Stability of finite difference schemes for hyperbolic systems in two space dimensions

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Abstract

We study the stability of some finite difference schemes for hyperbolic systems in two space dimensions. The grid is assumed to be cartesian, but the space steps in each direction are not necessarily equal. Our sufficient stability conditions are shown to be also necessary for one concrete example. We conclude with some numerical illustrations of our result.

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1 Introduction

Finite difference schemes are commonly used to approximate the solutions to hyperbolic systems of conservation laws. In this paper, we are interested in the stability of such finite difference schemes when applied to constant coefficients hyperbolic systems in two space dimensions. When applied to variable coefficients or nonlinear systems, the Courant-Friedrichs-Lewy condition that we derive can be seen as a local condition that needs to be satisfied in each cell of the grid.

We consider a symmetric hyperbolic system in two space dimensions:

$$\begin{cases} \partial_t u + A_1\, \partial_{x_1} u + A_2\, \partial_{x_2} u = 0\,, & t \geq 0\,,\, x \in \mathbb{R}^2\,, \\ u_{|t=0} = u_0\,, & x \in \mathbb{R}^2\,. \end{cases} \tag{1}$$

The matrices A_1, and A_2 belong to $M_d(\mathbb{R})$, and are symmetric, so that the Cauchy problem (1) is well-posed in $L^2(\mathbb{R}^2)$, see e.g. [3]. Moreover, the solution of (1) satisfies

$$\forall\, t \geq 0\,, \quad \|u(t)\|_{L^2(\mathbb{R}^2)} = \|u_0\|_{L^2(\mathbb{R}^2)}\,. \tag{2}$$

We introduce a finite difference approximation of (1). Let Δx_1, and Δx_2 denote some space steps in the x_1, and x_2 directions, and let Δt denote the time step. Then the vector $u^n_{j,k}$, where $(n, j, k) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{Z}$, denotes an approximation of $u(n\,\Delta t, j\,\Delta x_1, k\,\Delta x_2)$. Following [1], we define

$$\lambda_1 := \frac{\Delta t}{\Delta x_1}\,, \quad \lambda_2 := \frac{\Delta t}{\Delta x_2}\,.$$

We refer to [1, chapter IV.3], and [2, chapter 6] for a general description of finite difference schemes for two-dimensional hyperbolic systems, and we shall thus assume that the reader is familiar with the basic L^2 stability theory of finite difference schemes (see e.g. [1, page 348]). In this paper, we shall study the stability of four finite difference schemes:

- The *two-dimensional* Lax-Friedrichs scheme:

$$u_{j,k}^{n+1} = \frac{1}{4}\,(u_{j-1,k}^n + u_{j+1,k}^n + u_{j,k-1}^n + u_{j,k+1}^n) - \frac{\lambda_1}{2}\,A_1\,(u_{j+1,k}^n - u_{j-1,k}^n) - \frac{\lambda_2}{2}\,A_2\,(u_{j,k+1}^n - u_{j,k-1}^n)\,. \tag{3}$$

- The *dimensional-splitting* Lax-Friedrichs scheme:

$$\begin{aligned} u_{j,k}^{n+1/2} &= \frac{1}{2}\,(u_{j-1,k}^n + u_{j+1,k}^n) - \frac{\lambda_1}{2}\,A_1\,(u_{j+1,k}^n - u_{j-1,k}^n)\,, \\ u_{j,k}^{n+1} &= \frac{1}{2}\,(u_{j,k-1}^{n+1/2} + u_{j,k+1}^{n+1/2}) - \frac{\lambda_2}{2}\,A_2\,(u_{j,k+1}^{n+1/2} - u_{j,k-1}^{n+1/2})\,. \end{aligned} \tag{4}$$

- The *two-dimensional* Godunov scheme:

$$\begin{aligned} u_{j,k}^{n+1} = u_{j,k}^n &- \frac{\lambda_1}{2}\,A_1\,(u_{j+1,k}^n - u_{j-1,k}^n) - \frac{\lambda_1}{2}\,|A_1|\,(2u_{j,k}^n - u_{j+1,k}^n - u_{j-1,k}^n) \\ &- \frac{\lambda_2}{2}\,A_2\,(u_{j,k+1}^n - u_{j,k-1}^n) - \frac{\lambda_2}{2}\,|A_2|\,(2u_{j,k}^n - u_{j,k+1}^n - u_{j,k-1}^n)\,. \end{aligned} \tag{5}$$

- The *dimensional-splitting* Godunov scheme:

$$\begin{aligned} u_{j,k}^{n+1/2} &= u_{j,k}^n - \frac{\lambda_1}{2}\,A_1\,(u_{j+1,k}^n - u_{j-1,k}^n) - \frac{\lambda_1}{2}\,|A_1|\,(2u_{j,k}^n - u_{j+1,k}^n - u_{j-1,k}^n)\,, \\ u_{j,k}^{n+1} &= u_{j,k}^{n+1/2} - \frac{\lambda_2}{2}\,A_2\,(u_{j,k+1}^{n+1/2} - u_{j,k-1}^{n+1/2}) - \frac{\lambda_2}{2}\,|A_2|\,(2u_{j,k}^{n+1/2} - u_{j,k+1}^{n+1/2} - u_{j,k-1}^{n+1/2})\,. \end{aligned} \tag{6}$$

We do not know whether the terminology is really standard, but we hope that it is clear enough. Recall that in (5), and (6), the matrices $|A_{1,2}|$ are defined as follows: let $P_{1,2}$ denote orthogonal matrices that diagonalize $A_{1,2}$:

$$P_1^{-1}\,A_1\,P_1 = \text{diag }(\alpha_1, \dots, \alpha_d)\,, \quad P_2^{-1}\,A_2\,P_2 = \text{diag }(\beta_1, \dots, \beta_d)\,. \tag{7}$$

Then the matrices $|A_1|$, and $|A_2|$, are given by:

$$P_1^{-1}\,|A_1|\,P_1 = \text{diag }(|\alpha_1|, \dots, |\alpha_d|)\,, \quad P_2^{-1}\,|A_2|\,P_2 = \text{diag }(|\beta_1|, \dots, |\beta_d|)\,. \tag{8}$$

Observe that $|A_1|$, and $|A_2|$ are symmetric, nonnegative matrices. They are positive definite if A_1, and A_2 are nonsingular.

When $\lambda_1 = \lambda_2$, the stability of (3) was completely analyzed in [4], even in the case of variable coefficients. The extension to different space steps is easy, but we give it here to enlight the difference between the stability criteria for (3) and (5).

In all what follows, the spectral radius of a square matrix M with complex entries is denoted $\rho(M)$. Our main result is the following:

Theorem 1. • *The scheme* (3) *is stable in* $\ell^2(\mathbb{Z}^2)$ *if*

$$\forall\,\vartheta \in [0, 2\pi]\,, \quad \rho(\lambda_1\,\cos\vartheta\,A_1 + \lambda_2\,\sin\vartheta\,A_2) \le \frac{1}{\sqrt{2}}\,. \tag{9}$$

- *The scheme* (4) *is stable in* $\ell^2(\mathbb{Z}^2)$ *if, and only if*

$$\max(\lambda_1\,\rho(A_1)\,, \lambda_2\,\rho(A_2)) \le 1\,. \tag{10}$$

- *The scheme (5) is stable in $\ell^2(\mathbb{Z}^2)$ if*

$$\lambda_1\,\rho(A_1)+\lambda_2\,\rho(A_2)\le 1\,. \tag{11}$$

If A_1, and A_2 are nonsingular, and if $\lambda_1\rho(A_1)+\lambda_2\rho(A_2)<1$, then the scheme (5) is dissipative (in Kreiss' sense) of order 2. Namely, if $\mathbb{G}(\xi_1,\xi_2)$ denotes the symbol of the scheme (5), there exists a constant $c>0$ such that

$$\forall(\xi_1,\xi_2)\in\left[-\frac{\pi}{\Delta x_1},\frac{\pi}{\Delta x_1}\right]\times\left[-\frac{\pi}{\Delta x_2},\frac{\pi}{\Delta x_2}\right]\,,\quad \rho(\mathbb{G}(\xi_1,\xi_2))\le 1-c\left((\xi_1\,\Delta x_1)^2+(\xi_2\,\Delta x_2)^2\right)\,.$$

- *The scheme (6) is stable in $\ell^2(\mathbb{Z}^2)$ if, and only if*

$$\max(\lambda_1\,\rho(A_1)\,,\lambda_2\,\rho(A_2))\le 1\,. \tag{12}$$

For the schemes (3), and (5), Theorem 1 only gives sufficient stability conditions. For a particular system, one may hope to get less restrictive stability conditions. However, the following result shows that the conditions of Theorem 1 are optimal in the general case (that is, they can not be improved for all symmetric hyperbolic systems):

Theorem 2. *Let A_1, and A_2 be given by*

$$A_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\,,\quad A_2=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\,.$$

Then we have the following necessary and sufficient conditions:

- *The scheme (3) is stable in $\ell^2(\mathbb{Z}^2)$ if, and only if $\sqrt{2}\,\max(\lambda_1,\lambda_2)\le 1$, which is equivalent to (9).*
- *The scheme (5) is stable in $\ell^2(\mathbb{Z}^2)$ if, and only if $\lambda_1+\lambda_2\le 1$.*

The paper is organized as follows. In section 2, we prove the first two items of Theorem 1, and we also give a Lax-Friedrichs type scheme that is always unstable. We give this example in order to highlight the fact that one should be cautious when constructing two-dimensional schemes by simply *adding* one-dimensional schemes in each direction. Such an operation may yield instabilities. In section 3, we prove the last two items of Theorem 1. Then in section 4, we prove Theorem 2. Eventually, in section 5, we compare the dissipativity of the Lax-Friedrichs and Godunov schemes with the help of numerical simulations. We shall also discuss the choice of the space steps.

2 Stability of Lax-Friedrichs type schemes

2.1 An unstable Lax-Friedrichs type scheme

There are many possible ways to construct a finite difference schemes in two space dimensions. As a first guess, one could think that it is enough to add the one-dimensional Lax-Friedrichs fluxes in each direction. Such a procedure yields the following scheme (see e.g. [1, page 346]):

$$u_{j,k}^{n+1}=\frac{1}{2}\,(u_{j-1,k}^n+u_{j+1,k}^n+u_{j,k-1}^n+u_{j,k+1}^n-2u_{j,k}^n)-\frac{\lambda_1}{2}\,A_1\,(u_{j+1,k}^n-u_{j-1,k}^n)-\frac{\lambda_2}{2}\,A_2\,(u_{j,k+1}^n-u_{j,k-1}^n)\,.$$

The symbol G of this scheme is computed by using a Fourier transform in the space variables. We obtain:

$$G(\xi_1,\xi_2)=\left(\cos(\xi_1\Delta x_1)+\cos(\xi_2\Delta x_2)-1\right)I_d-i\,(\lambda_1\,\sin(\xi_1\Delta x_1)\,A_1+\lambda_2\,\sin(\xi_2\Delta x_2)\,A_2)\,.$$

In particular, when $\xi_1\Delta x_1=\xi_2\Delta x_2=\pi$, the symbol G equals $-3I_d$, and the scheme is unstable in $\ell^2(\mathbb{Z}^2)$.

2.2 Stability of Lax-Friedrichs scheme

We now study the scheme (3). Its symbol is computed by applying a Fourier transform in the space variables. We get

$$G^{LF}(\xi_1,\xi_2) = \frac{1}{2}\left(\cos(\xi_1\Delta x_1)+\cos(\xi_2\Delta x_2)\right) I_d - i\,(\lambda_1\,\sin(\xi_1\Delta x_1)\,A_1 + \lambda_2\,\sin(\xi_2\Delta x_2)\,A_2)\;. \quad (13)$$

The matrices $A_{1,2}$ are symmetric. Therefore, the matrix $G^{LF}(\xi_1,\xi_2)$ is normal for all (ξ_1,ξ_2). The scheme (3) is thus stable if, and only if:

$$\forall\,(\xi_1,\xi_2)\in\mathbb{R}^2\,,\quad I_d - G^{LF}(\xi_1,\xi_2)^*\,G^{LF}(\xi_1,\xi_2)\geq 0\,.$$

To simplify the computations, we denote $\zeta_k=\xi_k\,\Delta x_k$, $k=1,2$. Following [4], we compute

$$\begin{aligned}I_d - G^{LF}(\xi_1,\xi_2)^*\,G^{LF}(\xi_1,\xi_2) = &\left(\frac{1}{2}(\sin^2\zeta_1+\sin^2\zeta_2)+\frac{1}{4}(\cos\zeta_1-\cos\zeta_2)^2\right)\,I_d \\ &-\left(\lambda_1\,\sin\zeta_1\,A_1+\lambda_2\,\sin\zeta_2\,A_2\right)^2 .\end{aligned}$$

Choosing ϑ such that

$$\sin\zeta_1=\cos\vartheta\,\sqrt{\sin^2\zeta_1+\sin^2\zeta_2}\,,\quad \sin\zeta_2=\sin\vartheta\,\sqrt{\sin^2\zeta_1+\sin^2\zeta_2}\,,$$

we end up with

$$I_d - G^{LF}(\xi_1,\xi_2)^*\,G^{LF}(\xi_1,\xi_2)\geq(\sin^2\zeta_1+\sin^2\zeta_2)\left(\frac{1}{2}\,I_d-\left(\lambda_1\,\cos\vartheta\,A_1+\lambda_2\,\sin\vartheta\,A_2\right)^2\right)\,.$$

The first item of Theorem 1 follows, by recalling that for a hermitian matrix H (and more generally for a normal matrix), the hermitian norm of H (that is, the norm induced by the hermitian norm in $\mathbb{C}^d$) equals the spectral radius $\rho(H)$.

2.3 Stability of the dimensional-splitting Lax-Friedrichs scheme

We now study the scheme (4). Its symbol is given by

$$G^{LFs}(\xi_1,\xi_2)=\left[\cos(\xi_2\Delta x_2)\,I_d-i\lambda_2\,\sin(\xi_2\Delta x_2)\,A_2\right]\left[\cos(\xi_1\Delta x_1)\,I_d-i\lambda_1\,\sin(\xi_1\Delta x_1)\,A_1\right].\quad (14)$$

Choosing either $\xi_1=0$, or $\xi_2=0$, it is clear that the stability of (4) implies the stability of each corresponding one-dimensional Lax-Friedrichs schemes. Therefore, if (4) is stable, then $\lambda_1\rho(A_1)$, and $\lambda_2\rho(A_2)$ are both less than 1.

Assume now that both $\lambda_1\rho(A_1)$, and $\lambda_2\rho(A_2)$ are less than 1. From (14), we see that the symbol $G^{LFs}(\xi_1,\xi_2)$ is the product of two normal matrices, each of which has a spectral radius bounded by 1. For a normal matrix, the spectral radius coincides with the hermitian norm, which implies that the hermitian norm of $G^{LFs}(\xi_1,\xi_2)$ is less than 1. This ensures that (4) is stable.

3 Stability of Godunov type schemes

3.1 Stability of the two-dimensional Godunov scheme

The symbol of the Godunov scheme (5) is

$$\begin{aligned}\mathbb{G}(\xi_1,\xi_2)=I_d-2&\left(\lambda_1\,\sin^2(\frac{\xi_1\Delta x_1}{2})\,|A_1|+\lambda_2\,\sin^2(\frac{\xi_2\Delta x_2}{2})\,|A_2|\right)\\ &-i\,(\lambda_1\,\sin(\xi_1\Delta x_1)\,A_1+\lambda_2\,\sin(\xi_2\Delta x_2)\,A_2)\;.\quad (15)\end{aligned}$$

In general, the matrix $\mathbb{G}(\xi_1,\xi_2)$ is not normal for all values of (ξ_1,ξ_2). As a matter of fact, the reader can check that $\mathbb{G}(\xi_1,\xi_2)$ is normal if, and only if the matrices A_1, and A_2 satisfy

$$A_1\,|A_2| - |A_2|\,A_1 = A_2\,|A_1| - |A_1|\,A_2 = 0\,.$$

We shall not assume that these conditions are satisfied. Instead, we are going to show that under the condition (11), one has

$$\forall\, z \in \mathbb{C}\,, \quad |z| > 1\,, \qquad \|(\mathbb{G}(\xi_1,\xi_2) - z I_d)^{-1}\| \le \frac{1}{|z|-1}\,, \tag{16}$$

where $\|\cdot\|$ denotes the usual hermitian norm in $\mathbb{C}^d$, as well as the induced matrix norm.

From the well-known Kreiss' matrix Theorem, see e.g. [2, Theorem 5.2.4], the inequality (16) yields the stability of the difference scheme (5).

Assume first of all that, under the condition (11), we can prove the inequality

$$\forall\, X \in \mathbb{C}^d\,, \quad |X^*\,\mathbb{G}(\xi_1,\xi_2)\,X| \le \|X\|^2\,. \tag{17}$$

In particular, the spectral radius $\rho(\mathbb{G}(\xi_1,\xi_2))$ is less than 1. Furthermore, let $z \in \mathbb{C}$ with $|z| > 1$, let $Y \in \mathbb{C}^d$, and let $X \in \mathbb{C}^d$ be the unique solution to

$$(\mathbb{G}(\xi_1,\xi_2) - z I_d)\,X = Y\,.$$

We get

$$z\,\|X\|^2 = X^*\,\mathbb{G}(\xi_1,\xi_2)\,X - X^*\,Y\,. \tag{18}$$

Using (17), and the Cauchy-Schwarz' inequality, (18) yields

$$(|z|-1)\,\|X\|^2 \le \|X\|\,\|Y\|\,,$$

from which we obtain (16). We thus only need to prove (17).

It is convenient to define $\eta_k = \xi_k\,\Delta x_k/2$, $k = 1,2$. Then the symbol $\mathbb{G}(\xi_1,\xi_2)$ reads

$$\mathbb{G}(\xi_1,\xi_2) = I_d - 2\left(\lambda_1\,\sin^2\eta_1\,|A_1| + \lambda_2\,\sin^2\eta_2\,|A_2|\right) - i\left(\lambda_1\,\sin(2\eta_1)\,A_1 + \lambda_2\,\sin(2\eta_2)\,A_2\right)\,.$$

Let $X \in \mathbb{C}^d$, with $\|X\| = 1$. Using the symmetry of A_1, $|A_1|$, A_2, and $|A_2|$, we compute:

$$\begin{aligned}|X^*\,\mathbb{G}(\xi_1,\xi_2)\,X|^2 = &\left[1 - 2(\lambda_1\,\sin^2\eta_1\,X^*|A_1|X + \lambda_2\,\sin^2\eta_2\,X^*|A_2|X)\right]^2\\ &+ \left[\lambda_1\,\sin(2\eta_1)\,X^*A_1X + \lambda_2\,\sin(2\eta_2)\,X^*A_2X\right]^2\,.\end{aligned} \tag{19}$$

Observing that

$$|\lambda_1\,\sin(2\eta_1)\,X^*A_1X + \lambda_2\,\sin(2\eta_2)\,X^*A_2X| \le \lambda_1\,|\sin(2\eta_1)|\,X^*|A_1|X + \lambda_2\,|\sin(2\eta_2)|\,X^*|A_2|X\,,$$

we can expand (19), and derive the inequality:

$$\begin{aligned}|X^*\,\mathbb{G}(\xi_1,\xi_2)\,X|^2 \le 1 &- 4(\mu_1\,\sin^2\eta_1 + \mu_2\,\sin^2\eta_2) + 4(\mu_1\,\sin^2\eta_1 + \mu_2\,\sin^2\eta_2)^2\\ &+ (\mu_1\,|\sin 2\eta_1| + \mu_2\,\sin|2\eta_2|)^2\,,\end{aligned} \tag{20}$$

where we have set

$$\mu_k := \lambda_k\,X^*|A_k|X\,, \quad k = 1,2\,. \tag{21}$$

Expanding the right-hand side of (20), we obtain

$$\begin{aligned}|X^*\,\mathbb{G}(\xi_1,\xi_2)\,X|^2 \le 1 &+ 4(\mu_1^2 - \mu_1)\sin^2\eta_1 + 4(\mu_2^2 - \mu_2)\sin^2\eta_2\\ &+ 2\mu_1\mu_2(4\sin^2\eta_1\sin^2\eta_2 + |\sin 2\eta_1||\sin 2\eta_2|)\,.\end{aligned} \tag{22}$$

To complete the proof, we shall use the following Lemma:

Lemma 1. *Let* $(\eta_1, \eta_2) \in \mathbb{R}^2$*, and let* $\mathcal{T}$ *denote the triangle:*

$$\mathcal{T} := \{(y_1, y_2) \in \mathbb{R}^2 / y_1 \geq 0\,,\ y_2 \geq 0\,,\ y_1 + y_2 \leq 1\}\,.$$

Then for all $(y_1, y_2) \in \mathcal{T}$*, one has*

$$(y_1^2 - y_1)\,\sin^2\eta_1 + (y_2^2 - y_2)\,\sin^2\eta_2 + 2\,y_1\,y_2\,(\sin^2\eta_1\,\sin^2\eta_2 + \frac{1}{4}\,|\sin 2\eta_1|\,|\sin 2\eta_2|) \leq 0\,.$$

From the definition (21), we have $\mu_1 \geq 0$, and $\mu_2 \geq 0$. The inequality $\mu_1 + \mu_2 \leq 1$ follows from the condition (11). Then using Lemma 1 in (22), we obtain (17). (Note that it is sufficient to prove (17) on the unit sphere by homogeneity). We now prove Lemma 1. Define

$$g(y_1, y_2) := (y_1^2 - y_1)\sin^2\eta_1 + (y_2^2 - y_2)\sin^2\eta_2 + 2y_1y_2(\sin^2\eta_1\sin^2\eta_2 + \frac{1}{4}|\sin 2\eta_1||\sin 2\eta_2|)\,. \quad (23)$$

Using the inequality

$$\begin{aligned}\sin^2\eta_1\sin^2\eta_2 + \frac{1}{4}|\sin 2\eta_1||\sin 2\eta_2| &= |\sin\eta_1||\sin\eta_2|(|\sin\eta_1||\sin\eta_2| + |\cos\eta_1||\cos\eta_2|)\\ &\leq |\sin\eta_1||\sin\eta_2|\,,\end{aligned}$$

one easily checks that g is a convex function. Therefore, the maximum of g on the triangle $\mathcal{T}$ is attained on the edges of the triangle. We compute g on each edge of $\mathcal{T}$:

$$\begin{aligned}&g(y_1, 0) = (y_1^2 - y_1)\,\sin^2\eta_1 \leq 0\,,\\ &g(0, y_2) = (y_2^2 - y_2)\,\sin^2\eta_2 \leq 0\,,\\ &g(y_1, 1 - y_1) = (y_1^2 - y_1)\,\left[\sin^2\eta_1 + \sin^2\eta_2 - 2(\sin^2\eta_1\sin^2\eta_2 + \frac{1}{4}|\sin 2\eta_1||\sin 2\eta_2|)\right] \leq 0\,.\end{aligned}$$

Consequently, g is nonpositive on $\mathcal{T}$, and Lemma 1 is proved.

3.2 Dissipativity of the Godunov scheme

We now assume that the matrices A_1, and A_2 are nonsingular, and that $\lambda_1\rho(A_1) + \lambda_2\rho(A_2) < 1$. Consequently, there exists a constant $\delta > 0$ such that for all $X \in \mathbb{C}^d$, with $\|X\| = 1$, one has

$$\mu_1 := \lambda_1\,X^*|A_1|X \geq \delta\,, \quad \mu_2 := \lambda_2\,X^*|A_2|X \geq \delta\,, \quad \mu_1 + \mu_2 \leq 1 - \delta\,.$$

For such a positive constant δ, we define the triangle:

$$\mathcal{T}_\delta := \{(y_1, y_2) \in \mathbb{R}^2 / y_1 \geq \delta\,,\ y_2 \geq \delta\,,\ y_1 + y_2 \leq 1 - \delta\}\,.$$

In order to show the dissipativity of the scheme (5), we use the inequality:

$$\rho(\mathbb{G}(\xi_1, \xi_2))^2 \leq \max_{\|X\|=1} |X^*\mathbb{G}(\xi_1, \xi_2)X|^2 \leq 1 + 4\max_{(y_1,y_2)\in\mathcal{T}_\delta} g(y_1, y_2)\,,$$

see (22), and (23). It is therefore sufficient to derive an upper bound of the function g on the triangle $\mathcal{T}_\delta$. We have the following result:

Proposition 1. *Let* $\delta > 0$ *be fixed as above. Then for all* $(\eta_1, \eta_2) \in [-\pi/2, \pi/2]^2$*, one has*

$$\max_{(y_1,y_2)\in\mathcal{T}_\delta} g(y_1, y_2) \leq 0\,,$$

and the maximum is zero if, and only if $(\eta_1, \eta_2) = (0, 0)$*. Moreover, there exists a positive constant* c *such that for all* $(\eta_1, \eta_2) \in [-\pi/2, \pi/2]^2$*, one has*

$$\max_{(y_1,y_2)\in\mathcal{T}_\delta} g(y_1, y_2) \leq -c\,(\eta_1^2 + \eta_2^2)\,. \quad (24)$$

Proof. From the definition (23) of the function g, it is clear that we only need to prove the result when $(\eta_1, \eta_2) \in [0, \pi/2]^2$, which we assume from now on. Moreover, we already know that g is convex, so it is sufficient to estimate the maximum of g on the edges of the triangle $\mathcal{T}_\delta$.

When $y_2 \in [\delta, 1-2\delta]$, one has

$$\begin{aligned} g(\delta, y_2) &= (\delta^2-\delta)\,\sin^2\eta_1 + (y_2^2 - y_2)\,\sin^2\eta_2 + 2\,\delta\,y_2\,(\sin^2\eta_1\sin^2\eta_2 + \frac{1}{4}\sin 2\eta_1\,\sin 2\eta_2) \\ &\le (\delta^2-\delta)(\sin^2\eta_1+\sin^2\eta_2) + 2\,\delta\,(1-2\delta)\,(\sin^2\eta_1\sin^2\eta_2 + \frac{1}{4}\sin 2\eta_1\,\sin 2\eta_2) \\ &\le (\delta^2-\delta)(\sin\eta_1-\sin\eta_2)^2 - 2\delta^2\,(\sin^2\eta_1\sin^2\eta_2 + \frac{1}{4}\sin 2\eta_1\,\sin 2\eta_2) \le 0\,. \end{aligned}$$

In a completely similar, way, for $y_1 \in [\delta, 1-2\delta]$, we obtain

$$g(y_1, \delta) \le (\delta^2-\delta)(\sin\eta_1-\sin\eta_2)^2 - 2\delta^2\,(\sin^2\eta_1\sin^2\eta_2 + \frac{1}{4}\sin 2\eta_1\,\sin 2\eta_2) \le 0\,.$$

Eventually, for $y_1 \in [\delta, 1-2\delta]$, we compute

$$\begin{aligned} g(y_1, 1-\delta-y_1) &= -y_1(1-\delta-y_1)\left[\sin^2\eta_1+\sin^2\eta_2 - 2(\sin^2\eta_1\sin^2\eta_2+\frac{1}{4}\sin 2\eta_1\,\sin 2\eta_2)\right] \\ &\quad - \delta\,y_1\,\sin^2\eta_1 - \delta\,(1-\delta-y_1)\,\sin^2\eta_2 \\ &\le (2\delta^2-\delta)(\sin\eta_1-\sin\eta_2)^2 - \delta^2\,(\sin^2\eta_1+\sin^2\eta_2) \\ &\le -\delta^2\,(\sin^2\eta_1+\sin^2\eta_2) \le -\frac{4\,\delta^2}{\pi^2}\,(\eta_1^2+\eta_2^2)\,. \end{aligned}$$

Consequently, the maximum of g on $\mathcal{T}_\delta$ is nonpositive, and the maximum is zero if, and only if $\eta_1 = \eta_2 = 0$.

When $(\eta_1, \eta_2) \in [0, \pi/4]^2$, one has

$$\begin{aligned} g(\delta, y_2) &\le 4(\delta^2-\delta)\,\sin^2\frac{\eta_1-\eta_2}{2}\,\cos^2\frac{\eta_1+\eta_2}{2} - 2\delta^2\,\sin\eta_1\,\sin\eta_2\,\cos(\eta_1-\eta_2) \\ &\le -c(\delta)\,\big((\eta_1-\eta_2)^2+\eta_1\,\eta_2\big) \le -c(\delta)\,(\eta_1^2+\eta_2^2)\,, \end{aligned}$$

and similarly

$$g(y_1, \delta) \le -c(\delta)\,(\eta_1^2+\eta_2^2)\,.$$

We have thus obtained (24) when $(\eta_1, \eta_2) \in [0, \pi/4]^2$. When $(\eta_1, \eta_2) \in [0, \pi/2]^2 \setminus [0, \pi/4]^2$, we have

$$\max_{(y_1,y_2)\in\mathcal{T}_\delta} g(y_1, y_2) \le -c(\delta) \le -c(\delta)\,(\eta_1^2+\eta_2^2)\,,$$

so the proof of (24) is complete. $\square$

Using (24), we thus obtain:

$$c\,(\eta_1^2+\eta_2^2) \le 1 - \rho(\mathbb{G}(\xi_1,\xi_2))^2 \le 2\left[1-\rho(\mathbb{G}(\xi_1,\xi_2))\right]\,,$$

so the scheme (5) is dissipative (in Kreiss' sense) of order 2.

3.3 Stability of the dimensional-splitting Godunov scheme

The symbol of the scheme (6) is given by:

$$\mathbb{G}^s(\xi_1,\xi_2) = \Big(I_d - 2\lambda_2 \sin^2(\frac{\xi_2 \Delta x_2}{2})|A_2| - i\lambda_2 \sin(\xi_2\Delta x_2) A_2 \Big) \Big(I_d - 2\lambda_1 \sin^2(\frac{\xi_1 \Delta x_1}{2})|A_1| - i\lambda_1 \sin(\xi_1\Delta x_1) A_1 \Big) \, .$$

Observe that A_1, and $|A_1|$ commute, as well as A_2, and $|A_2|$. Therefore, $\mathbb{G}^s(\xi_1,\xi_2)$ is the product of two normal matrices. Then the proof of the last item of Theorem 1 follows exactly the arguments that we have used to study the dimensional-splitting Lax-Friedrichs scheme.

4 Necessary stability conditions for a particular system

In all this section, we study the finite difference schemes (3), and (5) when the matrices A_1, and A_2 are given by:

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, , \quad A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \, . \tag{25}$$

Observe that for all $(\xi_1,\xi_2) \in \mathbb{R}^2$, the eigenvalues of the matrix $\xi_1 A_1 + \xi_2 A_2$ are $\pm(\xi_1^2+\xi_2^2)^{1/2}$.

4.1 Lax-Friedrichs scheme

We first consider the two-dimensional Lax-Friedrichs scheme (3). According to Theorem 1, the scheme is stable if (9) holds. In the particular case (25), (9) is equivalent to

$$\forall \vartheta \in [0,2\pi] \, , \quad \lambda_1^2 \cos^2\vartheta + \lambda_2^2 \sin^2\vartheta \le \frac{1}{2} \, ,$$

and this condition is equivalent to $\max(\lambda_1,\lambda_2) \le 1/\sqrt{2}$.

Assume now that the scheme (3) is stable. Its symbol is given by (13). Using the notation $\zeta_k = \xi_k \, \Delta x_k$, $k = 1,2$, we get

$$\rho(G^{LF}(\xi_1,\xi_2))^2 = \frac{1}{4} (\cos\zeta_1 + \cos\zeta_2)^2 + \lambda_1^2 \sin^2\zeta_1 + \lambda_2^2 \sin^2\zeta_2 \, .$$

The spectral radius of $G^{LF}(\xi_1,\xi_2)$ is less than 1 for all (ξ_1,ξ_2). Consequently, the mapping

$$r : \quad (\zeta_1,\zeta_2) \longmapsto \frac{1}{4} (\cos\zeta_1 + \cos\zeta_2)^2 + \lambda_1^2 \sin^2\zeta_1 + \lambda_2^2 \sin^2\zeta_2 \, ,$$

has a global maximum at the origin, therefore its hessian matrix at the origin is nonpositive. We compute

$$D^2 r(0,0) = \begin{pmatrix} 2\lambda_1^2 - 1 & 0 \\ 0 & 2\lambda_2^2 - 1 \end{pmatrix} \, ,$$

and we get $\max(\lambda_1,\lambda_2) \le 1/\sqrt{2}$.

4.2 Godunov scheme

When the matrices $A_{1,2}$ are given by (25), one computes $|A_1| = |A_2| = I_2$, and the symbol of the Godunov scheme (5) is given by

$$\mathbb{G} = \Big(1 - 2\big(\lambda_1 \sin^2(\frac{\xi_1\Delta x_1}{2}) + \lambda_2 \sin^2(\frac{\xi_2\Delta x_2}{2})\big) \Big) \, I_2 - i\big(\lambda_1 \sin(\xi_1\Delta x_1) \, A_1 + \lambda_2 \sin(\xi_2\Delta x_2) \, A_2\big) \, .$$

In this case, the symbol $\mathbb{G}$ is normal for all (ξ_1, ξ_2), and the scheme is stable if, and only if the spectral radius $\rho(\mathbb{G})$ does not exceed 1 for all (ξ_1, ξ_2).

To simplify the subsequent calculations, we denote $\eta_k = \xi_k\, \Delta x_k/2$, $k = 1, 2$. The eigenvalues of the symbol $\mathbb{G}$ are

$$1 - 2(\lambda_1 \, \sin^2 \eta_1 + \lambda_2 \, \sin^2 \eta_2) \pm i\sqrt{\lambda_1^2 \sin^2(2\eta_1) + \lambda_2^2 \sin^2(2\eta_2)}\,.$$

After some simplifications, we thus compute

$$\rho(\mathbb{G})^2 = 1 + 4\,(\lambda_1^2 - \lambda_1)\, \sin^2 \eta_1 + 4\,(\lambda_2^2 - \lambda_2)\, \sin^2 \eta_2 + 8\,\lambda_1\,\lambda_2\, \sin^2 \eta_1\, \sin^2 \eta_2\,.$$

The scheme (5) is thus stable if, and only if the following inequality holds true for all $(\eta_1, \eta_2) \in \mathbb{R}^2$:

$$(\lambda_1^2 - \lambda_1)\, \sin^2 \eta_1 + (\lambda_2^2 - \lambda_2)\, \sin^2 \eta_2 + 2\,\lambda_1\,\lambda_2\, \sin^2 \eta_1\, \sin^2 \eta_2 \le 0\,. \tag{26}$$

Choosing $\eta_1 = \eta_2 = \pi/2$, (26) implies the necessary condition $\lambda_1 + \lambda_2 \le 1$.

When $\lambda_1 + \lambda_2 \le 1$, the scheme (5) is stable according to Theorem 1. Therefore, the condition $\lambda_1 + \lambda_2 \le 1$ is sufficient and necessary for the stability of (5).

5 Numerical results

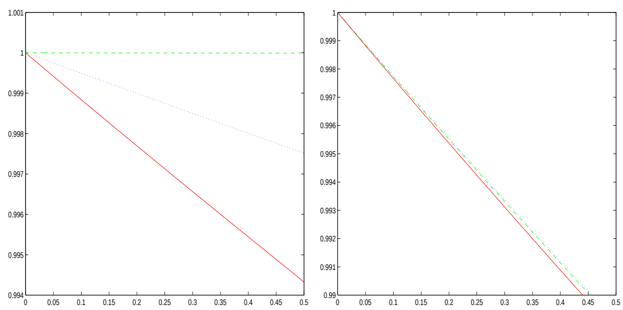

Figure 1: Left: scheme (3). Right: scheme (5)

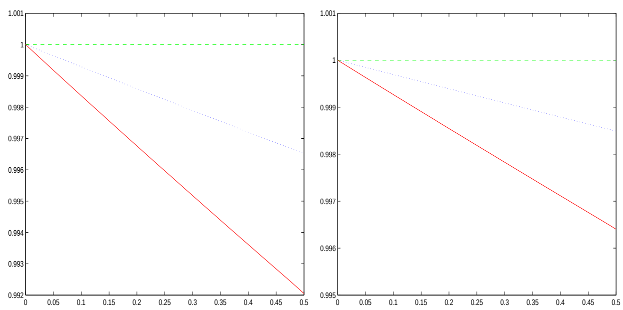

Figure 2: Left: schemes (4). Right: scheme (6).

In this section, we compare the dissipativity of the schemes (3), (5), (4), (6). We consider the system

$$\partial_t u + A_1\, \partial_{x_1} u + 2\, A_2\, \partial_{x_2} u = 0\,, \tag{27}$$

where the matrices $A_{1,2}$ are given by (25). (Observe the scaling in the x_2 variable). We run each of the four finite difference schemes on the square $[-2, 2] \times [-2, 2]$. To avoid the problem of boundary conditions, we choose some initial data that are supported in the square $[-1,1] \times [-1,1]$, and we stop the computations when the support of the solution reaches the boundary (that is, at time $T = 1/2$). The initial data are

$$u_0(x_1, x_2) = \begin{pmatrix} (\cos(x_1\,\pi/2)\,\cos(x_2\,\pi/2))^2 \\ \cos(x_1\,\pi/2)\,\cos(x_2\,\pi/2) \end{pmatrix} ,$$

when $(x_1, x_2) \in [-1,1] \times [-1,1]$, and 0 outside.

In the x_2 direction, we consider a space step $\Delta x_2 = 4/300$ (which corresponds to 300 points), while in the x_1 direction, the space step is first $\Delta x_1 = 4/450$, then $\Delta x_1 = 4/600$, and, at last, $\Delta x_1 = 4/750$. We always choose the maximal time step that ensures stability (see Theorem 1). In the first case, one has $\lambda_1 \rho(A_1) < \lambda_2 \rho(2A_2)$, in the second case, one has $\lambda_1 \rho(A_1) = \lambda_2 \rho(2A_2)$, and in the last case, one has $\lambda_1 \rho(A_1) > \lambda_2 \rho(2A_2)$. In figures 5, and 5, we plot the ratio $\|u(t)\|_{L^2} / \|u_0\|_{L^2}$ on the interval $[0, T]$. The dotted line represents the case $\Delta x_1 = 4/450$, the dashed line represents the case $\Delta x_1 = 4/600$, and the solid line represents the case $\Delta x_1 = 4/750$.

The numerical results show the following fact: the schemes (3), (4), and (6) do not diffuse when $\lambda_1 \rho(A_1) = \lambda_2 \rho(2A_2)$, and in any case, (6) is the less diffusive scheme. Surprisingly, the two-dimensional Godunov scheme (5) has a more and more diffusive behavior as Δx_1 decreases. In particular, it is still diffusive when $\lambda_1 \rho(A_1) = \lambda_2 \rho(2A_2)$.

These observations are easily explained by computing the modified equations of the finite difference schemes (3), and (5) (we shall not detail here the modified equations of the schemes (4), and (6)). For the test case (27), the modified equation of the scheme (3) is

$$\partial_t u + A_1\,\partial_{x_1} u + 2\,A_2\,\partial_{x_2} u = \Delta t \left[(\frac{1}{4\lambda_1^2} - \frac{1}{2}) \partial^2_{x_1 x_1} u + (\frac{1}{4\lambda_2^2} - 2) \partial^2_{x_2 x_2} u \right] .$$

In particular, when $\lambda_1 = 2\,\lambda_2 = 1/\sqrt{2}$, there is no diffusion in the modified equation.

For the test case (27), the modified equation of the scheme (5) is

$$\partial_t u + A_1\,\partial_{x_1} u + 2\,A_2\,\partial_{x_2} u = \Delta t \left[(\frac{1}{2\lambda_1} - \frac{1}{2}) \partial^2_{x_1 x_1} u + (\frac{1}{\lambda_2} - 2) \partial^2_{x_2 x_2} u \right] .$$

When $\lambda_1 = 2\,\lambda_2 = 1/2$, there is a positive definite diffusion tensor in the modified equation, and the scheme (5) is dissipative. Note that the ideal choice would be $\lambda_1 = 2\lambda_2 = 1$, but in this case the scheme is unstable according to Theorem 2 (this is confirmed by numerical simulations).

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