An Optimal Transportation Metric for Solutions of the Camassa-Holm Equation

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Dedicated to Prof. Joel Smoller in the occasion of his 65-th birthday

Abstract. In this paper we construct a global, continuous flow of solutions to the Camassa-Holm equation on the entire space H^1. Our solutions are conservative, in the sense that the total energy $\int(u^2 + u_x^2)\,dx$ remains a.e. constant in time. Our new approach is based on a distance functional $J(u, v)$, defined in terms of an optimal transportation problem, which satisfies $\frac{d}{dt}J(u(t), v(t)) \leq \kappa \cdot J(u(t), v(t))$ for every couple of solutions. Using this new distance functional, we can construct arbitrary solutions as the uniform limit of multi-peakon solutions, and prove a general uniqueness result.

1 - Introduction

The Camassa-Holm equation can be written as a scalar conservation law with an additional integro-differential term:

$$u_t + (u^2/2)_x + P_x = 0\,, \tag{1.1}$$

where P is defined as a convolution:

$$P \doteq \frac{1}{2}e^{-|x|} * \left(u^2 + \frac{u_x^2}{2}\right). \tag{1.2}$$

For the physical motivations of this equation we refer to [CH], [CM1], [CM2], [J]. Earlier results on the existence and uniqueness of solutions can be found in [XZ1], [XZ2]. One can regard (1.1) as an evolution equation on a space of absolutely continuous functions with derivatives $u_x \in \mathbf{L}^2$. In the smooth case, differentiating (1.1) w.r.t. x one obtains

$$u_{xt} + uu_{xx} + u_x^2 - \left(u^2 + \frac{u_x^2}{2}\right) + P = 0\,. \tag{1.3}$$

Multiplying (1.1) by u and (1.3) by u_x we obtain the two conservation laws with source term

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3} + u\,P\right)_x = u_x\,P\,, \tag{1.4}$$

$$\left(\frac{u_x^2}{2}\right)_t + \left(\frac{uu_x^2}{2} - \frac{u^3}{3}\right)_x = -u_x\, P\,. \tag{1.5}$$

As a consequence, for regular solutions the total energy

$$E(t) \doteq \int \left[u^2(t,x) + u_x^2(t,x)\right] dx \tag{1.6}$$

remains constant in time.

As in the case of scalar conservation laws, by the strong nonlinearity of the equations, solutions with smooth initial data can lose regularity in finite time. For the Camassa-Holm equation (1.1), however, the uniform bound on $\|u_x\|_{\mathbf{L}^2}$ guarantees that only the $\mathbf{L}^\infty$ norm of the gradient can blow up, while the solution u itself remains Hölder continuous at all times.

In order to construct global in time solutions, two main approaches have recently been introduced. On one hand, one can add a small diffusion term in the right hand side of (1.1), and recover solutions of the original equations as a vanishing viscosity limit [CHK1, CHK2]. An alternative technique, developed in [BC2], relies on a new set of independent and dependent variables, specifically designed with the aim of "resolving" all singularities. In terms of these new variables, the solution to the Cauchy problem becomes regular for all times, and can be obtained as the unique fixed point of a contractive transformation.

In the present paper, we implement yet another approach to the Camassa-Holm equation. As a starting point we consider all multi-peakon solutions, of the form

$$u(t,x) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|}\,. \tag{1.7}$$

These are obtained by solving the system of O.D.E's

$$\begin{cases} \dot q_i = \displaystyle\sum_j p_j\, e^{-|q_i-q_j|}\,, \\ \dot p_i = \displaystyle\sum_{j\neq i} p_i p_j\, \mathrm{sign}(q_i-q_j)\, e^{-|q_i-q_j|}\,. \end{cases} \tag{1.8}$$

It is well known that this can be written in hamiltonian form:

$$\begin{cases} \dot q_i &=& \dfrac{\partial}{\partial p_i} H(p,q)\,, \\ \dot p_i &=& -\dfrac{\partial}{\partial q_i} H(p,q)\,, \end{cases} \qquad\qquad H(p,q) \doteq \frac{1}{2}\sum_{i,j} p_i p_j e^{-|q_i-q_j|}\,.$$

If all the coefficients p_i are initially positive, then they remain positive and bounded for all times. The solution $u = u(t,x)$ is thus uniformly Lipschitz continuous. We stress, however, that here we are not making any assumption about the signs of the p_i. In a typical situation, two peakons can cross at a finite time τ. As $t\to\tau-$ their strengths p_i, p_j and positions q_i, q_j will satisfy

$$p_i(t)\to+\infty\,, \qquad p_j(t)\to-\infty\,, \qquad p_i(t)+p_j(t)\to\bar p\,, \tag{1.9}$$

$$q_i(t)\to\bar q\,, \quad q_j(t)\to\bar q\,, \qquad q_i(t)<q_j(t) \quad \text{for} \quad t<\tau\,, \tag{1.10}$$

for some $\bar{p}, \bar{q} \in I\!R$. Moreover, $\|u_x(t)\|_{\mathbf{L}^\infty} \to \infty$. In this case, we will show that there esists a unique way to extend the multi-peakon solution beyond the interaction time, so that the total energy is conserved.

Having constructed a set of "multi-peakon solutions", our main goal is to show that these solutions form a continuous semigroup, whose domain is dense in the space $H^1(I\!R)$. Taking the unique continuous extension, we thus obtain a continuous semigroup of solutions of (1.1), defined on the entire space H^1.

The particular metric, used to derive the Lipschitz continuity of the semigroup, is the most novel feature of our paper. Indeed, one easily checks that the flow map $\Phi_t : u(0) \mapsto u(t)$ cannot be continuous as a map from H^1 into itself, or from $\mathbf{L}^2$ into itself. Distances defined in terms of convex norms perform well in connection with linear problems, but occasionally fail when nonlinear features become dominant. In the present setting, we construct a new distance $J(u, v)$ between functions $u, v \in H^1$, defined by a problem of optimal transportation. Roughly speaking, $J(u, v)$ will be the minimum cost in order to transport the mass distribution with density $1 + u_x^2$ located on the graph of u onto the mass distribution with density $1 + v_x^2$ located on the graph of v. See Section 3 for details. With this definition of distance, our main result shows that

$$\left| \frac{d}{dt} J\big(u(t),\, v(t)\big) \right| \;\leq\; C \cdot J\big(u(t),\, v(t)\big) \tag{1.11}$$

for some constant C and any couple of multi-peakon solutions u, v. Moreover, $J(u_n, u) \to 0$ implies the uniform convergence $\|u_n - u\|_{\mathbf{L}^\infty} \to 0$. The distance functional J thus provides the ideal tool to measure continuous dependence on the initial data for solutions to the Camassa-Holm equation. Earlier applications of distances defined in terms of optimal transportation problems can be found in the monograph [V].

The issue of uniqueness of solutions must here be discussed in greater detail. For a multi-peakon solution, as long as all coefficients p_i remain bounded, the solution to the system of ODE's (1.8) is clearly unique. For each time t, call μ_t the measure having density $u^2(t) + u_x^2(t)$ w.r.t. Lebesgue measure. Consider a time τ where a positive and a negative peakon collide, according to (1.9)-(1.10). As $t \to \tau-$, we have the weak convergence $\mu_t \rightharpoonup \mu_\tau$ for some positive measure μ_τ which typically contains a Dirac mass at the point $\bar{q}$. By energy conservation, we thus have

$$\int \big[u^2(\tau, x) + u_x^2(\tau, x)\big]\, dx + \mu_\tau\big(\{\bar{q}\}\big) = \lim_{t \to \tau-} \int \big[u^2(\tau, x) + u_x^2(\tau, x)\big]\, dx = E(\tau-)\,.$$

There are now two natural ways to prolong the multi-peakon solution beyond time τ: a conservative solution, such that

$$E(t) = \int \big[u^2(t, x) + u_x^2(t, x)\big]\, dx = E(\tau-) \qquad\qquad t > \tau\,,$$

or a dissipative solution, where all the energy concentrated at the point $\bar{q}$ is lost. In this case

$$E(t) = \int \big[u^2(t, x) + u_x^2(t, x)\big]\, dx = E(\tau-) - \mu_\tau\big(\{\bar{q}\}\big) \qquad\qquad t > \tau\,.$$

For $t > \tau$, the dissipative solution is obtained by simply replacing the two peakons p_i, p_j with one single peakon of strength $\bar{p}$, located at $x = \bar{q}$. On the other hand, as we will show in Section 2, the conservative solution contains two peakons emerging from the point $\bar{q}$. As $t \to \tau+$, their strengths and positions satisfy again (1.9), while (1.10) is replaced by

$$q_i(t) \to \bar{q}\,, \quad q_j(t) \to \bar{q}\,, \qquad q_i(t) > q_j(t) \;\text{ for }\; t > \tau\,. \tag{1.12}$$

The vanishing viscosity approach in [CHK1, CHK2] singles out the dissipative solutions. These can also be characterized by the Oleinik type estimate

$$u_x(t,x) \le C(1+t^{-1}),$$

valid for $t>0$ at a.e. $x \in \mathbb{R}$. On the other hand, the coordinate transformation approach in [BC2] and the present one, based on optimal transport metrics, appear to be well suited for the study of both conservative and dissipative solutions.

In the present paper we focus on conservative solutions to the C-H equation in the spatially periodic case. The assumption of spatial periodicity allows us to concentrate on the heart of the matter, i.e. the uniqueness and stability of solutions beyond the time of singularity formation. It will spare us some technicalities, such as the analysis of the tail decay of u, u_x as $x \to \pm\infty$. The construction of an optimal transportation metric in connection with dissipative solutions to the C-H equation is significantly different, and will be carried out in a forthcoming paper. The main ingredients can already be found in the paper [BC1], devoted to dissipative solutions of the Hunter-Saxton equation.

As initial data, we take

$$u(0,x) = \bar u(x), \tag{1.13}$$

with $\bar u$ in the space $H^1_{\rm per}$ of of periodic, absolutely continuous functions u with derivative $u_x \in \mathbf{L}^2_{\rm per}$. To fix the ideas, we assume that the period is 1, so that

$$u(x+1) = u(x) \qquad\qquad x \in \mathbb{R}.$$

On $H^1_{\rm per}$ we shall use the norm

$$\|u\|_{H^1_{\rm per}} \doteq \left(\int_0^1 \big|u(x)\big|^2\,dx + \int_0^1 \big|u_x(x)\big|^2\,dx\right)^{1/2}.$$

Our main results can be stated as follows.

Theorem 1. *For each initial data $\bar u \in H^1_{\rm per}$, there exists a solution $u(\cdot)$ of the Cauchy problem (1.1), (1.13). Namely, the map $t \mapsto u(t)$ is Lipschitz continuous from $\mathbb{R}$ into $\mathbf{L}^2_{\rm per}$, satisfies (1.13) at time $t = 0$, and the identity*

$$\frac{d}{dt}u = -uu_x - P_x \tag{1.14}$$

is satisfied as an equality between elements in $\mathbf{L}^2_{\rm per}$ at a.e. time $t \in \mathbb{R}$. This same map $t \mapsto u(t)$ is continuously differentiable from $\mathbb{R}$ into $\mathbf{L}^p_{\rm per}$ and satisfies (1.14) at a.e. time $t \in \mathbb{R}$, for all $p \in [1,2[$. The above solution is conservative in the sense that, for a.e. $t \in \mathbb{R}$,

$$E(t) = \int_0^1 \big[u^2(t,x) + u_x^2(t,x)\big]\,dx = E^{\bar u} \doteq \int_0^1 \big[\bar u^2(x) + \bar u_x^2(x)\big]\,dx. \tag{1.15}$$

Theorem 2. *Conservative solutions to (1.1) can be constructed so that they constitute a continuous flow Φ. Namely, there exists a distance functional J on $H^1_{\rm per}$ such that*

$$\frac{1}{C}\cdot \|u-v\|_{\mathbf{L}^1_{\rm per}} \le J(u,v) \le C\cdot \|u-v\|_{H^1_{\rm per}} \tag{1.16}$$

for all $u, v \in H^1_{\rm per}$ *and some constant* C *uniformly valid on bounded sets of* $H^1_{\rm per}$ *. Moreover, for any two solutions* $u(t) = \Phi_t \bar u$*,* $v(t) = \Phi_t \bar v$ *of (1.1), the map* $t \mapsto J\big(u(t),\, v(t)\big)$ *satisfies*

$$J\big(u(t),\, \bar u\big) \le C_1 \cdot |t|\,, \tag{1.17}$$

$$J\big(u(t),\, v(t)\big) \ \le\ J(\bar u, \bar v)\cdot e^{C_2|t|} \tag{1.18}$$

for a.e. $t \in \mathbb{R}$ *and constants* C_1, C_2*, uniformly valid as* u, v *range on bounded sets of* H^1_{per} *.*

Somewhat surprisingly, all the properties stated in Theorem 1 are still not strong enough to single out a unique solution. To achieve uniqueness, an additional condition is needed.

Theorem 3. *Conservative solutions* $t \mapsto u(t)$ *of (1.1) can be constructed with the following additional property:*

For each $t \in \mathbb{R}$*, call* μ_t *the absolutely continuous measure having density* $u^2 + u_x^2$ *w.r.t. Lebesgue measure. Then, by possibly redefining* μ_t *on a set of times of measure zero, the map* $t \mapsto \mu_t$ *is continuous w.r.t. the topology of weak convergence of measures. It provides a measure-valued solution to the conservation law*

$$w_t + (uw)_x = (u^3 - 2uP^u)_x\,. \tag{1.19}$$

The solution of the Cauchy problem (1.1), (1.13) satisfying the properties stated in Theorem 1 and this additional condition is unique.

The plan of the paper is as follows. In Section 2 we derive some elementary properties of multi-peakon solutions and show that any initial data can be approximated in $H^1_{\rm per}$ by a finite sum of peakons. In Section 3 we introduce our distance functional $J(u, v)$ and study its relations with other distances defined by Sobolev norms. The continuity of the flow (1.1), together with the key estimates (1.17)-(1.18) are then proved in the following two sections. The proofs of Theorems 1 and 2 are completed in Section 6. The uniqueness result stated in Theorem 3 is proved in Section 7. As a corollary, we also show that in a multi-peakon solution the only possible interactions involve exactly two peakons: one positive and one negative. In particular, no triple interactions can ever occur.

2 - Multipeakon solutions

By a periodic peakon we mean a function of the form

$$u(x) = p\,\chi(x-q)\,, \qquad\qquad \chi(x) \doteq \sum_{n\in\mathbb{Z}} e^{-|x-n|}\,. \tag{2.1}$$

Observe that the periodic function χ satisfies

$$\begin{aligned} &\chi(-x) = \chi(x) = \chi(x+1) && x \in \mathbb{R}\,,\\ &\chi(x) = \frac{e^x + e^{1-x}}{e-1} && x \in [0,1]\,. \end{aligned} \tag{2.2}$$

We begin this section by observing that any periodic initial data can be approximated by a periodic multi-peakon.

Lemma 1. *Let $f \in H^1_{\rm per}$. Then for any $\varepsilon > 0$ there exists periodic multi-peakon g, of the form*

$$g(x) = \sum_{i=1}^{N} p_i \sum_{n\in\mathbb{Z}} e^{-|x-q_i-n|} = \sum_{i=1}^{N} p_i\, \chi(x-q_i) \tag{2.3}$$

such that

$$\|f-g\|_{H^1_{\rm per}} < \varepsilon\,.$$

Proof. By taking a suitable mollification, we can approximate f with a periodic function $\tilde f \in \mathcal{C}^\infty$, so that

$$\|f-\tilde f\|_{H^1_{\rm per}} < \varepsilon/2\,. \tag{2.4}$$

Next, we observe that

$$\frac{1}{2}\left(e^{-|x|} - \frac{\partial^2}{\partial x^2}e^{-|x|}\right) = \delta_0\,,$$

where δ_0 denotes the Dirac distribution concentrating a unit mass at the origin. We can thus write $\tilde f$ as a convolution:

$$\tilde f = \delta_0 * \tilde f = \frac{1}{2}\left(e^{-|x|} - \frac{\partial^2}{\partial x^2}e^{-|x|}\right) * \tilde f = e^{-|x|} * \left(\frac{\tilde f - \tilde f''}{2}\right)\,,$$

$$\tilde f(x) = \int_0^1 \chi(x-y)\cdot \frac{\tilde f(y) - \tilde f''(y)}{2}\,dy\,.$$

The above integral can now be approximated with a Riemann sum

$$g(x) = \sum_{i=1}^{N} p_i\,\chi(x-q_i)\,, \qquad\qquad p_i = \int_{(i-1)/N}^{i/N} \frac{\tilde f(y) - \tilde f''(y)}{2}\,dy\,.$$

Choosing N sufficiently large we obtain $\|\tilde f - g\|_{H^1_{per}} < \varepsilon/2$. Together with (2.4) this yields the result. $\square$

Next, we show how to construct a unique conservative solution, for multi-peakon initial data. As long as the locations q_i of the peakons remain distinct, this can be obtained by solving the Hamiltonian system of O.D.E's (1.8). However, at a time τ where two or more peakons interact, the corresponding strengths p_i become unbounded. A suitable transformation of variables is needed, in order to resolve the singularity and uniquely extend the solution beyond the interaction time.

Lemma 2. *Let $\bar u$ be any periodic, multi-peakon initial data. Then the Cauchy problem (1.1), (1.13) has a global, conservative multi-peakon solution defined for all $t \in \mathbb{R}$. The set $\mathcal{I}$ of times where two or more peakons interact is at most countable. Moreover, for all $t \notin \mathcal{I}$, the energy conservation (1.15) holds.*

Proof. The solution can be uniquely constructed by solving the hamiltonian system (1.8), up to the first time τ where two or more peakons interact. We now show that there exists a unique way to prolong the solution for $t > \tau$, in terms of two outgoing peakons. To fix the ideas, call

$$\bar q = \lim_{t\to\tau-} q_i(t) \qquad\qquad i = 1,\ldots,k\,,$$

the place where the interaction occurs, and let $p_1(t),\ldots,p_k(t)$ be the strengths of the interacting peakons. Later in this paper we will show that only the case $k = 2$ can actually occur, but at this stage we need to consider the more general case. We observe that the strengths $p_{k+1},\ldots,p_N$ of the peakons not involved in the interaction remain continuous at time τ. Moreover, by (1.8) there exists the limit

$$\bar p = \lim_{t\to\tau-} \sum_{i=1}^{k} p_i(t)\,.$$

We can thus write

$$u(\tau,x) \;=\; \lim_{t\to\tau-} \sum_{i=1}^{N} p_i(t)\, e^{-|x-q_i(t)|} \;=\; \bar p\, e^{-|x-\bar q|} + \sum_{i=k+1}^{N} p_i(\tau)\, e^{-|x-q_i(\tau)|}\,.$$

For $t > \tau$, we shall prolong the solution with two peakons emerging from the point $\bar q$. The strength of these two peakons will be uniquely determined by the requirement of energy conservation (1.15).

Call $\xi^-(t)$, $\xi^+(t)$ respectively the position of the smallest and largest characteristic curves passing through the point $(\tau,\bar q)$, namely

$$\begin{aligned} \xi^-(t) &\doteq \min\Big\{\xi(t)\,;\;\; \xi(\tau)=\bar q\,,\;\; \dot\xi(s) = u\big(s,\xi(s)\big) \qquad \text{for all } s\in[\tau-h,\tau+h]\Big\}\,,\\ \xi^+(t) &\doteq \max\Big\{\xi(t)\,;\;\; \xi(\tau)=\bar q\,,\;\; \dot\xi(s) = u\big(s,\xi(s)\big) \qquad \text{for all } s\in[\tau-h,\tau+h]\Big\}\,. \end{aligned} \tag{2.5}$$

Moreover, define

$$e_{(\tau,\bar q)} \doteq \lim_{t\to\tau-} \int_{\xi^-(t)}^{\xi^+(t)} u_x^2(t,x)\,dx\,. \tag{2.6}$$

The existence of this limit follows from the balance law (1.5). This describes how much energy is concentrated at the interaction point.

For $t > \tau$ the solution will contain the peakons $p_{k+1},\ldots,p_N$, located at $q_{k+1},\ldots,q_N$, together with the two outgoing peakons p_1,p_2, located at $q_1 < q_2$. The behavior of p_i,q_i for $i\in\{k+1,\ldots,N\}$ is still described by a system of O.D.E's as in (1.8). However, to describe the evolution of p_1,p_2,q_1,q_2 one has to use a different set of variables, resolving the singularity occurring at $(\tau,\bar q)$. As $t\to\tau+$ we expect (1.9), (1.12) to hold. To devise a suitable set of rescaled variables, we observe that, by (1.3),

$$\frac{d}{dt}\,u_x\big(t,\xi(t)\big) = -\frac12\,u_x^2\big(t,\xi(t)\big) + [u^2 - P] \tag{2.7}$$

along any characteristic curve $t\mapsto\xi(t)$ emerging from the point $\bar q$. Since u,P remain uniformly bounded, one has

$$u_x(t,x) \approx \frac{2}{t-\tau} \qquad\qquad t>\tau\,,\quad x\in\big[q_1(t)\,,\;q_2(t)\big]\,.$$

The total amount of energy concentrated in the interval between the two peakons is given by

$$\int_{q_1(t)}^{q_2(t)} \left[u^2(t,x) + u_x^2(t,x)\right] dx \approx \left(\frac{u(q_2) - u(q_1)}{q_2 - q_1}\right)^2 \cdot (q_2 - q_1) \approx \frac{\left[(p_2 - p_1)\left(1 - e^{-|q_2 - q_1|}\right)\right]^2}{q_2 - q_1}$$
$$\approx (p_2 - p_1)^2 (q_2 - q_1) \approx e_{(\tau,\bar{q})} .$$

The previous heuristic analysis suggests that, in order to resolve the singularities, we should work with the variables

$$z = p_1 + p_2 , \qquad w = 2\arctan(p_2 - p_1) , \qquad \eta = q_2 + q_1 , \qquad \zeta = (p_2 - p_1)^2 (q_2 - q_1) ,$$

together with $p_{k+1}, \ldots, p_N$, $q_{k+1}, \ldots, q_N$. To simplify the following calculations we here assume $0 < q_1 < q_2 < q_{k+1} < ... < q_N < 1$, which is not restrictive.

Let χ defined in (2.1) and $\tilde{\chi}(x) \doteq \frac{-e^x + e^{1-x}}{e-1}$, $x \in [0,1]$. From the original system of equations (1.8) it follows

$$\dot{z} = \cosh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right) z\cos^2\frac{w}{2} \sum_{j=k+1}^{N} p_j \chi\left(q_j - \frac{\eta}{2}\right) - \frac{\sinh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right)}{\frac{1}{\tan\frac{w}{2}}} \sum_{j=k+1}^{N} p_j \tilde{\chi}\left(q_j - \frac{\eta}{2}\right)$$

$$\dot{w} = \left(z^2\cos^2\frac{w}{2} - \sin^2\frac{w}{2}\right)\chi\left(\frac{\zeta}{\tan^2\frac{w}{2}}\right) + 2\cosh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right) z \sum_{j=k+1}^{N} p_j \chi\left(q_j - \frac{\eta}{2}\right)$$
$$+ 2\sinh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right)\sin w \sum_{j=k+1}^{N} p_j \tilde{\chi}\left(q_j - \frac{\eta}{2}\right)$$

$$\dot{\eta} = z\left[\chi(0) + \chi\left(\frac{\zeta}{\tan^2\frac{w}{2}}\right)\right] + \cosh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right) \sum_{j=k+1}^{N} p_j \chi\left(q_j - \frac{\eta}{2}\right)$$

$$\dot{\zeta} = \frac{\chi(0) - \chi\left(\frac{\zeta}{\tan^2\frac{w}{2}}\right)}{\frac{1}{\tan^2\frac{w}{2}}}\zeta + \chi\left(\frac{\zeta}{\tan^2\frac{w}{2}}\right)\frac{z^2\zeta}{\tan\frac{w}{2}} - \frac{\sinh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right)}{\frac{1}{\tan^2\frac{w}{2}}} \sum_{j=k+1}^{N} p_j \tilde{\chi}\left(q_j - \frac{\eta}{2}\right)$$
$$+ 2\frac{\zeta}{\tan\frac{w}{2}}\left[\cosh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right)\frac{z}{\tan\frac{w}{2}} \sum_{j=k+1}^{N} p_j \chi\left(q_j - \frac{\eta}{2}\right) - \sinh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right) \sum_{j=k+1}^{N} p_j \tilde{\chi}\left(q_j - \frac{\eta}{2}\right)\right]$$

$$\dot{p}_i = p_i\left[\cosh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right) z\chi\left(q_i - \frac{\eta}{2}\right) + \sinh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right)\tan\frac{w}{2}\tilde{\chi}\left(q_i - \frac{\eta}{2}\right)\right]$$
$$+ p_i \sum_{j=k+1}^{N} p_j \,\mathrm{sign}(q_i - q_j)\chi\left(|q_i - q_j|\right)$$

$$\dot{q}_i = \cosh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right) z\chi\left(q_i - \frac{\eta}{2}\right) + \sinh\left(\frac{\zeta}{2\tan^2\frac{w}{2}}\right)\tan\frac{w}{2}\tilde{\chi}\left(q_i - \frac{\eta}{2}\right) + \sum_{j=k+1}^{N} p_j \chi\left(|q_i - q_j|\right)$$

with initial data

$$z(\tau) = \bar{p} , \qquad w(\tau) = \pi , \qquad \eta(\tau) = 2\bar{q} , \qquad \zeta(\tau) = e_{(\tau,\bar{q})} ,$$

$$p_i(\tau) = \lim_{t\to\tau-} p_i(t)\,, \qquad q_i(\tau) = \lim_{t\to\tau-} q_i(t) \qquad i = k+1,\ldots,N\,.$$

For the above system of O.D.E's, a direct inspection reveals that the right hand side can be extended by continuity also at the value $w = \pi$, because all singularities are removable. This continuous extension is actually smooth, in a neighborhood of the initial data. Therefore, our Cauchy problem has a unique local solution. This provides a multi-peakon solution defined on some interval of the form $[\tau,\, \tau'[$, up to the next interaction time.

The case where two or more groups of peakons interact exactly at the same time τ, but at different locations within the interval $[0,1]$, can be treated in exactly the same way. Since the total number of peakons (on a unit interval in the x-variable) does not increase, it is clear that the number of interaction times is at most countable. The solution can thus be extended to all times $t > 0$, conserving its total energy. □

3 - A distance functional

In this section we shall construct a functional $J(u,v)$ which controls the distance between two solutions of the equation (1.1). All functions and measures on $I\!R$ are assumed to be periodic with period 1. Let $\mathbf{T}$ be the unit circle, so that $\mathbf{T} = [0, 2\pi]$ with the endpoints 0 and 2π identified. The distance $|\theta - \theta'|_*$ between two points $\theta, \theta' \in \mathbf{T}$ is defined as the smaller between the lengths of the two arcs connecting θ with θ' (one clockwise, the other counterclockwise). We now consider the product space

$$X \doteq I\!R \times I\!R \times \mathbf{T}$$

with distance

$$d^\diamond\Big((x,u,w),\ (\tilde x, \tilde u, \tilde w)\Big) \doteq \Big(|x-\tilde x| + |u - \tilde u| + |w - \tilde w|_*\Big) \wedge 1\,, \tag{3.1}$$

where $a\wedge b \doteq \min\{a,b\}$. Let $\mathcal{M}(X)$ be the space of all Radon measures on X which are 1-periodic w.r.t. the x-variable. To each 1-periodic function $u \in H^1_{\rm per}$ we now associate the positive measure $\sigma^u \in \mathcal{M}(X)$ defined as

$$\sigma^u(A) \doteq \int_{\left\{x\in I\!R:\ (x,\, u(x),\, 2\arctan\, u_x(x)\,)\in A\right\}} \big(1 + u_x^2(x)\big)\,dx \tag{3.2}$$

for every Borel set $A \subseteq I\!R^2 \times \mathbf{T}$. Notice that the total mass of $\sigma^{(u,\mu)}$ over one period is

$$\sigma^u\big([0,1]\times I\!R \times \mathbf{T}\big) = 1 + \int_0^1 u_x^2(x)\,dx\,.$$

On this family of positive, 1-periodic Radon measures, we now introduce a kind of Kantorovich distance, related to an optimal transportation problem. Given the two measures σ^u and $\sigma^{\tilde u}$, their distance $J(u,\tilde u)$ is defined as follows.

Call $\mathcal{F}$ the family of all strictly increasing absolutely continuous maps $\psi : I\!R \mapsto I\!R$ which have an absolutely continuous inverse and satisfy

$$\psi(x+n) = n + \psi(x) \qquad \text{for every} \quad n \in Z\!\!\!Z\,. \tag{3.3}$$

For a given $\psi \in \mathcal{F}$, we define the 1-periodic, measurable functions $\phi_1, \phi_2 : I\!R \mapsto [0,1]$ by setting

$$\begin{aligned} \phi_1(x) &\doteq \sup\left\{\theta \in [0,1]\,;\quad \theta\cdot\Big(1+u_x^2(x)\Big) \le \Big(1+\tilde u_x^2\big(\psi(x)\big)\Big)\,\psi'(x)\right\},\\ \phi_2(x) &\doteq \sup\left\{\theta \in [0,1]\,;\quad 1+u_x^2(x) \ge \theta\cdot\Big(1+\tilde u_x^2\big(\psi(x)\big)\Big)\,\psi'(x)\right\}. \end{aligned} \tag{3.4}$$

Observe that the above definitions imply $\max\big\{\phi_1(x),\,\phi_2(x)\big\} = 1$ together with

$$\phi_1(x)\,\Big(1+u_x^2(x)\Big) = \phi_2\big(\psi(x)\big)\,\Big(1+\tilde u_x^2\big(\psi(x)\big)\Big)\,\psi'(x) \tag{3.5}$$

for a.e. $x \in I\!R$. We now define

$$\begin{aligned} J^\psi(u,\tilde u) \doteq \int_0^1 & d^\diamondsuit\Big(\big(x,\,u(x),\,2\arctan u_x(x)\big)\,,\ \big(\psi(x),\,\tilde u(\psi(x)),\,2\arctan \tilde u_x(\psi(x))\big)\Big)\\ &\qquad\qquad\qquad\cdot\phi_1(x)\,\big(1+u_x^2(x)\big)\,dx\\ &+\int_0^1 \Big|\big(1+u_x^2(x)\big) - \big(1+\tilde u_x^2(\psi(x))\big)\,\psi'(x)\Big|\,dx\,. \end{aligned} \tag{3.6}$$

Of course, the integral is always computed over one period. Observe that $x \mapsto \psi(x)$ can be regarded as a **transportation plan**, in order to transport the measure σ^u onto the measure $\sigma^{\tilde u}$. Since these two positive measures need not have the same total mass, we allow the presence of some excess mass, not transferred from one place to the other. The penalty for this excess mass is given by the second integral in (3.6). The factor $\phi_1 \le 1$ in the first integral indicates the percentage of the mass which is actually transported. Integrating (3.5) over one period, we find

$$\int_0^1 \phi_1(x)\big(1+u_x^2(x)\big)\,dx = \int_0^1 \phi_2(y)\big(1+\tilde u_x^2(y)\big)\,dy\,.$$

We can thus transport the measure $\phi_1\,\sigma^u$ onto $\phi_2\,\sigma^{\tilde u}$ by a map $\Psi : \big(x,\,u(x)\ \arctan u_x(x)\big) \mapsto \big(y,\,\tilde u(y),\,\arctan \tilde u_x(y)\big)$, with $y = \psi(x)$. The associated cost is given by the first integral in (3.6). Notice that in this case the measure $\phi_2\,\sigma^{\tilde u}$ is obtained as the push-forward of the measure $\phi_1\,\sigma^u$. We recall that the **push-forward** of a measure σ by a mapping Ψ is defined as $(\Psi\#\sigma)(A) \doteq \sigma(\Psi^{-1}(A))$ for every measurable set A. Here $\Psi^{-1}(A) \doteq \big\{z\,;\ \Psi(z) \in A\big\}$.

Our distance functional J is now obtained by optimizing over all transportation plans, namely

$$J(u,\tilde u) \doteq \inf_{\psi\in\mathcal{F}} J^\psi(u,\tilde u)\,. \tag{3.7}$$

To check that (3.7) actually defines a distance, let $u, v, w \in H^1(I\!R)$ be given.

1. Choosing $\psi(x) = x$, so that $\phi_1(x) = \phi_2(x) = 1$, we immediately see that $J(u,u) = 0$. Moreover, if $J(u,\tilde u) = 0$, then by the definition of $d^\diamondsuit$ we have $\tilde u = u$.

2. Given $\psi \in \mathcal{F}$, define $\tilde\psi = \psi^{-1}$, so that $\tilde\phi_1 = \phi_2$, $\tilde\phi_2 = \phi_1$. This yields

$$J^{\tilde\psi}(\tilde u, u) = J^\psi(u,\tilde u)\,.$$

Hence $J(\tilde u, u) = J(u, \tilde u)$.

3. Finally, to prove the triangle inequality, let $\psi^\flat, \psi^\sharp : I\!R \mapsto I\!R$ be two increasing diffeomorphisms satisfying (3.3), and let $\phi_1^\flat, \phi_1^\sharp, \phi_2^\flat, \phi_2^\sharp : I\!R \mapsto [0,1]$ be the corresponding functions, defined as in (3.4). We now consider the composition $\psi \doteq \psi^\sharp \circ \psi^\flat$ and define the functions ϕ_1, ϕ_2 according to (3.4). Observing that

$$\phi_1(x) \geq \phi_1^\flat(x) \cdot \phi_1^\sharp\big(\psi^\flat(x)\big)\,,$$

$$\phi_2\big(\psi(x)\big) = \phi_2\Big(\psi^\sharp\big(\psi^\flat(x)\big)\Big) \geq \phi_2^\flat\big(\psi^\flat(x)\big) \cdot \phi_2^\sharp\Big(\psi^\sharp\big(\psi^\flat(x)\big)\Big)\,,$$

and recalling that the distance $d^\diamondsuit$ at (3.1) is always ≤ 1, we conclude

$$J^\psi(u,w) \leq J^{\psi^\flat}(u,v) + J^{\psi^\sharp}(v,w)\,.$$

This implies the triangle inequality $J(u,v) + J(v,w) \geq J(u,w)$. □

In the remainder of this section we study the relations between our distance functional J and the distances determined by various norms.

Lemma 3. *For any $u, v \in H^1_{\rm per}$ one has*

$$\frac{1}{C} \cdot \|u - v\|_{\mathbf{L}^1_{\rm per}} \leq J(u,v) \leq C \cdot \|u - v\|_{H^1_{\rm per}}\,, \tag{3.8}$$

with a constant C uniformly valid on bounded subsets of $H^1_{\rm per}$.

Proof. We shall use the elementary bound

$$\big|\arctan a - \arctan b\big| \cdot a^2 \leq 4\pi\big(|a| + |b|\big)\,|a - b|\,, \tag{3.9}$$

valid for all $a, b \in I\!R$. In connection with the identity mapping $\psi(x) = x$ we now compute

$$\begin{aligned} J^\psi(u,v) &\leq \int_0^1 \Big\{\big|u(x) - v(x)\big| + 2\big|\arctan u_x - \arctan v_x\big|\Big\}\,(1 + u_x^2)\,dx + \int_0^1 \big|u_x^2 - v_x^2\big|\,dx \\ &\leq \|u - v\|_{\mathbf{L}^\infty}\,\|1 + u_x^2\|_{\mathbf{L}^1} + (8\pi + 1)\int_0^1 |u_x + v_x|\,|u_x - v_x|\,dx \\ &\leq (8\pi + 3)\,\big(1 + \|u\|_{H^1} + \|v\|_{H^1}\big) \cdot \|u - v\|_{H^1}\,, \end{aligned}$$

proving the second inequality in (3.8).

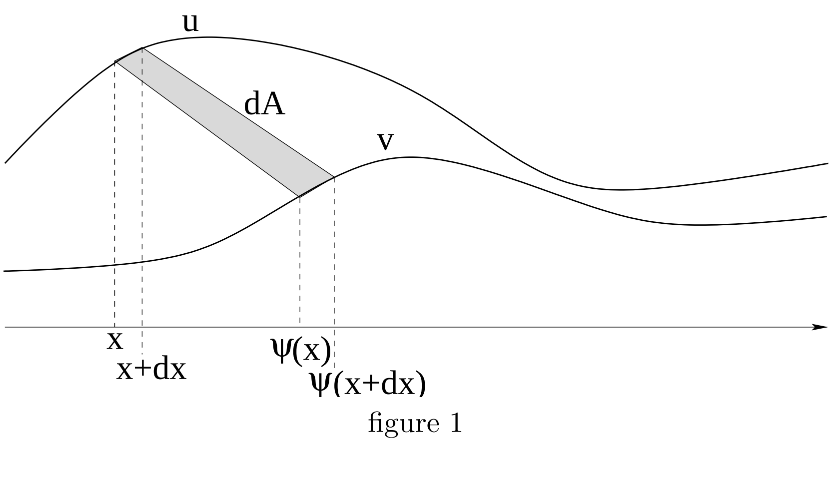

figure 1

To achieve the first inequality, choose any $\psi \in \mathcal{F}$. For $x \in [0,1]$, call γ^x the segment joining the point $P^x = \big(x, u(x)\big)$ with $Q^x = \big(\psi(x), v(\psi(x))\big)$. Clearly, the union of all these segments covers the region between the graphs of u and v. Moving the base point from x to $x+dx$, the corresponding segments sweep an infinitesimal area dA estimated by (fig. 1)

$$
\begin{aligned}
|dA| &\leq |P^x - Q^x| \cdot \big(|dP^x| + |dQ^x|\big) \\
&\leq \Big(\big|x-\psi(x)\big|^2 + \big|u(x) - v(\psi(x))\big|^2 + \Big)^{1/2} \cdot \Big[(1+u_x^2)^{1/2}dx + (1+v_x^2)^{1/2}\psi'(x)\,dx\Big].
\end{aligned}
$$

Integrating over one period we obtain

$$
\begin{aligned}
&\int_0^1 \big|u(x) - v(x)\big|\,dx \\
&\quad\leq \int_0^1 \Big(\big|x - \psi_{(t)}(x)\big| + \big|u(x) - v(\psi_{(t)}(x))\big|\Big) \cdot \Big[\big(1+u_x^2(x)\big) + \big(1+v_x^2(\psi(x))\big)^{1/2}\psi'_{(t)}(x)\Big]\,dx \\
&\quad\leq \big(2 + \|u\|_{H^1} + \|v\|_{H^1}\big) \cdot \Big[J^{\psi}(u,v) + J^{\psi^{-1}}(v,u)\Big] \;\leq\; C \cdot J(u,v)\,,
\end{aligned}
\tag{3.10}
$$

completing the proof of (3.8). □

Lemma 4. *Let $(u_n)_{n\geq 1}$ be a Cauchy sequence for the distance J, uniformly bounded in the $H^1_{\rm per}$ norm. Then*

(i) There exists a limit function $u \in H^1_{\rm per}$ such that $u_n \to u$ in $\mathbf{L}^\infty$ and the sequence of derivatives $u_{n,x}$ converges to u_x in $L^p_{\rm per}$, for $1 \leq p < 2$.

(ii) Let μ_n be the absolutely continuous measure having density $u^2_{n,x}$ w.r.t. Lebesgue measure. Then one has the weak convergence $\mu_n \rightharpoonup \mu$, for some measure μ whose absolutely continuous part has density u_x^2.

Proof. 1. By Lemma 3 we already know the convergence $u_n \to u$, for some limit function $u \in \mathbf{L}^1_{\rm per}$. By a Sobolev embedding theorem, all functions u_n, u are uniformly Hölder continuous. This implies $\|u_n - u\|_{\mathbf{L}^\infty} \to 0$.

To establish the convergence of derivatives, we first show that the sequence of functions

$$
v_n \doteq \exp\{2i \arctan u_{n,x}\}
$$

is compact in $\mathbf{L}^1_{\rm per}$. Indeed, fix $\varepsilon > 0$. Then there exists N such that $J(u_m, u_n) < \varepsilon$ for $m, n \geq N$. We can now approximate u_N in $H^1_{\rm per}$ with a piecewise affine function $\tilde u_N$ such that $J(\tilde u_N, u_N) \leq \varepsilon$. By assumption, choosing suitable transport maps ψ_n we obtain

$$
\int_0^1 \Big|\exp\big\{2i\arctan u_{n,x}(x)\big\} - \exp\big\{2i \arctan \tilde u_{N,x}(\psi_n(x))\big\}\Big|\,dx \;\leq\; 2\,J(u_n, \tilde u_N) \;\leq\; 4\varepsilon
$$

for all $n \geq N$. We now observe that all functions $x \mapsto \exp\big\{2i\arctan \tilde u_{N,x}(\psi_n(x))\big\}$ are uniformly bounded, piecewise constant with the same number of jumps: namely, the number of subintervals on which $\tilde u_N$ is affine. The set of all such functions is compact in $\mathbf{L}^1_{\rm per}$. This argument shows that the sequence $v_n \doteq \exp\{2i\arctan u_{n,x}\}$ eventually remains in an ε-neighborhood of a compact subset of $\mathbf{L}^1_{\rm per}$. Since $\varepsilon > 0$ can be taken arbitrarily small, by possibly choosing a subsequence we obtain the strong convergence $v_n \to v$ for some $v \in \mathbf{L}^1_{\rm per}$.

2. From the uniform H^1 bounds and the $\mathbf{L}^1$ convergence of the functions v_n, we now derive the $\mathbf{L}^p$ convergence of the derivatives. For a given $\varepsilon > 0$, define

$$M \doteq \sup_n \|u_n\|_{H^1_{\rm per}}, \qquad\qquad A_n \doteq \Big\{x \in [0,1]\,;\ \ \big|u_{n,x}(x)\big| > M/\varepsilon\Big\}.$$

The above definitions imply

$$\text{meas}(A_n) \leq \varepsilon^2 \tag{3.11}$$

We now have

$$\begin{aligned}\|u_{m,x} - u_{n,x}\|_{\mathbf{L}^p} &\leq \left(\int_{A_n\cup A_m} |u_{m,x} - u_{n,x}|^p\,dx\right)^{1/p} + \left(\int_{[0,1]\setminus(A_n\cup A_m)} |u_{m,x} - u_{n,x}|^p\,dx\right)^{1/p}\\ &\doteq I_1 + I_2\,.\end{aligned} \tag{3.12}$$

$$\begin{aligned}I_1 &\leq \left[\int_{A_m\cup A_n} 1\cdot dx\right]^{(2-p)/2p} \cdot \left[\int_{A_m\cup A_n} \big(|u_{m,x}| + |u_{n,x}|\big)^2\,dx\right]^{1/2}\\ &\leq \varepsilon^{(2-p)/p}\cdot 2M\,.\end{aligned} \tag{3.13}$$

Next, choosing a constant C_ε such that

$$\left|e^{2i\arctan a} - e^{2i\arctan b}\right| \geq C_\varepsilon |a-b| \qquad\qquad \text{whenever}\quad |a|,|b| \leq M/\varepsilon\,,$$

we obtain

$$I_2 \leq C_\varepsilon \left[\int \left|e^{2i\arctan u_{m,x}} - e^{2i\arctan u_{n,x}}\right|^p dx\right]^{1/p}. \tag{3.14}$$

Taking $\varepsilon > 0$ small, we can make the right hand side of (3.13) as small as we like. On the other hand, choosing a subsequence such that $v_\nu = e^{2i\arctan u_{\nu,x}}$ converges in $\mathbf{L}^1_{\rm per}$, the right hand side of (3.154) approaches zero. Hence, for this subsequence,

$$\limsup_{m,n\to\infty} \|u_{m,x} - u_{n,x}\|_{\mathbf{L}^p_{\rm per}} = 0\,.$$

Since $u_n \to u$ uniformly, in this case we must have

$$\|u_{n,x} - u_x\|_{\mathbf{L}^p_{\rm per}} \to 0\,. \tag{3.15}$$

We now observe that from any subsequence we can extract a further subsequence for which (3.15) holds. Therefore, the whole sequence $(u_{n,x})_{n\geq 1}$ converges to u_x in $\mathbf{L}^p_{\rm per}$.

3. To establish (ii), we consider the sequence of measures having density $1 + u^2_{n,x}$ w.r.t. Lebesgue measure. This sequence converges weakly, because our distance functional is stronger than the Kantorovich-Waserstein metric which induces the topology of weak convergence on spaces of measures. Therefore, $\mu_n \rightharpoonup \mu$ for some positive measure μ.

Since the sequence $1 + u_{n,x}$ converges to $1 + u_x$ in $\mathbf{L}^1_{\rm per}$, by possibly choosing a subsequence we achieve the pointwise convergence $u_{n,x}(x) \to u_x(x)$, for a.e. $x \in [0,1]$. For any $\varepsilon > 0$, by Egorov's theorem we have the uniform convergence $u_{n,x}(x) \to u_x(x)$ for all $x \in [0,1]\setminus V_\varepsilon$, for some set with meas$(V_\varepsilon) < \varepsilon$. Since $\varepsilon > 0$ can be taken arbitrarily small, this shows that the absolutely continuous part of the measure μ has density $u^2 + u_x^2$ w.r.t. Lebesgue measure. □

4 - Continuity in time of the distance functional

Here and in the next section we examine how the distance functional $J(\cdot,\cdot)$ evolves in time, in connection with multi-peakon solutions of the Camassa-Holm equation (1.1). We first provide estimates valid on a time interval where no peakon interactions occur. Then we show that the distance functional is continuous across times of interaction. Since the number of peakons is locally finite, this will suffice to derive the basic estimates (1.17)-(1.18), in the case of multi-peakon solutions.

Lemma 5. *Let $t \mapsto u(t) \in H^1_{\rm per}$ be a multi-peakon solution of (1.1). Assume that no peakon interactions occur within the interval $[0,\tau]$. Then*

$$J\big(u(s),\, u(s')\big) \leq C\cdot |s-s'|\,, \qquad\qquad \textit{for all } s,s'\in[0,\tau]\,, \tag{4.1}$$

for some constant C, uniformly valid as u ranges on bounded subsets of $H^1_{\rm per}$.

Proof. Assume $0\leq s<s'\leq\tau$. By the assumptions, the solution $u=u(t,x)$ remains uniformly Lipschitz continuous on the time interval $[0,\tau]$. Therefore, for each $s\in[0,\tau]$ and $x\in I\!R$, the Cauchy problem

$$\frac{d}{dt}\,\xi(t) = u\big(t,\,\xi(t)\big)\,, \qquad\qquad \xi(s)=x\,, \tag{4.2}$$

determines a unique characteristic curve $t\mapsto\xi(t;s,x)$ passing through the point (s,x). Given $s'\in[0,\tau]$, we can thus define a transportation plan by setting

$$\psi(x)\doteq\xi(s';s,x)\,. \tag{4.3}$$

Of course, moving mass along the characteristics is the most natural thing to do. We then choose ϕ_1,ϕ_2 to be as large as possible, according to (3.4). Namely:

$$\phi_1(x)\doteq\sup\left\{\theta\in[0,1]\,;\quad \theta\cdot\Big(1+u_x^2(s,x)\Big)\leq\psi'(x)\cdot\Big(1+u_x^2\big(s',\,\psi(x)\big)\Big)\right\},$$

$$\phi_2(x)\doteq\sup\left\{\theta\in[0,1]\,;\quad \theta\cdot\Big(1+u_x^2\big(s',\,\psi(x)\big)\Big)\,\psi'(x)\leq 1+u_x^2(s,x)\right\}.$$

The cost of this plan is bounded by

$$\begin{aligned}
J^\psi\big(u(s)\,,\ u(s')\big) \leq &\int_0^1 \Big\{\big|x-\xi(s';s,x)\big| + \big|u(s,x)-u(s',\xi(s';s,x))\big| \\
&\qquad + \big|2\arctan u_x(s,x) - 2\arctan u_x(s',\xi(s';s,x))\big|_*\Big\}\big(1+u_x^2(s,x)\big)\,dx \\
&+\int_0^1 \big(1-\phi_1(x)\big)\,\big(1+u_x^2(s,x)\big)\,dx \\
&+\int_0^1 \big(1-\phi_2(\psi(x))\big)\,\Big(1+u_x^2\big(s',\psi(x)\big)\Big)\,\psi'(x)\,dx\,.
\end{aligned} \tag{4.4}$$

To estimate the right hand side of (4.4), we first observe that, for all $u\in H^1_{\rm per}$,

$$\begin{aligned}
\|u\|_{\mathbf{L}^\infty} &\leq \int_0^1 \big|u(x)\big|\,dx + \int_0^1 \big|u_x(x)\big|\,dx \\
&\leq \|u\|_{\mathbf{L}^2} + \|u_x\|_{\mathbf{L}^2} \leq 2\|u\|_{H^1_{\rm per}} = 2(E^u)^{1/2}.
\end{aligned} \tag{4.5}$$

Using (4.5) in (4.2) we obtain

$$\big|\xi(s)-\xi(s')\big|\leq 2(E^{\bar u})^{1/2}\cdot|s-s'|\,. \tag{4.6}$$

Next, from the definition of the source term P at (1.2) it follows

$$\big\|P\|_{\mathbf{L}^\infty}\leq\frac{1}{2}\big\|e^{-|x|}\,\big\|_{\mathbf{L}^1(I\!R)}\cdot\left\|u^2+\frac{u_x^2}{2}\right\|_{\mathbf{L}^1([0,1])}\leq\|u\|^2_{H^1_{\rm per}}=E^u\,. \tag{4.7}$$

Similarly,

$$\big\|P_x\|_{\mathbf{L}^\infty}\leq\|u\|^2_{H^1_{\rm per}}=E^u\,. \tag{4.8}$$

Using (4.8) we obtain

$$\begin{aligned}\Big|u\big(s',\,\xi(s')\big)-u\big(s,\,\xi(s)\big)\Big|&\leq\int_s^{s'}\left|\frac{d}{dt}u\big(t,\,\xi(t)\big)\right|\,dt\\&=\int_s^{s'}\Big|P_x\big(t,\,\xi(t)\big)\Big|\,dt\ \leq\ E^{\bar u}\cdot|s'-s|\,.\end{aligned} \tag{4.9}$$

Concerning the term involving arctangents, recalling (1.3) we obtain

$$\frac{d}{dt}\Big[2\arctan u_x\big(t,\xi(t,x)\big)\Big]=\frac{2}{1+u_x^2}\,\left[u^2-\frac{u_x^2}{2}-P\right]\,.$$

The bounds (4.7) and (4.9) thus yield

$$\begin{aligned}\Big|2\arctan u_x\big(s',\xi(s')\big)-2\arctan u_x\big(s,\,\xi(s)\big)\Big|_*&\leq\Big(2\|u\|^2_{\mathbf{L}^\infty}+1+2\|P\|_{\mathbf{L}^\infty}\Big)\cdot|s'-s|\\&\leq\big(10\,E^{\bar u}+1\big)\cdot|s'-s|\,.\end{aligned} \tag{4.10}$$

This already provides a bound on the first integral on the right hand side of (4.4).

Next, call I_1,I_2 the last two integrals on the right hand side of (4.4). Notice that

$$I_1+I_2=\int_0^1\left|\Big(1+u_x^2(s,y)\Big)-\xi_y(s';s,y)\,\Big(1+u_x^2\big(s',\xi(s';s,y)\big)\Big)\right|\,dy \tag{4.11}$$

Indeed, I_1+I_2 measures the difference between the measure $\big(1+u_x^2(s',y)\big)\,dy$ and the push-forward of the measure $\big(1+u_x^2(s,x)\big)\,dx$ through the mapping $x\mapsto\xi(s';s,x)$.

Since the push-forward of the measure $u_x^2\,dy$ satisfies the linear conservation law

$$w_t+(uw)_x=0\,, \tag{4.12}$$

comparing (4.12) with (1.5) we deduce

$$\begin{aligned}\int_0^1\Big|u_x^2(s,y)-\xi_y(s,y)\,u_x^2\big(s',\xi(s';s,y)\big)\Big|\,dy\ &\leq\ \int_s^{s'}\int_0^1 2\big|(u^2-P)u_x\big|\,dx\,dt\\\leq 2\int_s^{s'}\big(\|u^2_{\mathbf{L}^\infty}+\|P\|_{\mathbf{L}^\infty}\big)\,\|u_x\|_{\mathbf{L}^1}\,dt\ &\leq\ 2\,\big(4E^{\bar u}+E^{\bar u}\big)\,E^{\bar u}\cdot|s'-s|\,,\end{aligned} \tag{4.13}$$

because of (4.5), (4.7) and (1.15). Finally, we need to estimate the remaining terms, describing by how much the Lebesgue measure fails to be conserved by the transformation $x \mapsto \xi(s'; s, x)$. Observing that

$$\frac{\partial}{\partial t}\,\xi_y(t,y) = u_x\big(t,\,\xi(t,y)\big)\,\xi_y(t,y)\,, \qquad\qquad \xi_y(0,y)=1\,, \tag{4.14}$$

we find

$$\begin{aligned}\int_0^1 \big|1-\xi_y(s';s,y)\,\big|\,dy &\leq \int_s^{s'}\int_0^1 \left|\frac{\partial}{\partial t}\big[\xi_y(t,s,y)\big]\right|\,dy\,dt\\ &\leq \int_s^{s'}\int_0^1 \xi_y(t;s,y)\,\big|u_x(t,\xi(t;s,y))\big|\,dy\,dt\,.\end{aligned} \tag{4.15}$$

To estimate the right hand side of (4.15), we use the decomposition $[0,1]=Y\cup Y'\cup Y''$, where

$$Y \doteq \Big\{y\,;\quad \xi_y(t;s,y)\in\big[(1/2)\,,\ 2\big]\quad\text{for all } t\in[s,s']\Big\}\,,$$

$$Y' \doteq \Big\{y\,;\quad \xi_y(t;s,y)<1/2\quad\text{for some } t\in[s,s']\Big\}\,,$$

$$Y'' \doteq \Big\{y\,;\quad \xi_y(t;s,y)>2\quad\text{for some } t\in[s,s']\Big\}\,.$$

Integrating over Y one finds

$$\int_s^{s'}\int_Y \xi_y(t;s,y)\,\big|u_x(t,\xi(t;s,y))\big|\,dy\,dt \;\leq\; 2\int_s^{s'}\big\|u_x(t)\big\|_{\mathbf{L}^1}\,dt\;\leq\; 2E^{\bar u}\cdot|s'-s|\,. \tag{4.15}$$

Next, if $y\in Y'$ we define

$$\tau(y)\doteq \inf\,\big\{t>s\,;\quad \xi_y(t;s,y)<1/2\big\}\,.$$

Observe that $y\in Y'$ implies

$$\int_s^{\tau(y)}\Big|u_x\big(t,\,\xi(t;s,y)\big)\Big|\,dt\geq \ln 2\,.$$

Therefore

$$\begin{aligned}\int_{Y'}dy &\leq \frac{1}{\ln 2}\int_{Y'}\left[\int_s^{\tau(y)}\Big|u_x\big(t,\,\xi(t;s,y)\big)\Big|\,dt\right]\,dy\\ &\leq \frac{2}{\ln 2}\int_s^{s'}\int_0^1\big|u_x\big(t,x\big)\big|\,dx\,dt\;\leq\; 4\,E^{\bar u}\cdot|s'-s|\,.\end{aligned} \tag{4.16}$$

The estimate for the integral over Y'' is entirely analogous, Indeed, the push-forward of the Lebesgue measure along characteristic curves from $t=s$ to $t=s'$ satisfies exactly the same type of estimates as the pull-back of the Lebesgue measure from $t=s'$ to $t=s$. All together, these three estimates imply

$$\int_{Y\cup Y'\cup Y''}\big|1-\xi_y(s';s,y)\,\big|\,dy\leq 10\,E^{\bar u}\cdot|s'-s| \tag{4.17}$$

Putting together the estimates (4.6), (4.9), (4.10), (4.3) and (4.17), the distance in (4.4) can be estimated by

$$J^{\psi}\big(u(s)\,,\ u(s')\big)\leq\Big[2(1+E^{\bar u})+E^{\bar u}+(10\,E^{\bar u}+1)+10\,(E^{\bar u})^2+10\,E^{\bar u}\Big]\cdot|s'-s|\,. \tag{4.18}$$

This establishes (4.1). □

According to Lemma 5, as long as no peakon interactions occur, the map $t \mapsto u(t)$ remains uniformly Lipschitz continuous w.r.t. our distance functional, with a Lipschitz constant that depends only on the total energy $E^{\bar u}$. Since interactions can occur only at isolated times, to obtain a global Lipschitz estimate it suffices to show that trajectories are continuous (w.r.t. the distance J) also at interaction times.

Lemma 6. *Assume that the multi-peakon solution $u(\cdot)$ contains two or more peakons which interact at a time τ. Then*

$$\lim_{h\to 0+} J\big(u(\tau-h),\, u(\tau+h)\big) = 0\,. \tag{4.19}$$

Proof. To fix the ideas, call $x=\bar q$ the place where the interaction occurs, and let $p_1,\ldots,p_k$ be the strengths of the peakons that interact at time τ. We here assume that $0<\bar q<1$. The case where two or more groups of peakons interact exactly at the same time τ, within the interval $[0,1]$, can be treated similarly.

For $|t-\tau|\le h$, call $\xi^-(t)$, $\xi^+(t)$ respectively the position of the smallest and largest characteristic curves passing through the point $(\tau,\bar q)$, as in (2.5). We observe that u is Lipschitz continuous in a neighborhood of each point (τ,x), with $x\neq\bar q$. Hence, for $x\in[0,1]\setminus\{\bar q\}$ there exists a unique characteristic curve $t\mapsto\xi(t;\tau,x)$ passing through x at time τ. For a fixed $h>0$, the transport map ψ is defined as follows. Consider the intervals $I_{-h}\doteq\big[\xi^-(\tau-h),\,\xi^+(\tau-h)\big]$ and $I_h\doteq\big[\xi^-(\tau+h),\,\xi^+(\tau+h)\big]$. On the complement $[0,1]\setminus I_{-h}$ we define

$$\psi\big(\xi(\tau-h;\tau,x)\big)=\xi(\tau+h;\tau,x)\,,$$

so that transport is performed along characteristic curves. It now remains to extends ψ as a map from I_{-h} onto I_h. Toward this goal, we recall that our construction of multi-peakon solutions in Section 2 was specifically designed in order to achieve the identity

$$e_{(\tau,\bar q)}=\lim_{h\to 0+}\int_{I_{-h}} u_x^2(\tau-h,x)\,dx \;=\; \lim_{h\to 0+}\int_{I_h} u_x^2(\tau+h,x)\,dx\,. \tag{4.20}$$

For $h>0$ we introduce the quantities

$$E(-h)\doteq\int_{I_{-h}}\big(1+u_x^2(\tau-h,\,x)\big)\,dx\,,\qquad\qquad E(h)\doteq\int_{I_h}\big(1+u_x^2(\tau+h,\,x)\big)\,dx\,,$$

$$e(h)\doteq 2\min\big\{E(-h),\,E(h)\big\}-\max\big\{E(-h),\,E(h)\big\}\,.$$

Notice that (4.20) implies $e(h)>0$ and

$$E(-h)\to e_{(\tau,\bar q)}\,,\qquad E(h)\to e_{(\tau,\bar q)}\,,\qquad e(h)\to e_{(\tau,\bar q)}\,, \tag{4.21}$$

as $h\to 0+$. Consider the point $x^*=x^*(h)$ inside the interval $I_{-h}=\big[\xi^-(\tau-h),\,\xi^+(\tau-h)\big]$, implicitly defined by

$$\int_{\xi^-(\tau-h)}^{x^*}\big(1+u_x^2(x)\big)\,dx=e(h)\,.$$

For $x\in\big[\xi^-(\tau-h),\;x^*\big]$ we define $\psi(x)$ as the unique point such that

$$\int_{\xi^-(\tau+h)}^{\psi(x)}\big(1+u_x^2(\tau+h,\,x)\big)\,dx=\int_{\xi^-(\tau-h)}^{x}\big(1+u_x^2(\tau-h,\,x)\big)\,dx\,. \tag{4.22}$$

We then extend ψ as an affine map from $\big[x^*,\, \xi^+(\tau-h)\big]$ onto $\big[\psi(x^*),\, \xi^+(\tau+h)\big]$, namely

$$\psi\Big(\theta\cdot\xi^+(\tau-h)+(1-\theta)\cdot x^*\Big)=\theta\cdot\xi^+(\tau+h)+(1-\theta)\cdot\psi(x^*)\qquad\qquad \theta\in[0,1]\,.$$

Finally, we prolong ψ to the whole real line according to (3.3). As usual, the 1-periodic functions ϕ_1,ϕ_2 are then chosen to be as large as possible, according to (3.4). As $h\to 0+$, we claim that the following quantity approaches zero:

$$\begin{aligned}
&J^\psi\big(u(\tau-h),\ u(\tau+h)\big)\\
&=\int_0^1 d^\diamondsuit\Big(\big(x,\,u(\tau-h,x),\,2\arctan u_x(\tau-h,x)\big)\,,\ \big(\psi(x),\,u(\tau+h,\,\psi(x)),\,2\arctan\tilde u_x(\tau+h,\,\psi(x))\big)\Big)\\
&\qquad\qquad\cdot\phi_1(x)\,\big(1+u_x^2(\tau-h,\,x)\big)\,dx\\
&\quad+\int_0^1\big(1-\phi_1(x)\big)\,\big(1+u_x^2(\tau-h,\,x)\big)\,dx\\
&\quad+\int_0^1\big(1-\phi_2(\psi(x))\big)\,\Big(1+u_x^2\big(\tau+h,\,\psi(x)\big)\Big)\,\psi'(x)\,dx\,.
\end{aligned}\tag{4.23}$$

It is clear that the restriction of all the above integrals to the complement $[0,1]\setminus I_{-h}$ approaches zero as $h\to 0$. We now prove that their restriction to I_{-h} also vanishes in the limit. As $h\to 0+$, for $x\in I_{-h}$ we have

$$d^\diamondsuit\Big(\big(x,\,u(\tau-h,x),\,2\arctan u_x(\tau-h,x)\big)\ ,\ \big(\psi(x),\,u(\tau+h,\psi(x)),\,2\arctan u_x(\tau+h,\psi(x))\big)\Big)\ \to\ 0\,,$$

because all points approach the same limit $\big(\bar q,\,u(\tau,\bar q),\,\pi\big)$. The first integral in (4.23) thus approaches zero as $h\to 0+$.

Concerning the last two integrals, by (4.22) it follows

$$\phi_1(x)=\phi_2\big(\psi(x)\big)=1\qquad\qquad\text{for all } x\in\big[\xi^-(\tau-h),\,x^*\big]\,.$$

Moreover, our choice of x^* implies

$$\begin{aligned}
&\int_{x^*}^{\xi^+(\tau-h)}\big(1+u_x^2(\tau-h,\,x)\big)\,dx+\int_{\psi(x^*)}^{\xi^+(\tau+h)}\Big(1+u_x^2\big(\tau+h,\,\psi(x)\big)\Big)\,\psi'(x)\,dx\\
&\qquad\le 2\max\big\{E(-h),\,E(h)\big\}-2\min\big\{E(-h),\,E(h)\big\}\,.
\end{aligned}\tag{4.24}$$

By (4.21), as $h\to 0+$ the right hand side of (4.24) approaches zero. Hence the same holds for the last two integrals in (4.23). This completes the proof of the lemma. □

5 - Continuity w.r.t. the initial data

We now consider two distinct solutions and study how the distance $J\big(u(t)\ \ v(t)\big)$ evolves in time. To fix the ideas, let $t\mapsto u(t)$ and $t\mapsto v(t)$ be two multi-peakon solutions of (1.1), and assume that no interaction occurs within a given time interval $[0,T]$. In this case, the functions u,v remain

Lipschitz continuous. We can thus define the characteristic curves $t \mapsto \xi(t,y)$ and $t \mapsto \zeta(t,\tilde y)$ as the solutions to the Cauchy problems

$$\begin{aligned} &\dot\xi = u(t,\xi), \qquad && \xi(0) = y\,,\\ &\dot\zeta = v(t,\zeta), \qquad && \zeta(0) = \tilde y\,, \end{aligned} \tag{5.1}$$

respectively. Let now $\psi_{(0)} \in \mathcal{F}$ be any transportation plan at time $t = 0$. For each $t \in [0,T]$ we can define a transportation plan $\psi_{(t)} \in \mathcal{F}$ by setting

$$\psi_{(t)}\big(\xi(t,y)\big) \doteq \zeta\big(t,\ \psi_{(0)}(y)\big). \tag{5.2}$$

The corresponding functions $\phi_1^{(t)}, \phi_2^{(t)}$ are then defined according to (3.4), namely

$$\phi_1^{(t)}(x) \doteq \sup\left\{\theta \in [0,1]\,;\ \ \theta\cdot\Big(1+u_x^2(t,x)\Big)\ \le\ \Big(1+v_x^2\big(t,\,\psi_{(t)}(x)\big)\Big)\,\psi'_{(t)}(x)\right\},$$

$$\phi_2^{(t)}(x) \doteq \sup\left\{\theta \in [0,1]\,;\ \ 1+u_x^2(t,x)\ \ge\ \theta\cdot\Big(1+v_x^2\big(t,\,\psi_{(t)}(x)\big)\Big)\,\psi'_{(t)}(x)\right\}.$$

If initially the point y is mapped into $\tilde y \doteq \psi_{(0)}(y)$, then at a later time $t > 0$ the point $\xi(t,y)$ along the u-characteristic starting from y is sent to the point $\zeta(t,\tilde y)$ along the v-characteristic starting from $\tilde y$. We thus transport mass from the point $\Big(\xi(t,y)\,,\ u\big(t,\xi(t,y)\big)\,,\ 2\arctan u_x\big(t,\xi(t,y)\big)\Big)$ to the corresponding point $\Big(\zeta(t,\tilde y)\,,\ v\big(t,\zeta(t,\tilde y)\big)\,,\ 2\arctan v_x\big(t,\zeta(t,\tilde y)\big)\Big)$,

In the following, our main goal is to provide an upper bound on the time derivative of the function

$$\begin{aligned} J^{\psi(t)}\big(u(t)\,,\ v(t)\big) \doteq &\int_0^1 d^\diamondsuit\Big(\big(x,\,u(t,x),\,2\arctan u_x(t,x)\big)\,,\\ &\qquad\big(\psi_{(t)}(x),\,v(t,\,\psi_{(t)}(x)),\,2\arctan v_x(t,\,\psi_{(t)}(x))\big)\Big)\cdot\phi_1^{(t)}(x)\,\big(1+u_x^2(t,x)\big)\,dx\\ &+\int_0^1 \big(1-\phi_1^{(t)}(x)\big)\,\big(1+u_x^2(t,x)\big)\,dx\\ &+\int_0^1 \big(1-\phi_2^{(t)}(\psi_{(t)}(x))\big)\,\Big(1+v_x^2\big(t,\,\psi_{(t)}(x)\big)\Big)\,\psi'_{(t)}(x)\,dx\,. \end{aligned} \tag{5.3}$$

Differentiating the right hand side of (5.3) one obtains several terms, due to

- Changes in the distance $d^\diamondsuit$ between the points $(\xi,\,u,\,2\arctan u_x)$ and $(\zeta,\,v,\,2\arctan v_x)$.
- Changes in the base measures $(1+u_x^2)\,dx$ and $(1+v_x^2)\,dx$.

Throughout the following, by $\mathcal{O}(1)$ we denote a quantity which remains uniformly bounded as u, v range in bounded subsets of $H^1_{\rm per}$. Using the elementary estimate

$$\min\big\{|u-v|,\,1\big\} \le \big(1+|u|+|v|\big)\,|u-v|\,,$$

we begin by deriving the bound

$$
\begin{aligned}
I_1 &\doteq \int_0^1 \frac{d}{dt}\left|x-\psi_{(t)}(x)\right|\cdot\phi_1^{(t)}(x)\left(1+u_x^2(t,x)\right)\,dx \\
&\le \int_0^1 \Big|u(t,x)-v\big(t,\,\psi_{(t)}(x)\big)\Big|\cdot\phi_1^{(t)}(x)\left(1+u_x^2(t,x)\right)\,dx \\
&\le \Big(1+\|u(t)\|_{\mathbf{L}^\infty}+\|v(t)\|_{\mathbf{L}^\infty}\Big)\cdot J^{\psi_{(t)}}\big(u(t),\,v(t)\big) \\
&= \mathcal{O}(1)\cdot J^{\psi_{(t)}}\big(u(t),\,v(t)\big)\,.
\end{aligned}
\tag{5.4}
$$

Here and in the sequel, the time derivative is computed along characteristics.

Next, recalling the basic equation (1.1), we consider

$$
\begin{aligned}
I_2 &\doteq \int_0^1 \frac{d}{dt}\Big|u(t,x)-v\big(t,\,\psi_{(t)}(x)\big)\Big|\cdot\phi_1^{(t)}(x)\left(1+u_x^2(t,x)\right)\,dx \\
&\le \int_0^1 \Big|P_x^u(t,x)-P_x^v\big(t,\psi_{(t)}(x)\big)\Big|\cdot\left(1+u_x^2(t,x)\right)\,dx\,.
\end{aligned}
\tag{5.5}
$$

In the spatially periodic case, by (1.2) and (2.2) we can write the source terms P_x^u, P_x^v as

$$
\begin{aligned}
P_x^u(t,x) &= \int_{x-1}^{x}\chi'(x-y)\cdot\left[u^2(t,y)+\frac{u_x^2(t,y)}{2}\right]dy\,, \\
P_x^v\big(t,\,\psi_{(t)}(x)\big) &= \int_0^1 \chi'\big(\psi_{(t)}(x)-\tilde y\big)\cdot v^2(t,\tilde y)\,d\tilde y\,, \\
&\qquad + \int_{x-1}^{x}\chi'\big(\psi_{(t)}(x)-\psi_{(t)}(y)\big)\cdot\frac{v_x^2\big(t,\psi_{(t)}(y)\big)}{2}\psi'_{(t)}(y)\,dy\,,
\end{aligned}
\tag{5.6}
$$

where, according to (2.2),

$$
\chi'(x)=\frac{e^x-e^{1-x}}{e-1}\qquad 0<x<1\,,\qquad\qquad \chi'(x)=\chi'(x+1)\qquad x\in I\!R\,.
\tag{5.7}
$$

In the next computation, we use the estimate

$$
\int_0^1\Big|\big(1+u_x^2(y)\big)-\Big(1+v_x^2\big(\psi(y)\big)\Big)\psi'(y)\Big|\,dy=\mathcal{O}(1)\cdot J^{\psi}(u,v)\,.
\tag{5.8}
$$

which holds because of the last two terms in the definition (3.6). Observing that χ' is Lipschitz continuous on the open interval $]0,1[$, we now compute (omitting explicit references to the time t)

$$\begin{aligned}
\Big|P_x^u(x)-P_x^v\big(\psi(x)\big)\Big| &\leq \int_0^1 \big|\chi'(x-y)\cdot u^2(y)-\chi'\big(\psi(x)-y\big)\cdot v^2(y)\big|\,dy\\
&+\mathcal{O}(1)\cdot\int_{x-1}^x \Big|x-y-\big(\psi(x)-\psi(y)\big)\Big|\cdot\frac{v_x^2\big(\psi(y)\big)}{2}\,\psi'(y)\,dy\\
&+\frac12\left|\int_{x-1}^x \chi'(x-y)\,\Big(u_x^2(y)-v_x^2\big(\psi(y)\big)\,\psi'(y)\Big)\,dy\right|\\
&=\mathcal{O}(1)\cdot\Big(\big|x-\psi(x)\big|+\|u^2-v^2\|_{\mathbf{L}^1}\Big)\\
&\qquad+\mathcal{O}(1)\cdot\left(\big|x-\psi(x)\big|+\int_{x-1}^x \big|y-\psi(y)\big|\cdot\frac{v_x^2\big(\psi(y)\big)}{2}\,\psi'(y)\,dy\right)\\
&\qquad+\mathcal{O}(1)\cdot\left(J^\psi(u,v)+\Big|\int_{x-1}^x \chi'(x-y)\cdot\big[\psi'(y)-1\big]\,dy\Big|\right)\\
&=\mathcal{O}(1)\cdot\big|x-\psi(x)\big|+\mathcal{O}(1)\cdot J^\psi(u,v)+\mathcal{O}(1)\cdot\left(\big|x-\psi(x)\big|+\int_{x-1}^x \chi''(x-y)\cdot\big[\psi(y)-y\big]\,dy\right)\\
&=\mathcal{O}(1)\cdot\big|x-\psi(x)\big|+\mathcal{O}(1)\cdot J^\psi(u,v)\,.
\end{aligned}\tag{5.9}$$

Integrating over one period we conclude

$$I_2=\mathcal{O}(1)\cdot J^\psi\big(u(t),\,v(t)\big)\,.\tag{5.10}$$

For future use, we observe that a computation entirely similar to (5.9) yields

$$\Big|P_x^u(x)-P_x^v\big(\psi(x)\big)\Big|=\mathcal{O}(1)\cdot\big|x-\psi(x)\big|+\mathcal{O}(1)\cdot J^\psi(u,v)\,.\tag{5.11}$$

Next, we look at the term

$$I_3\doteq\int_0^1 \frac{d}{dt}\Big|2\arctan\,u_x(t,x)-2\arctan\,v_x\big(t,\psi_{(t)}(x)\big)\Big|\cdot\phi_1^{(t)}(x)\,\big(1+u_x^2(t,x)\big)\,dx\,.\tag{5.12}$$

Along a characteristic, according to (1.3) one has

$$\frac{d}{dt}2\arctan u_x(t,\xi(t))=\frac{2}{1+u_x^2}\Big[u^2-\frac{u_x^2}{2}-P^u\Big]\,.$$

Call $\theta^u\doteq 2\arctan u_x$, $\theta^v\doteq 2\arctan v_x$, so that

$$\frac{1}{1+u_x^2}=\cos^2\frac{\theta^u}{2}\,,\qquad\qquad \frac{u_x}{1+u_x^2}=\frac12\sin\theta^u\,,\qquad\qquad \frac{u_x^2}{1+u_x^2}=\sin^2\frac{\theta^u}{2}\,.$$

We now have

$$\begin{aligned}
&\int_0^1\big(1+u_x^2(x)\big)\cdot\left|\frac{u_x^2(x)}{1+u_x^2(x)}-\frac{v_x^2(\psi(x))}{1+v_x^2(\psi(x))}\right|\,dx\\
&=\int_0^1\big(1+u_x^2(x)\big)\cdot\left|\sin^2\frac{\theta^u(x)}{2}-\sin^2\frac{\theta^v\big(\psi(x)\big)}{2}\right|\,dx\\
&\leq\int_0^1\big(1+u_x^2(x)\big)\cdot\Big|\theta^u(x)-\theta^v\big(\psi(x)\big)\Big|\,dx\\
&=\mathcal{O}(1)\cdot J(u,v)\,.
\end{aligned}\tag{5.13}$$

Next, using (5.11) we compute

$$
\begin{aligned}
\int_0^1 \big(1+u_x^2(x)\big)\cdot & \left|\frac{u^2(x)-P^u(x)}{1+u_x^2(x)} - \frac{v^2\big(\psi(x)\big)-P^v\big(\psi(x)\big)}{1+v_x^2\big(\psi(x)\big)}\right| dx \\
& \le \int_0^1 \Big|u^2(x)-v^2\big(\psi(x)\big)\Big|\,dx + \int_0^1 \Big|P^u(x)-P^v\big(\psi(x)\big)\Big|\,dx \\
& \qquad + \mathcal{O}(1)\cdot \int_0^1 \left|\frac{1}{1+u_x^2(x)} - \frac{1}{1+v_x^2\big(\psi(x)\big)}\right| \cdot \big(1+u_x^2(x)\big)\,dx \\
& = \mathcal{O}(1)\cdot J^{\psi}(u,v)\,,
\end{aligned}
\tag{5.14}
$$

where the last term was estimated by observing that

$$
\left|\frac{1}{1+u_x^2} - \frac{1}{1+v_x^2}\right| \le \big|2\arctan u_x - 2\arctan v_x\big|_*\,. \tag{5.15}
$$

Putting together all previous estimates we conclude

$$
I_1+I_2+I_3 = \mathcal{O}(1)\cdot J^{\psi}(u,v)\,. \tag{5.16}
$$

To complete the analysis, we have to consider the terms due to the change in base measures. From (1.5) it follows that the production of new mass in the base measures is described by the balance laws

$$
\left\{
\begin{array}{l}
(1+u_x^2)_t + \big[u(1+u_x^2)\big]_x = [2u^2+1-2P^u]u_x \doteq f^u\,,\\
(1+v_x^2)_t + \big[v(1+v_x^2)\big]_x = [2v^2+1-2P^v]v_x \doteq f^v\,.
\end{array}
\right.
\tag{5.17}
$$

This leads us to consider two further integrals I_4, I_5 :

$$
\begin{aligned}
I_4 & = \int_0^1 d^{\diamondsuit}\Big(\big(x,\, u(x),\, 2\arctan u_x(x)\big)\,,\ \big(\psi(x),\, v(\psi(x)),\, 2\arctan v_x(\psi(x))\big)\Big) \\
& \qquad\qquad \cdot \big|2u^2(x)+1-2P^u(x)\big|\,\big|u_x(x)\big|\,dx \\
& = \mathcal{O}(1)\cdot \int_0^1 d^{\diamondsuit}\Big(\big(x,\, u(x),\, 2\arctan u_x(x)\big)\,,\ \big(\psi(x),\, v(\psi(x)),\, 2\arctan v_x(\psi(x))\big)\Big) \\
& \qquad\qquad\qquad \cdot \big(1+u_x^2(x)\big)\,dx \\
& = \mathcal{O}(1)\cdot J^{\psi}(u,v)\,.
\end{aligned}
\tag{5.18}
$$

$$
\begin{aligned}
I_5 & = \int_0^1 \Big|\big[2u^2(x)+1-2P^u(x)\big]u_x(x) - \Big[2v^2\big(\psi(x)\big)+1-2P^v\big(\psi(x)\big)\Big]v_x\big(\psi(x)\big)\,\psi'(x)\Big|\,dx \\
& \le 2\int_0^1 \left\{\Big|u^2(x)-v^2\big(\psi(x)\big)\Big| + \Big|P^u(x)-P^v\big(\psi(x)\big)\Big|\right\}\big|u_x(x)\big|\,dx \\
& \qquad + \int_0^1 \Big|2v^2\big(\psi(x)\big)+1-2P^v\big(\psi(x)\big)\Big|\cdot\Big|u_x(x)-v_x\big(\psi(x)\big)\psi'(x)\Big|\,dx \\
& = I_5' + I_5''
\end{aligned}
\tag{5.19}
$$

Using (5.11) we easily obtain

$$
\begin{aligned}
I_5' &\leq \int_0^1 \left\{ \Big| u^2(x) - v^2\big(\psi(x)\big) \Big| + \Big| P^u(x) - P^v\big(\psi(x)\big) \Big| \right\} \big(1 + u_x^2(x)\big)\, dx \\
&= \mathcal{O}(1) \cdot J^\psi(u, v)\,.
\end{aligned} \tag{5.20}
$$

On the other hand, recalling (5.13) and using the change of variable $y = \psi(x)$, $x = \psi^{-1}(y)$, we find

$$
\begin{aligned}
I_5'' &= \mathcal{O}(1) \cdot \int_0^1 \Big| u_x(x) - v_x\big(\psi(x)\big)\psi'(x) \Big|\, dx \\
&= \mathcal{O}(1) \cdot \int_0^1 \left| \frac{u_x(x)}{1 + u_x^2(x)} - \frac{v_x\big(\psi(x)\big)}{1 + v_x^2\big(\psi(x)\big)} \right| \big(1 + u_x^2(x)\big)\, dx \\
&\qquad + \mathcal{O}(1) \cdot \int_0^1 \Big| v_x\big(\psi(x)\big)\, \psi'(x) \Big| \cdot \left| \frac{1 + u_x^2(x)}{\big(1 + v_x^2(\psi(x))\big)\, \psi'(x)} - 1 \right| \\
&\leq \mathcal{O}(1) \cdot J^\psi(u, v) + \int_0^1 \Big| \big(1 + u_x^2(x)\big) - \big(1 + v_x^2(\psi(x))\big)\, \psi'(x) \Big|\, dx \\
&= \mathcal{O}(1) \cdot J(u, v)\,.
\end{aligned} \tag{5.21}
$$

All together, the previous estimates show that

$$
\frac{d}{dt} J^{\psi(t)}\big(u(t),\, v(t)\big) \leq I_1 + I_2 + I_3 + I_4 + I_5' + I_5'' = \mathcal{O}(1) \cdot J^{\psi(t)}\big(u(t),\, v(t)\big)\,, \tag{5.22}
$$

where $\mathcal{O}(1)$ denotes a quantity which remains uniformly bounded as u, v range on bounded sets of H^1_{per}. As an immediate consequence we obtain

Lemma 7. *Let $t \mapsto u(t)$, $t \mapsto v(t)$ be two conservative, spatially periodic multipeakon solutions, as in Lemma 2. Then there exists a constant κ, depending only on $\max\big\{\|u\|_{H^1_{\mathrm{per}}},\ \|v\|_{H^1_{\mathrm{per}}}\big\}$, such that*

$$
J\big(u(t)\,,\ v(t)\big) \leq e^{\kappa|t-s|} \cdot J\big(u(s)\,,\ v(s)\big) \qquad\qquad s, t \in \mathbb{R}\,. \tag{5.23}
$$

Proof. For $t > s$ the estimate (5.23) follows from (5.22), taking the infimum among all transportation plans $\psi_{(s)}$ at time s. The case $t < s$ is obtained simply by observing that the Camassa-Holm equations are time-reversible. □

6 - Proof of the main theorems

Thanks to the analysis in the previous sections, we now all the ingredients toward a proof of Theorems 1 and 2.

The estimates in (1.16) follow from Lemma 3. Given an initial data $\bar u \in H^1_{\mathrm{per}}$, to construct the solution of the Camassa-Holm equation we consider a sequence of multi-peakons $\bar u_n$, converging to $\bar u$ in H^1_{per}. Then we consider the corresponding solutions $t \mapsto u_n(t)$, defined for all $n \geq 1$ and $t \in \mathbb{R}$. This is possible because of Lemmas 1 and 2.

We claim that the sequence $u_n(t)$ is Cauchy in $L^2_{\rm per}$. Indeed, by Lemma 3 and Lemma 5,

$$\big\|u_m(t)-u_n(t)\big\|_{\mathbf{L}^1_{\rm per}} \leq C\cdot J\big(u_m(t),\, u_n(t)\big) \leq C\cdot e^{\kappa|t|}\, J\big(u_m(0),\, u_n(0)\big) \leq C^2\cdot e^{\kappa|t|}\big\|u_m(t)-u_n(t)\big\|_{H^1_{\rm per}}\,.$$

Therefore, $u_n(t) \to u(t)$ in $\mathbf{L}^1_{\rm per}$, for some function $u : I\!R \mapsto H^1_{\rm per}$. By interpolation, the convergence $u_n \to u$ also holds in all spaces $\mathbf{L}^p_{\rm per}$, $1 \leq p \leq \infty$. The continuity estimates (1.17)-(1.18) now follow by passing to the limit in Lemma 5 and 6.

It remains to show that the limit function $u(\cdot)$ is actually a solution to the Camassa-Holm equation and its energy $E(t)$ in (1.15) is a.e. constant. Toward these goals, we observe that all solutions u_n are Lipschitz continuous with the same Lipschitz constant, as maps from $I\!R$ into $L^2_{\rm per}$. Indeed

$$\|u_{n,t}\|_{\mathbf{L}^2_{\rm per}} \leq \|u_n\|_{\mathbf{L}^\infty}\cdot\|u_{n,x}\|_{\mathbf{L}^2_{\rm per}} + \left\|\frac{1}{2}\,e^{-|x|}\right\|_{\mathbf{L}^2}\cdot\left\|u_n^2+\frac{u_{n,x}^2}{2}\right\|_{\mathbf{L}^1_{\rm per}}. \tag{6.1}$$

As a consequence, the map $t \mapsto u(t)$ has uniformly bounded $H^1_{\rm per}$ norm, and is Lipschitz continuous with values in $L^2_{\rm per}$. In particular, u is uniformly Hölder continuous as a function of t, x and the convergence $u_m(t,x) \to u(t,x)$ holds uniformly for t in bounded sets. Moreover, since $\mathbf{L}^2_{\rm per}$ is a reflexive space, the time derivative $u_t(t) \in \mathbf{L}^2_{\rm per}$ is well defined for a.e. $t \in I\!R$.

We now observe that, for each $n \geq 1$, both sides of the equality

$$\frac{d}{dt}u_n = -u_n\, u_{n,x} - P^{u_n}_x \tag{6.2}$$

are continuous as functions from $I\!R$ into $\mathbf{L}^1_{\rm per}$, and the identity holds at every time $t \in I\!R$, with the exception of the isolated times where a peakon interaction occurs.

At any time t where no peakon interaction occur in the solution u_n, we define $\mu^{(n)}_t$ to be the measure with density $u_n^2(t,\cdot) + \frac{1}{2}u_{n,x}(t,\cdot)$ w.r.t. Lebesgue measure. By Lemmas 5 and 4, the map $t \mapsto \mu^{(n)}_t$ can be extended by weak continuity to all times $t \in I\!R$. We can now redefine

$$P^{u_n}(t,x) \doteq \int \frac{1}{2}e^{-|x-y|}\, d\mu^{(n)}_t(y)\,, \qquad\qquad P^u(t,x) \doteq \int \frac{1}{2}e^{-|x-y|}\, d\mu_t(y)\,. \tag{6.3}$$

where μ_t is the weak limit of the measures $\mu^{(n)}_t$. Because of the convergence $J\big(u_n(t),\, u(t)\big) \to 0$, by Lemma 4 the map $t \mapsto \mu_t$ is well defined and continuous w.r.t. the weak topology of measures. Using again Lemma 4, we can take the limit of (6.2) as $n \to \infty$, and obtain the identity (1.14), for every $t \in I\!R$ and $P = P^u$ defined by (6.3).

For each n, the total energy $\mu^{(n)}_t\big(]0,1]\big) = E^{\bar u_n}$ is constant in time and converges to $E^{\bar u}$ as $n \to \infty$. Therefore we also have

$$\mu_t\big(]0,1]\big) = E^{\bar u} \doteq \int_0^1 \big[\bar u^2(x) + \bar u_x^2(x)\big]\, dx \qquad\qquad t \in I\!R\,. \tag{6.4}$$

To complete the proof of Theorem 1, it now only remains to prove that the measure μ_t is absolutely continuous with density $u^2(t,\cdot) + \frac{1}{2}u_x^2(t,\cdot)$ w.r.t. Lebesgue measure, for a.e. time $t \in I\!R$.

In this direction, we recall that, by (1.5), the function $w \doteq u^2_{n,x}$ satisfies the linear transport equation with source

$$w_t + (uw)_x = (u_n^2 - P^{u_n})u_{n,x}\,. \tag{6.5}$$

Moreover, along any characteristic curve $t \mapsto \xi(t)$ by (1.3) one has

$$\frac{d}{dt}\Big[2\arctan u_{n,x}\big(t,\xi(t,x)\big)\Big] \;=\; \frac{2}{1+u_{n,x}^2}\left[u_n^2 - \frac{u_{n,x}^2}{2} - P^{u_n}\right] \;\leq\; -\frac{1}{2}, \tag{6.6}$$

whenever $u_{n,x}^2$ is sufficiently large. Now fix any time interval $[\tau,\tau']$. By (6.5)-(6.6), for all $\varepsilon > 0$ small we have

$$\int_\tau^{\tau'}\int_{\{|\pi+2\arctan u_{n,x}|<\varepsilon\}} \big|u_{n,x}^2(t,x)\big|\,dxdt \leq C\cdot\varepsilon\,, \tag{6.7}$$

for some constant C depending only on $\tau'-\tau$ and uniformly valid as u_n ranges over bounded subsets of $H^1_{\rm per}$. Because of (6.7), the sequence of functions $u_{n,x}^2$ is equi-integrable on any domain of the form $[\tau,\tau']\times[0,1]$. Namely

$$\lim_{\kappa\to\infty}\int_\tau^{\tau'}\int_{\{x\in[0,1],\ u_{n,x}^2>\kappa\}} u_{n,x}^2(t,x)\,dxdt = 0\,,$$

uniformly w.r.t. n. Therefore, by possibly taking a subsequence, we have

$$u_{n,x}^2 \to u_x^2 \qquad \text{in}\quad \mathbf{L}^1\big([\tau,\tau']\times[0,1]\big)\,.$$

By Fubini's theorem, this implies

$$\lim_{n\to\infty}\int_0^1 u_{n,x}^2(t,x)\,dx = \int_0^1 u_x^2(t,x)\,dx$$

for a.e. $t\in[\tau,\tau']$. At every such time t, the measure μ_t is absolutely continuous and the definition (6.3) coincides with (1.2). This completes the proof of Theorem 1. □

7 - Uniqueness

Before proving Theorem 3, we remark that the solution satisfying all conditions in Theorem 1 need not be unique.

Example. Let $u = u(t,x)$ be a solution containing exactly two peakons of opposite strengths $p_1(t) = -p_2(t)$, located at points $q_1(t) = -q_2(t)$ (see fig. 2). We assume that initially $p_1(0) > 0\, q_1(0) < 0$. At a finite time $T > 0$, the two peakons interact at the origin. In particular, as $t\to T-$ there holds

$$p_1(t)\to\infty, \qquad p_2(t)\to-\infty, \qquad q_1(t)\to 0, \qquad q_2(t)\to 0\,.$$

Moreover, $\big\|u(t)\big\|_{\mathbf{L}^\infty}\to 0$, while the measure μ_t approaches a Dirac mass at the origin. We now have various ways to extend the solution beyond time T:

$$\tilde u(\tau,x) \doteq 0\,, \tag{7.1}$$

$$u(\tau,x) = -u(\tau-T,\,-x)\,, \tag{7.2}$$

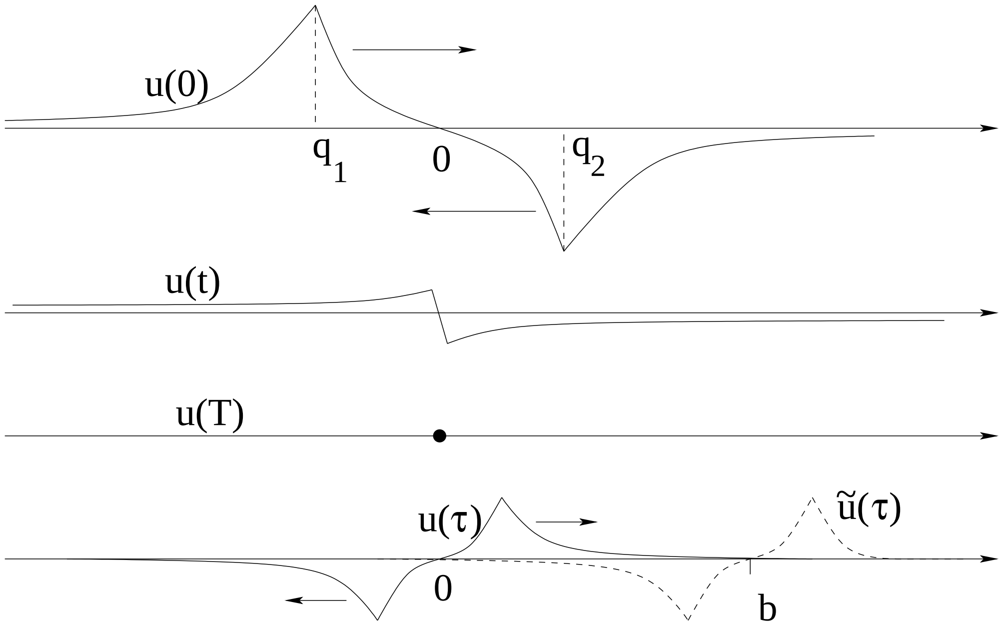

figure 2

Clearly, $\tilde{u}$ dissipates all the energy, and does not satisfy the identity (1.15). The function u in (7.2) is the one constructed by our algorithm in Section 2. However, there are infinitely many other solutions that still satisfy (1.15), for example

$$\tilde{u}(\tau, x) = u(\tau, x - b) \tag{7.3}$$

where u is as in (7.2) and $b \neq 0$. The additional condition in Theorem 3 rules out all of them, because as $\tau \to T+$, the corresponding measures $\tilde{\mu}_\tau$ approach a Dirac mass at the point $x = b$, not at the origin.

We can now give a proof of Theorem 3. As a first step, we extend our distance J to a larger domain $\mathcal{D}$, consisting of couples (u, μ), where $u \in H^1_{\rm per}$ and μ is a positive (spatially periodic) measure whose absolutely continuous part has density $u^2 + u_x^2$ w.r.t. Lebesgue measure. This extension is achieved by continuity:

$$J\big((u, \mu),\, (\tilde{u}, \tilde{\mu})\big) \doteq \liminf_{n \to \infty} J(u_n, \tilde{u}_n) \tag{7.4}$$

where the infimum is taken over all couple of sequences $(u_n, \tilde{u}_n)_{n \geq 1}$ such that

$$\|u_n - u\|_{\mathbf{L}^\infty} \to 0\,, \qquad \|\tilde{u}_n - \tilde{u}\|_{\mathbf{L}^\infty} \to 0\,,$$

$$u^2_{n,x} \rightharpoonup \mu\,, \qquad \tilde{u}^2_{n,x} \rightharpoonup \tilde{\mu}\,.$$

We observe that the flow Φ constructed in Theorem 2 can be continuously extended to a locally Lipschitz continuous group of transformations on the domain $\mathcal{D}$.

Now let $t \mapsto \tilde{u}(t)$ be a solution of the Cauchy problem (1.1), (1.13), satisfying all the required conditions. In particular, the map $t \mapsto \big(\tilde{u}(t), \tilde{\mu}_t\big)$ is Lipschitz continuous w.r.t. the distance J, with values in the domain $\mathcal{D}$.

Calling $t \mapsto \big(\tilde u(t), \tilde\mu_t\big) \doteq \Phi_t(\bar u, \bar u_x^2)$ the unique solution of the Cauchy problem obtained as limit of multi-peakon approximations, we need to show that $\tilde u(t) = u(t)$ for all t. To fix the ideas, let $t > 0$. By the Lipschitz continuity of the flow, we can use the error estimate

$$J\Big(\big(\tilde u(t), \tilde\mu_t\big)\,,\ \big(u(t), \mu_t\big)\Big) \leq e^{C_2 t}\int_0^t \liminf_{h\to 0} \frac{1}{h}\cdot J\Big(\big(\tilde u(\tau+h), \tilde\mu_{\tau+h}\big)\,,\ \Phi_h\big(\tilde u(\tau), \tilde\mu_\tau\big)\Big)\,d\tau \tag{7.5}$$

For a proof of (7.5), see pp. 25–27 in [B]. The conditions stated in Theorem 1 now imply that, at almost every time τ, the measure $\tilde\mu_t$ is absolutely continuous and the integrand in (7.5) vanishes. Therefore $\tilde u(t) = u(t)$ for all t. □

We can now prove that, in multi-peakon solutions, interactions involving exactly two peakons are the only possible ones.

Corollary. *Let $t \mapsto u(t, \dot)$ be a multi-peakon solution of the form (1.7), which remains regular on the open interval $]0, T[$. Assume that at time $T > 0$ an interaction occurs, say among the first k peakons, so that*

$$\lim_{t\to T-} q_i(t) = \bar q \qquad\qquad i = 1, \ldots, k\,.$$

Then $k = 2$.

Proof. We first observe that the Camassa-Holm equations (1.1) are time-reversible. In particular, our proof of Theorem 3 shows that the solution to a Cauchy problem is unique both forward and backward in time.

Now consider the data $\big(u(T),\, \mu_T\big) \in \mathcal{D}$, where μ_T is the weak limit of the measures μ_t having density $u^2(t) + u_x^2(t)$ w.r.t. Lebesgue measure, as $t \to T-$. By the analysis in Section 2, we can construct a backward solution of this Cauchy problem in terms of exactly two incoming peakons. By uniqueness, this must coincide with the given solution $u(\cdot)$ for all $t \in [0, T]$. □

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