{d\xi} = \sigma_{\pm} = \frac{\xi\eta \pm \sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\xi^2 - \bar{p}}, \tag{28}$$

$$[u] = \frac{\xi\bar{p} \pm \eta\sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\bar{p}(\xi^2 + \eta^2)}[p], \tag{29}$$

$$[v] = \frac{\eta\bar{p} \mp \xi\sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\bar{p}(\xi^2 + \eta^2)}[p]. \tag{30}$$

We require that the state $(p_2, u_2, v_2)$ be such that the velocity $(u_2, v_2)$ be parallel to the wall; that is,

$$v_2 = \tan\theta_w \, u_2. \tag{31}$$

This requirement and the Rankine-Hugoniot relation determine the state 2.

**Proposition 3.1.** (Regular reflection of the algebraic portion) *There exists a critical* $\theta_w = \theta_0 \in (0, \pi/2)$*, depending only on* $p_1/p_0$*, given by the formula*

$$\tan^2\theta_0 = \frac{8p_1(p_1 - p_0)}{(p_1 + p_0)^2}, \tag{32}$$

*such that there exist two states* $(p_2, u_2, v_2)$ *for each* $\theta_w > \theta_0$*, given by*

$$p_2 = p_1 + \bar{p}_{10}\tan^2\theta_w \pm \tan\theta_w\sqrt{\bar{p}_{10}^2\tan^2\theta_w - 2p_1(p_1 - p_0)}, \tag{33}$$

$$u_2 = \frac{p_2 - p_0}{\sqrt{\bar{p}_{10}}}\frac{1}{1 + \tan^2\theta_w}, \qquad v_2 = u_2\tan\theta_w. \tag{34}$$

*Both values of the pressure of the state 2 are greater than* $p_1$*; the larger one goes to infinity while the smaller one approaches*

$$p_2^* := p_1 + \frac{2p_1(p_1 - p_0)}{p_1 + p_0} \tag{35}$$

*as* $\theta_w \to \pi/2-$.

We comment that the $p_2^*$ in (35) is the pressure that corresponds to the planar shock hitting a vertical wall with zero velocity $(u_2, v_2) = (0, 0)$ between the reflected backward shock at $\xi = -\sqrt{(p_1 + p_2^*)/2}$ and the vertical wall.

*Proof.* We manipulate the second equation in (27) to yield

$$p_2 = \xi_{10}(1 + \tan^2\theta_w)u_2 + p_0. \tag{36}$$

Introducing the notation

$$\tilde{u}_2 := (1 + \tan^2\theta_w)u_2, \tag{37}$$

we have

$$\tilde{u}_2 = \frac{p_2 - p_0}{\sqrt{\bar{p}_{10}}}, \tag{38}$$

or equivalently

$$\tilde{u}_2 - u_1 = \frac{p_2 - p_1}{\sqrt{\bar{p}_{10}}}. \tag{39}$$

Manipulating the third equation of (27), we obtain

$$(p_2 - p_1)^2/\bar{p}_{12} = (u_2 - u_1)^2 + \tan^2\theta_w\, u_2^2. \tag{40}$$

This is a quadratic equation for $u_2$, from which we find one branch

$$\tilde{u}_2 - u_1 = \sqrt{1 + \tan^2\theta_w}\sqrt{(p_2 - p_1)^2/\bar{p}_{12} - \sin^2\theta_w\, u_1^2}. \tag{41}$$

We have discarded the minus sign branch since it is irrelevant here. We equate the two equations (39) and (41) to eliminate $\tilde{u}_2$ and end up with the equation

$$(p_2 + p_1)(p_2 - p_1)^2 = \bar{p}_{10}(1 + \tan^2\theta_w)[2(p_2 - p_1)^2 - (p_2 + p_1)\sin^2\theta_w\, u_1^2]. \tag{42}$$

Let $x := p_2 - p_1$, we rewrite the above equation as

$$x^3 + (p_1 - p_0 - (p_1 + p_0)\tan^2\theta_w)x^2 + (x + 2p_1)\tan^2\theta_w(p_1 - p_0)^2 = 0. \tag{43}$$

We observe by inspection that the equation has a solution $x = p_0 - p_1$, which helps to reduce the equation to

$$x^2 - (p_0 + p_1)\tan^2\theta_w x + 2p_1\tan^2\theta_w(p_1 - p_0) = 0. \tag{44}$$

We thus find the two roots easily when the discriminant is nonnegative. The asymptotic behavior for $p_2$ is obvious from the explicit formula. This completes the proof of the proposition. $\square$

We will use the weak reflection (with the smaller pressure value), and ignore the strong one (see p. 317 in [22] for a reason). We check easily to see that

$$\xi_{10}^2 + \eta_{10}^2 - \bar{p}_{12} > 0 \tag{45}$$

for the weak reflection as $\tan\theta_w \geq \tan\theta_0$, so the square root is well defined for the slope of the shock wave. In fact, we use the explicit formula (33) to first obtain $p_2 < p_1 + \bar{p}_{10}\tan^2\theta_w$, then use $\tan\theta_w > \tan\theta_0$ to derive

$$\xi_{10}^2 + \eta_{10}^2 - \bar{p}_{12} > \frac{(p_1 - p_0)(3p_1 - p_0)}{2(p_1 + p_0)} > 0. \tag{46}$$

Further, the slope satisfies

$$\frac{d\eta}{d\xi} = \sigma < \tan\theta_w \tag{47}$$

through direct comparison for as long as $p_2 > p_1$. So locally at the reflection point, the regular reflection is possible when $\theta_w \in [\theta_0, \pi/2)$.

The state 2 at the reflection point is also supersonic in the sense that

$$\xi^2 + \eta^2 - p > 0 \tag{48}$$

when $\theta_w$ is slightly more closer to $\pi/2$; i.e., $\tan\theta_w > \tan\theta_1$, where

$$\tan^2\theta_1 := \frac{p_1 - p_0}{(p_1+p_0)^2}\left(4p_1 + \sqrt{16p_1^2 + (p_1+p_0)^2}\right) \tag{49}$$

The proof is straightforward so we omit it. The state $(p_2, u_2, v_2)$ will remain supersonic in a neighborhood of the reflection point. So the constant state $(p_2, u_2, v_2)$ will be used as a solution in the sector formed by the reflected straight shock and the wall until it hits the sonic circle

$$\xi^2 + \eta^2 = p_2. \tag{50}$$

*Elliptic portion:* Beyond the sonic arc, the state becomes nonconstant, and the reflected shock curls downward to meet the ground, vertically at the ground. We propose the problem as a boundary value problem with a free boundary on which the Rankine-Hugoniot relation holds. We convert the Rankine-Hugoniot relation into a (degenerate) oblique derivative boundary value problem, see the next paragraph. On the wall, the boundary condition $v_2 = \tan\theta_w\, u_2$ becomes

$$\partial_{\mathbf{n}} p = 0 \tag{51}$$

where $\mathbf{n}$ denotes the exterior unit normal to the wall. On the ground, we can use the Neumann condition (51) or equivalently we convert the ramp problem into a wedge problem, so there is no boundary. On the sonic arc, we use the Dirichlet problem. The decoupled $p$-equation in the subsonic domain is derived from (23) and mentioned in (11):

$$(p-\xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p-\eta^2)p_{\eta\eta} + \frac{(\xi p_\xi + \eta p_\eta)^2}{p} - 2(\xi p_\xi + \eta p_\eta) = 0. \tag{52}$$

Here is the derivation of the (degenerate) oblique derivative boundary value on the reflected shock, which we call $\Sigma$ from now on. We require that all three equations (23) (taken the limit from the inside) and all three Rankine-Hugoniot relations (28-30) hold. We differentiate the last two equations in the Rankine-Hugoniot relations along the shock wave so we have five differential equations for six derivatives $(u_\xi, v_\xi, p_\xi, u_\eta, v_\eta, p_\eta)$. Holding $p_\eta$ fixed, we solve for the other five $(u_\xi, v_\xi, u_\eta, v_\eta, p_\xi)$. The relation between

$p_\xi$ and $p_\eta$ decouples from the other derivatives and is used as the oblique derivative condition on the shock wave:

$$\begin{aligned} &\dot{p}\left(\tfrac{[p]}{4\bar{p}}(\xi^2+\eta^2)-(\xi^2+\eta^2-\bar{p})\right) \\ &+(\xi\eta+\eta'(\bar{p}-\xi^2))\left\{(-\sigma_\pm p_\xi+p_\eta)+(\xi p_\xi+\eta p_\eta)\tfrac{\xi\sigma_\pm-\eta}{p}\right\} \;=\; 0, \end{aligned} \tag{53}$$

where $[p] = p - p_1$, and the term $\dot{p}$ denotes the tangential differentiation along the shock: $\dot{p} := p_\xi + \sigma_\pm p_\eta$. The other relations are used for determining $(u, v)$ once $p$ is obtained. The first equation in the R-H relation (28) is symbolically "reserved" for use to determine the location of $\Sigma$. The oblique derivative condition (53) is in the form $\mathbf{l}\cdot\nabla p = 0$ for a smooth vector field $\mathbf{l}(\xi, \eta, \eta')$, which is tangent to the shock boundary at its tip. The apparent $p$-dependence of $\mathbf{l}$ is removed by the formula

$$\bar{p} = \frac{(\eta-\xi\sigma_\pm)^2}{1+\sigma_\pm^2} \tag{54}$$

obtained by inverting (28).

The sufficiency of the oblique derivative boundary condition is seen as follows. The second-order equation for $p$ plus the two first-order equations for $(u, v)$ implies

$$(\xi\partial_\xi+\eta\partial_\eta)L+L=0, \tag{55}$$

for

$$L := (\xi\partial_\xi p+\eta\partial_\eta p)/p-u_\xi-v_\eta. \tag{56}$$

This ODE (55) in the form $r\partial_r L + L = 0$ has the only solution $L = 0$ if $L(r_0) = 0$ at any point $r = r_0 > 0$. Thus the third equation of (23) will hold if it holds on the boundary $\Sigma$. From our previous derivation of the oblique derivative boundary condition, we see that $L = 0$ on the free boundary, provided that the oblique derivative boundary condition holds along with the other four relations between $(u_\xi, v_\xi, u_\eta, v_\eta)$ and $p_\eta$ in obtaining $(u, v)$. We will use the four relations to find $(u, v)$. Thus the oblique derivative boundary condition is a condition that guarantees that solutions of the second-order equation for $p$ are solutions for the first-order systems for $(p, u, v)$ in the sectoral domain spanned by $\Sigma$. We point out that $L = 0$ holds automatically on the sectoral domain spanned by the sonic arc if $p$ is Lipschitz.

**The free boundary problem**: *Find p satisfying the degenerate elliptic equation (52) in the domain $\Omega$ bounded by the curved reflected shock $\Sigma$, the wall and the ground, and the sonic arc (50). On the wall and the ground, it is the Neumann condition (51). On the circular arc, it is the Dirichlet problem. On the free boundary $\Sigma$, it is the degenerate oblique derivative boundary condition (53). The free boundary satisfies (28).*

A trivial solution to the above free boundary problem is $p = p_2$ in the entire domain $\Omega$, whose boundary $\Sigma$ consists of two parts: The first part is the extension of the reflected shock $R$ to the tangent point with the circle $\xi^2 + \eta^2 = \bar{p}_{12}$ while the second part is the circle $\xi^2 + \eta^2 = \bar{p}_{12}$ between the tangent point and the ground. This constant solution does not result in a solution $(p, u, v)$ for the original problem. The reason is that the tangential oblique derivative boundary value condition is completely degenerate on the circular free boundary which fails to impart information from the second-order equation (52) for $p$ to the first-order equation for $p$ (i.e., the third equation in (23)). In short, the velocity field $(u, v)$ caused by a circular shock wave connecting two constant pressure is necessarily such that $u_\xi + v_\eta \neq 0$.

# 4 Result

**Main Theorem**. *There exists an (entropy) solution $(p, u, v)$ defined for all $(\xi, \eta) \in \mathbb{R}^2$, provided that the ramp angle $\theta_w$ is close to $\pi/2$. The shock curve is $C^1$ and piecewise $C^\infty$ smooth. The pressure p is $C^\infty$ smooth in the subsonic domain. The velocity is bounded and Hölder continuous on the closure of the subsonic domain.*

See Figure 1 for an illustration, where only half of the wedge is presented. Our result is consistent with the asymptotic results of Keller and Blank [41] and Hunter and Keller [40].

The proof is given in the next sections.

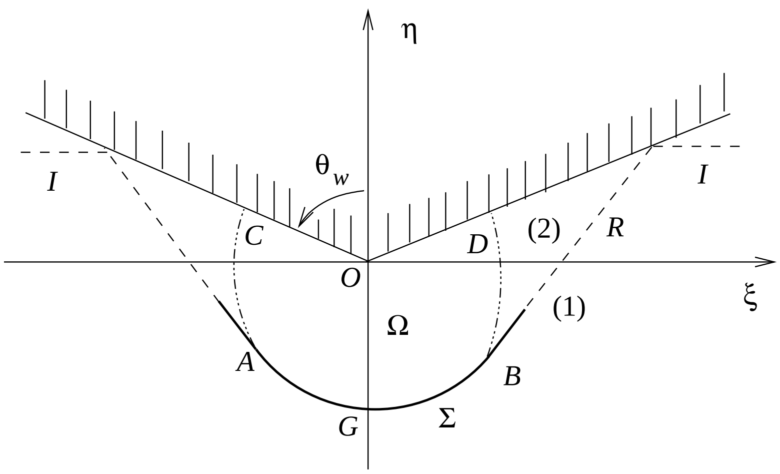


Figure 2. Domain for the linear theory

# 5 Linear theory with fixed boundary

We consider a linear problem with a fixed boundary that forms the basis for our nonlinear problem. The domain $\Omega$ is the region between the wedge and a pre-selected shock curve and its corresponding sonic arcs. See Figure 2, where the shock curve is denoted by the curve $AGB$, and also by $\Sigma$. The sonic arcs are $AC$ and $BD$. We are interested in the case where $AC$ and $BD$ do not degenerate to single pints.

More precisely, for a given $\theta_w \in (0, \pi/2)$, $p_1$, $p_m$ ($p_1 < p_m$), and a $\beta \in (0, \beta_0)$, where

$$\beta_0 := \frac{1}{\sqrt{\bar{p}_{m1}}}, \qquad (\bar{p}_{m1} := (p_1 + p_m)/2) \tag{57}$$

let

$$\mathbb{K} = \{\eta = \eta(\xi) \in C^{2,\gamma}(\mathbb{R}) \,|\, \eta(0) = \eta_{m1}, \eta'(0) = 0, \eta''(0) = \beta, 0 \le \eta'' \le \beta_0, \eta \text{ even } \}. \tag{58}$$

Here $\gamma \in (0,1)$ and

$$\eta_{m1} := -\sqrt{\bar{p}_{m1}}. \tag{59}$$

We introduce

$$\bar{p} = \frac{(\eta - \xi\eta'(\xi))^2}{1 + (\eta'(\xi))^2}, \tag{60}$$

where $\bar{p}$ stands for the average of $p_1$ and the $p$ from the inside of the shock. This $\bar{p}$ is always well-defined and belongs to $C^{1,\gamma}(\mathbb{R})$ and $C^{2,\gamma}$ at the point $\xi = 0$ magically. In fact, it has the expansion

$$\bar{p} = \bar{p}_{m1} + \beta\sqrt{\bar{p}_{m1}}(1 - \beta\sqrt{\bar{p}_{m1}})\xi^2 + O(\xi^{2+\gamma}) \tag{61}$$

at $\xi = 0$. So $\bar{p}$ has a local minimum at $\xi = 0$ for the choices of $\beta$ and $\beta_0$. Furthermore, we have

$$\xi^2 + \eta^2 - \bar{p} = \xi^2 \left((1 - \beta\sqrt{\bar{p}_{m1}})^2 + O(\xi^\gamma)\right) \tag{62}$$

at $\xi = 0$. Thus the square root function

$$F(\xi) = \begin{cases} \sqrt{\xi^2 + \eta^2 - \bar{p}}, & \xi > 0, \\ -\sqrt{\xi^2 + \eta^2 - \bar{p}}, & \xi < 0 \end{cases}$$

is a smooth function at $\xi = 0$ along the curve $\eta(\xi)$. Finally, through pure algebraic manipulations, we find

$$\bar{p}(\xi^2 + \eta^2 - \bar{p}) = (\eta'(\xi^2 - \bar{p}) - \xi\eta)^2. \tag{63}$$

We would like to invert (63) to find

$$\frac{d\eta}{d\xi} = \sigma_\pm = \frac{\xi\eta + \xi\sqrt{\bar{p}(1 + (\eta^2 - \bar{p})/\xi^2)}}{\xi^2 - \bar{p}} =: \sigma, \tag{64}$$

which is the standard Rankine-Hugoniot locus, but we need to prove that

$$\eta'(\xi^2 - \bar{p}) - \xi\eta > 0, \qquad \text{for } \xi > 0. \tag{65}$$

In fact, we use algebraic manipulations to find

$$\eta'(\xi^2 - \bar{p}) - \xi\eta = \frac{1}{1 + (\eta')^2}(\xi\eta' - \eta)(\xi + \eta\eta'). \tag{66}$$

The term $\xi\eta' - \eta > 0$ because it is positive at $\xi = 0$ and its derivative along the curve is $\xi\eta'' > 0$. The other term $\xi + \eta\eta'$ is positive:

$$\xi + \eta\eta' > 0, \tag{67}$$

because it is zero at $\xi = 0$ and its derivative is $1 + (\eta')^2 + \eta\eta''$ which is positive since $\eta'' \geq 0$ if $\eta > 0$, or $|\eta\eta''| \leq (\max|\eta|)\beta_0 = 1$. So indeed (65) holds and we have (64).

The pre-selected shock will be a curve from the set $\mathbb{K}$. It is possible that only a portion of an $\eta(\xi)$ in $\mathbb{K}$ is actually used: We use the elliptic portion, corresponding to $AGB$. Let us explain. For each $\eta(\xi) \in \mathbb{K}$, we use the equation $\eta' = \sigma_-$ for $\xi < 0$ and $\eta' = \sigma_+$ for $\xi \geq 0$ to locate $p$ on the boundary; i.e., formula (60). Along this curve as $\xi$ increases, both $p$ and $\xi^2 + \eta^2$ increase (see 67) until $p - \xi^2 - \eta^2 = 0$ from which point we stop using this boundary, i.e., the point $B$. Use $p_2$ to denote this value of $p$ (at $B$) along the boundary where the ellipticity first vanishes. Use this $p$ from (60) in the vector field $\mathbf{l}$ of the oblique derivative condition. We then have a linear tangential oblique derivative boundary value problem for $p^{(1)}$:

$$\begin{aligned} &(p^{(1)}_\xi + \eta' p^{(1)}_\eta)\left(\tfrac{[p]}{4\bar{p}}(\xi^2 + \eta^2) - (\xi^2 + \eta^2 - \bar{p})\right) \\ &+(\xi\eta + \eta'(\bar{p} - \xi^2))\left\{(-\eta' p^{(1)}_\xi + p^{(1)}_\eta) + (\xi p^{(1)}_\xi + \eta p^{(1)}_\eta)\tfrac{\xi\eta' - \eta}{p}\right\} \;=\; 0. \end{aligned} \tag{68}$$

The coefficients of (68) all enjoy $C^{1,\gamma}$ regularity or higher. Note carefully that we do *not* use the value $p$ in (60) as Dirichlet boundary value. We use Dirichlet value $p_2$ on the sonic line and $p_m$ at G:

$$p = p_2 \quad \text{on } \xi^2 + \eta^2 = p_2; \qquad p = p_m \quad \text{at } G. \tag{69}$$

The extra condition $p_m$ at $G$ will be justified later by identifying the problem as an *emergent type* (see [35][6]) since the obliqueness fails in an emergent type.

The direction field $\mathbf{l}(\xi, \eta, \eta'(\xi))$ of the oblique derivative condition is drawn for a flat shock in Figure 3. This vector field has a natural extension into the domain by simply letting $\eta$ free everywhere including in the formula for $\bar{p}$, while holding $\eta'(\xi)$ fixed as the slope of the fixed boundary.

We linearize equation (52). Fix an $\alpha \in (0, 1)$. And fix a positive constant $\epsilon_e > 0$. For any

$$Q \in C^{1,\alpha}(\bar{\Omega}), \quad p_m < Q < p_2 \quad \text{in } \Omega, \tag{70}$$

we consider the equation

$$\epsilon_e \Delta p + ((Q-\xi^2-\eta^2)^+ + \eta^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + ((Q-\xi^2-\eta^2)^+ + \xi^2)p_{\eta\eta} \qquad (71)$$
$$+\{\frac{\xi Q_\xi + \eta Q_\eta}{Q} - 2\}(\xi p_\xi + \eta p_\eta) = 0,$$

where the regularization term $\epsilon_e \Delta p$ is added to ensure uniform ellipticity. Or in short, we consider an equation like this:

$$Lu := \sum_{ij} a^{ij} u_{x_i x_j} + \sum_i b^i u_{x_i} = 0 \qquad \text{in } \Omega. \qquad (72)$$

We point out that we have to linearize the equation to utilize the established theory on (tangential) oblique derivative boundary value problems of Lieberman and Guan and Sawyer, although some quasilinear and even fully nonlinear theory, which unfortunately do not apply to our case, is available (see [32] [59]).

The boundary condition on the wall is the Neumann condition (51)

$$\partial_{\mathbf{n}} p^{(1)} = 0. \qquad (73)$$

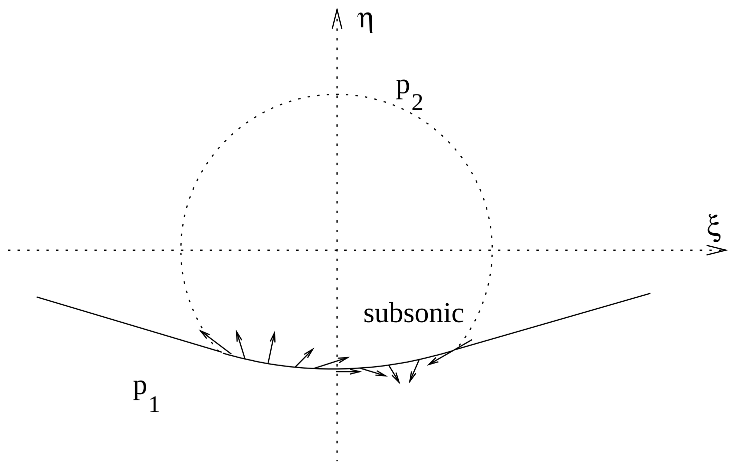


Figure 3. Directions of the tangential oblique derivatives.

We introduce a couple of notations. Let

$$d_x = \text{ distance } \{x, \{O, A, B, C, D\}\},$$

and for $\delta > 0$,

$$\Omega_\delta := \{x \in \Omega \,|\, d_x > \delta\}.$$

Let $|u|_a$ be the $C^a$ norms on the usual $C^a$ functions, $a \geq 0$.

**Theorem 5.1** (Existence for the linearized and fixed boundary problem). *There exists a classical smooth solution* $p^{(1)} \in C^{1,\alpha'}(\overline{\Omega}) \cap C^2(\Omega \cup \{\Sigma \backslash \{A, B\}\})$ *for some* $\alpha' > 0$ *to* *the linearized equation (71) with the Dirichlet (69), Neumann (73), and the tangential oblique derivative boundary conditions (68). The solution satisfies the following basic estimates:*

$$\begin{aligned} &p_m < p^{(1)} < p_2 \qquad \textit{in } \Omega, \\ &|p^{(1)} - p_m|_{1+\alpha'}(\overline{\Omega}) \le C|p_2 - p_m| \qquad \textit{for } \alpha' \le \alpha_1(\tfrac{\theta_w}{\pi - \theta_w}), \\ &|p^{(1)} - p_m|_{2+\gamma'}(\overline{\Omega}_\delta) \le C_\delta |p_2 - p_m| \qquad \textit{for } \gamma' \le \min\{\gamma, \alpha\}, 0 < \delta \ll 1. \end{aligned} \tag{74}$$

*Remark:* We remark that it is possible to express these estimates more neatly by using the intermediate spaces $H_a^{(b)}(\Omega)$, introduced in Gilbarg-Hörmander [27]. The $H_a^{(b)}(\Omega)$ consists of functions $u$ such that

$$|u|_a^{(b)}(\Omega) := \sup_{\delta > 0}\{\delta^{a+b}|u|_a(\overline{\Omega}_\delta)\} < \infty. \tag{75}$$

The numbers $a$ and $b$ satisfy $a \ge 0$ and $a + b \ge 0$.

*Proof.* The proof is based on a series of four papers [46][47][49][50] from Lieberman, one paper from Azzam [2], and one paper from Guan and Sawyer [32]. The paper of Guan and Sawyer helps us to solve the tangential oblique derivative problem at the point $G$ locally. The paper of Azzam provides higher regularity of solutions at corners C and D, where higher compatibility conditions than Lieberman's [50] are satisfied. All other difficulties are handled by the papers of Lieberman. Paper [46] provides the frame-work of Perron's method and handles the oblique derivative part. Paper [47] handles the mixed case and in particular points $A, B, C$, and $D$. Paper [50] gives optimal regularity at those points. And paper [49] handles the point $O$ where two oblique derivative boundary conditions are satisfied simultaneously. As for the interior and the Dirichlet boundary condition on the sonic arcs, they are classical, see Gilbarg and Trudinger [28] or Zheng [71].

We mention here the key points of the proof, details are given in the Appendix. First we verify the obliqueness of (68) on the boundary. An interior normal of $\Sigma$ is $(-\sigma_\pm, 1)$. The obliqueness is defined by the inner product of a unit (interior) normal with the oblique direction $(-\sigma_\pm, 1) + \frac{\xi\sigma_\pm - \eta}{p}(\xi, \eta)$. We ignore the tangential direction. We ignore the normalization of the interior normal for now as well. Thus

$$\text{Obliqueness } = (-\sigma_\pm, 1) \cdot \{(-\sigma_\pm, 1) + \frac{\xi\sigma_\pm - \eta}{p}(\xi, \eta)\}. \tag{76}$$

We do some simple algebra to yield,

$$\text{Obliqueness } = \frac{p - \xi^2}{p}\left\{(\sigma_\pm + \frac{\xi\eta}{p - \xi^2})^2 + \frac{p(p - \xi^2 - \eta^2)}{(p - \xi^2)^2}\right\}. \tag{77}$$

The obliqueness is obvious provided that $p - \xi^2 - \eta^2 > 0$ along the curve. It is true in a neighborhood of G, and recall we stop when it ceases to be true. The other factor

$\xi\eta + \eta'(\bar{p} - \xi^2)$, which we do not place in (76), has been shown in (65) to be less than zero for $\xi$ between $G$ and $B$.

Regarding point $O$, we realize that the interior angle $COD$ is not less than $\pi$, thus Lieberman's theory does not apply directly to yield $C^{1,\alpha}$ estimate. But, we realize that the solution is symmetric with respect to the $\eta$ axis. Hence, the left-half of the domain enjoys an interior angle $COG$ which is $\pi - \theta_w \leq \pi$. Therefore Lieberman's theory can apply in this case to yield $C^{1,\alpha_1}$ solutions where $\alpha_1$ is a positive number depending on $\frac{\theta_w}{\pi - \theta_w}$.

Regarding point $B$, where $p_2 = \xi^2 + \eta^2$, the oblique derivative boundary condition becomes

$$K^2(p_\xi + \eta' p_\eta) - \eta' p_\xi + p_\eta + (\xi p_\xi + \eta p_\eta)(\xi\eta' - \eta)/(\xi^2 + \eta^2) = 0, \tag{78}$$

where

$$K^2 = \frac{p_1}{p_1 + p_2}\sqrt{\frac{p_2 - p_1}{p_2 + p_1}}.$$

We rewrite (78)

$$p_\xi + H p_\eta = 0, \quad \text{where} \quad H := \frac{K^2\eta' + 1 + \eta(\xi\eta' - \eta)/(\xi^2 + \eta^2)}{K^2 - \eta' + \xi(\xi\eta' - \eta)/(\xi^2 + \eta^2)}. \tag{79}$$

We claim that the direction of the oblique derivative points in between the tangential directions of the shock curve and the sonic circle; I.e.,

$$\xi/(-\eta) > H > \eta', \tag{80}$$

assuming $\eta_B < 0$. To prove the claim, we consider the two separate cases $\eta_B \geq 0$ and $\eta_B < 0$. First let $\eta_B < 0$. Then the denominator of $H$ is positive since

$$-\eta' + \xi(\xi\eta' - \eta)/(\xi^2 + \eta^2) = -\eta(\xi + \eta\eta')/p_2 > 0.$$

(See (67) for $\xi + \eta\eta' > 0$). Using cross multiplication, the inequality $H > \eta'$ becomes $(\xi + \eta\eta')^2 > 0$, while the inequality $\xi/(-\eta) > H$ becomes $K^2(\xi + \eta\eta') > 0$. So they are both valid. For the case $\eta_B \geq 0$, we note that the numerator of $H$ is always positive: $1 + \eta(\xi\eta' - \eta)/p_2 = \xi(\xi + \eta\eta')/p_2 > 0$. When the denominator is positive or zero, the proof for inequality $H > \eta'$ is the same as before. When the denominator is negative, there holds $H < 0 < \eta'$. When the denominator is positive or zero, the inequality $\xi/(-\eta) < 0 < H$ is trivial. When the denominator is negative, then both denominators of $\xi/(-\eta)$ and $H$ are negative, thus the proof for the case $\eta < 0$ is still applicable. Thus, in all cases, we have proved our claim: The vector field $\mathbf{l}$ points from the exterior to the exterior of the domain, see Figure 3. By Lieberman's [47], the solution $p^{(1)}$ enjoys $C^{1,\alpha_2}$ regularity at the point for some $\alpha_2 > 0$. This $\alpha_2$ depends on $p_1, p_m$, and the upper bound $\beta_0$ of $\beta$, as well as $\epsilon_e$.

Regarding point $C$, Lieberman's yields only Hölder continuity $C^{\alpha_3}$ for any $\alpha_3 < 1$. But the constant value of $p$ on the sonic arc has zero tangential derivative while the normal of the wall is tangent to the arc, which allows for higher $C^{1,\alpha_4}$ regularity for some $\alpha_4 > 0$ by Azzam [2].

Regarding point $G$, we realize that Theorem 1.1 of Guan and Sawyer does not apply directly since its requirement include that the boundary $\Sigma$ be in $C^{3+\lambda}$ $(\lambda > 0)$. Fortunately, its Remark 1.3 indicates that the boundary regularity can be reduced to $C^{2+\gamma}$ when the structure of the tangential manifold is simple, which is our case.

We use the Perron method for existence, as framed in Lieberman [46]. The basic local existence at point $G$ is the only new case we need to provide a proof for. It is formulated as a Dirichlet and tangential oblique derivative boundary value problem, which is slightly different from a pure tangential oblique derivative problem. More precisely, let $B_2$ be a neighborhood of G with smooth boundary. Let $h$ be any continuous function on $\partial B_2 \cap \Omega$. Consider the problem

$$p = h \qquad \text{on } \partial B_2 \cap \Omega \tag{81}$$

for equation (71) restricted to the domain $B_2 \cap \Omega$ with the oblique derivative boundary conditions (68) restricted to $\Sigma \cap B_2$. The local existence for some such $B_2$ implies global existence of solution in $\Omega$ by Lieberman [46]. We know that the elliptic estimates are local, so Guan and Sawyer's estimate applies in this case. More details on the local existence are provided in the Appendix. □

We comment that there is a lot of interesting work on the topic of tangential oblique derivative boundary value problems, see [5][6][7][25][32][35][42][51][53][56][57][58][59][64] [68].

# 6 Quasilinear theory with fixed boundary

Now we take away the linearizing function $Q$. This step is simple by the estimates of the previous theorem. So we consider

$$Q \in C^{1,\alpha}(\overline{\Omega}), p_m < Q < p_2 \quad \text{in } \Omega. \tag{82}$$

Let $\alpha_1$ be the least of the three exponents given at the points $A$, $O$, and $C$. Recall that $\alpha_1$ is determined by the local geometry and is independent of $\alpha \in [0, 1)$, i.e., we can obtain $\alpha_1$ by setting $\alpha = 0$. We choose $\alpha$ so that $0 < \alpha < \alpha_1$. So our previous section yields a solution $p^{(1)} \in C^{1,\alpha_1}(\overline{\Omega})$ and satisfies the same lower and upper bounds in $\Omega$.

For each fixed positive $\varepsilon_e$, the estimates make the mapping $Q \in C^{1,\alpha} \to p^{(1)} \in C^{1,\alpha_1}$ compact.

We quote from Gilbarg and Trudinger [28] the following fixed point theorem: Corollary 11.2, p. 280:

*Let $\mathcal{C}$ be a closed convex set in a Banach space $\mathcal{B}$ and let $T$ be a continuous mapping of $\mathcal{C}$ into itself such that the image $T\mathcal{C}$ is precompact. Then $T$ has a fixed point.*

We let $\mathcal{B}$ be the Banach space $C^{1,\alpha}(\overline{\Omega})$. We let $\mathcal{C}$ be all $Q \in \mathcal{B}$ such that $p_m \leq Q \leq p_2$ in $\overline{\Omega}$, $Q = p_m$ at the point $G$, $Q = p_2$ on the sonic arcs $AC$ and $BD$, and the $\mathcal{B}$-distance from $p_2$ be less or equal to 1. This $\mathcal{C}$ is a bounded, closed, and convex set.

Let $T$ be the mapping from $Q \in \mathcal{C}$ to $\mathcal{B}$ constructed above. Each $TQ$ satisfies $p_m < TQ < p_2$ in $\Omega$ and the corresponding boundary values at $G$, $AC$ and $BD$. The image $T\mathcal{C}$ is in $C^{1,\alpha_1}(\overline{\Omega})$ for $\alpha_1 > \alpha$ and the $C^{1,\alpha_1}(\overline{\Omega})$-norm is bounded by a constant times the factor $p_2 - p_m$. Letting $p_2 - p_m$ be sufficiently small, we obtain $T\mathcal{C} \subset\subset \mathcal{C}$. The continuity of $T$ follows from the precompactness and the linearity. So there exists a fixed point $Q = p^{(1)} \in C^{1,\alpha_1}(\overline{\Omega})$.

The maximum principle still holds for the quasilinear problem:

$$p_m < p^{(1)} < p_2 \qquad \text{in } \Omega. \tag{83}$$

We still have that $p^{(1)}$ is $C^{2,\gamma'}$ in $\Omega \cup \{\Sigma \backslash \{A, B\}\}$. The ellipticity holds

$$p^{(1)} - \xi^2 - \eta^2 > 0 \quad \text{in } \Omega. \tag{84}$$

To prove this, we write an equation for the variable $w =: p^{(1)} - \xi^2 - \eta^2$ and show that there is no interior minimum for $w$. Thus the sign restriction is redundant and our solution satisfies

$$\varepsilon_e \Delta p + (p - \xi^2) p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2) p_{\eta\eta} + \frac{(\xi p_\xi + \eta p_\eta)^2}{p} - 2(\xi p_\xi + \eta p_\eta) = 0. \tag{85}$$

The solution $p^{(1)}$ is monotone on either arms of the edge $\Sigma$, $\xi < 0$, or $\xi > 0$. The proof is similar to that for Proposition 2.4 of [13]. The idea is to use contradiction method. Suppose $p^{(1)}$ is not monotone on the right arm. Then a typical case would be that there will be a local max at $\xi_N$ and a local min at a $\xi_n > \xi_N$ in the interior of the arm, with the local max greater than the local min. Since these values are not permitted to be global extremes from the closure of the domain, we can find a curve leading from the max into the domain along which the value of $p^{(1)}$ will be increasing; Similarly we find a curve starting from the min and leading into the domain along which the value of $p^{(1)}$ will be decreasing. These two curves will have no where to end due to the geometry and data we have for our problem. A more elaborate proof is available in [13].

The solution satisfies

$$|\nabla p^{(1)}| \leq C_1(p_2 - p_m), \qquad \text{in } \overline{\Omega} \tag{86}$$

for some constant $C_1 = C_1(\varepsilon_e)$.

# 7 Free boundary

We pay most of our attention to the free boundary problem. We would consider a mapping from $\mathbb{K}$ to itself, establish its compactness, and thus obtain a fixed point. However, there is an alternative.

Let us first consider an idealized situation. We take two values of pressure $p_2 > p_1 > 0$ and a small number $\delta \in (0, \delta_0]$ where

$$\delta_0 := 1 - \frac{\sqrt{\bar{p}_{12}}}{\sqrt{p_2}}. \tag{87}$$

We introduce the point $B$ with coordinates

$$(\xi, \eta)|_B = (\sqrt{(p_2 - p_1)/2}, -\sqrt{(p_2 + p_1)/2}) =: (\xi_{12},\ \eta_{12}), \tag{88}$$

and $A$ is its symmetric counterpart. Let $B_\delta$ be the intersection point of the circle

$$\xi^2 + \eta^2 = p_2 \tag{89}$$

with a straight line of slope $\delta$ that is tangent to the circle

$$\xi^2 + \eta^2 = \bar{p}_{12}. \tag{90}$$

Denote the coordinates of $B_\delta$ by $(\xi_{12\delta}, \eta_{12\delta})$ that is the closest to the point $B$. Let $A_\delta$ be its symmetric part. We consider the pressure equation in the domain $\Omega$ bounded by the upper boundary (89) and the free boundary connecting $A_\delta$ and $B_\delta$. The $\delta$ will be the slope of the free boundary at the point $B_\delta$. See Figure 4, where $a$ is inside or on the circle (89) for $\delta \le \delta_0$.

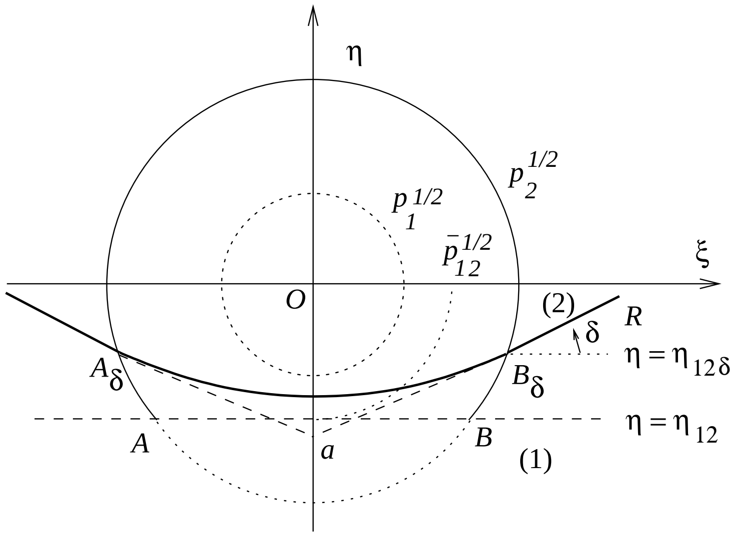


Figure 4. Domain for the free boundary.

On the upper boundary we impose $p = p_2$; Below the shock wave we take $p = p_1$. The free boundary is required to start at point $B_\delta$, have slope $\delta$ at point $B_\delta$, be even and convex in $\xi$. A typical approximate free boundary is this

$$\eta = \eta^X(\xi) := \eta_{12\delta} + \frac{\delta}{2\xi_{12\delta}}(\xi^2 - \xi_{12\delta}^2), \quad \xi \in [-\xi_{12\delta},\ \xi_{12\delta}]. \tag{91}$$

We now introduce a set

$$\begin{aligned} \mathbb{B} = \ & \{\eta \in C^{2,\gamma}[-\xi_{12\delta}, \xi_{12\delta}] \mid \\ & \eta(\xi_{12\delta}) = \eta_{12\delta}, \eta'(\xi_{12\delta}) = \delta, 0 \le \eta'' \le \tfrac{1}{4}p_2^{-1/2}, [\eta'']_\gamma \le 1, \eta(\xi) = \eta(-\xi)\} \end{aligned} \tag{92}$$

where $\gamma \in (0,1)$, $\delta \in (0, \delta_0]$, and $[\eta'']_\gamma$ denotes the $\gamma$-Hölder modulus of continuity. The set is closed and convex in $C^{2,\gamma}[-\xi_{12\delta}, \xi_{12\delta}]$.

For any $\eta \in \mathbb{B}$, we find $\bar{p}$ on it by (60). We estimate the minimum of $p$: Called $p_m^\delta$. We differentiate (60) along the boundary to find

$$\frac{d\bar{p}}{d_\Sigma \xi} = \frac{2\sigma'}{(1+\sigma^2)^2}(\xi\sigma - \eta)(\xi + \sigma\eta). \tag{93}$$

We show that this derivative is nonnegative for $\xi \in [0, \xi_{12\delta}]$. First $\sigma' \ge 0$ is given. Second $\xi\sigma - \eta \ge 0$ trivially, assuming $\eta < 0$. Third we estimate that $\sigma \le p_2^{-1/2}\xi$ and

$$|\eta| \le \delta_0 \xi_{12\delta} + |\eta_{12\delta}| \le \delta_0 \sqrt{p_2} + |\eta_{12}| \le p_2^{1/2}$$

by the choice of $\delta_0$, thus $\xi + \sigma\eta \ge 0$. So the derivative is nonnegative. And the derivative is bounded from above by $C\sigma'$ from which we integrate (93) in $\xi$ to find that $p_m^\delta$ deviates from $p_2$ at most by $C\delta$. In terms of the previous sections, we have $p_m = p_m^\delta = p_2 - O(\delta)$. This is a crucial estimate as it is the starting point for a mapping to be defined to take $\mathbb{B}$ to itself. Thus, there exists a $\delta^* \in (0, \delta_0]$, such that $p_m - p_1 > C\delta_0$ for all $\delta \in (0, \delta^*)$.

We show that $p - \xi^2 - \eta^2 > 0$ between $A_\delta$ and $B_\delta$. At $B_\delta$ it is zero. We show that its derivative is nonpositive for $\xi \in (0, \xi_{12\delta})$. The derivative along the shock is

$$\frac{d}{d_\Sigma \xi}(p - \xi^2 - \eta^2) = \frac{4\eta''}{(1+(\eta')^2)^2}(\xi\eta' - \eta)(\xi + \eta\eta') - 2(\xi + \eta\eta'). \tag{94}$$

Its nonpositivity is equivalent to

$$2\eta''(\xi\eta' - \eta) < (1 + (\eta')^2)^2.$$

We use the bound $\eta'' \le \frac{1}{4}p_2^{-1/2}, \delta \le \delta_0$ to derive

$$2\eta''(\xi\eta' - \eta) < 2 \cdot \frac{1}{4}\frac{1}{\sqrt{p_2}}(\sqrt{p_2}\delta + \sqrt{p_2}) < \frac{1}{2}(\delta + 1) < \frac{1}{2}(\delta_0 + 1) < 1.$$

Thus the nonpositivity holds and $p - \xi^2 - \eta^2 > 0$ between $A_\delta$ and $B_\delta$.

And the choice of the upper bound of the second-order derivative of $\eta$ is such that $\xi\eta + \eta'(\bar{p} - \xi^2) < 0$ for $\xi > 0$, see (65)(66), thus the obliqueness holds.

For completeness we mention that $\xi^2 + \eta^2 - \bar{p} > 0$ for $\xi \in (0, \xi_{12\delta})$ because the second-order derivative of $\eta$ is less than $\beta_0$ defined in (57), see (65)(63)(62).

We impose the condition that the solution $p$ takes on the old value at the point $G$: $p_m = p_m^\delta$, which is part of the condition for the emergent type of tangential oblique derivative boundary condition. By the previous sections, there exists a $\delta^{**} > 0$ such that a solution $p$ exists for all $\delta \in (0, \delta^{**})$. The solution is smooth every where in the closure of the domain including $A$ and $B$ since the higher order compatibility condition is satisfied. The oscillation $\max_\Omega |p - p_2| \le p_2 - p_m^\delta$ is bounded by $K_1\delta$ where $K_1$ is determined by the upper bound of the $C^{2,\gamma}$ norm of $\eta \in \mathbb{B}$.

Now we define the mapping on $\mathbb{B}$: The mapping $J$. Given a boundary $\Sigma^{(0)}$ from $\mathbb{B}$, we let $p^{(1)}$ denote the unique solution. Restricted to the old $\Sigma^{(0)}$, we can regard $p^{(1)}$ as a function of the single variable $\xi$. We use $p^{(1)}(\xi)$ to define $\Sigma^{(1)}$ in the standard formula:

$$\frac{d\eta^{(1)}}{d\xi} = \frac{\xi\eta^{(1)} \pm \sqrt{\bar{p}^{(1)}(\xi^2 + (\eta^{(1)})^2 - \bar{p}^{(1)})}}{\xi^2 - \bar{p}^{(1)}}; \qquad \eta^{(1)}(\xi_{12\delta}) = \eta_{12\delta}, \tag{95}$$

up to a point $\xi_9 \in (0, \xi_{12\delta})$ where either

$$\eta^{(1)} = -\sqrt{\bar{p}^{(1)} - 4K_3\delta\xi^2}, \tag{96}$$

or

$$\eta^{(1)} = -\sqrt{\bar{p}^{(1)}(0)}. \tag{97}$$

The constant $K_3$ is explained below.

First, the tangential oblique derivative boundary condition (68) implies

$$|p_\xi^{(1)} + \eta^{(0)'} p_\eta^{(1)}|\ |\frac{[p]}{4\bar{p}}(\xi^2 + \eta^2) - (\xi^2 + \eta^2 - \bar{p})| \le |\xi\eta + \eta'(\bar{p} - \xi^2)|\, K_2\ |\nabla p^{(1)}|. \tag{98}$$

Recalling (63), we can move the term

$$\xi^2 + \eta^2 - \bar{p} = O(|\xi\eta + \eta'(\bar{p} - \xi^2)|)$$

to the other side to obtain

$$|p_\xi^{(1)} + \eta^{(0)'} p_\eta^{(1)}|\ \frac{[p]}{4\bar{p}}(\xi^2 + \eta^2) \le |\xi\eta + \eta'(\bar{p} - \xi^2)|\, K_2'\ |\nabla p^{(1)}|. \tag{99}$$

We have

$$|\xi\eta + \eta'(\bar{p} - \xi^2)| \le |\xi|(\sqrt{p_2} + \frac{1}{4\sqrt{p_2}}(\bar{p} + p_2)) \le K_2|\xi|.$$

Thus

$$|p^{(1)}_\xi + \eta^{(0)'} p^{(1)}_\eta| \ \le K'_2|\xi|\ |\nabla p^{(1)}| \tag{100}$$

for all $\xi \in [0, \xi_{12\delta}]$ and all $\delta \in (0, \delta^{**})$. Now, our solution $p^{(1)}$ is a smooth solution satisfying

$$|\nabla p^{(1)}| \le K_2(p_2 - p^\delta_m) \tag{101}$$

where $K_2$ is independent of $\delta$, but it may depend on the parameter $\epsilon_e$, the size of the domain $\Omega$, and the $C^2$ norm of $\Sigma^{(0)}$ which is bounded. So,

$$|p^{(1)}_\xi + \eta^{(0)'} p^{(1)}_\eta| \ \le K_3\xi\delta \tag{102}$$

for all $\xi \in [0, \xi_{12\delta}]$ and all $\delta \in (0, \delta^{**})$.

Thus in the definition of the map, if the curve $\eta^{(1)}$ hits the upper boundary (96), then it is still a distance away from the singularity boundary $\eta^{(1)} = -\sqrt{\bar{p}^{(1)} - \xi^2}$, assuming $4K_3\delta < 1$. So we can and will continue the curve from the upper boundary (96) to $\xi = 0$, $C^{3,\gamma}$−smoothly, keeping convexity, $(\eta^{(1)})'(0) = 0$, and with second-order derivative bounded by $\beta_0$. This can be done, we omit the details. See the upper thin curve in Figure 5.

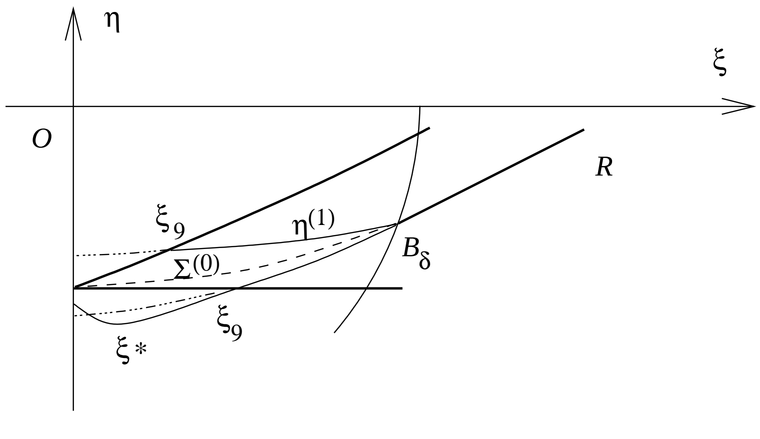


Figure 5. Definition of the mapping J.

Second, if the curve goes below the line $\eta = \bar{p}^{(1)}(0)$ at a point $\xi_9$, see the lower thin curve in Figure 5, then from the definition formula expressed as

$$\frac{d\eta^{(1)}}{d\xi} = \frac{\bar{p}^{(1)} - (\eta^{(1)})^2}{\sqrt{\bar{p}^{(1)}(\xi^2 + (\eta^{(1)})^2 - \bar{p}^{(1)})} + \xi(-\eta^{(1)})} > 0, \quad \text{for } \xi > \xi_9, \tag{103}$$

we find that there will be a point $\xi^* \in (0, \xi_9)$ such that the curve $\eta^{(1)}$ has zero derivative at that point. We find that the second-order derivative formula

$$\frac{d^2\eta^{(1)}}{d\xi^2} = \frac{1}{\bar{p}^{(1)} - \xi^2}\left(\frac{2\bar{p}^{(1)} - (\xi^2 + (\eta^{(1)})^2)}{\sqrt{\bar{p}^{(1)}(\xi^2 + (\eta^{(1)})^2 - \bar{p}^{(1)})}} - \frac{d\eta^{(1)}}{d\xi}\right)\frac{d\bar{p}^{(1)}}{d_{\Sigma^{(0)}}\xi} \tag{104}$$

and the estimate (102) yield a uniform estimate

$$0 \leq \frac{d^2\eta^{(1)}}{d\xi^2} \leq K_5\delta, \qquad \text{for all} \xi \geq \xi^* \tag{105}$$

for all $\xi^* \geq 0$. We can then start at the point $\xi_9$ (not $\xi^*$) and modify the curve so that it remains convex with convexity less than or equal to $\beta_0$, gets to $\xi = 0$ with ending slope zero.

From the construction it is clear that the mapping is defined on $\mathbb{B}$ and takes $\mathbb{B}$ to itself when $\delta$ is small enough. In addition, the solution $p^{(1)}$ is in $C^{2,\gamma}$, thus $\eta^{(1)}$ is $C^{3,\gamma}$. Hence the mapping $J$ is pre-compact. We obviously have continuity of $J$, assuming that our modification at the upper and lower boundaries are smooth. So $J$ is a continuous mapping from a closed and convex set $\mathbb{B}$ of the Banach space $C^{2,\gamma}$ to itself whose image $J\mathbb{B}$ is pre-compact, thus, by Schauder Fixed Point Theorem ([28], Corollary 11.2, p. 280) quoted already earlier, we have a fixed point for $J$, which is the existence of the free boundary. $\square$

Since the corners $C, D, O$ are a fixed distance away from the free boundary, we can similarly obtain the existence of a free boundary for the wedge problem. The idealization mentioned at the beginning of this section is an equivalent way to the mapping of $\mathbb{K}$ to itself. The mild dependence of $p_2$ on $\tan\theta_w$ is trivially allowed.

# 8 De-regularization $-\epsilon_e$ and Lipschitz continuity

We obtain uniform estimates with respect to $\epsilon_e > 0$. We use barriers to find the bound of the gradient of $p$ on the sonic boundary. Then we use maximum principle to establish global bound of the gradient of $p$ on the entire closure of the subsonic domain. Thus the solution $p$ is Lipschitz at the sonic arcs.

First, the maximum principle implies $p \leq p_2$. Second, the uniform ellipticity implies $p \geq \xi^2 + \eta^2$. Combining the two, we obtain that $0 \leq \partial_{\mathbf{n}} p \leq 2\sqrt{p_2}$ on the sonic arcs where $\mathbf{n}$ is the unit exterior normal to the arcs, independent of $\epsilon_e \in (0, 1]$.

We let $(r, \theta)$ denote the polar coordinates of the $(\xi, \eta)$ plane. In polar coordinates, the $p$ equation can be written as

$$(p - r^2)p_{rr} + \frac{p}{r^2}p_{\theta\theta} + \frac{p}{r}p_r + \frac{1}{p}(rp_r)^2 - 2rp_r = 0. \tag{106}$$

We omit the term $\epsilon_e \Delta p$ since it does not present any help or trouble. Multiplying with $\frac{r^2}{p}$, we obtain

$$r^2(1 - \frac{r^2}{p})p_{rr} + p_{\theta\theta} + rp_r + \frac{r^2}{p^2}(rp_r)^2 - \frac{2r^3}{p}p_r = 0. \tag{107}$$

Taking the derivative $\partial_r$, letting $Z := p_r$, we obtain

$$r^2(1-\frac{r^2}{p})Z_{rr} + Z_{\theta\theta} + rZ_r + \frac{2r^4}{p^2}p_r Z_r - \frac{2r^3}{p}Z_r + \partial_r(r^2 - \frac{r^4}{p})Z_r - ZF(Z,r,p) = 0, \quad (108)$$

where

$$F(Z,r,p) := \frac{6r^2}{p} - 1 - \frac{6r^3}{p^2}Z + \frac{2r^4}{p^3}Z^2. \quad (109)$$

Near $p = r^2 = p_2$, the function $F$ is bounded from below by $1/2$ for all $Z \in \mathbb{R}$. We select a thin shell domain immediately inside the sonic arc. We choose $\delta$ small, so all the derivatives are small depending on $\delta$ and $\epsilon_e$. We then apply the maximum/minimum principle on $Z$. Thus $|Z| < 2\sqrt{p_2}$ in a small neighborhood of the sonic arc.

In the interior of the domain $\Omega$, the limit $\epsilon_e \to 0+$ does not cause trouble since there holds uniform ellipticity.

# 9 Recovering the velocity

Now that we have obtained $p$ in the entire subsonic domain, we can integrate the first two linear equations in (23) for $u$ and $v$ to obtain $(u,v)$. More precisely, we integrate

$$ru_r = \xi u_\xi + \eta u_\eta = p_\xi, \qquad u = u_2 \text{ at } r = \sqrt{p_2} \quad (110)$$

in the subsonic sector spanned by radial rays of the sonic arc to find

$$u(r\cos\alpha, r\sin\alpha) = \int_{\sqrt{p_2}}^{r} p_\xi(\tau\cos\alpha, \tau\sin\alpha)/\tau\, d\tau + u_2 \quad (111)$$

for $0 < r < \sqrt{p_2}$ and $\alpha$ stretching between $\theta_w$ and $\theta_A$ (the polar angle of the point $A$, see Figure 2). For $\alpha$ between $\theta_A$ and $\pi$, we use in (110) as data the velocity obtained by (29)(30) on the free shock to obtain $u$ in the subsonic sector spanned by radial rays of the free boundary $\Sigma$. We evaluate $v$ similarly. Thus $(u,v)$ are defined, and the first two equations in (23) are satisfied in the subsonic domain.

We verify the third equation in (23). As previously discussed, the second-order equation written in the form $r\partial_r L + L = 0$ implies the third equation if the third equation holds at a single point on each ray. Thus it suffices to show that $L = 0$ (defined in (56)) on the inside edge of the sonic arc and the free boundary $\Sigma$. To achieve that for the sonic arc we first note that our $p$ is Lipschitz up to the boundary from the previous section. Our $(u,v)$ are also Lipschitz at the boundary. Note further that $(u,v)$ are constant on the arc, thus their derivatives along the sonic arc are zero:

$$-\eta u_\xi + \xi u_\eta = 0, \qquad -\eta v_\xi + \xi v_\eta = 0. \quad (112)$$

Using these relations in the equation

$$\xi u_\xi + \eta u_\eta = p_\xi, \qquad \xi v_\xi + \eta v_\eta = p_\eta \tag{113}$$

we obtain

$$(\xi^2 + \eta^2)(u_\xi + v_\eta) = \xi p_\xi + \eta p_\eta. \tag{114}$$

Recall that $\xi^2 + \eta^2 = p$ on the sonic arc, we obtain

$$L := (\xi p_\xi + \eta p_\eta)/p - (u_\xi + v_\eta) = 0 \tag{115}$$

on the sonic arc! Thus the third equation of (23) holds in the entire sector spanned by the sonic arc.

The oblique derivative boundary condition ensures, as previously mentioned, that $L = 0$ on $\Sigma$. Recall that we derived the oblique derivative boundary condition by assuming that $L = 0$ on $\Sigma$. Now we want to reverse the process; I.e., we want to derive that $L = 0$ on $\Sigma$ from $p$ and the oblique derivative boundary condition. From the three facts that (i) $p$ is known and the free boundary satisfies equation (28), (ii) the inside edge values $(u, v)$ are obtained from (29)(30), (iii) the first two equations of (23) hold on the edge, we can differentiate (29)(30) along (28) to solve five variables $(u_\xi, u_\eta, v_\xi, v_\eta, p_\xi)$ from the five equations in terms of the sixth $p_\eta$, where we used the condition $\xi^2 + \eta^2 - \bar{p} \neq 0$ for $\eta > 0$. It can be verified that the five variables depend on the sixth linearly, and the relation $L = 0$ holds identically on the inside edge of $\Sigma$, where we use the fact that $(p_1, u_1, v_1)$ is a constant state. Then the second-order equation $r\partial_r L + L = 0$ implies $L = 0$ in the interior of the domain. Thus the solution $p$ of the second-order equation is a solution to the system of first order equations in the sectoral domain spanned by $\Sigma$.

Furthermore, the velocity $(u, v)$ are smooth in the interior of the subsonic domain. On the wall and the ground we show that the velocity is parallel to the solid surface. Our $p$ satisfies $\partial_{\mathbf{n}} p = 0$. Our $(u, v)$ from (113) satisfies the equation

$$\xi(u_{\mathbf{n}})_\xi + \eta(u_{\mathbf{n}})_\eta = \partial_{\mathbf{n}} p$$

where $u_{\mathbf{n}} := (u, v) \cdot \mathbf{n}$. Thus, $u_{\mathbf{n}} =$ constant . The constant is zero in the state (2) and at the tip of the bow shock. So $u_{\mathbf{n}} = 0$. Thus, the velocity is parallel to the solid surface. The velocity is bounded at the origin $O$ since $\nabla p$ is Hölder continuous and vanishes at $O$.

# 10 Fine properties of the velocity

One might ask if there is a stagnation point of the flow in the subsonic domain, or at the corner in particular. One might also ask if the velocity remains uniformly small to fulfill the condition posteriorly, under which the model is derived. Furthermore, one might ask what exactly determines the position of the free boundary shock, if the pressure equation decouples from the velocity field.

To answer these questions, we find it is necessary to study qualitatively the velocity field of the solution in the subsonic domain.

**Proposition 10.1** (State 2) *As* $\cos\theta_w \to 0+$, *we have*

$$p_2 = p_2^* + \frac{p_1^2(p_1 - p_0)^2}{2\bar{p}_{10}^3 \tan^2\theta_w} + O(\tan^{-4}\theta_w), \tag{116}$$

*where* $p_2^*$ *is defined in (35).*

*Proof.* This is an easy consequence from the explicit formula in Proposition 3.1. □

**Proposition 10.2** (State 2 on $\Sigma$) *The velocity on the free boundary* $\Sigma$ *is nonnegative*

$$u \geq 0, \qquad v \geq 0 \qquad\qquad \text{on } \Sigma \tag{117}$$

*for large wedge angles.*

*Proof.* In fact, we find from (30) that

$$\frac{[v]}{[p]} = \frac{\bar{p} - \xi^2}{\eta\bar{p} - \xi\sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}} > 0$$

on the shock, thus $v > 0$ on the inside of the shock. We present the calculation for $u$, at the point $G$ only for brevity. At the point $G$, we use the R-H relation (29) to find

$$u_{2G} = u_1 - \frac{p_{2G} - p_1}{\sqrt{(p_{2G} + p_1)/2}}. \tag{118}$$

In (118), we see quickly that $u_{2G}$ is a decreasing function of $p_{2G}$, holding $p_1$ and $u_1$ fixed and unrelated to $p_{2G}$. In addition, if we use $p_{2G} = p_2^*$ from (35), we find $u_{2G} = 0$. So, to show $u_{2G} > 0$, it suffices to show that $p_{2G} < p_2^*$ for all large wedge angles.

From Proposition 10.1, we have $p_2 = p_2^* + O(\cos^2\theta_w)$. On the other hand, we show that $p_{2G}$ will be smaller than $p_2$ by an amount $C_2\delta$ at the least, which is $C_2 \cot\theta_w$, $C_2 \neq 0$. This follows from the oblique derivative boundary condition written in the form (in the swapped coordinate where the free boundary shock is basically horizontal)

$$\frac{\eta''(\bar{p} - \xi^2)\{\frac{[p]}{\bar{p}}(\xi^2 + \eta^2) - 4(\xi^2 + \eta^2 - \bar{p})\}}{2\bar{p} - \xi^2 - \eta^2 + 2\eta'\{\xi\eta + \eta'(\bar{p} - \xi^2)\}} = -\eta' p_\xi^{(1)} + p_\eta^{(1)} + (\xi p_\xi^{(1)} + \eta p_\eta^{(1)})\frac{\xi\eta' - \eta}{p}. \tag{119}$$

We have $0 \leq p_\eta \leq C_2\delta$ since $p_2 - p_{2G} \leq C_2\delta$. So $\eta''(\xi) \leq C_2\delta$. From formula (93) we have

$$\frac{d\bar{p}}{d_\Sigma \xi} \geq C_2 \xi \sigma' \tag{120}$$

which implies

$$\bar{p}(B) - \bar{p}(G) \geq C_2 \int_0^{\xi_B} \xi\sigma'\, d\xi = C_2(\xi_B\delta - \int_0^{\xi_B} \sigma\, d\xi). \tag{121}$$

But we have

$$\int_0^{\xi_B} \sigma\, d\xi \leq \int_0^{1/C_2} \xi\sigma'(\xi)\, d\xi + \int_{1/C_2}^{\xi_B} \delta\, d\xi \leq \frac{1}{2}(\frac{1}{C_2})^2(C_2\delta) + (\xi_B - \frac{1}{C_2})\delta = \xi_B\delta - \frac{1}{2C_2}\delta. \tag{122}$$

Thus $p_{2G} = p_2 - C_2 \cos\theta_w < p_2^*$. Thus $u_{2G} > 0$. This completes the proof. □

As fluid particles flow toward the wedge wall, the build-up of fluid raises the pressure in the subsonic region. In the extreme case of a vertical wall, there is no escape for the fluid, thus the velocity $(u, v) = (0, 0)$ and the pressure is $p = p_2^*$ in the subsonic region. For a wall with $\theta_w \in (0, \pi/2)$, fluid particles can slide along the wedge surface: The fluid particles get deflected at the corner. Although the pressure gradient is required to be zero into the walls, no extra condition at the corner is placed to send a signal to the decoupled second-order pressure gradient equation to take care of the build-up of fluid at the corner. Thus, the wall is the primary reason for the reflected bow shock.

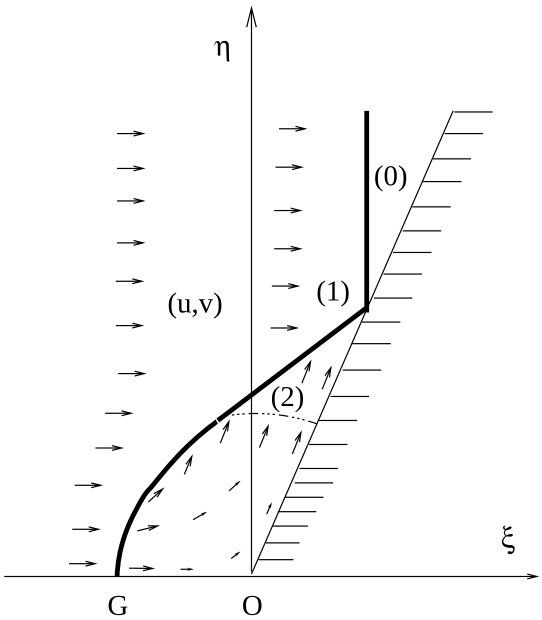


Figure 6. Flow pattern of the regular reflection

To explain the changes of velocity in the subsonic domain in response to the pressure gradient, we find that our $(u, v)$ from (113) satisfies the equation

$$\xi(u_\alpha)_\xi + \eta(u_\alpha)_\eta = \partial_\alpha p$$

where $u_\alpha := u\cos\alpha + v\sin\alpha$ and $\partial_\alpha p = p_\xi \cos\alpha + p_\eta \sin\alpha$. Thus, for example $u_\alpha$ for $\alpha = \pi/4$ is decreasing from the shock wave toward the origin, provided that the pressure is increasing in the direction $(1, 1)$. See Figure 6. Overall, the velocity remains small. And existence of a stagnation point seems rather exceptional under these general circumstances, so we go deeper to the vorticity level to find out.

## 10.1 Vorticity

We study the vorticity in the subsonic region to reveal more refined properties of the velocity. As is usual, the vorticity is defined by $\omega = \partial_x v - \partial_y u$. We can show that the vorticity is zero at time $t = 0$ for our configuration. Furthermore, it is easy to check that

$$\partial_t \omega = 0 \qquad (t > 0)$$

for smooth solutions of the pressure gradient system. So we expect that our solution has zero vorticity.

We show that our solution indeed has zero vorticity. Although it might seem unnecessary to do the proof, we feel that our construction of the solution has been a guess work, and piecewise, so it is a good idea to verify the end result. In addition, curved oblique shock waves in a gas are known to generate vorticity, see e.g. the appendix of [33], or p.431 of [52]. Thus, first, from (112)(113) we obtain

$$v_\xi = \frac{\xi}{p} p_\eta, \quad u_\eta = \frac{\eta}{p} p_\xi \tag{123}$$

on the sonic arc, and thus

$$v_\xi - u_\eta = \frac{\xi p_\eta - \eta p_\xi}{p} = 0 \tag{124}$$

on the sonic arc, since $p = p_2$ is a constant there. Thus, there is no vorticity along the sonic arc in the subsonic region.

In the self-similar plane we call for convenience $\omega = v_\xi - u_\eta$ although the physical quantity $t(\partial_x v - \partial_y u)$ scales to be $v_\xi - u_\eta$. We use the equations (23) to easily find that $\omega$ satisfies the equation

$$-\xi \omega_\xi - \eta \omega_\eta + \omega = 0 \tag{125}$$

in any region where the solution is smooth. Using the equation (125) and the boundary data (124), we obtain that vorticity is zero in the subsonic domain spanned by the sonic arc.

We show then that the vorticity is zero on the free boundary shock. From the original form of Rankine-Hugoniot relation we can obtain that

$$[u] = -\frac{\sigma}{\eta - \xi\sigma}[p], \quad [v] = \frac{1}{\eta - \xi\sigma}[p]. \tag{126}$$

We differentiate the $[u]$ equation along the shock to find

$$\frac{d[u]}{d_\Sigma \xi} = \frac{-\eta\sigma'}{(\eta-\xi\sigma)^2}[p] - \frac{\sigma}{\eta-\xi\sigma}\frac{dp}{d_\Sigma \xi} \tag{127}$$

Using the fact that $u_1$ is a constant and the first equation in (23) we obtain

$$(\xi\sigma-\eta)u_\eta = -p_\xi - \frac{\xi\eta\sigma'}{(\eta-\xi\sigma)^2}[p] - \frac{\xi\sigma}{\eta-\xi\sigma}\frac{dp}{d_\Sigma \xi}. \tag{128}$$

Similarly we obtain

$$(\eta-\xi\sigma)v_\xi = \frac{\xi\eta\sigma'}{(\eta-\xi\sigma)^2}[p] + \frac{\eta}{\eta-\xi\sigma}\frac{dp}{d_\Sigma \xi} - \sigma p_\eta. \tag{129}$$

Combining the two terms we obtain $v_\xi - u_\eta = 0$ on the shock. Thus vorticity is zero in the subsonic domain spanned by the free boundary. So the flow is irrotational in the entire subsonic domain.

We can now use $v_\xi = u_\eta$ to manipulate the first two equations of (23) into

$$u = (\xi u + \eta v - p)_\xi, \qquad v = (\xi u + \eta v - p)_\eta. \tag{130}$$

We call

$$\psi = \xi u + \eta v - p \tag{131}$$

the potential. Thus we have a nice formula

$$p + \psi = \xi u + \eta v. \tag{132}$$

So we have

$$\Delta\psi = u_\xi + v_\eta = \frac{\xi p_\xi + \eta p_\eta}{p} \tag{133}$$

by the third equation of (23). Replacing the derivatives $p_\xi$ and $p_\eta$ from the first two equations of (23), we obtain

$$p\Delta\psi - (\xi^2\psi_{\xi\xi} + 2\xi\eta\psi_{\xi\eta} + \eta^2\psi_{\eta\eta}) = 0. \tag{134}$$

We can replace $p$ to obtain a decoupled equation

$$(\xi\psi_\xi + \eta\psi_\eta - \psi)\Delta\psi - (\xi^2\psi_{\xi\xi} + 2\xi\eta\psi_{\xi\eta} + \eta^2\psi_{\eta\eta}) = 0. \tag{135}$$

The boundary data for $\psi$ are the Dirichlet on the free boundary and the sonic arc, and Neumann $\nabla\psi\cdot\mathbf{n} = 0$ on the ground and the surface of the wall. Since we have obtained $p$ already, we choose to use equation (134). We are mainly interested in regularity of the solution. First we find that there is a stagnation point at the origin. Second, the solution $\psi$ is in $C^\alpha(\bar{\Omega})$ where $\alpha = \pi/(\pi-\theta_w) \in (1,2)$, and the solution $\psi$ is unique, see Grisvard [31]. Further, we have the asymptotic formula

$$\psi = r^\alpha \cos(\alpha(\theta-\theta_w)) + o(r^\alpha) \tag{136}$$

in polar coordinates $(r,\theta)$, at the origin. This formula is useful for explaining the existence of a stationary point in the pseudo-velocity $(u-\xi, v-\eta)$ of the full Euler system on the wall (but not the corner), see [74].

# 11 Further discussions

It remains open to calculate the threshold at which regular reflection gives way to Mach reflection in the whole (not just locally at the Mach stem). Further in depth questions include solutions to the Riemann problems and the Euler system.

# Appendix: Linear theory

## A1. Local existence

We provide details on the issue of local existence. Recall that $B_2$ is a neighborhood of G with smooth boundary and $h$ is a continuous function on $\partial B_2 \cap \Omega$. Consider the problem

$$p = h \qquad \text{on } \partial B_2 \cap \Omega \tag{137}$$

for equation (71) restricted to the domain $B_2 \cap \Omega$ with the tangential oblique derivative boundary condition (68) restricted to $B_2 \cap \Sigma$. This is a mixed-type boundary value problem on which we do not find any clear literature. So we give an existence proof here. We chop off the tip $G$; i.e., we replace $\Omega$ by $\Omega_\delta$ which is $\delta$-distance shorter than $\Omega$ from the point $G$ upward. See Figure A1. On the bottom straight boundary of $\Omega_\delta$, we impose the Dirichlet boundary condition

$$p = p_m \qquad \text{on bottom of } \Omega_\delta.$$

Now by Lieberman [47], there exists a solution $p_\delta$ in $C(\bar{\Omega}_\delta \cap \bar{B}_2) \cap C^{2,\alpha}(\Omega_\delta \cap B_2)$. The maximum principle holds for $p_\delta$, thus there is a subsequence of $p_\delta$, which converges locally in $C^2(\Omega \cap B_2)$ to a solution in $C^{2,\alpha}(\Omega \cap B_2)$ as $\delta \to 0+$.

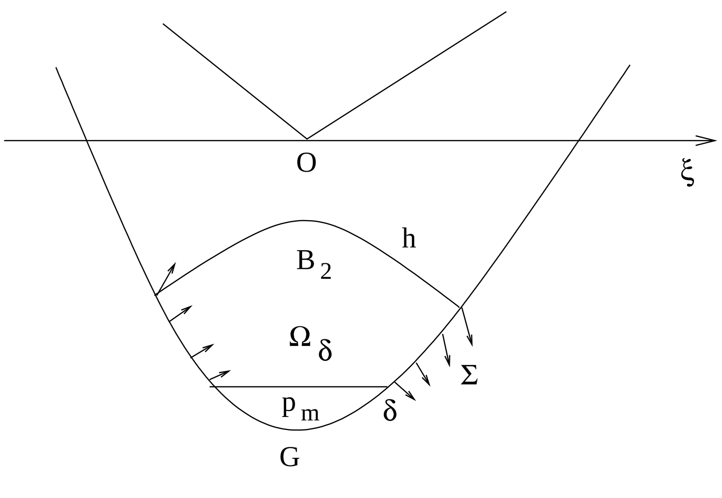


Figure A1. Domain with tip G removed.

We use a barrier function at $G$ to obtain continuity of $p$ at $G$. We note that our $\mathbf{l}$ is not degenerate ($|\mathbf{l}| \neq 0$) and well-defined for $(\xi, \eta) \in B_2$ provided that $B_2$ is sufficiently small and the variable $\bar{p}$ is regarded as given by formula (60) where $\eta'(\xi)$ is given, but $\eta$ can be free (well, we choose $\eta = \eta(\xi)$ to be given). So we can introduce integral curves in $B_2$ by

$$(\frac{d\xi}{ds}, \frac{d\eta}{ds}) = \mathbf{l}(\xi, \eta(\xi), \eta'(\xi)). \tag{138}$$

The vector field enjoys $C^{1,\gamma}(B_2)$ regularity, so the integral curves enjoy $C^{2,\gamma}$ regularity. We now perform a coordinate transformation to straighten the vector field so that the boundary condition becomes

$$\frac{\partial p}{\partial \xi} = 0 \tag{139}$$

and the equation becomes

$$a^{ij} D_{ij} p + b^i D_i p = 0 \tag{140}$$

where the usual summation convention is used. We omit introducing new notation for $p$ for the new coordinate system. We consider the auxiliary function

$$v = p_m + c(1 - e^{-N(\eta - \bar{\eta}_{m1})}) \tag{141}$$

where $c > 0$ and $N > 0$ are to be chosen large. This $v$ satisfies the oblique boundary condition and is greater than $p_m$ on each of the bottom boundary of $\Omega_\delta$, $\delta > 0$. For the equation, we have

$$a^{ij} D_{ij} v + b^i D_i v = -ce^{-N(\eta - \bar{\eta}_{m1})}(N^2 a^{11} - N b^1)$$

which can be made less than a negative constant by choosing $N > |b^1|_0/\lambda$ where $\lambda \leq a^{11}(\xi, \eta)$. Now we can choose $c$ so that $v$ is greater than $\sup |h|$. Thus all our solutions $p_\delta$ are bounded from above by the super solution $v$ and below by the constant $p_m$. (We consider the case $h \geq p_m$ only.) So $p$ is continuous at the point $G$. The continuity of $p$ at other points follows from Lieberman's aforementioned work.

We establish the $C^{2,\alpha'}(\bar{\Omega} \cap B_2)$ regularity for $p$ for $\alpha' = \min(\alpha, \gamma)$. Take a domain $B_3 \subset\subset B_2$ so that $\partial B_3$ intersects $\Sigma$ with infinite order of contact. Extend the vector field $\mathbf{l}$ from the (only two) contact points to $\partial B_3 \cap B_2$ $C^\infty$ smoothly so that there is only one point on the boundary $\partial B_3 \cap B_2$ that is tangentially degenerate, thus this portion is rather like the $\Sigma$. Propose the corresponding tangential oblique derivative boundary value problem on $B_3 \cap \Omega$: The equation is the same, the boundary condition on $\Sigma$ is the same as before, but the (nonhomogeneous) tangential oblique derivative boundary condition on $\partial B_3 \cap B_2$ is the newly-invented one. We know that this problem has a solution $p$ that is continuous on the closure of $B_3 \cap \Omega$ by design. But this problem has maximum principle, so any $C^{2,\alpha}$ solution (kept the nonhomogeneous data unchanged) will coincide with the solution $p$. The homogeneous problem will have only the zero solution, thus the null space has dimension zero. Hence, there is no compatibility condition for this problem, since this problem is Fredholm (p. 158, [25]). Thus smooth data imply existence of smooth solutions by Theorem 5.3 of Egorov and Kondrat'ev [25]; i.e., there exists a solution $p$ in the Sobolev space $W^{1,2}(B_3\cap\Omega)$. By the smoothness theorem, Theorem 5.2, of the same paper [25], the solution is as regular as the data, which we can make very smooth to start with the curve $\eta(\xi)$ and the linearization $Q$. For smooth solutions, we then use Guan and Sawyer [32] to obtain uniform estimates in the space $C^{2,\gamma}(B_2\cap\bar{\Omega})$ in terms of $C^{2,\gamma}$ regularity of $\eta(\xi)$ and $Q$.

## A2: Corner O

The angle $2\theta_w$ will be close to $\pi$, but strictly less than $\pi$. Thus the interior angle is more than $\pi$, so Lieberman [49] does not apply. We extract an example from Grisvard [31].

**Example**. Consider the Laplace equation

$$\Delta p = 0$$

in a sector of angle $\omega \in (0, 2\pi)$ in $\mathbb{R}^2$, in the polar coordinate $(r, \phi)$. See Figure A2. Consider the function

$$p_1 = r^{\pi/\omega}\cos(\phi\pi/\omega).$$

It is harmonic in the sector and satisfies $\partial_\phi p = 0$ on the edges of the sector. See Grisvard [31]. If $\omega \in (\pi, 2\pi)$, this function is only Hölder continuous at the origin. This example illustrates the requirement $0 < \phi \le \pi$ in Theorem 1 of [49] in order for $p$ to be in $\in C^1(\bar{\Omega}) \cup C^2(\Omega)$. $\square$

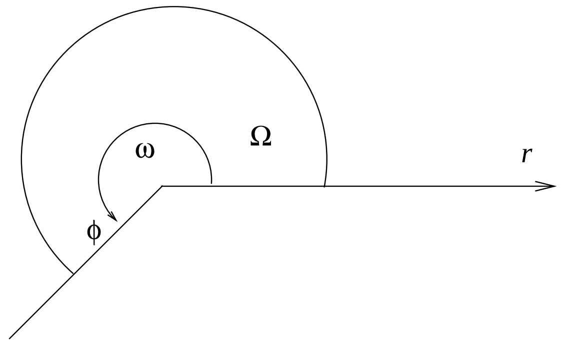


Figure A2. Laplace at a big corner.

So we use the symmetry of the problem with respect to $\xi$ to cut the large angle in half. Then the solution at the corner has Hölder continuous gradient: $p \in C^{1,\theta_w/(\pi-\theta_w)}$.

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