# On the convergence of SPH method for Scalar Conservation Laws with boundary conditions

Bachir Ben Moussa *

**Abstract**

This paper is the third of a series where the convergence analysis of SPH method for multidimensional conservation laws is analyzed. In this paper, two original numerical models for the treatment of boundary conditions are elaborated. To take into account nonlinear effects in agreement with Bardos, LeRoux and Nedelec boundary conditions ([1], [14]), the state at the boundary is computed by solving appropriate Riemann problems. The first numerical model is developed around the idea of boundary forces in surrounding walls, recently initiated in [33] by Monaghan in his simulation of gravity currents. The second one extends the well-known approach of ghost particles for plane boundaries to the case of general curved boundaries. The convergence analysis in $L^p_{loc}$ $(p < \infty)$ is achieved thanks to the uniqueness result of measure-valued solutions recently established in [3] for $L^\infty$ initial and boundary data.

*Keywords:* SPH method, numerical modeling, boundary conditions, conservation laws, measure valued solutions, convergence.

## 1 Introduction

In this paper, we continue the investigation of the convergence analysis of SPH (Smoothed Particle Hydrodynamics) method for scalar nonlinear conservation laws. We consider the initial boundary value problem which remains a very challenging question for deriving fitting and efficient numerical methods to model boundary conditions with their convergence analysis. In ([4],[6]), this analysis is performed for the Cauchy problem by deriving some new features of the SPH scheme in connection with finite volume methods. The resulting hybrid formulation turned out to be well-fitted for constructing conservative and weak consistent SPH schemes. For the stability of the resulting scheme (i.e. when additional time-explicit discretizations is applied), we have developed an original strategy making use of a robust class of upwinding schemes based on approximate Riemann solvers instead of the classical approach employing an artificial viscosity (see [19]) to deal with shocks.

*Department of Applied Mathematics University of Crete, GR-714 09 Heraklion , Crete, Greece.

In the present work, we elaborate two numerical models for the treatment of boundary conditions without compromising the specific SPH formalism which has contributed until now to its attractiveness and its popularity among the so-called Gridless or Meshless methods. This popularity has increasingly achieved (see [29],[27] for the pioneer works) through some specific domains of physics as astronomy and astrophysics, where fluids are mostly contained by their self-gravity. Thereby, boundary conditions are taken as vanishing at the infinity and do not interfere in the effective numerical simulation. In contrast, when dealing with problems restricted to a domain as those we encounter in industrial applications, commercial codes have been mostly developed using only finite volume or finite element approaches because of their heavy study since the 1950's and their best understanding. Nevertheless, in the last two decades, SPH method, due to its Lagrangian character and its simple and self-adaptive formalism, has known a great success in the simulation of a variety of complex industrial applications. However, to achieve its maturity, SPH is still on a great demand of a mathematical background to increase its accuracy and efficiency, and particularly to derive efficient numerical models to treat boundary conditions.

Despite their wide use, it is worth indicating that, Eulerian based methods are facing to increasing difficulties hampering the advances of large-scale numerical simulations of some specific and sophisticated problems where obviously they are not suited. For example those with multiple phases, which require an additional complex internal mathematical modeling to represent these phases. The result is that a large part of the overall computational effort is expended on technical details connected with mesh adaptation and grid generation. Moreover, changes in the domain geometry and/or topology, are more difficult, if not impossible, to accommodate with existing meshing techniques. By contrast, SPH method, based on its Lagrangian and grid-free characteristics, has shown a great ability to handle most of these mechanical engineering applications. Since then, some efforts have been made to treat some types of boundary conditions for wall and free boundaries by using essentially two techniques. The first one models wall boundaries by using the so-called boundary particles which interact with the fluid particles through boundary forces that prevent the fluid from passing through the boundary. This approach is first used by Monaghan [33] in his simulation of gravity currents for nearly incompressible fluid flow. The second one employs the so-called ghost particles (generated in a neighborhood of the boundary outside of the domain) well-known for plane boundaries when approximating specular reflection boundary conditions. From the physical point of view, the two types of particles are endowed with similar physical properties to those of the particles that represent the flow and interact with them in a way such that the necessary boundary conditions are satisfied. For both approaches, we deal here with general curved boundaries, in particular we give new treatments of polyhedral boundaries as those encountered in industrial problems. We develop a mathematical framework for deriving these numerical models by taking into account nonlinear effects and by computing the state at the boundary by solving appropriate Riemann problems. As far as the stability analysis is concerned, we introduce a nice formulation based on the equilibrium property for uniform fields. For the numerical simulation of these models in continuum mechanics, we refer to the forthcoming paper [8]. We also refer to Randles and Libersky's paper [38] and to the recent book by Lui and Lui [30].

The convergence analysis which is our second contribution in this paper, is derived by using the uniqueness result of measure valued solutions established in [3] for $L^\infty$ initial and boundary data. As showed by Diperna in [13] and by Szepessy in [42], this convergence follows by proving that the approximate solutions are uniformly bounded in $L^\infty$, weakly consistent with all entropy inequalities and consistent with the initial data. For the weak consistency with all entropy inequalities, one needs to show in particular the weak consistency with the boundary integral term associated to Bardos, LeRoux and Nedelec's boundary conditions formulated in [1]. For that purpose, we introduce a new definition of measure valued solutions equivalent to the one proposed by Szepessy in [41]. The two definitions differ as far as the formulation of boundary conditions is concerned. Our formulation requires less information in terms of measure-valued solutions and turns out to be well-adapted for the convergence of numerical schemes in bounded domains.

To carry out this program, we start in section 2 with a brief review of SPH method and show how to adapt Raviart's standard accuracy results in ([39],[32]) to the case of bounded domains. We end this section by describing our two numerical models of boundary conditions for a given boundary data. Section 3 will however be concerned with the derivation of the SPH scheme for scalar nonlinear equation (1). In section 4, we state the main convergence result (Theorem 4.1) as well as the uniqueness result of measure-valued solutions (Theorem 4.4). The remainder of the paper is roughly devoted to the existence proof of measure-valued solutions as weak star limit of the approximate solutions provided by the SPH scheme.

## 2 SPH method and boundary conditions

Let $\Omega$ be a bounded open set in $I\!R^d$ with a smooth boundary $\partial\Omega$ and an outward unit normal vector $n$. Consider for $u : \Omega \times I\!R_+ \to I\!R$ the following model of scalar nonlinear conservation law

$$\mathcal{L}_{\mathbf{a}} u + \mathrm{div} F(u,x,t) = S(u,x,t), \qquad (x,t) \in \Omega \times I\!R^+, \quad u(x,t) \in I\!R \tag{1}$$

with the Bardos, LeRoux and Nedelec ([1], [14]) boundary condition on $\partial\Omega \times I\!R_+$, for all $k \in I\!R$,

$$(\mathrm{sgn}(u(\bar{x},t)-k) - \mathrm{sgn}(b(\bar{x},t)-k))(F(u(\bar{x},t),\bar{x},t) - F(k,\bar{x},t)) \cdot n(\bar{x}) \geq 0, \tag{2}$$

and the initial condition

$$u(\cdot,0) = u_0, \quad \text{on } \Omega, \tag{3}$$

where $F = (F_1, ..., F_d) : I\!R \to I\!R^d$, $\mathrm{div}_x F(u,x,t) = \sum_{i=1}^{d} \partial F_i(u(x,t),x,t)/\partial x_i$ and $I\!R_+ \equiv (0,\infty)$. The transport operator $\mathcal{L}_{\mathbf{a}} u$ and its adjoint operator $-\mathcal{L}^*_{\mathbf{a}} u$ via the usual $L^2$ scalar product are defined by

$$\mathcal{L}_{\mathbf{a}} u := \frac{\partial u}{\partial t} + \mathrm{div}(\mathbf{a}(x,t)u). \quad \mathcal{L}^*_{\mathbf{a}} u = \frac{\partial u}{\partial t} + \sum_{i=1}^{d} \mathbf{a}^i(x,t)\frac{\partial u}{\partial x^i} := \frac{d}{dt}(u).$$

In the present model (1), the advection field $\mathbf{a}$ is a given smooth vector say $\mathbf{a} \in L^\infty(\mathbb{R}^+, W^{2,\infty}(\Omega))$ and which verifies $\mathbf{a}(\bar{x},t).n(\bar{x}) = 0$ along the boundary $\partial\Omega$ . This assumption is very important in our convergence analysis since it avoids in the limit process the difficult problem related to the behavior of a regularized no smooth physical advection field (see [45]) since it can lead to a nonlocal dispersive equation. Alternativelly, in our methodology, the model equation (1) suggests to treat the physical advection term in the generic nonlinear flux term $F(u,x,t)$ by well-known arguments based on relevant entropy admissibility criteria to select the physical solution in the field of nonlinear conservation laws. Thereby, we keep track of all the properties inherited from the physical velocity as those regarding the boundary conditions in the sense of inequality (2) as well as in the entropy formulation (47). As a matter of fact, from the numerical point of view, this process of regularization (by convolution) of the physical velocity is of current use in the SPH literature for Euler equations where the density, the velocity and the energy are tansported by the velocity itself. The mathematical analysis of the resulting scheme is however a very open question.

In what follows, to develop our program of performing the SPH scheme of (1-3), let us first give a brief review of the method.

## 2.1 Review of SPH method

The method was introduced at the end of the seventies by Lucy in [15] and Gingold and Monaghan in [20] as an alternative to classical methods (based on grid technique) to solve compressible Euler equations. The method still uses computational nodes called particles to be sprinkled through the domain, but do not require any pre-specified connectivity of these particles, or locally regular topological structure as is needed for traditional meshing. Basically, the approximate solutions of equation (1) are computed for any time $t$ with respect to a set denoted by $K$ ($K \subset \mathbb{Z}^d$) of moving particles provided by a suitable quadrature formula $(x_k(t), w_k(t))_{k\in K}$[1]. In this formula, $x_k(t)$, which is the position of the particle $k \in K$ and $w_k(t)$ which is its effective weight, are solutions to the following systems of differential equations

$$(i) \quad \begin{array}{l} \dfrac{d}{dt}x_k = \mathbf{a}(x_k,t) \\ x_k(0) = x_k^0 \end{array} \qquad (ii) \quad \begin{array}{l} \dfrac{d}{dt}w_k = div(\mathbf{a}(x_k,t))\, w_k \\ w_k(0) = w_k^0. \end{array} \tag{4}$$

The quadrature formula $(x_k^0, w_k^0)_{k\in K}$ defines the initial particle distribution through the domain $\Omega$. It is worth indicating that the solutions of the system $(i)$ are the classical characteristic curves of the field $\mathbf{a}$ while those of equation

[1] in the sequel of this paper, we omit the time dependence when there is no ambiguity

$(ii)$ reflect the evolution of the weights (i.e. the deformation of the particle distribution) in the change of coordinates. Thereby, the accuracy of the SPH approximation is in part connected to the quadrature rule over $\Omega$

$$\int_{\Omega} g(x)dx \approx \sum_{k\in K} w_k g(x_k). \tag{5}$$

On account of the regularity of the field $\mathbf{a}$, this approximation is accurate for any $t > 0$ as soon as it is accurate initially and the particles and their weights move according to (4). In most of practical computations, the particles are initially distributed on a regular grid (for instance cubic grid) and it is quite easy to find suitable weights such that the error in (5) is of "infinite order" (see Raviart [39]) when applied to $\mathcal{C}^\infty$ functions $g$ that vanish sufficiently rapidly at the infinity. In our setting, the set $K$ of the initial distribution of particles could be performed by using finite elements triangulation denoted by $P_\Omega$ and by taking that the particles are initially distributed on the center of the elements $B_k^0 \in P_\Omega$ of this triangulation with their initial weights $w_k^0 = meas(B_k^0)$. If we denote by $B_k(t)$ (or simply $B_k$) the image at time $t$ of the cell $B_k^0$ by the vector field $\mathbf{a}$ such that $meas(B_k) = w_k$, then the $P^1$ finite element approximation gives

$$|\mathcal{E}^h(g)| \leq C(\Omega)\, h \quad \text{with} \quad \mathcal{E}^h(g) = \sum_{k\in K} \mathcal{E}_k(g) := \sum_{k\in K} \left( \int_{B_k} g(x)dx - w_k g(x_k) \right). \tag{6}$$

The SPH method takes advantage of the particle distribution to provide by convolution, a discrete derivative operator denoted by $D_\varepsilon$ approximating first derivatives as follows

$$\nabla g \approx D_\varepsilon g(x) = \sum_{k\in K} w_k g(x_k) \nabla \zeta^\varepsilon (x - x_k), \qquad \zeta^\varepsilon(x) = 1/\varepsilon^d \zeta(x/\varepsilon), \tag{7}$$

where the smooth function $\zeta$ is a nonnegative with compact supported and verifies

$$(i) \quad \int_{I\!R^d} \zeta(x)dx = 1 \qquad (ii) \quad \int_{I\!R^d} x^i \zeta(x)dx = 0 \quad (i \in 1, \dots, d). \tag{8}$$

For the conservativity of the SPH scheme, it is convenient to work with a symmetrized version of $D_\varepsilon$ denoted by $D_{\varepsilon,s}$ (s: refers to this symmetry)

$$D_{\varepsilon,s} g(x) := D_\varepsilon g(x) - g(x).D_\varepsilon 1(x) \tag{9}$$

and obtained by substituting the following vanishing error term

$$g(x).D_\varepsilon 1(x) \approx g(x) \nabla 1 = 0. \tag{10}$$

Accuracy results due to Raviart and Mas-Gallic are available in case of $\Omega = I\!R^d$ in ([39]) and ([32]). To make use of these results, one needs to restrict the validity of the approximations of (7) and (10) to the subset of $x \in \Omega$ such that $\mathrm{supp}(\zeta^{\epsilon}(x - x_k) \cap \partial\Omega = \emptyset$ for all $k \in K$. Thus, if one considers the subset $\Omega^{\kappa'} = \{y \in \Omega : \text{distance}\,(y, \partial(\Omega)) \geq \kappa' > 2\varepsilon\}$, then, for all $\varphi \in W^{2,\infty}(\Omega)$ and for all $T > 0$

$$\|D_\varepsilon \varphi(x) - D\varphi(x)\|_\infty \leq C(T, \Omega^{\kappa'}) \left( \varepsilon |\varphi|_{1,\infty} + \frac{h}{\varepsilon^2} \|\varphi\|_{2,\infty} \right), \quad \forall x \in \Omega^{\kappa'}. \quad (11)$$

In particular, this last estimate implies

$$\|D_\varepsilon 1(x)\|_\infty \leq C(T, \Omega^{\kappa'}) \frac{h}{\varepsilon^2}, \qquad \forall x \in \Omega^{\kappa'}. \quad (12)$$

Therefore, the combination of (11) and (12) yields

$$\forall x \in \Omega^{\kappa'} \qquad \|D_{\varepsilon,s} \varphi(x) - D\varphi(x)\|_\infty \leq C(T, \Omega^{\kappa'}) \left( \varepsilon |\varphi|_{1,\infty} + \frac{h}{\varepsilon^2} \|\varphi\|_{2,\infty} \right). \quad (13)$$

Notice that, on account of the compact support of the cut-off $\zeta$ and the regularity of the field $\mathbf{a}$, a straightforward calculation proves the following estimates

$$(i) \quad \mathrm{card}\{k \in K \; ; \; \|\nabla \zeta^\varepsilon(x - x_k)\| \neq 0\} \leq C(\frac{\varepsilon}{h})^d \quad (ii) \quad C_1 h^d \leq w_k \leq C_2 h^d$$

$$(iii) \qquad \|\nabla \zeta^\varepsilon(x - x_k)\| \leq \frac{C}{\varepsilon^{d+1}} \qquad (iv) \qquad \sum_{k \in K} w_k \, \|\nabla \zeta^\varepsilon(x - x_k)\| \leq \frac{C}{\varepsilon}. \quad (14)$$

These estimates will be on a systematic use when evaluating errors terms following on our SPH scheme of equation (1).

**Comments** It is apparent from the estimates (11-13) that in the SPH setting, the convergence of the approximations (7) and (10) is obtained by letting simultaneously $h \to 0$, $\varepsilon \to 0$ and the ratio $h/\varepsilon^2 \to 0$ (which in the sequel will be expressed by $\Delta(\varepsilon, h) \to 0$). This clearly shows that the parameter discretization $h$ has to be taken much smaller than the smoothing lenght $\varepsilon$, precisely $h = o(\varepsilon^2)$. As a result, the ratio $\upsilon = \varepsilon/h$ becomes much bigger and tends to infinity. To show the central role of this resulting scaling $\upsilon$ in the SPH formalism, note that by (14) $(i)$, it provides the appropriate relatively constant number $N_{sph}$ of the neighboring particles inside the smoothing lenght $\varepsilon$ of each particle so that the local approximation (7) makes sense. Moreover, the fact that $\upsilon$ tends to infinity can be seen as a necessary condition to make it possible the passage limit from the discrete SPH model to the continuous fluid flow model. We refer to [26] for some convergence tests in the simulation of fracture analysis. We should point out that by (14) $(i)$, $N_{sph}$ also depends on the space dimension $d$. So, in practice, the parameters $h$ and $\varepsilon$ have to be chosen such that $N_{sph}$ is around 25 for $d = 2$ computations and 50 in $d = 3$.

## 2.2 Particle formulation of boundary conditions

The question of deriving an appropriate SPH scheme of equation (1) in unbounded domains is treated in [4]. On account of the fact that the solution develops singularities in finite time, even with smooth initial data, the right approximation is performed from the weak formulation of (1). Herein for "nice" test functions, this formulation reads

$$\int_{\Omega\times I\!R_+} (u\mathcal{L}^*_{\mathbf{a}}\varphi + F(u,x,t).\nabla\varphi + S(u,x,t)\varphi)dxdt \\ - \int_{\partial\Omega\times I\!R_+} F(u,x,t).n\varphi d\sigma(x)dt = 0.$$

This clearly shows the remaining difficulty we have to deal with which concerns the derivation of a suitable particle formulation of the boundary integral term by taking into account nonlinear effects of the solution itself. To motivate our investigation of this question, recall that in the SPH formalism, the fluid flow is represented by fluid pseudo-particles. These individual particles interact with one another, moving with the flow and carrying with them all of the computational information about the fluid. Fluid properties are then interpolated between the particles. In other words, for the model equation (1) (see the scheme (26)), the term div$F$ is interpreted as an internal volume force while the right hand side $S$ acts as an external volume force. On the light of these features, one needs to find a way to associate to the boundary contribution an appropriate volume approximation making relevant the interaction with the fluid particles such that the necessary boundary conditions are satisfied. In this direction, we shall study two solutions

- The technique based on the so-called boundary particles and boundary forces. It was first used by Monaghan [33] in the simulation of gravity currents. Here, we deal with the derivation of a mathematical framework for deriving an efficient numerical model modeling this phenomena. This numerical model will be set in a general and flexible way such that it can be used in variety of problems including solid friction, multiphase flows as well as when coupling SPH method with Eulerian based methods. Moreover, the nonlinear effects at the boundary will be taken into account by solving appropriate Riemann problems. In our formalism, the necessary numerical requirements such as the conservativity and the stability of the resulting scheme will be performed by using the equilibrium condition for a uniform field.

- The quite well-known approach of the so-called ghost particles used to model specular reflection boundary conditions in case of plane boundaries. Herein, we deal with the treatment of general curved boundaries and provide under the same machinery as in the previous case, an efficient numerical model to handle different types of boundary conditions including free boundaries (see [8]). In particular, our numerical model yields new treatments of polyhedral boundaries as those encountered in many industrial applications.

In what follows, we are going to focus on describing the general formalism of these numerical models while the question concerning nonlinear effects at the boundary will be treated in section 3.

### 2.2.1 Boundary forces

Let us introduce in a neighborhood of $\partial\Omega$, the change of coordinates

$$\Omega \ni x \to (\bar{x}, y) \in \partial\Omega \times (-\kappa', 0)$$

$$\bar{x} = x - yn(\bar{x}), \quad \text{for some} \quad \kappa' > 0. \tag{15}$$

Consider a finite element type triangulation $E_{\partial\Omega}$ of the boundary $\partial\Omega$. To this triangulation, we associate a finite element interpolation $R_E(g)$ of any boundary function $g(\bar{x}, t)$

$$R_E(g)(\bar{x}, t) = \sum_{i \in N_E} g(\bar{x}_i, t)\Psi_i(\bar{x}), \tag{16}$$

where the summation is taken over the degrees of freedom $N_E$ (respectively located at $\bar{x}_i \in \partial\Omega$, $i \in N_E$) of the finite element, associated with the basic polynomial functions $\Psi_i(\bar{x})$. These degrees of freedom, located on the boundary, can be considered either as boundary particles (moving or fixed) or as points of a fixed grid depending on the problem under consideration. Our weak model is derived as follows:

let $\beta$ a regular function of the real variable $y \in [0, 1]$, such that

$$(i),\ 0 \le \beta(y), \qquad (ii),\ \beta(y) = 0,\ \text{for} y \ge 1, \quad (iii),\ \int_0^1 \beta(y)dy = 1. \tag{17}$$

Next, we are going to construct an appropriate extension to the whole domain $\Omega$ of any $g(\bar{x}, t) \in \partial\Omega$ in the weak sense. So, take $\varphi \in \mathcal{C}^1(\overline{\Omega} \times I\!R_+)$, then the change of variables gives

$$\begin{aligned} \int_{\partial\Omega} g(x,t)\varphi d\sigma(x) &= \lim_{\kappa' \to 0} \int_{\partial\Omega} \int_0^1 g(x(\bar{x}, 0), t)\beta(y)\varphi(x(\bar{x}, \kappa' y), t)J_{\partial\Omega}(\bar{x})d\bar{x}dy \\ &= \lim_{\kappa' \to 0} \int_{\Omega} g(x(\bar{x}, 0), 0), t)\beta^{\kappa'}(y)\varphi(x,t)J_{\partial\Omega}(\bar{x})J(x)dx, \end{aligned}$$

where $J$ and $J_{\partial\Omega}$ are the Jacobian associated with the changes of coordinates while the kernel $\beta^{\kappa'}(y) = \frac{1}{\kappa'}\beta(y/\kappa')$. Consequently, for $\kappa'$ sufficiently small,

$$\int_{\partial\Omega} g(\bar{x}, t)\varphi d\sigma(x) \approx \int_{\Omega} g(\bar{x}, t)\beta^{\kappa'}(y)\varphi(x,t)J_{\partial\Omega}(\bar{x})J(x)dx.$$

Hence, an appropriate extension of $g(\bar{x}, t)$ to the whole domain $\Omega$ is given by

$$g_{\kappa'}(x(\bar{x}, y), t) = g(x(\bar{x}, 0), t)J_{\partial\Omega}(\bar{x})J(x(\bar{x}, y))\beta^{\kappa'}(y). \tag{18}$$

Moreover, one may write

$$\int_\Omega g_{\kappa'}(x,t)\varphi dx \underbrace{\approx}_{\text{by}(16)} \sum_{j\in N_E} g(\bar{x}_j,t)\int_\Omega J_{\partial\Omega}(\bar{x})\Psi_j(\bar{x})\beta^{\kappa'}(y)J(x)\varphi dx$$
$$\underbrace{\approx}_{\text{by}(5)} \sum_{k\in K, j\in N_E} w_k\varphi_k J(x_k)J_{\partial\Omega}(\bar{x}_k)g(\bar{x}_j,t)\Psi_j(\bar{x}_k)\beta^{\kappa'}(y_k).$$

In terms of external forces acting on the particles $k \in K$, this last formula can be interpreted by associating to each boundary particle (or grid point) $j$ a volume boundary force field $f_j(x)$ at the point $x$ defined by

$$f_j(x(\bar{x},y)) = J(x)J_{\partial\Omega}(\bar{x})g(\bar{x}_j)\Psi_j(\bar{x})\beta^{\kappa'}(y).$$

The resulting boundary force acting on each fluid particle $k \in K$ moving on the boundary layer $(-\kappa',0)\times\partial\Omega$ is then, given by

$$f(x) = \sum_{j\in N_T} f_j(x).$$

**Remark 2.1** *By construction, the conditions (17) on the shape function $\beta^{\kappa'}$ are devoted to ensure the weak consistency of our numerical model of boundary forces. Nevertheless, we should point out that the final choice of $\beta^{\kappa'}$ depends on the physical problem at hand, the type of boundary conditions as well as the compatibility relation between the initial and boundary data. In our model (1-72), one possible choice is to take simply $\beta(y) = 1$ for $0 \le y \le 1$.*

*Note that by our general formalism, the three models used by Monaghan in [33] based on central forces and Leonard-Jones inter-molecular force with a repulsive core and attractive well, could be improved to get weak consistent and wellposed models.*

**Remark 2.2** *The finite element approximation (16) is well fitted when modelling physical problems as solid friction or when coupling SPH method with Eulerian methods in such a way that the best aspects of both approaches can be incorporated into a single model. We refer to [8] for more details.*

#### 2.2.2 Ghost particles

This technique is formulated so that the local conservativity and consistency of the SPH scheme near the boundary is satisfied by means of the so-called ghost particles generated outside the domain $\Omega$. For the well-posedness of this approach, one needs to equip a neighborhood of $\partial\Omega$ outside to $\Omega$ with a new and appropriate quadrature formula in accordance with the one defining the real particles inside $\Omega$. To this end, since the boundary $\partial\Omega$ is assumed to be smooth enough, then by using the local system of coordinates $(\bar{x},y)$ over $\partial\Omega\times(-\kappa',0)$, it suffices to construct an appropriate extension $\tilde{\Omega} \subset I\!R^d$ of the set $\Omega$. In practice, this construction can be performed by means of a suitable diffeomorphism $M$ which maps to any point $x = \bar{x} - yn(\bar{x}) \in \partial\Omega \times (-\kappa', 0)$ the point

$$M(x) = \bar{x} + yn(\bar{x}) \in \Omega_M := \partial\Omega \times (0, \kappa'). \quad (19)$$

It turns out that, with this map, one may associate to each particle $k \in K$ of position $x_k \in K$, sufficiently close to the boundary $\partial\Omega$, a new particle called ghost particle located at the position $M(x_k)$ outside of $\Omega$. We thus get a new quadrature formula over $\Omega_M$ by taking as a weight of the ghost particle the weight of the particle that it is ghost multiplied by $J_M(x_k)$ the Jacobian determinant $|det(DM(x_k))|$ at point $x_k$. Consequently, by denoting by $G$ the set of the new ditribution of ghost particles i.e.

$$G = card\{k \in \mathbb{Z}^d, x_k(t) \in \Omega_M\},$$

one gets a quadrature formula over $\tilde{\Omega} = \Omega \cup \Omega_M$ with its associate quadrature rule defined by

$$\int_{\tilde{\Omega}} g(x)dx \approx \sum_{k \in K \cup G} \tilde{\omega}_k(t) g(x_k(t)), \quad (20)$$

with

$$\tilde{\omega}_k(t) = \begin{cases} \omega_k(t) \text{ if } x_k \in \Omega \\ \omega_{M^{-1}(k)}(t) J \circ M^{-1}(x_k(t)) \text{ if } x_k \in \Omega_M. \end{cases}$$

To this new quadrature formula, one associates a discrete derivative operator denoted by $\tilde{D}_\varepsilon$, well-defined in the whole domain $\Omega$ for any $g \in \mathcal{C}^1(\tilde{\Omega})$ by

$$\forall x \in \Omega, \quad \tilde{D}_\varepsilon g(x) = \sum_{k \in K \cup G} \tilde{\omega}_k(t) g(x_l) \nabla \zeta^\varepsilon (x - x_k).$$

We also define

$$\tilde{D}_{\varepsilon,s} g(x) = \tilde{D}_\varepsilon g(x) - g(x) \tilde{D}_\varepsilon 1(x) \quad (21)$$

so that, the accuracy results (11-13) can be extended to the whole domain $\Omega$ instead of $\Omega^{\kappa'}$.

**Remark 2.3** *There exists another technique to provide the local conservativity of SPH method called the semi-analytic approach (see [10]). It is based on the use of the exact values of the integrals of the shape function and its derivatives outside the domain $\Omega$ instead of generating a new distribution of ghost particles, by computing the integrals*

$$\int_{\mathbb{R}^d/\Omega} \zeta^\varepsilon(x-y)dy, \quad \int_{\mathbb{R}^d/\Omega} \nabla\zeta^\varepsilon(x-y)dy.$$

*For the half-plane or polyhedral boundaries, one may compute these integrals in case of polynomial shape functions $\zeta$ by using formal calculus computer codes such as the results given by Herand in [18].*

**Remark 2.4** *Note that the two approaches of modeling boundary conditions we have described, seem close as far as the basic principle is concerned, that is preventing the fluid particles from passing through the boundary. However, from the implementation point of view, these approaches are very different. Indeed, the technique of ghost particles needs the creation of a ghost particles' distribution at any time $t$ so that the number of particles inside the radius $\varepsilon$ is equal to the constant value $N_{sph}$ for all the particles near the boundary $\partial\Omega$. For the boundary forces' technique, the boundary particles are set initially.*

# 3 The weak form of the SPH scheme

In this section, we are going to derive the SPH scheme of equation (1) by using the results of the previous section.

## 3.1 Case of boundary forces

We proceed into three steps. First, we derive the particle scheme in the interior domain. Secondly, we make use of the upwind particle scheme developed in ([4]) to overcome the problem of the stability of the method. In the third step, we treat the boundary contribution in equation (1) by using the numerical model developed in the above section.

**Step 1 (The interior domain)** Consider the adjoint operator $D^*_{\varepsilon,s}$ of $D_{\varepsilon,s}$ according to the $L^2$ discrete scalar product as

$$(D_{\varepsilon,s}\varphi, \Psi)_h = -\left(\varphi, D^*_{\varepsilon,s}\Psi\right)_h \qquad (\varphi, \Psi)_h := \sum_{i\in K} w_i \varphi_i . \Psi_i . \tag{22}$$

Take also the hybrid particle approximation $\bar{u}_h(x,t)$ introduced in ([4]) in order to extend finite volume techniques to the particle scheme. It is defined by

$$\bar{u}_h(x,t) = \sum_{k\in K} u_k \chi_{B_k}(x), \tag{23}$$

where $u_k$ stands for an approximation of the exact solution of problem (1) and $\chi_{B_k}$ is the characteristic function of the set $B_k$. To get the particle scheme of equation (1) inside $\Omega$, let us take the following weak model ($\kappa = O(\kappa')$)

$$\int_{[0,T]} \left[ (\bar{u}_h, \mathcal{L}^*_a(\varphi))_h + (F(\bar{u}_h, x, t), \chi^\kappa D_{\varepsilon,s}\varphi)_h + (S(\bar{u}_h, x, t), \varphi)_h \right] dt = 0, \tag{24}$$

where $\varphi$ is a test function and $\chi^\kappa$ is a suitable regularization of the characteristic function $\chi$ of the domain $\Omega$, for instance

$$\chi^\kappa(x(\bar{x}, y)) = \begin{cases} 0 & 0 \le y < \kappa \\ 1/2 + 3(y-2\kappa)/(4\kappa) - (1/4)((y-2\kappa)/\kappa)^3 & \kappa \le y < 3\kappa \\ 1 & y \ge 3\kappa. \end{cases} \tag{25}$$

In view of (22) together with an integration by parts with respect to $t$, one easily gets that (24) is equivalent to the system of differential equations

$$\forall k \in K, \qquad \frac{d}{dt}(w_k u_k) + w_k (D^*_{\varepsilon,s})(\chi^\kappa F)_{x=x_k} = w_k S(u_k, x_k, t), \tag{26}$$

where

$$D^*_{\varepsilon,s}(\chi^\kappa F)_{x=x_k} = \sum_{l\in K} w_l^n (\chi_k^\kappa F(u_k, x_k, t)\mathcal{A}_{kl} - \chi_l^\kappa F(u_l, x_l, t)\mathcal{A}_{lk})$$

and the abbreviation $\mathcal{A}_{ij} = (\nabla \zeta^\varepsilon)_{x=x_i}(x_i - x_j)$ has been used.

Note that the result of our model (24) is that the weak consistency of the numerical scheme (26) is by definition a direct consequence of the quadrature error (6) and the accuracy result (13). The global conservativity is also ensured by the use of the derivative operator $D_{\varepsilon,s}$ defined in (9) instead of $D_\varepsilon$. Indeed, neglecting the boundary contribution and taking in the scheme (26), the sum over $k \in K$, we find that

$$\frac{d}{dt}(\sum_{k\in K} w_k u_k) + \sum_{k\in K} w_k D^*_{\varepsilon,s}(\chi^\kappa F)_{x=x_k} = \sum_{k\in K} w_k S(u_k, x_k, t). \tag{27}$$

This is nothing but the discrete equivalent inside $\Omega$ of

$$\frac{d}{dt}\int_\Omega u(x,t)\; dx = \int_\Omega S(u(x,t), x, t)\; dx, \tag{28}$$

since $\sum_{k\in K} w_k D^*_{\varepsilon,s}(\chi^\kappa F)_{x=x_k} = 0$. By symmetry, this last point can be seen by switching the indices $k$ and $l$ in one of the two terms composing it.

Note in passing that the scheme (26) is well-defined for all known versions of SPH method developed for compressible Euler equations to remedy particle distortion by taking the smoothing length $\varepsilon$ as an adaptive one $\varepsilon(x)$ such that the number of particles inside the smoothing radius $\varepsilon(x_k)$ for all $k \in K$ is equal to a suitable constant value $N_{sph}$. For instance, Gather and Scatter approximations where respectively in the scheme (26) $\mathcal{A}_{ij} = (\nabla \zeta^{\varepsilon(x_i)})(x_i - x_j)$ and $\mathcal{A}_{ij} = (\nabla \zeta^{\varepsilon(x_j)})(x_i - x_j)$ or the symmetric version in which $\varepsilon = (\varepsilon(x_i) + \varepsilon(x_j))/2$. This last form is the most popular of them (see [33],[19],[20], [34]).

**Step 2 (The upwind particle scheme)** As already underlined in ([4],[6]), the scheme (26) is somehow a generalized finite difference centered scheme well known to be unconditionally unstable whenever a time-explicit discretization is used. To lift this difficulty, we have developed an original approach using nonlinear upwinding and Riemann approximate solvers well known in the field of finite difference schemes for nonlinear hyperbolic equations (see for instance [16], [17]). Indeed, the form of the scheme (26) computing the interaction between any pair of particles $(k, l)$ along the direction $n_{kl}$ connecting $x_k$ with $x_l$, suggests the introduction at the point $x_{kl} = (x_k + x_l)/2$ of the Riemann problem

$$\begin{cases} \dfrac{\partial}{\partial t}(u) + \dfrac{\partial}{\partial x}(F(u, x_{kl}, t).n_{kl}) = 0, \quad \text{with} \quad n_{kl} = \mathcal{A}_{kl}/\|\mathcal{A}_{kl}\|. \\ u(x,0) = \begin{cases} u_k \text{ if } x < 0 \\ u_l \text{ if } x > 0. \end{cases} \end{cases} \tag{29}$$

Therefore, a suitable approximation can be performed by the introduction of a 1-dimensional finite difference scheme $g$ in a conservation form associated to (29). This numerical scheme $g$ is consistent with the nonlinearity $F.n(x)$ and conservative, i.e, that $g(n(x), u, u) = F(u, x, t).n(x)$ and that $g(n, u, v) + g(-n, v, u) = 0$ respectively. The corresponding numerical viscosity $Q(n(x), u, v)$ and incremental coefficient $C(n(x), u, v)$ are then classically defined by

$$\begin{aligned} Q(n(x), u, v) &= \frac{F(x,t,u).n(x) - 2g(n(x), u, v) + F(x,t,v).n(x)}{v-u} \\ C(n(x), u, v) &= \frac{F(u,x,t).n(x) - g(n(x), u, v)}{v-u}. \end{aligned} \tag{30}$$

Thus, the upwind numerical scheme which consists in finding functions $t \in \mathbb{R}^+ \longrightarrow u_k(t) \in \mathbb{R}$, $k \in K$ reads

$$\frac{d}{dt}(w_k u_k) + w_k \sum_{l \in K} w_l \Big( \chi_k^\kappa g(n_{kl}, u_k, u_l)\|\mathcal{A}_{kl}\| - \chi_l^\kappa g(n_{lk}, u_l, u_k)\|\mathcal{A}_{lk}\| \Big) = w_k S_k.$$

There exist a lot of numerical fluxes $g$ well fitted for such an upwinding. One can quote the Lax Friedrich and the Godunov schemes. They are in fact monotone finite difference schemes (see Crandall and Majda [12] and Kuznetsov and Volosin [25]) which belong to the widest class of E-schemes (Osher [36]). In the following, we suppose that $g$ is such an E-flux (i.e. its numerical viscosity satisfies $Q(n, u, v) \geq Q_G(n, u, v)$ where $Q_G$ is the numerical viscosity of the Godunov scheme).

**Step 3 (The boundary contribution)** According to result (18) and Remark (2.1), let $b_{\kappa'}(x,t)$ be the extension to $\Omega$ of the boundary data $b(\bar{x}, t)$ satisfying (2). In our formulation of boundary forces for equation (1), the boundary contribution will be computed in the boundary layer $[-\kappa', 0] \times \partial\Omega$ by taking into account nonlinear effects. With this end in view, the correct formalism consists in solving an appropriate Riemann problem similar to (29). Thus, for each particle $k \in K$ of position $x_k$ moving in this boundary layer, one considers the additional Riemann problem at $x_k$ along the direction $\tilde{n}_k$ (to be determined a posteriori) in the following way

$$\begin{cases} \dfrac{\partial}{\partial t}(v) + \dfrac{\partial}{\partial x}(F(v, x_k, t).\tilde{n}_k) = 0 \\ v(x,0) = \begin{cases} u_k \text{ if } x < 0 \\ b_k = b_{\kappa'}(x_k) \text{ if } x > 0. \end{cases} \end{cases}$$

Accordingly, in a similar way to the interior domain, one may use the E-scheme $g$ for the approximate solutions of this Riemann problem. Moreover, for the stability of the scheme, one needs to introduce a function $\theta(x)$ (also to be computed) with $\text{supp}(\theta) \subset (-\kappa', 0) \times \partial\Omega$ so that the global upwind scheme reads

$$\frac{d}{dt}(w_k u_k) + w_k \sum_{l \in K} w_l \Big( \chi_k^\kappa g(n_{kl}, u_k, u_l) \|\mathcal{A}_{kl}\| - \chi_l^\kappa g(n_{lk}, u_l, u_k) \|\mathcal{A}_{lk}\| \Big) \\ + w_k \theta(x_k) g(\tilde{n}_k, u_k, b_k) = w_k S_k. \tag{31}$$

Note that the scheme (31) is well-posed if in particular the equilibrium condition for a uniform field is satisfied (i.e. with $F(u,x,t) = cte$ and $S(u,x,t) = 0$). Accordingly, (31) reads

$$\begin{aligned} \frac{d}{dt}(w_k u_k) = 0 \Longrightarrow \theta_k \tilde{n}_k &= -\sum_{l \in K} w_l (\chi_k^\kappa \mathcal{A}_{kl} - \chi_l^\kappa \mathcal{A}_{lk}) \\ &= -(D_{\varepsilon,s}^* \chi^\kappa)_{x = x_k} \equiv -D_{\varepsilon,s}^* \chi_k^\kappa, \end{aligned} \tag{32}$$

which yields the following suitable choice

$$\tilde{n}_k = -D_{\varepsilon,s}^* \chi_k^\kappa / \|D_{\varepsilon,s}^* \chi_k^\kappa\|, \qquad \theta_k := \theta(x_k) = \|D_{\varepsilon,s}^* \chi_k^\kappa\|.$$

Therefore,

$$\theta(x) = \begin{cases} \|D_{\varepsilon,s}^* \chi^\kappa(x)\| & \text{if } x \in [-\kappa', 0] \times \partial\Omega \\ 0 & \text{elsewhere.} \end{cases}$$

To show the weak consistency with the boundary integral term in the new model (31), let us take for simplicity the case of symmetry shaped function $\zeta(x)$. Consequently, $\mathcal{A}_{kl} = -\mathcal{A}_{lk}$ and then the accuracy result (11) yields

$$-(D_{\varepsilon,s}^* \chi^\kappa)(x) \underset{\Delta(h,\varepsilon) \to 0}{\longrightarrow} \nabla(1 - \chi^\kappa)(x), \qquad \text{in} \quad L_{loc}^\infty(\Omega). \tag{33}$$

Moreover, replace the numerical flux $g(\tilde{n}_k, u_k, b_k)$ by its expression from (30) and take the contribution coming from the boundary forces, then a straightforward calculation using (33) and the quadrature rule (6) gives (with $\kappa' = 3\kappa$)

$$\begin{aligned} &\lim_{\kappa' \to 0} \lim_{\Delta(h,\varepsilon) \to 0} -\Big( \sum_{k \in K} w_k \varphi_k F(b_{\kappa'}(x_k, t), x_k, t) D_{\varepsilon,s}^* \chi_k^\kappa \Big) \\ &= \lim_{\kappa' \to 0} \int_\Omega F(b_{\kappa'}(x,t), x, t) \nabla(1 - \chi^\kappa)(x) \varphi(x) dx = \int_{\partial\Omega} F(b, x, t).n \ \varphi \ d\sigma(x). \end{aligned} \tag{34}$$

Note however, that since the hybrid particle approximation $\bar{u}_h$ is not smooth enough, the evaluation of the other terms requires more sophisticated arguments. These arguments use in particular the concept of measure-valued solutions and the convergence for measures in the weak topology $\sigma(M_b, \mathcal{C}_c)$ where $M_b$ denotes the set of bounded Radon measures. This last step constitutes the main difficulty in the present convergence analysis.

In the sequel, to make the reading easier, we assume that the cut off function $\zeta(x)$ is in addition symmetric, therefore the scheme (31) reads

$$\frac{d}{dt}(w_k u_k)+w_k \sum_{l\in K} w_l(\chi_k^\kappa+\chi_l^\kappa)g(n_{kl}, u_k, u_l)\|\mathcal{A}_{kl}\|+w_k\theta(x_k)g(\tilde{n}_k, u_k, b_k) = w_k S_k$$

Moreover, without loss of generality one may replace the average $(\chi_k^\kappa + \chi_l^\kappa)/2$ by the value $\chi^\kappa(x_{kl})$ at the mean point $x_{kl}$ denoted by $\chi_{kl}^\kappa$ to get

$$\frac{d}{dt}(w_k u_k) + w_k \sum_{l\in K} w_l \chi_{kl}^\kappa 2g(n_{kl}, u_k, u_l)\|\mathcal{A}_{kl}\| + w_k\theta(x_k)g(\tilde{n}_k, u_k, b_k) = w_k S_k. \tag{35}$$

Finally, by using the forward Euler scheme in time, we get the following algorithm

$$\begin{aligned}
&(i) && u_k^0 = \frac{1}{meas(B_k)} \int_{B_k} u(x,0)dx \\
&(ii) && \tilde{u}_k^{n+1} = u_k^n - \tau^n 2\sum_{l\in K} w_l^n \chi_{kl}^\kappa g(n_{kl}, u_k^n, u_l^n)\|\mathcal{A}_{kl}^n\| \\
& && \qquad\qquad -\tau^n(\theta_k^n g(\tilde{n}_k, u_k^n, b_k^n) - S_k^n) \\
&(iii) && \frac{w_k^{n+1}}{w_k^n} u_k^{n+1} = \tilde{u}_k^{n+1}
\end{aligned} \tag{36}$$

In the above algorithm the position and the effective weight of any particle $k \in K$ are computed by integrating in (4) the system $(i)$ and the equation $(ii)$ i.e.

$$x_k^n = x_k(t^n) \qquad w_k^n = w(x_k(t^n)). \tag{37}$$

In particular, we have the following relation connecting $w_k^{n+1}$ with $w_k^n$

$$w_k^{n+1} = w_k^n \exp(\int_{t^n}^{t^{n+1}} \mathrm{div}(a(x_k(t), t)dt) := w_k^n Da_k^n. \tag{38}$$

**Remark 3.1** *We should point out that our analysis can be done for the general case of an adaptive smoothing length $\varepsilon(x)$ subject to a slight adaptation of Raviart's approximation results (11) (see [39],[32]) together with the additional bound $C_1 \leq \varepsilon(x)/\varepsilon_0 \leq C_2$. For more details on the derivation of these approximation results, we refer to [26] or [7].*

**Remark 3.2** *In the frame of Euler equations, one may compute a suitable local approximation at the boundary of the pressure using the neighboring fluid particles. Indeed, in view of the above formalism at any time $t^n$, for any boundary* *particle $k \in N_E$, the pressure of the boundary particles can be computed by the formula*

$$p^n(\bar{x}_k) = \sum_{l \in K} w_l^n p_l^n \|\mathcal{A}_{kl}^n\| / \sum_{l \in K} w_l^n \chi_{kl}^{\kappa} \|\mathcal{A}_{kl}^n\|.$$

*It turns out (see [8]) that this approximation produces nice volume repulsive boundary forces acting on the fluid particles inside $\Omega$. It also has the advantage to keep a perfect equilibrium for moving particles with a velocity parallel to plan boundaries.*

## 3.2 Case of ghost particles

Equipped with the quadrature formula over $\tilde{\Omega}$ (20) and the diffeomorphism $M$ (19) defined in section 2.2.2, one may extend the flux $F(u,x,t)$ and the boundary data $b(x,t)$ outside to $\Omega$ in $\Omega_M = \partial\Omega \times (0,\kappa')$ in the following appropriate way

$$\Omega_M \ni x = \bar{x} + yn(\bar{x}) \longrightarrow \begin{cases} F(u,x,t) = F(u, M^{-1}(x), t) \\ \\ b(x,t) = b(\bar{x},t), \end{cases} \tag{39}$$

with $M^{-1}(x) = \bar{x} - yn(\bar{x}) \in \partial\Omega \times (-\kappa', 0)$.

Let also denote by $\tilde{D}^*_{\varepsilon,s}$ the adjoint operator associated to $\tilde{D}_{\varepsilon,s}$ provided by (21) in which the scalar product (22) is taken over $\tilde{\Omega}$. For the wellposedness of the model below, we also take $\bar{u}_h(x,t) = b(x,t)$ for $x \in \Omega_M$. So, the particle model of (1) using the ghost particles' approach is given by

$$\int_{[0,T]} \Big[ (\bar{u}_h, \mathcal{L}_a^* \varphi)_h - \Big( \tilde{D}^*_{\varepsilon,s} F(\bar{u}_h, x, t) - S(\bar{u}_h, x, t), \varphi \Big)_h \Big] \, dt = 0. \tag{40}$$

An integration by parts with respect to $t$ shows that this model consists in finding the sequence $t \longrightarrow u_k$ for all $k \in K$ solution of the system of differential equations

$$\frac{d}{dt}(w_k u_k) \quad + \quad w_k \sum_{l \in K} w_l (F(u_k, x_k, t) + F(u_l, x_l, t)) \mathcal{A}_{kl} \tag{41}$$

$$+ \quad w_k \sum_{l \in G} \tilde{w}_l (F(u_k, x_k, t) + F(b_l, x_l, t)) \mathcal{A}_{kl} = w_k S_k. \tag{42}$$

In this scheme, the term in the right-hand side of (41) represents the SPH formulation of div$F$ in the interior domain while the term in the left-hand side of (42) provides the suitable volume formulation associated with the boundary contribution $F.n$ in the weak sense (see Appendix A for the proof of this last fact in case of smooth solutions).

Using the numerical scheme $g$ introduced in the previous case, then the upwind particle scheme may be written

$$\frac{d}{dt}(w_k u_k) \quad + \quad w_k \sum_{l \in K} w_l 2g(n_{kl}, u_k, u_l) \|\mathcal{A}_{kl}\| \tag{43}$$

$$+ \quad w_k \sum_{l \in G} \tilde{w}_l 2g(n_{kl}, u_k, b_l) \|\mathcal{A}_{kl}\| = w_k S_k. \tag{44}$$

Hence, using an explicit-time discretization, the numerical particle scheme is then given by

$$\begin{aligned}
&(i) \qquad u_k^0 = \frac{1}{meas(B_k)} \int_{B_k} u(x,0)dx \\
&(ii)' \qquad \tilde{u}_k^{n+1} = u_k^n - \tau^n 2 \sum_{l\in K} w_l^n g(n_{kl}, u_k^n, u_l^n) \|\mathcal{A}_{kl}^n\| \\
&\qquad\qquad -\tau^n \sum_{l\in G} \tilde{w}_l^n 2g(n_{kl}, u_k^n, b_l^n) \|\mathcal{A}_{kl}^n\| + \tau^n S_k^n \\
&(iii) \qquad \frac{w_k^{n+1}}{w_k^n} u_k^{n+1} = \tilde{u}_k^{n+1},
\end{aligned} \tag{45}$$

where the positions and the effective weights $(x_k^n, w_k^n)$ of the fluid particles $k \in K$ are computed by (37) and (38).

# 4 Statement of the main results

We are now ready to state the convergence results of the two approximate solutions given by the schemes (36) and (45). To that purpose, let us denote by $\tau^+ = \max_{t^n \leq T} \tau^n$ and $\tau^- = \min_{t^n \leq T} \tau^n$ and by (FS) and (DFS) the following assumptions

$$\text{(FS)} \qquad \begin{aligned} &F \in [\mathcal{C}(I\!R \times \Omega \times I\!R_+)]^d, \qquad F(u,.,.) \in [\mathcal{C}^1(\Omega \times I\!R_+)]^d \\ &S \in \mathcal{C}(I\!R \times \Omega \times I\!R_+), \end{aligned}$$

$$\text{(DFS)} \qquad F^i(0,.,.), \partial_{x_j} F^i(0,.,.), S(0,.,.) \in L^1(\Omega \times I\!R^+). \text{ for } 1 \leq i,j \leq d.$$

**Theorem 4.1** *Assume that the assumptions* (FS) *and* (DFS) *hold and that the flux $F$ and $S$ are Lipschitz with respect to $u$ uniformly on $(x,t)$. Suppose also that the initial and boundary data $(u_0, b)$ belong to $L^\infty(\Omega) \times L^\infty(\partial\Omega \times I\!R^+)$. Let $\bar{u}_h(x,t)$ be approximate solutions of* (1)*, defined by* (23) *and computed by either the scheme* (36) *or* (45)*. Suppose in addition that $\kappa' = O(\varepsilon)$ and that the following CFL condition is satisfied for some constant $\beta$,* $0 < \beta < 1$

$$\tau^+ = (1-\beta) \underset{\substack{|u|,|v| \leq C_0 \\ k,l \in K}}{Sup} \frac{\varepsilon}{|\, C(n_{kl}, u, v)\, |}, \tag{46}$$

*where the constant $C_0 = C(T, \|u_0\|_\infty, \|b\|_\infty)$. Then $\bar{u}_h(x,t)$ converges in $L^p_{loc}(\Omega \times I\!R^+)$ (for $1 \leq p < \infty$) towards $u$, the unique weak entropy solution of (1) in Otto's sense* [37]*, when $\Delta(\varepsilon, h) \to 0$, $\dfrac{\varepsilon}{\sqrt{\tau_-}} \to 0$ and $\kappa' \to 0$.*

In this convergence result, the additional assumption $\varepsilon = o(\sqrt{\tau_-})$ is used to get the suitable control of the global dissipation of the scheme (see [4] for more details in case $\Omega = I\!R^d$).

The proof of this theorem will be given in the case of boundary forces. The case of ghost particles can be done in a similar way (see [6] for the detailed proof). For both cases, the proof is based on the use of the concept of measure valued solutions (based on Young measures) and their uniqueness for equation (1). To define this concept of solutions for equation (1), let us take, by the end of this section (for an easy presentation), that the transport field $\mathbf{a} \equiv 0$, since it does not play here any special role, except involving additional terms that can be included in the flux $F$.

**Definition 4.2** *A Young measure $\nu$, with its trace $\gamma\nu$ (see lemma 1.1 in [41]), is a measure solution to problem (1-3) if for any entropy-entropy flux pair $(\eta(u), \hbar(u,x,t))$ such that $\partial_u H^i(u,x,t) = \eta'(u)\partial_u F^i(u,x,t)$ and for all non-negative test function $\varphi \in \mathcal{C}^1_c(\overline{\Omega} \times I\!R_+)$,*

$$\begin{aligned}
&\int_{\Omega\times I\!R_+} \{\langle \nu_{x,t}(\lambda), \eta(\lambda)\rangle \partial_t\varphi + \langle \nu_{x,t}(\lambda), \hbar(\lambda,x,t)\rangle \nabla_x \varphi\}\, dxdt \\
&+ \int_{\Omega\times I\!R_+} \langle \nu_{x,t}(\lambda), \sum_{i=1,d} (\partial_{x^i}(H^i(\lambda,x,t)) - \eta'(\lambda)\partial_{x^i}F^i(\lambda,x,t))\rangle \varphi dxdt \\
&+ \int_{\Omega\times I\!R_+} \langle \nu_{x,t}(\lambda), \eta'(\lambda)S(\lambda,x,t)\rangle \varphi dxdt \\
&\qquad - \int_{\partial\Omega\times I\!R_+} \langle \gamma\nu_{s,t}(\lambda), B(\lambda,b,s,t)\rangle .n(s)\varphi(s,t) dsdt \geq 0,
\end{aligned} \tag{47}$$

*and with*

$$\lim_{t\to 0+} \int_\Omega \langle \nu_{(x,t)}(\lambda), |\lambda - u_0(x)|\rangle dx = 0. \tag{48}$$

*The boundary entropy flux is defined by*

$$B(\lambda,b,s,t) = \hbar(b(s,t),s,t) - \eta'(b(s,t))(F(b(s,t),s,t) - F(\lambda,s,t)). \tag{49}$$

**Remark 4.3** *Note that the well-known boundary formulation introduced by Bardos, LeRoux and Nedelec in [1] using the Kruskov entropy $|.-c|$ can be recovered from the above one (49) by taking $\eta'(.) = sgn(.-c)$ to get*

$$B(\lambda,b,s,t) = sgn(b-c)(F(\lambda,s,t) - F(c,s,t)).$$

*However, the inequality (47) is not well-defined with the Kruskov entropy-entropy flux pair*

$$\Big(|\lambda - c|, q(\lambda,c,x,t) = sgn(\lambda - c)(F(\lambda,x,t) - F(c,x,t))\Big). \tag{50}$$

*This comes from the fact that for example the term*

$$\langle \nu_{x,t}(\lambda), \eta'(\lambda)S(\lambda,x,t)\rangle = \langle \nu_{x,t}(\lambda), sgn(\lambda - c)S(\lambda,x,t)\rangle$$

*is not well-defined for discontinuous functions. To remedy this problem, we consider a regularized entropy-entropy flux pair $(\eta^\delta(\lambda - c), H^\delta(\lambda,c,x,t)$ of (50)* *provided by regularizing the sgn function*

$$sgn_\delta(x) = \begin{cases} 1 & if\ x \geq \delta \\ \frac{x}{\delta} & if\ -\delta < x < \delta \\ -1 & if\ x \leq -\delta \end{cases} \qquad (\eta^\delta)'(\lambda - c) = sgn_\delta(\lambda - c),$$

*then*

$$H^\delta(\lambda, c, x, t) = \int_c^\lambda \eta_\delta'(v)\partial_v F(v, x, t)dv = q^\delta(\lambda, c, x, t) + q_r^\delta(\lambda, c, x, t),$$

*with*

$$\begin{aligned} q^\delta(\lambda, c, x, t) &= sgn_\delta(\lambda - c)(F(\lambda, x, t) - F(c, x, t)) \\ q_r^\delta(\lambda, c, x, t) &= \int_c^\lambda \Big(sgn_\delta(v - c) - sgn_\delta(\lambda - c)\Big)\partial_v F(v, x, t)dv. \end{aligned}$$

*Moreover, a direct computation proves that*

$$|q_r^\delta(\lambda, c, x, t)| \leq C\delta. \tag{51}$$

*Thus*

$$(\eta^\delta(\lambda - c), H^\delta(\lambda, c, x, t) \longrightarrow (|\lambda - c|, q(\lambda, c, x, t)) \quad a.e \quad in \quad \Omega \times I\!R^+.$$

**Theorem 4.4** *Assume that* $(u_0, b)$ *belongs to* $L^\infty(\Omega) \times L^\infty(\partial\Omega \times I\!R^+)$*, that the assumption* $(FS)$ *holds and that* $\nu$ *and* $\sigma$ *are Young measure solutions to* (1-3)*, in the sense of definition* (4.2)*, then, the inequality*

$$\partial_t \int_\Omega \langle \nu_{x,t}(\lambda) \otimes \sigma_{x,t}(\mu), |\lambda - \mu|\rangle dx \\ \leq - \int_\Omega \langle \nu_{x,t}(\lambda) \otimes \sigma_{x,t}(\mu), sgn(\lambda - \mu)(S(\lambda, x, t) - S(\mu, x, t))\rangle dx$$

*holds in the distribution sense on* $I\!R_+$*. If in addition* $\nu$ *and* $\sigma$ *satisfy the same initial condition (48) and* $S$ *is Lipschitz with respect to* $u \in I\!R$ *uniformly on* $(x, t)$*, then there exists a unique solution* $u \in L^\infty(\Omega \times I\!R_+)$ *such that*

$$\nu_y = \sigma_y = \delta_{u(y)}, \qquad for\ a.e.\ y \in \Omega \times I\!R_+$$

*and* $u$ *is the unique weak entropy solution to* (1-3) *in the sense of Otto* [37].

Recall that the measure tensor product $\nu_y \otimes \sigma_y$ is defined for all $g \in \mathcal{C}(I\!R^2)$ by

$$\langle \nu_y \otimes \sigma_y, g(\lambda, \mu)\rangle \equiv \int_{I\!R}\int_{I\!R} g(\lambda, \mu)d\nu_y(\lambda)d\sigma_y(\mu).$$

This uniqueness result is established in [3] in case where $F(u, x, t) = f(u)$ and $S(u, x, t) = 0$. The proof of the present general result is a slight adaptation of the one of the previous case combined with Gronwall Lemma. The main difficulty lies in the treatment of boundary conditions which is completely fulfilled in [3].

We next give an equivalent definition to (4.2) which is well-fitted for the analysis of the scheme (36) since it requires less information than (4.2) for the weak formulation of boundary conditions. The proof of this equivalence is postponed to the end of the paper (Appendix B).

**Definition 4.5** *A Young measure $\nu$ is a measure solution to problem* (1-3) *if and only if*

*(*In the interior domain*) For any entropy-entropy flux pair* $(\eta(u), \hbar(u,x,t))$ *and* $\forall \varphi \in \mathcal{C}_c^1(\Omega \times I\!R_+)$

$$\mathcal{M}^\eta(\nu_{x,t}, \varphi) \geq 0, \tag{52}$$

*with*

$$\mathcal{M}^\eta(\nu_{x,t}, \varphi) := \int_{\Omega\times I\!R_+} \{\langle \nu_{x,t}(\lambda), \eta(\lambda)\rangle \partial_t \varphi + \langle \nu_{x,t}(\lambda), \hbar(\lambda,x,t)\rangle \nabla_x \varphi dx dt +$$
$$+ \int_{\Omega\times I\!R_+} \langle \nu_{x,t}(\lambda), \sum_{i=1,d} \big(\partial_{x^i} H^i(\lambda,x,t) - \eta'(\lambda)\partial_{x^i} F^i(\lambda,x,t)\big)\rangle \varphi dx dt$$
$$+ \int_{\Omega\times I\!R_+} \langle \nu_{x,t}(\lambda), S(\lambda,x,t)\rangle \varphi dx dt + \int_\Omega \langle \nu_{x,t}(\lambda), \eta(\lambda)\rangle \varphi(x,0) dx$$

*(*The weak formulation of boundary conditions*) There exists a Radon measure* $\vartheta_{s,t} \in \mathcal{M}_b(\partial\Omega \times I\!R^+)$ *such that* $\forall \varphi \geq 0 \in \mathcal{C}_c^1(\bar\Omega \times I\!R^+)$, $\forall\, c \in I\!R$

$$\lim_{\delta\to 0} \mathcal{M}^{\eta^\delta}(\nu_{x,t}, \varphi) - \int_{\partial\Omega\times I\!R^+} sgn(b-c)\varphi(s,t)\, d\vartheta_{s,t}$$
$$+ \int_{\partial\Omega\times I\!R^+} F(c,x,t).n sgn(b-c)\varphi d\sigma(x) dt \;\geq 0. \tag{53}$$

One emphasizes that the crucial point making this formulation equivalent to Szepessy's one (47), is the existence of the term $\mathrm{sgn}(b-c)$ in the weak formulation of both boundary terms (see Appendix B for the proof).

In view of the uniqueness result given by Theorem (4.4), the remainder of this paper will be mainly concerned with the existence proof of a measure valued solution of equation (1) in the sense of definition (4.5) as a weak star limit of the approximate solutions $\bar u_h(x,t)$. This existence result follows by proving that $\bar u_h$, provided either by the scheme (36) or (45), are uniformly bounded in $L^\infty$ (section 6), weakly consistent with all entropy inequalities in the sense of definition (4.5) and finally consistent with the initial data (section 7). For this program, one needs to derive in the next section some preliminary properties of the particle scheme.

# 5 Basic properties of the scheme and entropy production

In connection with finite volume schemes ([2], [11], [22]), we have obtained in [4] a new interpretation of the hybrid particle scheme (36) in terms of one-dimensional finite difference schemes as a sum of convex decomposition up to some additional terms (see (64) below). This interpretation allows us to get a suitable discrete maximum principle result yielding the derivation of the $L^\infty$ stability of $\bar{u}_h$.

For sake of clarity when there is no ambiguity, we will use the notations for any function $G(u, x, t)$

$$G^n_{k,kl} = G(u^n_k, x_{kl}, t^n), \qquad G^n_{k,l} = G(u^n_k, x_l, t^n), \qquad G_{\bar{u}_h} = G(\bar{u}_h, x, t).$$

## 5.1 Properties of the upwind particle scheme

Define the positive constant $\lambda^n_k$ by

$$\lambda^n_k = \frac{\tau^n}{\Delta x^n_k}, \qquad \Delta x^n_k = \frac{1}{\theta^n_k + \sum_{l\in K} \Gamma^n_{kl}}, \qquad \begin{cases} \Gamma^n_{kl} = 2w^n_l \chi^\kappa_{kl} \|\mathcal{A}_{kl}\| \\ \theta^n_k = \|(D^*_{\varepsilon,s}\chi^\kappa)^n_k\|. \end{cases} \tag{54}$$

Let also recall the numerical three point scheme which is purely one-dimensional, introduced in [4], defining the interaction between any pair $(k, l)$ of particles

$$u^{n+1,l}_k = u^n_k - \lambda^n_k (g(n_{kl}, u^n_k, u^n_l) - F^n_{k,kl}.n_{kl}) \equiv W(\lambda^n_k, u^n_k, u^n_l). \tag{55}$$

Define in a similar way the interaction with boundary forces by

$$u^{n+1,f}_k = u^n_k - \lambda^n_k (g(\tilde{n}_k, u^n_k, b^n_k) - F^n_{k,k}.\tilde{n}_k). \tag{56}$$

So, to get the appropriate convex form of the scheme, one needs to find some positive constants $\alpha^{l,n}_k, \alpha^{k,n}_k$ with $\sum_{l\in K} \alpha^{l,n}_k + \alpha^{f,n}_k = 1$ such that the scheme $(ii)$ in (36) could be written as

$$\tilde{u}^{n+1}_k = \sum_{l\in K} \alpha^{l,n}_k u^{n+1,l}_k + \alpha^{f,n}_k u^{n+1,f}_k + \tau^n \Big( S^n_k - \mathcal{G}^n_k - \mathcal{B}^n_k \Big). \tag{57}$$

Indeed, as in [4], a straightforward computation using the following choice

$$\alpha^{f,n}_k = \frac{\theta^n_k}{\theta^n_k + \sum_{l\in K} \Gamma^n_{kl}}, \qquad \alpha^{l,n}_k = \frac{\Gamma^n_{kl}}{\theta^n_k + \sum_{l\in K} \Gamma^n_{kl}}, \tag{58}$$

proves that (36) and (57) are equivalent if and only if

$$\mathcal{B}^n_k = F^n_{k,k} \sum_{l\in K} w^n_l 2\chi^\kappa_{kl} \mathcal{A}^n_{kl}$$

$$
\begin{aligned}
&= F^n_{k,k}(D^*_{\varepsilon,s}\chi^\kappa)^n_k + F^n_{k,k}\sum_{l\in K} w^n_l\,(2\chi^\kappa_{kl} - \chi^\kappa_k - \chi^\kappa_l)\,\mathcal{A}^n_{kl} \\
&:= F^n_{k,k}(D^*_{\varepsilon,s}\chi^\kappa)^n_k + F^n_{k,k}\mathcal{R}(\chi^\kappa)^n_k
\end{aligned}
\tag{59}
$$

$$
\begin{aligned}
\mathcal{G}^n_k &= \sum_{l\in K} w^n_l 2\chi^\kappa_{kl} F^n_{k,kl}\mathcal{A}^n_{kl} \\
&= (\partial^*_{\varepsilon,s}\chi^\kappa F_{\bar{u}_h})^n_k + \sum_{l\in K} w^n_l\left(2\chi^\kappa_{kl}F^n_{k,kl} - \chi^\kappa_k F^n_{k,k} - \chi^\kappa_l F^n_{k,l}\right)\mathcal{A}^n_{kl} \\
&:= (\partial^*_{\varepsilon,s}\chi^\kappa F_{\bar{u}_h})^n_k + \mathcal{R}(\chi^\kappa F_{\bar{u}_h})^n_k.
\end{aligned}
$$

The discrete partial derivative term $\partial^*_{\varepsilon,s}(\chi^\kappa F_{\bar{u}_h})^n_k$ in the last equality denotes the adjoint operator associated with $\partial_{\varepsilon,s}(\chi^\kappa F_{\bar{u}_h})^n_k$ according to (22). The latter one is nothing but the SPH approximation of the partial derivative

$$
\left(\sum_{i=1,d}\partial_{x^i}[\chi^\kappa(x)F(\bar{u}_h,x,t^n)]\right)_{x=x^n_k}
$$

while the generic remainder $\mathcal{R}(.)^n_k$ is defined for any smooth function $g$ by

$$
\mathcal{R}(g)^n_k = \sum_{l\in K} w^n_l\,(2g(x^n_{kl}) - g(x^n_k) - g(x^n_l))\,\mathcal{A}^n_{kl}. \tag{60}
$$

To get the final form of the decomposition (57), we claim that

$$
\mathcal{B}^n_k + \mathcal{G}^n_k = \chi^\kappa_k\partial^*_{\varepsilon,s}(F_{\bar{u}_h})^n_k + \mathcal{N}(\chi^\kappa F)^n_k \quad \text{with} \quad \lim_{\Delta(\varepsilon,h)\to 0}\mathcal{N}(\chi^\kappa F)^n_k = 0.
$$

Indeed, on the one hand, a direct computation gives for any smooth functions $f$ and $g$ that

$$
D^*_{\varepsilon,s}(fg) = fD^*_{\varepsilon,s}g + gD^*_{\varepsilon,s}f + \Xi(f,g), \tag{61}
$$

with

$$
\Xi(f,g)(x) = \sum_{l\in K} w_l(g(x) - g(x_l)(f(x) - f(x_l))\nabla\zeta^\varepsilon(x - x_l) - 2g(x)f(x)D_\varepsilon 1(x).
$$

Thus, the estimate (12) and the bounds (14) imply that

$$
|\Xi(f,g)(x)| \leq C(\varepsilon + h/\varepsilon^2).
$$

On the other hand, successive applications of Taylor expansion in $[x_k, x_{kl}]$ and $[x_l, x_{kl}]$ combined with the bounds (14) yield the following bounds

$$
\|\mathcal{R}(\chi^\kappa)^n_k\| \leq C\varepsilon \qquad\qquad \|\mathcal{R}(\chi^\kappa F_{\bar{u}_h})^n_k\| \leq C\varepsilon. \tag{62}
$$

Consequently, the combination of (61) applied to $\chi^\kappa$ and $F_{\bar{u}_h}$ and (62) ends the proof of the claim with

$$
\mathcal{N}(\chi^\kappa F)^n_k = \mathcal{R}(\chi^\kappa F)^n_k + F^n_{k,k}.\mathcal{R}(\chi^\kappa)^n_k + \Xi(\chi^\kappa, F_{\bar{u}_h}). \tag{63}
$$

Hence, the identity (57) reads

$$\tilde{u}_k^{n+1} = \sum_{l \in K} \alpha_k^{l,n} u_k^{n+1,l} + \alpha_k^{f,n} u_k^{n+1,f} + \tau^n \Big( S_k^n - \chi_k^\kappa \partial_{\varepsilon,s}^* (F_{\bar{u}_h})_k^n + \mathcal{N}(\chi^\kappa F)_k^n \Big). \tag{64}$$

To make appear the CFL condition (46), one first needs to rewrite the schemes (55) and (56) in terms of the incremental coefficient $C(n_{kl}, u_k^n, u_l^n)$ defined in (30). Secondly, on account of (14), one may show that there exist some constants $C$, $C_-$ and $C_+$, depending only on the velocity field $a$ and the kernel $\zeta$ such that

$$(i) \;\; 0 \le \theta_k^n \le C/\varepsilon, \qquad (ii) \;\; C_- \frac{\tau_-}{\varepsilon} \le \lambda_k^n \le C_+ \frac{\tau_+}{\varepsilon}. \tag{65}$$

## 5.2 Entropy production

In practice, we require that, for any convex entropy function $\eta$, there exists a numerical entropy flux $\hbar(n,u,v)$ satisfying similar requirements as the flux $g$ i.e.

$$(i) \quad \hbar(n(x),u,u) = H(u,x,t).n(x) \qquad (ii) \quad \hbar(n,u,v) = -\hbar(-n,v,u),$$

where $H$ is an entropy flux associated with $(F,\eta)$, such that $\partial_u H^i = \eta'(u)\partial_u F^i$. Moreover, such an entropy-entropy flux pair $(\eta, H)$ verifies a certain entropy inequality. Since we have an E-scheme, the incremental coefficient $C(n,u,v)$ is positive. Therefore, by using the three point scheme (55), it follows from Proposition 3.3 in [2] (the $(x,t)$ dependence is omitted below) that for the entropy-entropy flux pair $(\eta = u^2, H)$,

$$\eta(W(\lambda,u,v)) - \eta(u) + \lambda[\hbar(n,u,v) - H(u)n] \le -\frac{\beta}{2} \mid u - v \mid^2 (C(n,u,v))^2 \lambda^2, \tag{66}$$

provided the CFL condition $\lambda Q \le 1 - \beta$ is satisfied. Note however that (66) is valid for any entropy-entropy flux pair if we take $\beta = 0$, in which case, we have

$$\hbar(n,u,v) - H(u).n \le \eta'(u)(g(n,u,v) - F(u).n). \tag{67}$$

Moreover, in view of the treatment of the boundary contribution, we also recall that (67) for $\eta = \eta_c = |u - c|$ is equivalent to

$$\mathrm{sgn}(v-c)(g(u,v) - F(c)) \le \hbar_c(u,v) \le \mathrm{sgn}(u-c)(g(u,v) - F(c)). \tag{68}$$

Let us now turn to the derivation of the entropy dissipation corresponding to our hybrid particle scheme.

*In the interior domain:* The inequality (66) reads

$$\begin{aligned} &\eta(u_k^{n+1,l}) - \eta(u_k^n) + \lambda_k^n (\hbar(n_{kl}, u_k^n, u_l^n) - H_{k,kl}^n . n_{kl}) \\ &\qquad\qquad \le -\frac{\beta}{2} \mid u_k - u_l \mid^2 (C(n_{kl}, u_k^n, u_l^n))^2 (\lambda_k^n)^2. \end{aligned} \tag{69}$$

By adding and substituting appropriate terms, this last inequality reads

$$\begin{aligned}
&\eta(u_k^{n+1,l}) - \eta(u_k^n) + \frac{\lambda_k^n}{2}(H_{k,k}^n + H_{l,l}^n)n_{kl} - \lambda_k^n H_{k,kl}^n.n_{kl} \\
&\leq -\frac{\lambda_k^n}{2}(2\hbar(n_{kl}, u_k^n, u_l^n) - (H_{k,kl}^n + H_{l,lk}^n)n_{kl}) \\
&-\frac{\lambda_k^n}{2}(H_{k,kl}^n + H_{l,lk}^n - H_{k,k}^n - H_{l,l}^n).n_{kl} - \frac{\beta}{2} \mid u_k - u_l \mid^2 (C(n_{kl}, u_k^n, u_l^n))^2 (\lambda_k^n)^2.
\end{aligned}$$

Multiplying this last inequality by $\alpha_k^{l,n}$, summing it over $(l \in K)$ and using (54) and (58), we then get

$$\begin{aligned}
&\sum_{l\in K} \alpha_k^{l,n}(\eta(u_k^{n+1,l}) - \eta(u_k^n)) + \tau^n D_{\varepsilon,s}^*(\chi^\kappa H_{\bar{u}_h})_k^n \\
&\quad -\tau^n \partial_{\varepsilon,s}^*(\chi^\kappa H_{\bar{u}_h})_k^n \leq \mathcal{R}(\chi^\kappa H)_k^n - \sum_{i=1,3}(\mu_{h,\varepsilon}^i)_k^n \\
&\qquad - \sum_{l\in K} \tau^n w_l^n \lambda_k^n \mid u_k^n - u_l^n \mid^2 (C(n_{kl}, u_k^n, u_l^n))^2 \|\mathcal{A}_{kl}\|,
\end{aligned} \tag{70}$$

where the remainder $\mathcal{R}(\chi^\kappa H)_k^n$ is defined according to (60) and the above measure terms are given by

$$(\mu_{h,\varepsilon}^1)_k^n = \tau^n \sum_{l\in K} w_l^n \chi_{kl}^\kappa [2\hbar(n_{kl}, u_k^n, u_l^n) - (H_{k,kl}^n - H_{l,lk}^n).n_{kl}]\|\mathcal{A}_{kl}\|,$$

$$(\mu_{h,\varepsilon}^2)_k^n = \tau^n \sum_{l\in K} w_l^n \chi_{kl}^\kappa [(H_{k,kl}^n + H_{l,lk}^n) - (H_{k,k}^n + H_{l,l}^n))]\mathcal{A}_{kl},$$

$$(\mu_{h,\varepsilon}^3)_k^n = \tau^n \sum_{l\in K} w_l^n \Big((\chi_{kl}^\kappa - \chi_k^\kappa)H_{k,k}^n + (\chi_{kl}^\kappa - \chi_l^\kappa)H_{l,l}^n\Big)\mathcal{A}_{kl}.$$

*The boundary contribution:* In a similar way, by using the three point scheme (56) and the entropy inequality (66), one gets

$$\begin{aligned}
&\alpha_k^{f,n}[\eta(u_k^{n+1,f}) - \eta(u_k^n)] + \tau^n \theta_k^n \hbar(\tilde{n}_k, u_k^n, b_l^n) - \tau^n H_{k,k}^n.D_{\varepsilon,s}^*(\chi^\kappa)_k^n \\
&\qquad \leq \tau^n H_{k,k}^n.\mathcal{R}(\chi^\kappa)_k^n - \frac{\beta}{2}\tau^n \theta_k^n \mid u_k - b_l \mid^2 (C(\tilde{n}_k, u_k, b_k))^2 (\lambda_k^n).
\end{aligned} \tag{71}$$

*The resulting interaction:* By combining the two inequalities (70) and (71) and using the identity (61), it yields

$$\begin{aligned}
&\sum_{l\in K} \alpha_k^{l,n}(\eta(u_k^{n+1,l}) + \alpha_k^{f,n}(\eta(u_k^{n+1,f}) - \eta(u_k^n) \\
&\qquad + D_{\varepsilon,s}^*(\chi^\kappa H_{\bar{u}_h})_k^n - \tau^n \chi_k^\kappa(\partial_{\varepsilon,s}^* H_{\bar{u}_h})_k^n + \tau^n \theta_k^n \hbar(\tilde{n}_k, u_k^n, b_k^n) \\
&\qquad\qquad \leq \tau^n \mathcal{N}(\chi^\kappa, H)_k^n - \sum_{i=1,3}(\mu_{h,\varepsilon}^i)_k^n - \beta\Delta_k^n(u),
\end{aligned} \tag{72}$$

where the remainder $\mathcal{N}(\chi^\kappa, H)^n_k$ is defined according to (63) and

$$\Delta^n_k(u) = 2\sum_{l\in K} \tau^n w^n_l \lambda^n_k \mid u^n_k - u^n_l \mid^2 (C(n_{kl}, u^n_k, u^n_l))^2 \|\mathcal{A}_{kl}\| \\ +2\tau^n \theta^n_k \lambda^n_k \mid u^n_k - b^n_k \mid^2 (C(\tilde{n}_k, u_k, b_k))^2. \tag{73}$$

So, by using successively the convexity inequaliy of $\eta$, the new form (64), the inequality $\eta(u+v) \le \eta(u) + v\eta^{'}(u+v)$ together with Jensen's inequality, we then obtain the final form of the entropy inequality

$$\eta(\tilde{u}^{n+1}_k) - \eta(u^n_k) + D^*_{\varepsilon,s}(\chi^\kappa H_{\bar{u}_h})^n_k + \tau^n \theta^n_k \hbar(\tilde{n}_k, u^n_k, b^n_k) \\ -\tau^n \left(\chi^\kappa_k (\partial_{\varepsilon,s} H_{\bar{u}_h})^n_k - \eta'(u^n_k)\left(S^n_k - \chi^\kappa_k \partial_{\varepsilon,s}(F_{\bar{u}_h})^n_k\right)\right) \\ \le -\sum_{i=1,5} (\mu^i_{h,\varepsilon})^n_k - \beta \Delta^n_k(u), \tag{74}$$

where the additional measure terms are defined by

$$(\mu^4_{h,\varepsilon})^n_k = \tau^n(\eta'(\tilde{u}^{n+1}_k) - \eta'(u^n_k))\left(S^n_k - \chi^\kappa_k D^x_{\varepsilon,s}(F_{\bar{u}_h})^n_k\right)$$

$$(\mu^5_{h,\varepsilon})^n_k = \tau^n \Big(\mathcal{N}(\chi^\kappa, H)^n_k - \eta'(\tilde{u}^{n+1}_k)\mathcal{N}(\chi^\kappa, F)^n_k\Big).$$

## 6 $L^\infty$ Stability and weak BV estimate

Let us denote by the sequence $v^{n+1}_k$ the convex part in the scheme (64) i.e.

$$v^{n+1}_k = \sum_{l\in K} \alpha^{l,n}_k u^{n+1,l}_k + \alpha^{f,n}_k u^{n+1,f}_k.$$

**Proposition 6.1** *Assume that* $(u_0, b) \in L^\infty(\Omega) \times L^\infty(\partial\Omega \times I\!R_+)$ *and* $\bar{u}_h(x,t)$ *is computed by the scheme* (36-38) *with E-fluxes, then for any* $T > 0$*, provided the time step* $\tau^n$ *satisfies the CFL condition* (46)*, we successively have*

$$\begin{aligned} &(i) && \min_{\substack{i\in K\\ j\in N_e}}(u^n_i, b^n_j) \le v^{n+1}_k \le \max_{\substack{i\in K\\ j\in N_e}}(u^n_i, b^n_j) \\ &(ii) && \|\bar{u}_h(.,t)\|_\infty \le K_\infty := C(T, \|u_0\|_\infty, \|b\|_\infty) \\ &(iii) && \sum_{\substack{k\in K\\ n;\, t^n\le T}} w^n_k \Delta^n_k(u) \le C_\beta \equiv \frac{1}{\beta} \times \left(\|u_0\|^2_{L^2} + C(T, \|u_0\|_\infty, \|b\|_\infty)\right), \end{aligned} \tag{75}$$

*where the local dissipation term* $\Delta^n_k(u)$ *is given by* (73).

**Proof of proposition 6.1**

*$L^\infty$ stability:* Rewriting the schemes (55) and (56) by means of the incremental coefficient $C$ given by (30), then the CFL condition (46) implies that $u_k^{n+1,l} \in I(u_k^n, u_l^n)$ and $u_k^{n+1,f} \in I(u_k^n, b_l^n)$ with

$$I(f,g) := \{w; w = \theta f + (1-\theta)g, \theta \in [0,1]\}.$$

Therefore, the inequality $(i)$ follows directly. To prove the $L^\infty$ stability, consider the increasing function $v \longrightarrow G(v,T)$ defined by

$$G(v,T) = \max_{\substack{|u|\leq v \\ x\in\Omega, t\leq T}} \left(\|\partial_x F(u,x,t)\| + \|F(u,x,t)\| + \|S(u,x,t)\|\right),$$

and start with the new form of the scheme (64). Thus, by using successively the estimates (13) and (63), we find that

$$|u_k^{n+1}| \leq \frac{w_k^n}{w_k^{n+1}} \times \left(\|u^n\|_\infty + \|b\|_\infty + \tau^n G(\|u^n\|_\infty, T) C(1+\varepsilon+\frac{h}{\varepsilon^2})\right).$$

Take $C_{\mathbf{a}} = \|\text{div}\mathbf{a}\|_\infty \exp(\|\text{div}\mathbf{a}\|_\infty T)$ then, the identity (38) yields $w_k^n / w_k^{n+1} \leq (1 + C_{\mathbf{a}}\tau^n)$ and consequently,

$$|u_k^{n+1}| \leq (1 + C_{\mathbf{a}}\tau^n) \times \left(\|u^n\|_\infty + \|b\|_\infty + \tau^n G(\|u^n\|_\infty, T) C(1+\varepsilon+\frac{h}{\varepsilon^2})\right). \tag{76}$$

Next, as in [4], define the sequence $v^n$ by

$$\begin{cases} v^0 = \|u^0\|_\infty, \\ v^{n+1} + \|b\|_\infty = (1 + C_{\mathbf{a}}\tau^n)\left(v^n + \|b\|_\infty + \tau^n G(v^n,T) C(1+\varepsilon+\frac{h}{\varepsilon^2})\right). \end{cases}$$

By construction, $\|u^n\|_\infty \leq v^n$ and from the inequality (76) we infer that

$$\begin{aligned} \frac{v^{n+1} - v^n}{\tau^n} &\leq C(C_{\mathbf{a}}, T)(v^n + \|b\|_\infty) + C(1+\varepsilon+\frac{h}{\varepsilon^2}) G(v^n, T) \\ &\equiv \phi(v^n). \end{aligned}$$

Let $\Phi(v) := \displaystyle\int_{v^0}^{v} \frac{1}{\phi(x)} dx$ ; $\Phi$ is a smooth and increasing function, its converse function also. We easily get that : $\Phi(v^{n+1}) \leq (t^{n+1} - t^0) + \Phi(v^0)$ and the following $L^\infty$ estimate holds

$$\|u^n\|_\infty \leq \Phi^{-1}\left((t^n - t^0) + \Phi(v^0)\right).$$

*Weak BV estimate $(iii)$:* Take the inequality (74) with $\beta \neq 0$ and $\eta(u) = u^2/2$, then the multiplication by $w_k^n$ and the summation over $k \in K$ yield

$$\begin{aligned} &\sum_{k\in K} w_k^n \left[\eta(\tilde{u}_k^{n+1}) - \eta(u_k^n) + D^*_{\varepsilon,s}(\chi^\kappa H_{\bar{u}_h})_k^n + \tau^n \theta_k^n \hbar(\tilde{n}_k, u_k^n, b_k^n)\right] \\ &\qquad -\tau^n \sum_{k\in K} w_k^n \left(\chi_k^\kappa (\partial_{\varepsilon,s} H_{\bar{u}_h})_k^n - \eta'(u_k^n)\left(S_k^n - \chi_k^\kappa \partial_{\varepsilon,s}(F_{\bar{u}_h})_k^n\right)\right) \\ &\qquad \leq -\sum_{i=1,5}\sum_{k\in K} w_k^n (\mu_h^i)_k^n - \beta \sum_{k\in K} w_k^n \Delta_k^n(u). \end{aligned} \tag{77}$$

Notice that, by switching the indices $k$ and $l$ and using that $\mathcal{A}_{kl} = -\mathcal{A}_{lk}$ together with $\hbar(n_{kl}, u_k^n, u_l^n) = -\hbar(n_{lk}, u_l^n, u_k^n)$, one deduces

$$\sum_{k\in K} w_k^n D^*_{\varepsilon,s}(\chi^\kappa H_{\bar{u}_h})_k^n = 0 = \sum_{i=1}^{3} \mid \sum_{k\in K} w_k^n (\nu_h^i)_k^n \mid .$$

Moreover, by adding and substituting the term $\sum_{k\in K} w_k^{n+1}\eta(u_k^{n+1})$ and summing up the inequality (77) over $t^n \leq T = t^N$, we find that

$$\begin{aligned}
&\sum_{k\in K}(w_k^N|u_k^N|^2 - w_k^0|u_k^0|^2) + \beta \sum_{\substack{k\in K\\ n;\, t^n\leq T}} w_k^n \Delta_k^n(u) + \sum_{i=4,5}\sum_{k\in K} w_k^n(\mu_h^i)_k^n \\
&\quad \leq \sum_{\substack{k\in K\\ n;\, t^n\leq T}} \tau^n w_k^n \left(\chi_k^\kappa(\partial^*_{\varepsilon,s}H_{\bar{u}_h})_k^n - u_k^n\left(S_k^n - \chi_\kappa^k \partial^*_{\varepsilon,s}(F_{\bar{u}_h})_k^n\right)\right) \\
&\quad\quad + \sum_{\substack{k\in K\\ n;\, t^n\leq T}} \tau^n w_k^n \theta_k^n \hbar(\tilde{n}_k, u_k^n, b_k^n) + \sum_{\substack{k\in K\\ n;\, t^n\leq T}} (w_k^{n+1}|u_k^{n+1}|^2 - w_k^n|\tilde{u}_k^{n+1}|^2) \\
&\quad \equiv R^2 + R^3 + R^4.
\end{aligned} \tag{78}$$

Thus, straightforward computations using successively the identities (36) $(iii)$ and (38), the Lipschitz continuity of the numerical flux $\hbar(n,u,v)$, the fact that $\mathrm{supp}(\theta(x)) \subset \partial\Omega \times [0, \kappa']$ together with the estimates (11-13) and (65) $(ii)$ imply the following estimates

$$|R^2| + |R^3| \leq C(T, \varepsilon, h/\varepsilon^2, \kappa'/\varepsilon, \|u^0\|_\infty, \|b\|_\infty), \quad |R^4| \leq C(T, \frac{\kappa'}{\varepsilon}, \|\bar{u}_h\|_{L^2}). \tag{79}$$

On the other hand, since $\eta(u) = u^2/2$, we have

$$\eta'(\tilde{u}_k^{n+1}) - \eta'(u_k^n) = \tilde{u}_k^{n+1} - u_k^n,$$

therefore, in view of (36) $(ii)$, the combination of the bounds (14), (62) with the CFL condition (46) gives

$$\sum_{i=4,5} \mid \sum_{k\in K} w_k^n (\mu_h^i)_k^n \mid \leq C(T, \varepsilon, \|u^0\|_\infty, \|b\|_\infty). \tag{80}$$

By taking the results (79) and (80) into account, the inequality (78) yields the weak estimate (75) $(iii)$.

# 7 Proof of Theorem 4.1

The proof of this theorem will be split into three steps. In the first step, we derive the global weak entropy formulation of the scheme (36) while the last two steps are devoted to the existence proof of a measure-valued solution in the sense of the definition (4.5).

**First step: Weak entropy form of the scheme** In connection with the discrete scalar product used in (22), we are going to use the following notation whenever a time integration is added

$$(g,f)_{h,T} = \sum_{\substack{\{n,\ t^n \leq T\} \\ k\in K}} w_k^n g_k^n f_k^n = \int_{\mathcal{Q}_T} g(x,t) f(x,t) dx dt + \mathcal{E}^{h,T}(gf), \qquad (81)$$

where $\mathcal{Q}_T = \Omega \times [0,T]$ and $\mathcal{E}^{h,T}(gf)$ denotes the resulting quadrature error. Thus, we have

**Proposition 7.1** *For all nonnegative test function $\varphi \in \mathcal{C}_c^1(\bar{\Omega} \times I\!\!R_+)$, the approximate solutions $\bar{u}_h$ defined by* (23) *and computed by the scheme* (36)-(38) *satisfy*

$$\begin{aligned} \mathcal{M}^\eta(\delta_{\bar{u}_h(x,t)}, \chi^\kappa, \varphi) - \Gamma(\varphi)^{\kappa'}_{h,T} \quad &\geq \quad \sum_{i=1,7} < \mu^i_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T} + \sum_{m=1,4} \mathcal{R}_m^{h,\varepsilon} \\ &\equiv \quad < \mu_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T}, \end{aligned} \qquad (82)$$

*where*

$$\begin{aligned} \mathcal{M}^\eta(\delta_{\bar{u}_h(x,t)}, \mathbf{a}, \chi^\kappa, \varphi) := &\int_{\Omega\times I\!\!R_+} [\eta(\bar{u}_h)\mathcal{L}^*_{\mathbf{a}}(\varphi) + \chi^\kappa \hbar(\bar{u}_h, x, t).D_{\varepsilon,s}\varphi]\, dxdt \\ &+ \int_{\Omega\times I\!\!R_+} \left[\left(\eta(\bar{u}_h) - \eta^{'}(\bar{u}_h)\bar{u}_h\right) div(\mathbf{a}(x,t)\right] \varphi dxdt \\ &+ \int_{\Omega\times I\!\!R_+} \chi^\kappa \left(\partial^*_{\varepsilon,s} H_{\bar{u}_h} - \eta^{'}(\bar{u}_h)\partial^*_{\varepsilon,s} F_{\bar{u}_h}\right) \varphi dxdt \\ &+ \int_{\Omega\times I\!\!R_+} \eta^{'}(\bar{u}_h) S_{\bar{u}_h} \varphi dxdt + \int_\Omega \eta(\bar{u}_h(x,0))\varphi(x,0)dx, \end{aligned}$$

$$\Gamma(\varphi)^{\kappa'}_{h,T} = \sum_{\substack{\{n,\ t^n \leq T\} \\ k\in K}} \tau^n w_k^n \theta_k^n \hbar(\tilde{n}_k, u_k^n, b_k^n)\varphi_k^n$$

*while the right hand side $< \mu_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T}$ will be made precise below.*

**Proof of proposition 7.1** As in [4], start with the inequality (74) for all convex entropy $\eta$ (i.e. $\beta = 0$), multiply it by $\varphi_k^n w_k^n$ and take the double sum over $\{n,\ t^n \leq T\}$ and $k \in K$, on the one hand. On the other hand, making an integration by parts and using that $\mathcal{A}_{kl} = -\mathcal{A}_{lk}$ together with the notation (81), one gets

$$\begin{aligned} - \quad &\sum_{\substack{\{n,\ t^n \leq T\} \\ k\in K}} \varphi_k^n w_k^n \left(\eta(\tilde{u}_k^{n+1}) - \eta(u_k^n)\right) + \left(\hbar(\bar{u}_h, x, t), \chi^\kappa D_{\varepsilon,s}\varphi\right)_{h,T} \qquad (83) \\ + \quad &\left(\varphi, \chi^\kappa(\partial^*_{\varepsilon,s}(H_{\bar{u}_h})_k^n - \eta'(\bar{u}_h)(\chi_k^\kappa \partial^*_{\varepsilon,s}(F_{\bar{u}_h})_k^n - S_{\bar{u}_h})\right)_{h,T} \\ - \quad &\Gamma(\varphi)^{\kappa'}_{h,T} \leq \sum_{i=1}^5 < \mu^i_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T}, \end{aligned}$$

where the right hand side is given by

$$< \mu^1_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T} = - \sum_{\substack{\{n,\ t^n \le T\} \\ (k,l) \in K^2}} \tau^n w^n_k w^n_l \chi^\kappa_{kl} (\varphi^n_k - \varphi^n_l) [\hbar(n_{kl}, u^n_k, u^n_l) - H^n_{kl}.n_{kl}] \| \mathcal{A}_{kl} \|$$

$$< \mu^2_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T} = - \sum_{\substack{\{n,\ t^n \le T\} \\ (k,l) \in K^2}} \tau^n w^n_k w^n_l \chi^\kappa_{kl} \Big( \varphi^n_k - \varphi^n_l \Big) (H^n_l - H^n_{l,lk}).\mathcal{A}_{kl}$$

$$< \mu^3_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T} = \sum_{\substack{\{n,\ t^n \le T\} \\ (k,l) \in K^2}} \tau^n w^n_k \tau^n w^n_k w^n_l (\varphi^n_k - \varphi^n_l)(\chi^\kappa_{kl} - \chi^\kappa_k) H^n_k.\mathcal{A}_{kl}$$

$$< \mu^4_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T} = - \sum_{\substack{\{n,\ t^n \le T\} \\ k \in \bar{K}}} \tau^n w^n_k \varphi^n_k (\eta'(\tilde{u}^{n+1}_k) - \eta'(u^n_k)) \left[ -\chi^\kappa_k \partial^*_{\varepsilon,s} (F_{\bar{u}_h})^n_k + S^n_k \right]$$

$$< \mu^5_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T} = - \sum_{\substack{\{n,\ t^n \le T\} \\ k \in \bar{K}}} \tau^n w^n_k \varphi^n_k \Big( \mathcal{N}(\chi^\kappa, H)^n_k - \eta'(\tilde{u}^{n+1}_k) \mathcal{N}(\chi^\kappa, F)^n_k \Big).$$

Let us now denote by $\mathcal{D}$ the first term in the left hand side of (83) ; then, one establishes with similar arguments as before (see [4] for the detailed proof) that

$$\begin{aligned} \mathcal{D} \ge & \Big( \eta(\bar{u}_h), \mathcal{L}^*_{\mathbf{a}} \varphi \Big)_{h,T} + \Big( \eta(\bar{u}_h), \varphi \Big)_h \\ & + \Big( \Big( \eta(\bar{u}_h) - \eta^{'}(\bar{u}_h) \bar{u}_h \Big) \operatorname{div} \mathbf{a}, \varphi \Big)_{h,T} + < \mu^6_{h,\varepsilon} + \mu^7_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T}, \end{aligned} \tag{84}$$

where the above measure terms are defined as follows

$$< \mu^6_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T} = - \sum_{\substack{n,\ t^n \le T \\ k \in \bar{K}}} \left[ \omega^{n+1}_k \eta(u^{n+1}_k) - \omega^n_k \eta(u^n_k) \right] (\varphi^{n+1}_k - \varphi^n_k)$$

$$< \mu^7_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T} =$$

$$= \sum_{\substack{n,\ t^n \le T \\ k \in \bar{K}}} \varphi^n_k \left[ (\eta^{'}(u^{n+1}_k) \tilde{u}^{n+1}_k - \eta(\tilde{u}^{n+1}_k)) - (\eta^{'}(u^n_k) u^n_k - \eta(u^n_k)) \right] \left( \omega^{n+1}_k - \omega^n_k \right).$$

Thus, the combination of the inequalities (84) and (83) yields

$$\begin{aligned} & \Big( \eta(\bar{u}_h), \mathcal{L}^*_{\mathbf{a}} \varphi \Big)_{h,T} + \Big( \eta(\bar{u}_h) - \eta^{'}(\bar{u}_h) \bar{u}_h \operatorname{div} \mathbf{a}, \varphi \Big)_{h,T} \\ & + \Big( H_{\bar{u}_h}, \chi^\kappa D_{\varepsilon,s} \varphi \Big)_{h,T} + \Big( \varphi, \chi^\kappa \partial^*_{\varepsilon,s} (H_{\bar{u}_h}) - \eta'(\bar{u}_h)(\chi^\kappa \partial^*_{\varepsilon,s} (F_{\bar{u}_h}) - S_{\bar{u}_h}) \Big)_{h,T} \\ & \quad - (\mu^0_{h,\varepsilon}, \varphi)^{\kappa'}_{h,T} + \Big( \eta(\bar{u}_h), \varphi \Big)_h \le \sum_{i=1}^{7} < \mu^i_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T} . \end{aligned} \tag{85}$$

Finally, the desired inequality (82) follows by using the decomposition in (81) and denoting by $\mathcal{R}_i^{h,\varepsilon}$ for $i \in \{1,..4\}$ the corresponding quadrature error terms on $\mathcal{Q}_T$ and on $\Omega$ for the initial data

$$\mathcal{R}_1^{h,\varepsilon} = \mathcal{E}^{h,T}\Big(\eta(\bar{u}_h)\mathcal{L}^*_{\mathbf{a}}\varphi + (\eta(\bar{u}_h) - \eta'(\bar{u}_h)\bar{u}_h)\varphi \mathrm{div}\mathbf{a}\Big), \;\; \mathcal{R}_2^{h,\varepsilon} = \mathcal{E}^{h,T}\Big(H_{\bar{u}_h}\chi^\kappa D_{\varepsilon,s}\varphi\Big)$$

$$\mathcal{R}_3^{h,\varepsilon} = \mathcal{E}^{h,T}\Big(\varphi(\chi^\kappa \partial^*_{\varepsilon,s} H_{\bar{u}_h} - \eta'(\bar{u}_h)(\chi^\kappa \partial^*_{\varepsilon,s} F_{\bar{u}_h} - S_{\bar{u}_h}))\Big), \qquad \mathcal{R}_4^{h,\varepsilon} = \mathcal{E}^h\Big(\eta(\bar{u}_h)\varphi\Big).$$

**Second step: Derivation of (47)** On account of the $L^\infty$ stability result of $\bar{u}_h$ (75) $(ii)$ and following [43] and [13], one can extract a subsequence $\{\bar{u}_{h_j}\}$ with an associated Young measure-valued mapping $\nu_{(\cdot)} : \Omega \times I\!R_+ \to \mathrm{Prob}([-K_\infty, K_\infty])$ (see (75) (ii) for the value of the constant $K_\infty$), such that

$$\lim_{\substack{\Delta(h,\varepsilon)\longrightarrow 0\\ \kappa'\longrightarrow 0}} \mathcal{M}^\eta(\delta_{\bar{u}_h(x,t)}, \mathbf{a}, \chi^\kappa, \varphi) = \mathcal{M}^\eta(\nu_{x,t}, \mathbf{a}, \varphi), \tag{86}$$

where $\mathcal{M}^\eta(\nu_{x,t}, \mathbf{a}, \varphi)$ is given by

$$\begin{aligned}\mathcal{M}^\eta(\nu_{x,t}, \mathbf{a}, \varphi) := & \int_{\Omega\times I\!R_+} \{\langle \nu_{x,t}(\lambda), \eta(\lambda)\rangle \mathcal{L}^*_{\mathbf{a}}(\varphi) + \langle \nu_{x,t}(\lambda), \hbar(\lambda, x, t)\rangle \nabla_x \varphi dx dt \\ & + \int_{\Omega\times I\!R_+} \langle \nu_{x,t}(\lambda), \Big(\eta(\lambda) - \eta'(\lambda)\lambda\Big)\rangle \mathrm{div}(\mathbf{a}(x,t))\varphi dx dt \\ & + \int_{\Omega\times I\!R_+} \langle \nu_{x,t}(\lambda), \sum_{i=1,d} \big(\partial_{x^i} H^i(\lambda, x, t) - \eta'(\lambda)\partial_{x^i} F^i(\lambda, x, t)\big)\rangle \varphi dx dt \\ & + \int_{\Omega\times I\!R_+} \langle \nu_{x,t}(\lambda), S(\lambda, x, t)\rangle \varphi dx dt + \int_\Omega \langle \nu_{x,t}(\lambda), \eta(\lambda)\rangle \varphi(x,0) dx.\end{aligned}$$

Moreover, the combination of the bounds (14), the approximation results (6) and (13) together with the available regularity of $\chi^\kappa$, $\varphi$ and the flux $H$ with respect to the space variable, implies

$$\lim_{\Delta(h,\varepsilon)\longrightarrow 0} \sum_{i=2,7} | < \mu^i_{h,\varepsilon}, \varphi >_{\mathcal{Q}_T} | = \lim_{\Delta(h,\varepsilon)\longrightarrow 0} \sum_{m=1,4} |\mathcal{R}_m^{h,\varepsilon}| = 0. \tag{87}$$

We recall that the main difficulty encountered in [4] lies in the evaluation of the dissipative term $< \mu^1_{h,\varepsilon}, \varphi >$. We have then proved that

$$\liminf_{\substack{\Delta(\varepsilon,h)\to 0\\ \frac{\varepsilon}{\sqrt{\tau^-}}\to 0}} < \mu^1_{h,\varepsilon}, \varphi > \geq 0. \tag{88}$$

The proof combines similar arguments as those used in (87), the weak $BV$ estimate $(iii)$ in (75) together with the inequality (67) and the following suitable decomposition (used in order to control the sign of the difference $\varphi^n_k - \varphi^n_l$ with quantities that go to zero as $\varepsilon \to 0$ )

$$\varphi^n_k - \varphi^n_l = \underbrace{(\varphi^n_k - \varphi^n_l + A - B)}_{-C\varepsilon\leq \dots\dots \leq 0} + \underbrace{(B - A)}_{0\leq\dots\dots\leq C\varepsilon}$$

with

$$A = \inf_{k \in K} \inf_{x \in B(x_k, C\varepsilon)} (\varphi(x,t) - \varphi(x_k,t)), \quad B = \sup_{k \in K} \sup_{x \in B(x_k, C\varepsilon)} (\varphi(x,t) - \varphi(x_k,t)).$$

By contrast, in the present convergence, we still have to deal with the evaluation of the volume approximation $(\mu^0_{\kappa',\varepsilon}, \varphi)_{h,T}$ which needs a more careful treatment. With this end in view, note that on account of the estimate (65)$(i)$ and the fact that $\mathrm{supp}(\theta(x)) \subset \partial\Omega \times [0, \kappa']$ together with the Lipschitz continuity of the numerical flux $\hbar$, we arrive at

$$|\ \Gamma(\varphi)^{\kappa'}_{h,T}\ | \leq C\mathrm{meas}(\partial\Omega)(\|\bar{u}_h\|_\infty, \|b\|_\infty)\frac{\kappa'}{\varepsilon}. \tag{89}$$

This clearly shows the boundedness of this term according to the assumption $\kappa' = O(\varepsilon)$.

Let us now denote by

$$\vartheta^{\kappa'}_{h,\varepsilon}(x,t) = \theta(x)\hbar(\bar{n}(x), \bar{u}_h(x,t), b_{\kappa'}(x,t)), \quad \bar{\varphi} = \varphi(x(\bar{x},0),t),$$

thus, one may write

$$\begin{aligned}
\Gamma(\varphi)^{\kappa'}_{h,T} \quad \underbrace{=}_{\text{by}(81)} \quad & \sum_{\substack{\{n,\ t^n \leq T\} \\ k \in K}} \chi_{B_k}\chi_{[t^n,t^{n+1}[}\langle \vartheta^{\kappa'}_{h,\varepsilon}, \varphi\rangle + \mathcal{E}^{h,T}(\vartheta^{\kappa'}_{h,\varepsilon}\varphi) \\
= \quad & \int_{\Omega\times I\!R_+} \vartheta^{\kappa'}_{h,\varepsilon}(x,t)\varphi dx dt + \mathcal{E}^{h,T}(\vartheta^{\kappa'}_{h,\varepsilon}\varphi) \\
= \quad & \int_{\Omega\times I\!R_+} \vartheta^{\kappa'}_{h,\varepsilon}(x,t)\Big(\bar{\varphi} + (\varphi - \bar{\varphi})\Big) dx dt + \mathcal{E}^{h,T}(\vartheta^{\kappa'}_{h,\varepsilon}\varphi) \\
\underbrace{=}_{\text{by}(15)} \quad & \int_{\partial\Omega\times I\!R_+} \underbrace{\left(\int_0^{3\kappa} \vartheta^{\kappa'}_{h,\varepsilon}(x(\bar{x},y),t)J(\bar{x},y)dy\right)}_{\equiv \bar{\vartheta}^{\kappa'}_{h,\varepsilon}(\bar{x},t)} \bar{\varphi}(\bar{x},t) d\bar{x} dt \\
& + \int_{\Omega\times I\!R_+} \vartheta^{\kappa'}_{h,\varepsilon}(x,t)(\varphi - \bar{\varphi}) dx dt + \mathcal{E}^{h,T}(\vartheta^{\kappa'}_{h,\varepsilon}\varphi) \\
= \quad & \int_{\partial\Omega\times I\!R_+} \bar{\vartheta}^{\kappa'}_{h,\varepsilon}(\bar{x},t)\bar{\varphi}(\bar{x},t) d\bar{x} dt \\
& + \int_{\Omega\times I\!R_+} \vartheta^{\kappa'}_{h,\varepsilon}(x,t)(\varphi - \bar{\varphi}) dx dt + \ \mathcal{E}^{h,T}(\vartheta^{\kappa'}_{h,\varepsilon}\varphi).
\end{aligned} \tag{90}$$

Denote by $\mathcal{R}(\vartheta^{\kappa'}_{h,\varepsilon}, \varphi - \bar{\varphi})$ the second integral term in the above inequality. Thus, the quadrature error (5) with similar arguments used in (89) implies

$$(i) \quad |\ \mathcal{E}^{h,T}(\vartheta^{\kappa'}_{h,\varepsilon}\varphi)\ | \leq C\frac{h}{\varepsilon}, \quad (ii) \quad |\ \mathcal{R}(\vartheta^{\kappa'}_{h,\varepsilon}, \varphi - \bar{\varphi})\ | \leq C\kappa' \times \frac{\kappa'}{\varepsilon}. \tag{91}$$

The assumption $h = o(\varepsilon^2)$ (in particular $h = o(\varepsilon)$), together with (91) $(i)$ and (89) yield that the sequence $\vartheta^{\kappa'}_{h,\varepsilon} \in L^1(\Omega \times [0,T])$. Moreover, on account of (91) $(ii)$, the sequence $\bar{\vartheta}^{\kappa'}_{h,\varepsilon}$ is bounded in $\in L^1(\partial\Omega \times [0,T])$. Thus, there exists a subsequence which still is denoted by $\bar{\vartheta}^{\kappa'}_{h,\varepsilon}$ converging to some bounded Radon measure $\vartheta(\bar{x},t)$ for the topology $\sigma(M_b, C_c)$ i.e.

$$\int_{\partial\Omega \times I\!R_+} \bar{\vartheta}^{\kappa'}_{h,\varepsilon}(\bar{x},t)\bar{\varphi}(\bar{x},t)d\bar{x}dt \underset{\substack{\Delta(h,\varepsilon)\longrightarrow 0\\ \kappa'\longrightarrow 0}}{\longrightarrow} \langle \vartheta(\bar{x},t), \bar{\varphi}\rangle, \qquad \forall \bar{\varphi} \in C_c(\partial\Omega \times [0,T]). \tag{92}$$

Consequently,

$$\Gamma(\varphi)^{\kappa'}_{h,T} \underset{\substack{\Delta(h,\varepsilon)\longrightarrow 0\\ \kappa'\longrightarrow 0}}{\longrightarrow} \langle \vartheta(\bar{x},t), \bar{\varphi}\rangle, \qquad \forall \bar{\varphi} \in C_c(\partial\Omega \times [0,T]), \tag{93}$$

In view of the results (86-88) and the above limit, the inequality (82) implies

$$\mathcal{M}^\eta(\nu_{x,t}, \mathbf{a}, \varphi) - \langle \vartheta(\bar{x},t), \bar{\varphi}\rangle \geq 0.$$

Therefore, the inequality (47) follows by taking $\varphi \in \mathcal{C}^1_c(\Omega \times I\!R_+)$.

**Third step: Derivation of (53)** Take $(\eta^\delta, H^\delta)$ defined in Remark (4.3) as an entropy-entropy flux pair $(\eta, H)$ in (82) and denote by $\hbar^\delta$ the numerical flux associated with the flux $H^\delta$, then one first applies the inequality (68) to get

$$\hbar^\delta(n,u,v) \geq q^\delta_r(u,c,x,t).n + \operatorname{sgn}_\delta(v-c)(g(n,u,v) - F(c,x,t).n) \tag{94}$$

with

$$q^\delta_r(u,c,x,t) = \int_c^u \Big( \operatorname{sgn}_\delta(w-c) - \operatorname{sgn}_\delta(u-c)\Big)\partial_w F(w,x,t)dw.$$

Secondly, as before, denoting by $\psi_\delta = \varphi \operatorname{sgn}_\delta(b-c)$ and

$$\vartheta^{\kappa'}_{h,\varepsilon}(x,t) = \theta(x) g(\bar{n}(x), \bar{u}_h(x,t), b_{\kappa'}(x,t)), \quad \bar{\psi}_\delta = \psi_\delta(x(\bar{x},0),t),$$

one may write

$$\begin{aligned}
\Gamma(\varphi)^{\kappa'}_{h,T} \underbrace{\leq}_{\text{by}(94)} & \underbrace{\Gamma(\varphi)^{\kappa',\delta}_{h,T}}_{\text{see below}} - \Big(\operatorname{sgn}_\delta(b_{\kappa'}-c)F(c,x,t)D^*_{\varepsilon,s}\chi^\kappa, \varphi\Big)_{h,T} \\
& + \Big(q^\delta_r(\bar{u}_h,c,x,t)D^*_{\varepsilon,s}\chi^\kappa, \varphi\Big)_{h,T} \\
\underbrace{=}_{\text{by}(81)} & \Gamma(\varphi)^{\kappa',\delta}_{h,T} - \int_{\Omega\times I\!R^+} F(c,x,t)\psi_\delta \nabla(1-\chi^\kappa)dxdt \\
& + \int_{\Omega\times I\!R^+} q^\delta_r(\bar{u}_h,c,x,t)\varphi\nabla(1-\chi^\kappa)dxdt \\
& + \mathcal{E}^{h,T}\Big(F(c,x,t)\nabla(1-\chi^\kappa)\psi_\delta\Big) \\
& + \mathcal{E}^{h,T}\Big(q^\delta_r(\bar{u}_h,c,x,t)\nabla(1-\chi^\kappa)\varphi\Big)
\end{aligned} \tag{95}$$

with

$$\Gamma(\varphi)^{\kappa',\delta}_{h,T} := \sum_{k\in K}\tau^n w^n_k\varphi^n_k\theta^n_k \mathrm{sgn}_\delta(b^n_k-c)g(\tilde{n}_k,u^n_k,b^n_k)$$
$$\underbrace{=}_{\text{as in } (90)} \int_{\partial\Omega\times I\!R^+}\bar{\vartheta}^{\kappa'}_{h,\varepsilon}(\bar{x},t)\bar{\psi}_\delta d\bar{x}dt + \mathcal{E}^{h,T}(\vartheta^{\kappa'}_{h,\varepsilon}\psi_\delta)$$
$$+\mathcal{R}(\vartheta^{\kappa'}_{h,\varepsilon},\psi_\delta-\bar{\psi}_\delta).$$

So that, as in the previous step, one deduces, that there exists a bounded Radon measure $\vartheta(\bar{x},t)$ such that

$$\Gamma(\varphi)^{\kappa',\delta}_{h,T} \underset{\substack{\Delta(h,\varepsilon)\longrightarrow 0\\ \kappa'\longrightarrow 0}}{\longrightarrow} \langle\vartheta(\bar{x},t),\mathrm{sgn}_\delta(b-c)\bar{\varphi}\rangle, \qquad \forall\bar{\varphi}\in C_c(\partial\Omega\times[0,T]). \tag{96}$$

On the other hand, since we have in the weak star limit that

$$\lim_{\Delta(\varepsilon,h)\to 0} q^\delta_r(\bar{u}_h,c,x,t) = \langle\nu_{x,t},q^\delta_r(\lambda,c,x,t)\rangle,$$

then, by using the following splitting

$$q^\delta_r(\bar{u}_h,c,x,t) = \langle\nu_{x,t},q^\delta_r(\lambda,c,x,t)\rangle + (q^\delta_r(\bar{u}_h,c,x,t) - \langle\nu_{x,t},q^\delta_r(\lambda,c,x,t)\rangle)$$

together with the fact that $\kappa' = O(\kappa)$ and Szepessy's weak trace limit (Lemma 1.1 in [41]), one gets

$$\lim_{\substack{\Delta(\varepsilon,h)\to 0\\ \kappa'\to 0}} \int_{\Omega\times I\!R^+} q^\delta_r(\bar{u}_h)\varphi\nabla(1-\chi^\kappa)dxdt = \int_{\partial\Omega\times I\!R^+}\langle\gamma\nu_{x,t},q^\delta_r(\lambda)\rangle\varphi d\sigma(x)dt. \tag{97}$$

We also have

$$\lim_{\kappa'\to 0}\int_{\Omega\times I\!R^+} F(c,x,t)\psi_\delta\nabla(1-\chi^\kappa)dxdt = \int_{\partial\Omega\times I\!R^+}\mathrm{sgn}_\delta(b-c)F(c,x,t)\varphi d\sigma(x)dt. \tag{98}$$

In view of the results (96-98), the inequality (95) yields

$$\begin{aligned}\lim_{\substack{\Delta(\varepsilon,h)\to 0\\ \kappa'\to 0}}(\Gamma(\varphi)^{\kappa'}_{h,T} \le & \int_{\partial\Omega\times I\!R^+}\mathrm{sgn}_\delta(b-c)\varphi \;\; d\vartheta(\sigma(x),t)\\ & -\int_{\partial\Omega\times I\!R^+}\mathrm{sgn}_\delta(b-c)F(c,x,t)\varphi d\sigma(x)dt\\ & +\int_{\partial\Omega\times I\!R^+}\langle\gamma\nu_{x,t},q^\delta_r(\lambda,c,x,t)\rangle\varphi d\sigma(x)dt.\end{aligned} \tag{99}$$

On account of the results (86-88) with $\eta = \eta^\delta$ and the bound (51), the inequality (82) gives

$$\begin{aligned}\mathcal{M}^{\eta^\delta_c}(\nu_{x,t},\mathbf{a},\varphi) - \int_{\partial\Omega\times}\mathrm{sgn}_\delta(b-c)\varphi \;\; d\vartheta(\sigma(x),t)\\ +\int_{\partial\Omega\times}\mathrm{sgn}_\delta(b-c)F(c,x,t)\varphi d\sigma(x)dt \ge C\delta\|\varphi\|_\infty.\end{aligned} \tag{100}$$

By letting $\delta \to 0$ and using Lebesgue Theorem, the proof of (53) is completed.

**Appendix A**

Denoting by $F_i = F(b(x_i,t), x_i, t)$ (with $b$ a smooth function on $\bar{\Omega}$), one has to evaluate the term

$$\mathcal{K}^{h,\varepsilon} = \sum_{(k,l)\in(K\times G)} w_k \varphi_k \tilde{w}_l (F_k + F_l) \mathcal{A}_{kl}.$$

To this end, let us introduce the characteristic function $\chi$ of the domain $\Omega$ and use the notation $\chi_i = \chi(x_i)$; then, one may write, on the one hand,

$$\begin{aligned}
\sum_{(k,l)\in(K\times G)} w_k \varphi_k \tilde{w}_l F_l \mathcal{A}_{kl} &= \sum_{(k,l)\in(K\cup G)^2} \chi_k (1-\chi_l) \tilde{w}_k \tilde{w}_l \varphi_k F_l \mathcal{A}_{kl} \\
&= \sum_{k\in K} \chi_k w_k \varphi_k (\tilde{D}_{\varepsilon,s} F)_{x=x_k} - \sum_{(k,l)\in(K\times K)} w_k w_l \chi_k \chi_l \varphi_k F_l \mathcal{A}_{kl} \\
&\underbrace{\approx}_{\text{by (11),(6)}} \int_\Omega \varphi \operatorname{div} F(b(x,t),x,t) dx - \sum_{(k,l)\in(K\times K)} w_k w_l \chi_k \chi_l \varphi_k F_l \mathcal{A}_{kl} \\
&= \int_\Omega \varphi \operatorname{div} F(b(x,t),x,t) dx + I;
\end{aligned}$$

on the other hand,

$$\begin{aligned}
&\sum_{(k,l)\in(K\times G)} w_k \varphi_k \tilde{w}_l F_k \mathcal{A}_{kl} \\
&= \sum_{(k,l)\in(K\times G)} w_k \tilde{w}_l \varphi_l F_k \mathcal{A}_{kl} + \sum_{(k,l)\in(K\times G)} w_k \tilde{w}_l (\varphi_k - \varphi_l) F_k \mathcal{A}_{kl} \\
&= \sum_{(k,l)\in(K\cup G)^2} \chi_k (1-\chi_l) \tilde{w}_k \tilde{w}_l \varphi_l F_k \mathcal{A}_{kl} + \sum_{(k,l)\in(K\times G)} w_k \tilde{w}_l (\varphi_k - \varphi_l) F_k \mathcal{A}_{kl} \\
&\underbrace{\approx}_{\text{by (11),(6)}} \int_\Omega F(b(x,t),x,t) \nabla \varphi dx - \sum_{(k,l)\in(K\times K)} w_k w_l \chi_k \chi_l \varphi_l F_k \mathcal{A}_{kl} \\
&\qquad + \sum_{(k,l)\in(K\times G)} w_k \tilde{w}_l (\varphi_k - \varphi_l) F_k \mathcal{A}_{kl} \\
&= \int_\Omega F(b(x,t),x,t) \nabla \varphi dx + I' + II.
\end{aligned}$$

Therefore, one combines the two results to get

$$\mathcal{K}^{h,\varepsilon} \approx \int_{\partial\Omega} F(b(x,t),x,t).n \ \varphi(x) d\sigma(x) + I' + I + II.$$

By switching the indices $k$ and $l$ and using that $\mathcal{A}_{kl} = -\mathcal{A}_{lk}$, one gets that $I' + I = 0$. Moreover, the estimates (14) imply that $\mid II \mid \leq C \text{meas}(\partial\Omega)\kappa'$.

Consequently,

$$\mathcal{K}^{h,\varepsilon} \underset{\substack{\Delta(h,\varepsilon)\longrightarrow 0\\ \kappa'\longrightarrow 0}}{\longrightarrow} \int_{\partial\Omega} F(b(x,t),x,t).n\ \varphi(x)d\sigma(x)$$

and the proof is completed.

## Appendix B

To prove the equivalence between definitions (4.2) and (4.5), it is sufficient to show that (4.5) implies (4.2), since the converse is obvious. To this end, the main difficulty lies in establishing that the Radon measure $\vartheta_{s,t}$ is related to the Young measure $\gamma\nu_{s,t}$ in the following sense

$$d\vartheta_{s,t} = \langle \gamma\nu_{s,t}, F(\lambda,s,t)\rangle d\sigma(s)dt. \tag{101}$$

Indeed, using the notation $\mathcal{M}^{\eta}(\nu_{x,t},\varphi)$ in (52), the inequality (47) to be proved reads

$$\mathcal{M}^{\eta}(\nu_{x,t},\varphi) - \int_{\partial\Omega\times I\!R_+} \langle \gamma\nu_{s,t}, B(\lambda,s,t)\rangle.n(s)\varphi(s,t)dsdt \geq 0. \tag{102}$$

Taking the following decomposition

$$\varphi(x,t) = \varphi(x,t)\chi_{\kappa}(x(\bar{x},y)) + \varphi(x,t)(1-\chi_{\kappa}(x(\bar{x},y))),$$

where $\chi_\kappa$ is defined by (25), the inequality (102) becomes

$$\begin{aligned} &\mathcal{M}^{\eta}\Big(\nu_{x,t},\varphi(x,t)(\chi_{\kappa}(x(\bar{x},y))\Big) + \mathcal{M}^{\eta}\Big(\nu_{x,t},\varphi(x,t)(1-\chi_{\kappa}(x(\bar{x},y)))\Big) \\ &\qquad - \int_{\partial\Omega\times I\!R_+} \langle \gamma\nu_{s,t}, B(\lambda,s,t)\rangle.n(s)\varphi(s,t)dsdt \geq 0. \end{aligned} \tag{103}$$

Since $\varphi\chi_\delta \in \mathcal{C}^1_c(\Omega\times I\!R_+)$, then, the term $\mathcal{M}^{\eta}\Big(\nu_{x,t},\varphi(x,t)(\chi_{\kappa}(x(\bar{x},y))\Big)$ is positive thanks to (52). Thereby, to prove (102), it suffices to show that

$$\begin{aligned} &\mathcal{M}^{\eta}\Big(\nu_{x,t},\varphi(x,t)(1-\chi_{\kappa}(x(\bar{x},y)))\Big) \\ &\qquad - \int_{\partial\Omega\times I\!R_+} \langle \gamma\nu_{s,t}, B(\lambda,s,t)\rangle.n(s)\varphi(s,t)dsdt \geq 0. \end{aligned}$$

By developing the derivative $\nabla(\varphi(1-\chi_\kappa))$, a straightforward calculation proves that

$$\lim_{\kappa\to 0^+} \mathcal{M}^{\eta}\Big(\nu_{x,t},\varphi(1-\chi_{\kappa})\Big) = \int_{\partial\Omega\times I\!R_+} \langle \gamma\nu, \hbar(\lambda,x,t)\rangle.n\ \varphi\ d\sigma(x)dt.$$

Therefore, by tending $\kappa \to 0^+$ in the previous inequality, we get

$$\int_{\partial\Omega\times I\!R_+} \langle \gamma\nu_{x,t}, \hbar(\lambda,x,t)\rangle.n(x)d\sigma(x)dt \\ - \int_{\partial\Omega\times I\!R_+} \langle \gamma\nu_{s,t}, B(\lambda,s,t)\rangle.n(s)\varphi(s,t)dsdt \geq 0. \tag{104}$$

To prove this inequality, one first starts with (53) and introduces the decomposition (103). Secondly, by letting $\kappa \to 0^+$ and using the same arguments as before, one obtains

$$\int_{\partial\Omega\times I\!R^+} \Big(\langle \gamma\nu, q(\lambda,c,x,t).n\rangle + \mathrm{sgn}(b-c)F(c,x,t).n\Big)\varphi d\sigma(x)dt \\ - \int_{\partial\Omega\times I\!R^+} \mathrm{sgn}(b-c)\varphi \; d\vartheta(\sigma(x),t) \geq 0. \tag{105}$$

As in [42], taking consecutively, $c = 1 + \max\{\|u_0\|_\infty, \|b\|_\infty)\}$ and $c = -1 - \max\{\|u_0\|_\infty, \|b\|_\infty)\}$, we find that

$$\int_\Gamma \langle \gamma\nu_{x,t}, F(\lambda,x,t)\rangle.n\varphi d\sigma(x)dt = \int_\Gamma \varphi \; d\vartheta(\sigma(x),t), \tag{106}$$

which yields (101). By plugging this last identity into the inequality (105), one gets

$$\int_{\partial\Omega\times I\!R^+} \langle \gamma\nu, q(\lambda,c,x,t)\rangle.n\varphi d\sigma(x)dt \\ - \int_{\partial\Omega\times I\!R^+} \langle \gamma\nu, \mathrm{sgn}(b-c)(F(\lambda,x,t)-F(c,x,t))\rangle\varphi d\sigma(x)dt \geq 0.$$

To obtain (104), it suffices to use the decomposition

$$\mathrm{sgn}(b-c)(F(\lambda,x,t)-F(c,x,t)) = \mathrm{sgn}(b-c)(F(\lambda,x,t)-F(b,x,t)) - q(b,c,x,t)$$

and to approximate any entropy function $\eta \in \mathcal{C}^1$ by the following functions $\eta_n(s) = \sum_{i=1}^{n} \alpha_i^{(n)}|s-k_i^{(n)}|$.

To end the proof of (4.5) $\Longrightarrow$ (4.2), one needs to show that (52) implies the initial condition (48). This is achieved by Theorem 2.2 in [2].

**Acknowledgement.** The author is grateful to two referees for valuable comments and suggestions. This work has been partly supported by the European network HYKE, funded by the EC contract HPRN-CT-2002-00282.

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