Helically Symmetric Solutions to the 3-D Navier-Stokes Equations for Compressible Isentropic Fluids [1]

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Abstract: We prove the existence of global weak solutions to the Navier-Stokes equations for compressible isentropic fluids for any $\gamma > 1$ when the Cauchy data are helically symmetric, where the constant γ is the specific heat ratio. Moreover, a new integrability estimate of the density in any neighborhood of the symmetry axis (the singularity axis) is obtained.

Keywords: 3-D compressible Navier-Stokes equations, helical symmetry, global weak solutions, isentropic fluids.

1 Introduction

This paper is mainly concerned with the global existence of weak solutions to the Cauchy problem for the compressible isentropic Navier-Stokes with helically symmetric initial data in $\mathbb{R}^3$:

$$\begin{cases} \varrho_t + \mathrm{div}\,(\varrho \boldsymbol{U}) = 0, \\ (\varrho \boldsymbol{U})_t + \mathrm{div}\,(\varrho \boldsymbol{U} \otimes \boldsymbol{U}) + a\nabla(\varrho^\gamma) = \mu\Delta \boldsymbol{U} + \tilde{\mu}\nabla \mathrm{div}\, \boldsymbol{U}, \end{cases} \tag{1.1}$$

with initial data

$$\varrho(\boldsymbol{x}, 0) = \varrho_0, \quad (\varrho \boldsymbol{U})(\boldsymbol{x}, 0) = \boldsymbol{M}_0, \qquad \boldsymbol{x} \in \mathbb{R}^3 \tag{1.2}$$

that are helically symmetric, i.e., ϱ_0 and $\boldsymbol{M}_0$ are periodic in x_3 of period $2\pi/\alpha$ ($0 < \alpha \in \mathbb{R}$), where ϱ and $\boldsymbol{U} = (U_1, U_2, U_3)$ are the density and velocity, respectively, $a\varrho^\gamma$ is the pressure with $\gamma > 1$ being the specific heat ratio and $a > 0$ being constant, $\mu, \tilde{\mu} > 0$ are constant viscosity coefficients.

For helically symmetric flow, in cylindrical coordinates (r, θ, z) ($0 < r < \infty$, $0 \le \theta \le 2\pi$, $-\infty < z < \infty$), the velocity vector $\boldsymbol{U}$ and the pressure $a\varrho^\gamma$ do not depend on θ and z independently, but only on the linear combination $\xi = n\theta + \alpha z$ where n is a given even integer. Namely, for helically symmetric flow,

$$\varrho(t, \boldsymbol{x}) = \rho(t, r, \xi), \;\; \boldsymbol{U}(t, \boldsymbol{x}) = \Big(\frac{x_1}{r} u_1(t, r, \xi) - \frac{x_2}{r} u_2(t, r, \xi), \frac{x_2}{r} u_1(t, r, \xi) + \frac{x_1}{r} u_2(t, r, \xi), u_3(t, r, \xi)\Big) \tag{1.3}$$

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for some $\rho(t,r,\xi)$ and $\boldsymbol{u}(t,r,\xi) = \big(u_1(t,r,\xi), u_2(t,r,\xi), u_3(t,r,\xi)\big)$, where ρ and $\boldsymbol{u}$ are periodic in ξ of period 2π, $\boldsymbol{x} = (x_1,x_2,x_3) \in \mathbb{R}^3$ and $r = \sqrt{x_1^2 + x_2^2}$. Then, the helical symmetry of the initial data $(\varrho_0, \boldsymbol{M}_0)$ means that

$$\varrho_0(\boldsymbol{x}) \equiv \rho_0(r,\xi),\ \ \boldsymbol{M}_0(\boldsymbol{x}) \equiv \Big(\tfrac{x_1}{r} m_1^0(r,\xi) - \tfrac{x_2}{r} m_2^0(r,\xi), \tfrac{x_2}{r} m_1^0(r,\xi) + \tfrac{x_1}{r} m_2^0(r,\xi), m_3^0(r,\xi)\Big) \quad (1.4)$$

for some $\rho_0(r,\xi)$ and $(m_1^0, m_2^0, m_3^0)(r,\xi) \equiv \boldsymbol{m}_0(r,\xi)$, where $\rho_0(r,\xi)$ and $\boldsymbol{m}_0(r,\xi)$ are periodic in ξ of period 2π.

The Navier-Stokes equations for compressible fluids have been studied by many authors. The question concerning the global existence and the time-asymptotic behavior of solutions for large initial data has been largely solved in one dimension. The mathematical theory, however, is far from being complete in more than one dimension. In the case of sufficiently small initial data, there is an extensive literature on the global existence and the asymptotic behavior of solutions which is originated by the papers of Matsumura and Nishida[15, 16] (also see, e.g., [7] on recent progress). For large initial data, Lions[13] used the weak convergence method and first obtained the existence of global weak solutions for isentropic flow under the assumption that $\gamma \ge 3/2$ if the dimension $N = 2$ and $\gamma \ge 9/5$ if $N = 3$. In [6, 9, 10, 11] the global existence of spherically symmetric and axisymmetric weak solutions (without swirls) to the Cauchy problem for any $\gamma \ge 1$ are proved (also cf. [8] on an exterior problem for the full compressible Navier-Stokes equations). By modifying Lions' arguments, and using delicately the Div-Curl Lemma and an idea from [9], Feireisl, Novotný and Petzeltová extended Lions' global existence result in $\mathbb{R}^3$ to the case $\gamma > 3/2$ (see [4]). We also mention that the existence of weak time-periodic solutions was proved in [3] under a condition on γ similar to that of Lions [13] and the global existence of strong large solutions in [18] under the condition that the viscosity depends on ρ in a very specific way, while in [19], non-existence results of global smooth solutions were discussed for initial density with vacuum.

In this paper, we shall combine the ideas in [4, 9, 13] to prove the global existence of helically symmetric weak solutions to the 3-D compressible isentropic Navier-Stokes equations for any $\gamma > 1$. Comparing with the axisymmetric case in [9], the difficulties here lie in the following: First, for helically symmetric flows, there are three components in the velocity field and some swirls are allowed in the flows, and hence, the equations in the symmetric form become much more complex and contain the new cross terms, such as $\frac{n}{r}(\rho u_i u_j)$ $(i,j = 1,2,3)$ and $\frac{n^2}{r^2}\partial_\xi^2 u_i$ $(i = 1,2,3)$, which induce new difficulties and have to be dealt with carefully in weak convergence; then, in the first glance, when using the effective viscous pressure $P_{\text{eff}} \equiv P - \mu(\nabla_{\boldsymbol{x}} \Delta_{\boldsymbol{x}}^{-1} \nabla_{\boldsymbol{x}}) : \Delta_{\boldsymbol{x}} \boldsymbol{U}$ to derive higher estimates for the density ϱ, one should use three equations for the velocities (2.2)–(2.4). Unfortunately, this could bring difficulties in defining properly the inverse of a (degenerate) elliptic operator needed in the derivation of the higher estimates. Instead, we actually use only two equations (2.2), (2.4) to obtain the higher estimates for ϱ (cf. Section 3).

For the sake of the simplicity of the presentation, we may assume $\tilde{\mu} = 0$ without loss of generality. It is easy to see, from the proof of this paper, that the case $\tilde{\mu} \neq 0$ will not arise any new difficulties.

Now we modify the definition of the so-called finite energy solutions to the system (1.1), (1.2) in [4] in the following way (the notation below will be given at the end of this section):

Definition 1.1 *We call $(\varrho, \boldsymbol{U})$ a finite energy weak solution of $\frac{2\pi}{\alpha}$-period in x_3 to (1.1) and (1.2), if*

(1) $\varrho \geq 0$ a.e., and for any $T > 0$,

$$\begin{aligned}
&\varrho \in L^\infty(0,T;L^\gamma(G)), \quad \boldsymbol{U} \in L^2(0,T;H^1_{\mathrm{loc}}(\bar{G})),\\
&\varrho \in C^0([0,T];L^\gamma_{\mathrm{loc}}(\bar{G})-w), \quad \varrho\boldsymbol{U} \in C^0([0,T];L^{2\gamma/(\gamma+1)}_{\mathrm{loc}}(\bar{G})-w),\\
&(\varrho,\varrho\boldsymbol{U})(\boldsymbol{x},0) = (\varrho_0,\boldsymbol{M}_0)(\boldsymbol{x}) \quad \text{weakly in } L^\gamma_{\mathrm{loc}}(\bar{G}) \times L^{2\gamma/(\gamma+1)}_{\mathrm{loc}}(\bar{G}),
\end{aligned} \tag{1.5}$$

and $(\varrho, \boldsymbol{U})$ is periodic in x_3 of period $2\pi/\alpha$, where $G = \{\boldsymbol{x} \in \mathbb{R}^3 \mid 0 < x_3 < 2\pi/\alpha\}$ and $\bar{G} = \{\boldsymbol{x} \in \mathbb{R}^3 \mid 0 \leq x_3 \leq 2\pi/\alpha\}$.
(2) Eqs. (1.1) are satisfied in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$; moreover for any $b \in C^1(\mathbb{R})$ such that $|b(s)| + |b'(s)s| \leq C$ for all $s \in \mathbb{R}$, there holds:

$$\partial_t b(\varrho) + \operatorname{div}(b(\varrho)\boldsymbol{U}) + (b'(\varrho)\varrho - b(\varrho))\operatorname{div}\boldsymbol{U} = 0 \tag{1.6}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$, i.e., $(\varrho, \boldsymbol{U})$ is a renormalized solution of (1.1)$_1$ (see DiPerna and Lions [1]).
(3) The following energy inequality

$$\frac{d}{dt}\int_G \left(\varrho|\boldsymbol{U}|^2 + \frac{a\varrho^\gamma}{\gamma-1}\right)(x,t)dx + \int_G (\mu|\nabla\boldsymbol{U}|^2 + \tilde{\mu}|\operatorname{div}\boldsymbol{U}|^2)(x,t)dx \leq 0 \tag{1.7}$$

holds in the sense of distributions.

Thus, the main result of this paper reads:

Theorem 1.1 *Let $\gamma > 1$. Assume that $\varrho_0, \boldsymbol{M}_0$ are helically symmetric (i.e., (1.4) holds), and that $0 \leq \rho_0 \in \mathcal{L}^1(\mathbb{R}^+ \times (0,2\pi)) \cap \mathcal{L}^\gamma(\mathbb{R}^+ \times (0,2\pi))$, $\boldsymbol{m}_0/\sqrt{\rho_0} \in \mathcal{L}^2(\mathbb{R}^+ \times (0,2\pi))$ and $\rho_0, \boldsymbol{m}_0$ are periodic in ξ of 2π-period. Then, there exists a global finite energy weak solution $(\varrho, \boldsymbol{U})$ of $\frac{2\pi}{\alpha}$-period to (1.1), (1.2) which is helically symmetric, i.e., (1.3) holds for some $(\rho, \boldsymbol{u})$ that is a weak solution of (2.1)–(2.7). Moreover, for any $T > 0$ and $\beta \in (0,1)$, we have*

$$\int_0^T\int_0^1\int_0^{2\pi} (\rho^\gamma + \rho u_1^2 + \rho u_2^2) r^\beta dr d\xi dt \leq C. \tag{1.8}$$

Remark 1.1 *A similar existence of strong solutions in the incompressible fluid case has been proved by Mahalov, Titi and Leibovich in [14].*

We will prove Theorem 1.1 by showing first that there exists a global weak solution $(\rho, \boldsymbol{u})$ to (2.1)-(2.7), and then that $(\varrho, \boldsymbol{U})$ of the form (1.3) satisfies Definition 1.1, thus obtaining a global helically symmetric weak solution.

The paper is organized as follows: In Section 2 we derive a priori estimates for the approximate weak solutions of (2.1)–(2.7) and give the proof of Theorem 1.1 in Section 3. Section 4 is devoted to the study of the global existence of the approximate weak solutions to (2.1)-(2.7).

Notation (*used throughout this paper):* Let m be an integer and $1 \leq p \leq \infty$. By $W^{m,p}(\mathcal{O})$ ($W^{m,p}_0(\mathcal{O})$) we denote the usual Sobolev space defined over a domain $\mathcal{O}$. $W^{m,2}(\mathcal{O}) \equiv H^m(\mathcal{O})$ ($W^{m,2}_0(\mathcal{O}) \equiv H^m_0(\mathcal{O})$), $W^{0,p} \equiv L^p(\mathcal{O})$ with norm $\|\cdot\|_{L^p(\mathcal{O})}$. For $\Omega \subset \mathbb{R}^2$ we define

$$\mathcal{L}^p(\Omega) := \Big\{f \in L^1_{\mathrm{loc}}(\Omega) : \int_\Omega |f(r,\xi)|^p r dr d\xi < \infty\Big\}, \quad \|\cdot\|_{\mathcal{L}^p(\Omega)} := \Big(\int_\Omega |f(r,\xi)|^p r dr d\xi\Big)^{1/p}.$$

$\mathcal{L}^p_{\rm loc}(\Omega)$ and $\mathcal{H}^1_{\rm loc}(\Omega)$ are defined similarly to $L^p_{\rm loc}(\Omega)$ and $H^1_{\rm loc}(\Omega)$, respectively. In particular, we use the following abbreviations:

$$\begin{aligned}
&\mathbb{R}^+ := (0,\infty),\ \mathbb{R}^+_0 := [0,\infty), \quad \|\cdot\|_{\mathcal{L}^p} \equiv \|\cdot\|_{\mathcal{L}^p(\mathbb{R}^+\times(0,2\pi))}, \quad \|\cdot\|_{L^p} \equiv \|\cdot\|_{L^p(\mathbb{R}^+\times(0,2\pi))},\\
&\nabla := (\partial_r, \frac{n}{r}\partial_\xi, \alpha\partial_\xi), \quad \tilde{\Delta} := \partial_r^2 + \alpha^2\partial_\xi^2, \quad \tilde{\nabla} := (\partial_r, \alpha\partial_\xi) \equiv (\partial_1, \partial_2),\\
&\tilde{\boldsymbol{u}} =: (u_1, u_3), \quad \tilde{\rm div}\, \tilde{\boldsymbol{u}} := \partial_r u_1 + \alpha\partial_\xi u_3.
\end{aligned}$$

$L^p(I,B)$ respectively $\|\cdot\|_{L^p(I,B)}$ denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p=\infty$) functions from I to B respectively its norm, $I \subset R$ an interval, B a Banach space. $C(I, B-w)$ is the space of all functions which are in $L^\infty(I,B)$ and continuous in t with values in B endowed with the weak topology.

The same letter C (sometimes used as $C(X)$ to emphasize the dependence of C on X) will denote various positive constants which do not depend on ϵ and δ.

2 Approximate solutions and a priori estimates

The helically symmetric form of the compressible Navier-Stokes equations (1.1) for the unknowns $\rho(t,r,\xi)$ and $\boldsymbol{u}(t,r,\xi)$ reads (cf. [14] for a derivation in the incompressible fluid case):

$$\partial_t\rho + \frac{1}{r}\partial_r(r\rho u_1) + \frac{n}{r}\partial_\xi(\rho u_2) + \alpha\partial_\xi(\rho u_3) = 0, \tag{2.1}$$

$$\begin{aligned}
&\partial_t(\rho u_1) + \frac{1}{r}\partial_r(r\rho u_1^2) + \frac{n}{r}\partial_\xi(\rho u_1 u_2) + \alpha\partial_\xi(\rho u_1 u_3) - \frac{\rho u_2^2}{r} + \partial_r P\\
&\quad = \mu\Big[\frac{1}{r}\partial_r(r\partial_r u_1) + \Big(\alpha^2 + \frac{n^2}{r^2}\Big)\partial_\xi^2 u_1 - \frac{u_1}{r^2} - \frac{2n}{r^2}\partial_\xi u_2\Big],
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
&\partial_t(\rho u_2) + \frac{1}{r}\partial_r(r\rho u_1 u_2) + \frac{n}{r}\partial_\xi(\rho u_2^2) + \alpha\partial_\xi(\rho u_2 u_3) + \frac{\rho u_1 u_2}{r} + \frac{n}{r}\partial_\xi P =\\
&\quad \mu\Big[\frac{1}{r}\partial_r(r\partial_r u_2) + \Big(\alpha^2 + \frac{n^2}{r^2}\Big)\partial_\xi^2 u_2 - \frac{u_2}{r^2} + \frac{2n}{r^2}\partial_\xi u_1\Big],
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\partial_t(\rho u_3) + \frac{1}{r}\partial_r(r\rho u_1 u_3) + \frac{n}{r}\partial_\xi(\rho u_2 u_3) + \alpha\partial_\xi(\rho u_3^2) + \alpha\partial_\xi P\\
&\quad = \mu\Big[\frac{1}{r}\partial_r(r\partial_r u_3) + \Big(\alpha^2 + \frac{n^2}{r^2}\Big)\partial_\xi^2 u_3\Big],
\end{aligned} \tag{2.4}$$

together with initial values

$$\rho(0,r,\xi) = \rho_0(r,\xi), \quad (\rho\boldsymbol{u})(0,r,\xi) = \boldsymbol{m}_0(0,r,\xi), \qquad (r,\xi) \in \mathbb{R}^+ \times \mathbb{R}, \tag{2.5}$$

and boundary conditions

$$u_1(t,0,\xi) = u_2(t,0,\xi) = \partial_r u_3(t,0,\xi) = 0, \quad t>0,\ \xi \in \mathbb{R}, \tag{2.6}$$

$$\rho \text{ and } \boldsymbol{u} \text{ are periodic in } \xi \text{ of period } 2\pi. \tag{2.7}$$

Here $P = a\rho^\gamma$, $\boldsymbol{m}_0 = (m_1^0, m_2^0, m_3^0)$, and for simplicity, *we have assumed that n is an even integer* (cf. Remark 2.1).

Remark 2.1 *(i) The choice of the boundary conditions (2.6) follows from the fact that when n is even, a smooth helically symmetric solution to (1.1), (1.2) satisfies (2.6) at $r=0$ automatically. Furthermore, test functions with $2\pi/\alpha$-period in x_3 in (2) of Definition 1.1 automatically satisfy*

(2.6) when $(\varrho, \boldsymbol{U})$ is helically symmetric (cf. the calculations at the end of Section 3).
(ii) When n is odd, we have to impose the following boundary conditions, instead of (2.6),

$$(u_1 + n\partial_\xi u_2)(t,0,\xi) = (u_2 - n\partial_\xi u_1)(t,0,\xi) = \partial_\xi u_3(t,0,\xi) = 0$$

because of the same reason as in (i). In this case, we have to modify (2) in Definition 2.1 appropriately (cf. Remark 2.2). A similar theorem can be obtained without essential changes in the arguments for n being even.

In this section we first construct the approximate solutions of (2.1)–(2.7) by adding an artificial pressure term $\epsilon^\lambda \rho^\beta$ ($\beta > \max\{4,\gamma\}, \lambda > \frac{3\beta}{\gamma} - 3$) and cutting off the singularity induced by the axis $r = 0$ in (2.1)–(2.4), then we derive a priori estimates for the approximate solutions. Before we do this, we first give the definition of the global weak solution to problem (2.1)-(2.7).

Definition 2.1 *Let $\Omega = \mathbb{R}^+ \times [0, 2\pi]$, we call $(\rho, \boldsymbol{u})(r,\xi)$ a finite energy weak solution of (2.1)-(2.7), if*
(1) $\rho \geq 0$ a.e., and for any $T > 0$,

$$\begin{aligned}
&\rho \in L^\infty(0,T;\mathcal{L}^\gamma(\Omega)), \quad \boldsymbol{u} \in L^2(0,T;\mathcal{H}^1_{\mathrm{loc}}(\Omega)),\\
&\rho \in C^0([0,T];\mathcal{L}^\gamma_{\mathrm{loc}}(\Omega) - w), \quad \rho\boldsymbol{u} \in C^0([0,T];\mathcal{L}^{2\gamma/(\gamma+1)}_{\mathrm{loc}}(\Omega) - w),\\
&(\rho, \rho\boldsymbol{u})(r,\xi,0) = (\rho_0, \boldsymbol{u}_0)(r,\xi) \quad \textit{weakly in } \mathcal{L}^\gamma_{\mathrm{loc}}(\Omega) \times \mathcal{L}^{2\gamma/(\gamma+1)}_{\mathrm{loc}}(\Omega).
\end{aligned} \tag{2.8}$$

where $(\rho, \boldsymbol{u})$ is 2π-periodic in the variable ξ for all $\xi \in \mathbb{R}$.
(2) For any $t_2 \geq t_1 \geq 0$, and the test functions $\zeta, \phi, \varphi \in C^1_0(\Omega \times [t_1,t_2])$ with $\phi(t,0,\xi) = \partial_r\varphi(t,0,\xi) = \partial_\xi\phi(t,0,\xi) = \partial_\xi\varphi(t,0,\xi) = 0$, there hold:

$$\int_\Omega \rho\zeta r dr d\xi|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_\Omega \{\rho\zeta_t + \rho u_1 \zeta_r + (\frac{n\rho u_2}{r} + \alpha\rho u_3)\zeta_\xi\} r dr d\xi dt = 0, \tag{2.9}$$

$$\begin{aligned}
&\int_\Omega \rho u_1 \phi r dr d\xi|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_\Omega \{\rho u_1\phi_t + \rho u_1(u_1\phi_r + (\frac{nu_2}{r} + \alpha u_3)\phi_\xi)\} r dr d\xi dt\\
&= \int_{t_1}^{t_2}\int_\Omega \{a\rho^\gamma(\partial_r\phi + \frac{\phi}{r}) + \frac{\rho u_2^2}{r}\phi - \mu(\partial_r u_1 \phi_r + (\frac{w_2}{r} + \alpha^2\partial_\xi u_1)\phi_\xi + \frac{w_1}{r}\phi)\} r dr d\xi dt,
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&\int_\Omega \rho u_2 \phi r dr d\xi|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_\Omega \{\rho u_2\phi_t + \rho u_2(u_1\phi_r + (\frac{nu_2}{r} + \alpha u_3)\phi_\xi)\} r dr d\xi dt\\
&= \int_{t_1}^{t_2}\int_\Omega \{a\rho^\gamma \frac{n\phi_\xi}{r} - \frac{\rho u_1 u_2}{r}\phi - \mu(\partial_r u_2 \phi_r + (\frac{w_1}{r} + \alpha^2\partial_\xi u_2)\phi_\xi - \frac{w_2}{r}\phi)\} r dr d\xi dt,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&\int_\Omega \rho u_3 \varphi r dr d\xi|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_\Omega \{\rho u_1\varphi_t + \rho u_3(u_1\varphi_r + (\frac{nu_2}{r} + \alpha u_3)\varphi_\xi)\} r dr d\xi dt\\
&= \int_{t_1}^{t_2}\int_\Omega \{\alpha a\rho^\gamma\varphi_\xi - \mu(\partial_r u_3\varphi_r + (\frac{n^2}{r^2}\partial_\xi u_3 + \alpha^2\partial_\xi u_3)\varphi_\xi\} r dr d\xi dt,
\end{aligned} \tag{2.12}$$

where we denote $w_1 = \frac{n\partial_\xi u_2}{r} + \frac{u_1}{r}, w_2 = -\frac{n\partial_\xi u_2}{r} + \frac{u_1}{r}$.
(3) For any $b \in C^1(\mathbb{R})$ such that $|b(s)| + |b'(s)s| \leq C$ for all $s \in \mathbb{R}$, there holds:

$$\begin{aligned}
&\partial_t b(\rho) + \frac{1}{r}\partial_r(rb(\rho)u_1) + \frac{n}{r}\partial_\xi(b(\rho)u_2) + \alpha\partial_\xi(b(\rho)u_3)\\
&\quad + (b'(\rho)\rho - b(\rho))\{\frac{1}{r}\partial_r(ru_1) + \frac{n}{r}\partial_\xi u_2 + \alpha\partial_\xi u_3\} = 0.
\end{aligned}$$

in $\mathcal{D}'((0,T)\times\Omega)$, i.e., $(\rho,\boldsymbol{u})$ is a renormalized solution of (2.1)$_1$.
(4) The following energy inequality

$$E(t)+\mu\int_0^t\int_{\mathbb{R}^+}\int_0^{2\pi}\Big\{|\partial_r\boldsymbol{u}|^2+\alpha^2|\partial_\xi\boldsymbol{u}|^2+\frac{n^2}{r^2}|\partial_\xi u_3|^2$$
$$+\Big(\frac{u_1}{r}+\frac{n}{r}\partial_\xi u_2\Big)^2+\Big(\frac{u_2}{r}-\frac{n}{r}\partial_\xi u_1\Big)^2\Big\}(\tau)rdrd\xi d\tau\le E(0),\qquad \forall t>0,\qquad (2.13)$$

is satisfied, where

$$E(t)=\int_{\mathbb{R}^+}\int_0^{2\pi}(\frac{\rho|\boldsymbol{u}|^2}{2}+\frac{a}{\gamma-1}\rho^\gamma)rdrd\xi.$$

Remark 2.2 *If n is odd, we modify (2) in Definition 2.1 in the following way: For any $t_2\ge t_1\ge 0$, and test functions $\zeta,\zeta^1,\zeta^2\in C_0^1(\Omega\times[t_1,t_2])$ of $2\pi-$period in the variable ξ, $\zeta_\xi(t,0,\xi)=(\zeta^1-n\partial_\xi\zeta^2)(t,0,\xi)=(\zeta^2+n\partial_\xi\zeta^1)(t,0,\xi)=0$, the equations (2.1) and (2.4) are satisfied in the weak sense with test function ζ, and the following holds.*

$$\int_\Omega(\rho u_1\zeta^1-\rho u_2\zeta^2)rdrd\xi\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\int_\Omega\{\rho u_1\zeta_t^1-\rho u_2\zeta_t^2+\rho(u_1)^2\zeta_r^1-\rho u_1u_2\zeta_r^2+\frac{\rho u_1u_2}{r}(\zeta^2+n\zeta_\xi^1)$$
$$+\alpha(\rho u_1u_3\zeta_\xi^1-\rho u_2u_3\zeta_\xi^2)+\frac{\rho(u_1)^2}{r}(\zeta^1-n\zeta_\xi^2)\}rdrd\xi dt=\int_{t_1}^{t_2}\int_\Omega\{a\rho^\gamma(\zeta_r^1+\frac{\zeta^1-n\zeta_\xi^2}{r})$$
$$-\mu[(\partial_ru_1\zeta_r^1-\partial_ru_2\zeta_r^2)+\alpha^2(\partial_\xi u_1\zeta_\xi^1-\partial_\xi u_2\zeta_\xi^2)+\frac{w_1}{r}(\zeta^1-n\zeta_\xi^2)-\frac{w_2}{r}(\zeta^2+n\zeta_\xi^1)]\}rdrd\xi dt,$$

$$\int_\Omega(\rho u_1\zeta^2+\rho u_2\zeta^1)rdrd\xi\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\int_\Omega\{\rho u_1\zeta_t^2+\rho u_2\zeta_t^1+\rho(u_1)^2\zeta_r^2+\rho u_1u_2\zeta_r^1-\frac{\rho u_1u_2}{r}(\zeta^1-n\zeta_\xi^2)$$
$$+\alpha(\rho u_1u_3\zeta_\xi^2+\rho u_2u_3\zeta_\xi^1)+\frac{\rho(u_1)^2}{r}(\zeta^2+n\zeta_\xi^1)\}rdrd\xi dt=\int_{t_1}^{t_2}\int_\Omega\{a\rho^\gamma(\zeta_r^2+\frac{\zeta^2+n\zeta_\xi^1}{r})$$
$$-\mu[(\partial_ru_1\zeta_r^2+\partial_ru_2\zeta_r^1)+\alpha^2(\partial_\xi u_1\zeta_\xi^2+\partial_\xi u_2\zeta_\xi^1)+\frac{w_1}{r}(\zeta^2+n\zeta_\xi^1)+\frac{w_2}{r}(\zeta^1-n\zeta_\xi^2)]\}rdrd\xi dt,$$

$$\int_\Omega\rho u_3\zeta rdrd\xi|_{t_1}^{t_2}-\int_{t_1}^{t_2}\int_\Omega\{\rho u_3\zeta_t+\rho u_3[u_1\zeta_r+(\frac{nu_2}{r}+\alpha u_3)\zeta_\xi]\}rdtdrd\xi$$
$$=\int_{t_1}^{t_2}\int_\Omega\{\alpha a\rho^\gamma\zeta_\xi-\mu[\partial_ru_3\zeta_r+(\frac{n^2}{r^2}\partial_\xi u_3+\alpha^2\partial_\xi u_3)\zeta_\xi]\}rdtdrd\xi.$$

We start with the construction of approximation to the initial data $\rho_0,\boldsymbol{m}_0$. Let $\chi_1^\epsilon,\chi_2^\epsilon\in C^\infty(\mathbb{R})$ satisfy $\chi_1^\epsilon(x)=1$ for $x\le\epsilon^{-2}$, $\chi_2^\epsilon(r)=1$ for $r\ge 3\epsilon$, and $\chi_1^\epsilon(x)=0$ for $x\ge 2\epsilon^{-2}$, $\chi_2^\epsilon(r)=0$ for $r\le 2\epsilon$. Similar to [11], we define

$$\rho_0^\epsilon(r,\xi):=r^{-1/\gamma}[(r^{1/\gamma}\rho_0)*j_{\epsilon/2}](r,\xi)\chi_1^\epsilon(r^2),$$
$$\boldsymbol{m}_0^\epsilon(r,\xi):=\chi_2^\epsilon(r)\cdot\begin{cases}[(\boldsymbol{m}_0/\sqrt{\rho_0})*j_\epsilon](r,\xi)\sqrt{\rho_0^\epsilon(r,\xi)}, & \rho_0(r,\xi)>0,\\ 0, & \rho_0(r,\xi)=0,\end{cases}$$

where $j_\epsilon=\frac{1}{\epsilon}j(r/\epsilon,\xi/\epsilon)$ with $\int_0^\infty\int_0^{2\pi}j(r,\xi)rdrd\xi=1$. It is easy to see that $(\rho_0^\epsilon,\boldsymbol{m}_0^\epsilon)$ is periodic in ξ of period 2π.

Thus, the approximate solutions of (2.1)–(2.7) are obtained by solving the following initial boundary value problem in the domain $(\epsilon,\infty)\times\mathbb{R}$:

$$\partial_t\rho^\epsilon+\frac{1}{r}\partial_r(r\rho^\epsilon u_1^\epsilon)+\frac{n}{r}\partial_\xi(\rho^\epsilon u_2^\epsilon)+\alpha\partial_\xi(\rho^\epsilon u_3^\epsilon)=0, \tag{2.14}$$

$$\begin{aligned}&\partial_t(\rho^\epsilon u_1^\epsilon)+\frac{1}{r}\partial_r(r\rho^\epsilon(u_1^\epsilon)^2)+\frac{n}{r}\partial_\xi(\rho^\epsilon u_1^\epsilon u_2^\epsilon)+\alpha\partial_\xi(\rho^\epsilon u_1^\epsilon u_3^\epsilon)-\frac{\rho^\epsilon(u_2^\epsilon)^2}{r}\\ &\qquad+a\partial_r(\rho^\epsilon)^\gamma+\epsilon^\lambda\partial_r(\rho^\epsilon)^\beta=\mu\Big[\frac{1}{r}\partial_r(r\partial_r u_1^\epsilon)+(\alpha^2+\frac{n^2}{r^2})\partial_\xi^2u_1^\epsilon-\frac{u_1^\epsilon}{r^2}-\frac{2n}{r^2}\partial_\xi u_2^\epsilon\Big],\end{aligned} \tag{2.15}$$

$$\begin{aligned}&\partial_t(\rho^\epsilon u_2^\epsilon)+\frac{1}{r}\partial_r(r\rho^\epsilon u_1^\epsilon u_2^\epsilon)+\frac{n}{r}\partial_\xi(\rho^\epsilon (u_2^\epsilon)^2)+\alpha\partial_\xi(\rho^\epsilon u_2^\epsilon u_3^\epsilon)+\frac{\rho^\epsilon u_1^\epsilon u_2^\epsilon}{r}\\ &\qquad+\frac{n}{r}\partial_\xi\big(a(\rho^\epsilon)^\gamma+\epsilon^\lambda(\rho^\epsilon)^\beta\big)=\mu\Big[\frac{1}{r}\partial_r(r\partial_r u_2^\epsilon)+(\alpha^2+\frac{n^2}{r^2})\partial_\xi^2u_2^\epsilon-\frac{u_2^\epsilon}{r^2}+\frac{2n}{r^2}\partial_\xi u_1^\epsilon\Big],\end{aligned} \tag{2.16}$$

$$\begin{aligned}&\partial_t(\rho^\epsilon u_3^\epsilon)+\frac{1}{r}\partial_r(r\rho^\epsilon u_1^\epsilon u_3^\epsilon)+\frac{n}{r}\partial_\xi(\rho^\epsilon u_2^\epsilon u_3^\epsilon)+\alpha\partial_\xi(\rho^\epsilon (u_3^\epsilon)^2)+\alpha\partial_\xi\big(a(\rho^\epsilon)^\gamma+\epsilon^\lambda(\rho^\epsilon)^\beta\big)\\ &=\mu\Big[\frac{1}{r}\partial_r(r\partial_r u_3^\epsilon)+(\alpha^2+\frac{n^2}{r^2})\partial_\xi^2u_3^\epsilon\Big]\end{aligned} \tag{2.17}$$

together with initial values:

$$\rho^\epsilon(0,r,\xi)=\rho_0^\epsilon(r,\xi),\ \ (\rho^\epsilon\boldsymbol{u}^\epsilon)(0,r,\xi)=\boldsymbol{m}_0^\epsilon(r,\xi),\quad (r,\xi)\in(\epsilon,\infty)\times\mathbb{R}, \tag{2.18}$$

and boundary conditions:

$$u_1^\epsilon(t,\epsilon,\xi)=u_2^\epsilon(t,\epsilon,\xi)=\partial_r u_3^\epsilon(t,\epsilon,\xi)=0,\quad t>0,\ \xi\in\mathbb{R}, \tag{2.19}$$

$$\rho^\epsilon \text{ and } \boldsymbol{u}^\epsilon \text{ are periodic in } \xi \text{ of period } 2\pi, \tag{2.20}$$

where $\boldsymbol{u}^\epsilon=(u_1^\epsilon,u_2^\epsilon,u_3^\epsilon)$, $\beta>\max\{4,\gamma\}$ and $\lambda>\frac{3\beta}{\gamma}-3$ are constants.

From the construction of $\rho_0^\epsilon,\boldsymbol{m}_0^\epsilon$, one easily sees that $\rho_0^\epsilon\in\mathcal{L}^\beta_{\rm loc}(\mathbb{R}^+\times[0,2\pi])$, $\rho_0^\epsilon\ge 0$ a.e., and

$$\begin{aligned}&\|\rho_0^\epsilon-\rho_0\|_{\mathcal{L}^\gamma(\mathbb{R}^+\times(0,2\pi))}\to 0,\qquad \Big\|\frac{\boldsymbol{m}_0^\epsilon}{\sqrt{\rho_0^\epsilon}}-\frac{\boldsymbol{m}_0}{\sqrt{\rho_0}}\Big\|_{\mathcal{L}^2(\mathbb{R}^+\times(0,2\pi))}\to 0,\ \ \text{as } \epsilon\to 0,\\ &\int_{(\epsilon,\infty)\times(0,2\pi)}\rho_0^\epsilon r\,dr\,d\xi\le C\int_{\mathbb{R}^+\times(0,2\pi)}\rho_0 r\,dr\,d\xi.\end{aligned} \tag{2.21}$$

Therefore, by virtue of Theorem 4.3 in Section 4, the problem (2.14)–(2.20) has a global weak solution $(\rho^\epsilon,\boldsymbol{u}^\epsilon)$ on $\mathbb{R}_0^+\times[\epsilon,\infty)\times\mathbb{R}$ with $\rho^\epsilon\ge0$ a.e., such that

$$\begin{aligned}&E_\epsilon(t)+\mu\int_0^t\int_\epsilon^\infty\int_0^{2\pi}\Big\{|\partial_r\boldsymbol{u}^\epsilon|^2+\alpha^2|\partial_\xi\boldsymbol{u}^\epsilon|^2+\frac{n^2}{r^2}|\partial_\xi u_3^\epsilon|^2\\ &\qquad+\Big(\frac{u_1^\epsilon}{r}+\frac{n}{r}\partial_\xi u_2^\epsilon\Big)^2+\Big(\frac{u_2^\epsilon}{r}-\frac{n}{r}\partial_\xi u_1^\epsilon\Big)^2\Big\}(\tau)r\,dr\,d\xi\,d\tau\le E_\epsilon(0)\quad\forall\, t>0,\end{aligned} \tag{2.22}$$

$$\int_\epsilon^\infty\int_0^{2\pi}\rho^\epsilon r\,dr\,d\xi\le\int_{\mathbb{R}^+\times(0,2\pi)}\rho_0 r\,dr\,d\xi\qquad\forall\, t>0, \tag{2.23}$$

where

$$E_\epsilon(t):=\int_\epsilon^\infty\int_0^{2\pi}\Big[\frac{\rho^\epsilon|\boldsymbol{u}^\epsilon|^2}{2}+\frac{a}{\gamma-1}(\rho^\epsilon)^\gamma+\frac{\epsilon^\lambda}{\beta-1}(\rho^\epsilon)^\beta\Big](t)r\,dr\,d\xi.$$

Notice that $\beta > 2$, then by the proof in [13], ρ^ϵ is in fact a renormalized solution of (2.14), i.e., for any $b \in C^1(\mathbb{R})$, $|b(s)| \le C$ and $|b'(s)s| \le C$, one has

$$\partial_t b(\rho^\epsilon) + \frac{1}{r}\partial_r[rb(\rho^\epsilon)u_1^\epsilon] + \frac{n}{r}\partial_\xi[b(\rho^\epsilon)u_2^\epsilon] + \alpha\partial_\xi[b(\rho^\epsilon)u_3^\epsilon]$$
$$+ \left[b'(\rho^\epsilon)\rho^\epsilon - b(\rho^\epsilon)\right]\left[\frac{1}{r}\partial_r(ru_1^\epsilon) + \frac{n}{r}\partial_\xi u_2^\epsilon + \alpha\partial_\xi u_3^\epsilon\right] = 0. \tag{2.24}$$

Moreover, using Hölder's inequality and recalling $\lambda > \frac{3\beta}{\gamma} - 3$, one concludes that

$$\begin{aligned}\epsilon^\lambda \|\rho_0^\epsilon\|^\beta_{\mathcal{L}^\beta((\epsilon,\infty)\times[0,2\pi])} &\le C\epsilon^{1-\frac{\beta}{\gamma}+\lambda}\|(r\rho_0^\gamma) * j_{\epsilon/2}\|^{\beta/\gamma}_{L^{\beta/\gamma}}\\ &\le C\epsilon^{3-\frac{3\beta}{\gamma}+\lambda}\|\rho_0\|^{\beta-\gamma}_{\mathcal{L}^\gamma}\|(r\rho_0^\gamma) * j_{\epsilon/2}\|_{L^1}\\ &\le C\epsilon^{3-\frac{3\beta}{\gamma}+\lambda}\|\rho_0\|^\beta_{\mathcal{L}^\gamma} \to 0, \quad \text{as } \epsilon \to 0,\end{aligned} \tag{2.25}$$

which combined with (2.21) and (2.22) gives the energy estimate:

$$\text{L.H.S. of (2.22)} \le C\left\{1 + \int_{\mathbb{R}^+\times(0,2\pi)}\Big(\frac{|\boldsymbol{m}_0|^2}{2\rho_0} + \frac{a{\rho_0}^\gamma}{\gamma-1}\Big)rdrd\xi\right\}, \qquad \forall t \ge 0. \tag{2.26}$$

Using the estimate (2.22) and approximating the function $b(\rho)$ in (2.24) by ρ^θ with $0 < \theta \le \gamma - 1$ (i.e., taking some $|b_R(s)| + |b_R'(s)s| \le C$ and $b_R(s) \to \rho^\theta$ as $R \to \infty$), we find that (also cf. [13, pp. 25,19]),

$$\partial_t(\rho^\epsilon)^\theta + \frac{1}{r}\partial_r[r(\rho^\epsilon)^\theta u_1^\epsilon] + \frac{n}{r}\partial_\xi[(\rho^\epsilon)^\theta u_2^\epsilon] + \alpha\partial_\xi[(\rho^\epsilon)^\theta u_3^\epsilon]$$
$$+ (\theta-1)(\rho^\epsilon)^\theta\left[\frac{1}{r}\partial_r(ru_1^\epsilon) + \frac{n}{r}\partial_\xi u_2^\epsilon + \alpha\partial_\xi u_3^\epsilon\right] = 0. \tag{2.27}$$

Then, if we multiply the equation (2.15), (2.17) by $\phi(r) \in C_0^\infty(\epsilon,\infty)$ and employ the equation (2.27), we obtain by calculations similar to those in [13, Chapter 5] that

$$\begin{aligned}&\phi(r)(\rho^\epsilon)^\theta\{a(\rho^\epsilon)^\gamma + \epsilon^\lambda(\rho^\epsilon)^\beta - \mu(\partial_r u_1 + \alpha\partial_\xi u_3)\}\\ &= \partial_t\{(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}[\partial_r(\phi(r)\rho^\epsilon u_1^\epsilon) + \alpha\partial_\xi(\phi(r)\rho^\epsilon u_3^\epsilon)]\}\\ &+(-\tilde{\Delta})^{-1}[\partial_r(\phi(r)\rho^\epsilon u_1^\epsilon) + \alpha\partial_\xi(\phi(r)\rho^\epsilon u_3^\epsilon)]\Big\{\frac{1}{r}\partial_r[r(\rho^\epsilon)^\theta u_1^\epsilon] + \frac{n}{r}\partial_\xi[(\rho^\epsilon)^\theta u_2^\epsilon] + \alpha\partial_\xi[(\rho^\epsilon)^\theta u_3^\epsilon]\\ &+(\theta-1)(\rho^\epsilon)^\theta\big[\frac{1}{r}\partial_r(ru_1^\epsilon) + \frac{n}{r}\partial_\xi u_2^\epsilon + \alpha\partial_\xi u_3^\epsilon\big]\Big\} + (\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\Big\{\partial_r^2\big(\phi(r)\rho^\epsilon(u_1^\epsilon)^2\big)\\ &+\frac{n}{r}\partial^2_{r\xi}(\phi(r)\rho^\epsilon u_1^\epsilon u_2^\epsilon) + \alpha\partial^2_{r\xi}(\phi(r)\rho^\epsilon u_1^\epsilon u_3^\epsilon) + \alpha^2\partial_\xi^2(\phi(r)(u_3^\epsilon)^2) + \frac{n\alpha}{r}\partial_\xi^2(\phi(r)\rho^\epsilon u_2^\epsilon u_3^\epsilon)\Big\}\\ &-(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\Big\{\partial_r\Big[r\rho^\epsilon(u_1^\epsilon)^2\partial_r(\frac{\phi}{r}) + \frac{\phi\rho^\epsilon(u_2^\epsilon)^2}{r} + \big(a(\rho^\epsilon)^\gamma + \epsilon^\lambda(\rho^\epsilon)^\beta\big)\phi'(r)\Big]\\ &+\alpha\partial_\xi\big(r\rho^\epsilon u_1^\epsilon u_3^\epsilon\partial_r(\frac{\phi}{r})\big)\Big\} - \mu(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\Big\{\partial_r\big[\frac{n^2}{r^2}\partial_\xi^2(\phi u_1^\epsilon)\big] + \alpha\partial_\xi\big[\frac{n^2}{r^2}\partial_\xi^2(\phi u_3^\epsilon)\big]\Big\}\\ &+\mu(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\Big\{\partial_r\Big[r\partial_r u_1^\epsilon\partial_r(\frac{\phi}{r}) + (\frac{u_1^\epsilon}{r^2} + \frac{2n}{r^2}\partial_\xi u_2^\epsilon)\phi\Big] + \alpha\partial_\xi\Big[r\partial_r u_3^\epsilon\cdot\partial_r(\frac{\phi}{r})\\ &+\partial_r(\phi'(r)u_3^\epsilon)\Big] - \alpha^2\partial_\xi^2(\phi'(r)u_1^\epsilon)\Big\} \qquad (0 < \theta \le \gamma - 1),\end{aligned} \tag{2.28}$$

where $(-\tilde{\Delta})^{-1}$ stands for the inverse of the operator $\tilde{\Delta} = \partial_r^2 + \alpha^2\partial_\xi^2$ on $\mathbb{R}\times[0,2\pi]$. To handle the terms on the right hand side of the equation (2.28), we set

$$w_1^\epsilon := \frac{u_1^\epsilon}{r} + \frac{n}{r}\partial_\xi u_2^\epsilon, \quad w_2^\epsilon := \frac{u_2^\epsilon}{r} - \frac{n}{r}\partial_\xi u_1^\epsilon$$

to see that by virtue of (2.26), $w_1^\epsilon, w_2^\epsilon \in L^2(0,T;\mathcal{L}^2((\epsilon,+\infty)\times(0,2\pi))$. Thus,

$$
\begin{aligned}
\frac{n^2}{r^2}\partial_\xi^2(\phi u_1^\epsilon) - (\frac{u_1^\epsilon}{r^2} + \frac{2n}{r^2}\partial_\xi u_2^\epsilon)\phi &= \frac{n}{r}\partial_\xi(\frac{n}{r}\partial_\xi u_1^\epsilon - \frac{u_2^\epsilon}{r})\phi - (\frac{u_1^\epsilon}{r^2} + \frac{2n}{r^2}\partial_\xi u_2^\epsilon)\phi \\
&= -(\frac{w_1^\epsilon}{r} + \frac{n}{r}\partial_\xi w_2^\epsilon)\phi(r). \qquad (2.29)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(-\tilde{\Delta})^{-1}[\partial_r(\phi(r)\rho^\epsilon u_1^\epsilon) + \alpha\partial_\xi(\phi(r)\rho^\epsilon u_3^\epsilon)]\Big\{\partial_r[(\rho^\epsilon)^\theta u_1^\epsilon] + \frac{n}{r}\partial_\xi[(\rho^\epsilon)^\theta u_2^\epsilon] + \alpha\partial_\xi[(\rho^\epsilon)^\theta u_3^\epsilon]\Big\} \\
&\quad = \mathrm{div}\,\Big\{(-\tilde{\Delta})^{-1}\Big[\partial_r(\phi(r)\rho^\epsilon u_1^\epsilon) + \alpha\partial_\xi(\phi(r)\rho^\epsilon u_3^\epsilon)\Big](\rho^\epsilon)^\theta \boldsymbol{u}^\epsilon\Big\} \\
&\qquad -(\rho^\epsilon)^\theta \boldsymbol{u}^\epsilon\cdot\nabla\Big\{(-\tilde{\Delta})^{-1}\Big[\partial_r(\phi(r)\rho^\epsilon u_1^\epsilon) + \alpha\partial_\xi(\phi(r)\rho^\epsilon u_3^\epsilon)\Big]\Big\}. \qquad (2.30)
\end{aligned}
$$

So, multiplying (2.28) with $\theta = \gamma - 1$ by r and integrating, and following the same process as in the proof of Theorem 7.1 in [13, Chapter 7], we can deduce that

$$
\int_0^T \int_{\mathbb{R}^+\times(0,2\pi)} \phi(r)\big\{a(\rho^\epsilon)^{\gamma+\theta} + \epsilon^\lambda(\rho^\epsilon)^{\beta+\theta}\big\} r dr d\xi \le C, \qquad \theta = \gamma - 1, \qquad (2.31)
$$

where C is a positive constant depending only on θ, ρ_0 and $\boldsymbol{m}_0$.

Next, we exploit the pressure term in (2.15) to derive a (better) integrability estimate (2.33) of ρ^ϵ near $r = 0$, which will be needed in the exclusion of singularity concentration on the axis $r = 0$ at the end of Section 3.

For $h > 0$, let $\varphi \in C_0^1(\mathbb{R}^+\times\mathbb{R}^+\times[0,2\pi])$ be the $2\pi-$periodic function in ξ and $\chi^h \in C^\infty(\epsilon,\infty)$ be non-negative such that

$$
\begin{aligned}
&\varphi(t,r,\xi) = 1 \quad \text{for} \quad (t,r,\xi) \in [0,T]\times[0,1]\times[0,2\pi], \\
&\chi^h(r) = \begin{cases} 0, & \epsilon \le r \le \epsilon + h, \quad 0 \le \partial_r\chi^h(r) \le ch^{-1}, \\ 1, & \epsilon + 2h \le r. \end{cases}
\end{aligned}
$$

Taking $\psi(t,r,\xi) = r(r-\epsilon)^{2/3}\varphi(t,r,\xi)\chi^h(r)$ as a test function for (2.15), i.e., multiplying (2.15) by ψ and integrating over $(0,T)\times G_\epsilon$ with $G_\epsilon := (\epsilon,\infty)\times(0,2\pi)$, we deduce that

$$
\begin{aligned}
&\int_0^T \int_{G_\epsilon} \Big\{r\rho^\epsilon(u_1^\epsilon)^2\partial_r(\frac{\psi}{r}) + \big[a(\rho^\epsilon)^\gamma + \epsilon^\lambda(\rho^\epsilon)^\beta\big]\psi_r + \rho^\epsilon(u_2^\epsilon)^2\frac{\psi}{r}\Big\} dr d\xi dt \\
&\quad = \int_{G_\epsilon} \rho^\epsilon u_1^\epsilon\psi\Big|_0^t dr d\xi + \int_0^t \int_{G_\epsilon} \Big\{-\rho^\epsilon u_1^\epsilon\partial_t\psi - \big[\frac{n}{r}\rho^\epsilon u_1^\epsilon u_2^\epsilon + \alpha\rho^\epsilon u_1^\epsilon u_3^\epsilon\big]\psi_\xi\Big\} dr d\xi dt \\
&\qquad + \mu\int_0^T \int_{G_\epsilon} \Big\{r\partial_r u_1^\epsilon\partial_r\big[\frac{\psi}{r}\big] + \big(\frac{n^2}{r^2} + \alpha^2\big)\partial_\xi u_1^\epsilon\psi_\xi + \big(\frac{u_1^\epsilon}{r^2} + \frac{2n}{r^2}\partial_\xi u_2^\epsilon\big)\psi\Big\} dr d\xi dt.
\end{aligned}
$$

In view of $\psi_r \ge \frac{2}{3}r^{2/3}\varphi\chi^h + r(r-\epsilon)\chi^h\varphi_r$, $\big[\frac{\psi}{r}\big]_r \ge (r-\epsilon)^{2/3}\chi^h\varphi_r + \frac{2}{3}(r-\epsilon)^{-1/3}\chi^h\varphi$, and

$$
\begin{aligned}
&\int_0^T \int_{G_\epsilon} \Big\{-\frac{n^2}{r^2}\partial_\xi^2 u_1^\epsilon\psi + \big(\frac{u_1^\epsilon}{r^2} + \frac{2n}{r^2}\partial_\xi u_2^\epsilon\big)\psi\Big\} dr d\xi dt \\
&\quad = \int_0^T \int_{G_\epsilon} \Big\{-\frac{n}{r}\partial_\xi\big(\frac{u_2^\epsilon}{r} - w_2^\epsilon\big)\psi + \big(\frac{u_1^\epsilon}{r^2} + \frac{2n}{r^2}\partial_\xi u_2^\epsilon\big)\psi\Big\} dr d\xi dt \\
&\quad = \int_0^T \int_{G_\epsilon} \Big\{-\frac{n}{r}w_2^\epsilon\psi_\xi + \frac{1}{r}w_1^\epsilon\psi\Big\},
\end{aligned}
$$

we see that

$$
\begin{aligned}
&\frac{2}{3}\int_0^T\int_{G_\epsilon}\{\rho^\epsilon(u_1^\epsilon)^2+a(\rho^\epsilon)^\gamma+\epsilon^\lambda(\rho^\epsilon)^\beta\}r^{2/3}\varphi\chi^h drd\xi dt+\int_0^T\int_{G_\epsilon}\rho^\epsilon(u_2^\epsilon)^2(r-\epsilon)^{\frac{2}{3}}\varphi\chi^h drd\xi dt\\
&\le 2\sup_{0\le t\le T}\int_{G_\epsilon}\rho^\epsilon|u_1^\epsilon\psi|(t,r,\xi)drd\xi+\int_0^t\int_{G_\epsilon}\Big\{\rho^\epsilon|u_1^\epsilon\partial_t\psi|+|(\frac{n}{r}\rho^\epsilon u_1^\epsilon u_2^\epsilon+\alpha\rho^\epsilon u_1^\epsilon u_3^\epsilon)\psi_\xi|\\
&\quad+\mu\alpha^2|\partial_\xi u_1^\epsilon\partial_\xi\psi|+\mu\big(|\frac{n}{r}w_2^\epsilon\psi_\xi|+|\frac{1}{r}w_1^\epsilon\psi|\big)\Big\}drd\xi dt\\
&\quad+\int_0^t\int_{G_\epsilon}\Big\{(a(\rho^\epsilon)^\gamma+\epsilon^\lambda(\rho^\epsilon)^\beta)|\varphi_r|\chi^h(r)+\mu|\partial_r u_1^\epsilon||\partial_r(\varphi\chi^h)+\frac{2\varphi\chi^h}{3(r-\epsilon)}|(r-\epsilon)^{2/3}\Big\}rdrd\xi dt\\
&\le C\sup_{0\le t\le T}\int_{G_\epsilon}\{(\rho_\epsilon)^\gamma+\epsilon^\lambda(\rho_\epsilon)^\beta+\rho^\epsilon+\rho^\epsilon|\boldsymbol{u}^\epsilon|^2\}rdrd\xi\\
&\quad+C\int_0^T\int_{G_\epsilon}\Big\{|\nabla\boldsymbol{u}^\epsilon|^2+|w_1^\epsilon|^2+|w_2^\epsilon|^2+\Big[|\partial_r(\varphi\chi)|+\frac{\varphi\chi}{r-\epsilon}\Big]^2(r-\epsilon)^{4/3}\Big\}rdrd\xi dt\\
&\le C+C\Big\{\int_0^T\int_0^{2\pi}\int_{h+\epsilon}^{2h+\epsilon}\frac{(r-\epsilon)^{4/3}}{h^2}\varphi rdrd\xi dt+\int_0^T\int_0^{2\pi}\int_\epsilon^\infty\frac{1}{(r-\epsilon)^{2/3}}\varphi rdrd\xi dt\Big\}\\
&\le C
\end{aligned}
\tag{2.32}
$$

with C being independent of ϵ and h. Hence letting $h\to 0$ in (2.32), we obtain

$$
\int_0^T\int_\epsilon^1\int_0^{2\pi}\rho^\epsilon(u_2^\epsilon)^2(r-\epsilon)^{\frac{2}{3}}drd\xi dt+\int_0^T\int_\epsilon^1\int_0^{2\pi}\{(\rho^\epsilon)^\gamma+\epsilon^\lambda(\rho^\epsilon)^\beta+\rho^\epsilon(u_1^\epsilon)^2\}r^{\frac{2}{3}}drd\xi dt\le C \tag{2.33}
$$

for any $T>0$, where the constant C is independent of ϵ.

3 Proof of the precompactness

In this section we extract a subsequence from the approximate weak solution sequence $(\rho^\epsilon,\boldsymbol{u}^\epsilon)$ of (2.14)–(2.17) and prove that its weak limit $(\rho,\boldsymbol{u})$ is indeed a global weak solution of (2.1)–(2.7).

First we extend $(\rho^\epsilon,\boldsymbol{u}^\epsilon)$ to the whole domain $\mathbb{R}^+\times\mathbb{R}$ by setting ρ^ϵ as well as $u_1^\epsilon,u_2^\epsilon$ to be zero and u_3^ϵ to be $u_3^\epsilon(t,\epsilon,\xi)$ for $(t,r,\xi)\in\mathbb{R}_0^+\times[0,\epsilon)\times\mathbb{R}$. For simplicity, we still denote by $(\rho^\epsilon,\boldsymbol{u}^\epsilon)$ this extension, we note that $(\rho^\epsilon,\boldsymbol{u}^\epsilon)$ is periodic with period 2π in the variable ξ. *Throughout this section, we denote* $\Omega:=\mathbb{R}^+\times[0,2\pi]$.

It follows from (2.26) and (5.91) in [13, p. 43] that $\|\boldsymbol{u}^\epsilon\|_{L^2(0,T;\mathcal{H}^1_{\mathrm{loc}}(\Omega))}$ is uniformly bounded with respect to ϵ, and hence, we can extract a subsequence of $(\rho^\epsilon,\boldsymbol{u}^\epsilon)$, still denoted by $(\rho^\epsilon,\boldsymbol{u}^\epsilon)$, with

$$
\rho^\epsilon\in L^\infty(0,T;\mathcal{L}^\gamma(\Omega)\cap\mathcal{L}^\beta(\Omega)),\qquad\tilde\nabla\boldsymbol{u}^\epsilon\in L^2(0,T;\mathcal{L}^2(\Omega)),
$$

such that

$$
\begin{cases}
w_1^\epsilon\rightharpoonup w_1,\quad w_2^\epsilon\rightharpoonup w_2\quad\text{weakly in } L^2(0,T;\mathcal{L}^2(\Omega)),\\
\rho^\epsilon\rightharpoonup\rho\ \ \text{weak-}*\text{ in}\quad L^\infty(0,T;\mathcal{L}^\gamma(\Omega)),\\
\boldsymbol{u}^\epsilon\rightharpoonup\boldsymbol{u}\ \ \text{weakly in}\quad L^2(0,T;\mathcal{H}^1_{\mathrm{loc}}(\Omega)).
\end{cases}
\tag{3.1}
$$

Using (2.14) and (2.26), we see that $\partial_t\rho^\epsilon\in L^2(0,T;W^{-1,p}_{\mathrm{loc}}(\Omega))$ for any $1<p<\gamma$. So, by Appendix C in [12], one obtains

$$
\rho^\epsilon\to\rho\qquad\text{in }\ C^0([0,T];L^p_{\mathrm{loc}}(\Omega)-w)\text{ for any }1<p<\gamma. \tag{3.2}
$$

On the other hand, since $\gamma > 1$, we can take $1 < p < \gamma$ such that $L^p_{\rm loc}(\Omega) \hookrightarrow\hookrightarrow H^{-1}_{\rm loc}(\Omega)$. Hence, for any $T < \infty$, $\rho^\epsilon \to \rho$ in $C^0([0,T]; H^{-1}_{\rm loc}(\Omega))$ as $\epsilon \to 0$. This together with (3.2) implies that

$$\rho^\epsilon \boldsymbol{u}^\epsilon \rightharpoonup \rho \boldsymbol{u} \quad \text{in} \quad \mathcal{D}'((0,T)\times\Omega). \tag{3.3}$$

Moreover, by (2.26), (2.15)–(2.17),

$$\rho^\epsilon \boldsymbol{u}^\epsilon \in L^\infty(0,T; \mathcal{L}^{2\gamma/(\gamma+1)}(\Omega)) \cap L^2(0,T; L^p_{\rm loc}(\Omega)), \quad \text{for any } p < \gamma,$$

and

$$\partial_t(\rho^\epsilon \boldsymbol{u}^\epsilon) \in L^2(0,T; W^{-1,\tilde{p}}_{\rm loc}(\Omega)) \quad \text{for some } 1 < \tilde{p} < \min\{2,\gamma\}.$$

Hence, Appendix C of [12] and (3.3) imply immediately

$$\begin{aligned} &\rho^\epsilon \boldsymbol{u}^\epsilon \to \rho \boldsymbol{u} \text{ weak-}* \text{ in } L^\infty(0,T; \mathcal{L}^{2\gamma/(\gamma+1)}(\Omega)) \\ &\qquad\qquad \text{and weakly in } L^2(0,T; L^p_{\rm loc}(\Omega)) \quad \text{for any } 1 < p < \gamma, \\ &\rho^\epsilon \boldsymbol{u}^\epsilon \to \rho \boldsymbol{u} \text{ in } C^0([0,T]; L^p_{\rm loc}(\Omega) - w) \quad \text{for any } p < 2\gamma/(\gamma+1). \end{aligned} \tag{3.4}$$

From (3.1), (3.3) and (3.4), we get

$$\rho^\epsilon \boldsymbol{u}^\epsilon \otimes \boldsymbol{u}^\epsilon \rightharpoonup \rho \boldsymbol{u} \otimes \boldsymbol{u} \qquad \text{weakly in } L^2(0,T; L^p_{\rm loc}(\Omega)) \text{ for any } 1 < p < \frac{2\gamma}{\gamma+1}. \tag{3.5}$$

Furthermore, from (3.1), (3.5), and (2.21), (2.22), (2.25) and the lower semicontinuity of weak convergence, the estimate (2.8) follows.

By (2.31) and Hölder's inequality, we conclude that for any $K \subset\subset \mathbb{R}^+ \times \mathbb{R}$,

$$\epsilon^\lambda \int_0^T \int_K (\rho^\epsilon)^\beta dr d\xi dt \le C(K) \epsilon^{\lambda\theta/(\beta+\theta)} \Big\{ \epsilon^\lambda \int_0^T \int_K (\rho^\epsilon)^{\beta+\theta} r dr d\xi dt \Big\}^{\beta/(\beta+\theta)} \to 0 \text{ as } \epsilon \to 0. \tag{3.6}$$

If we sum up (3.1)–(3.6) and take $\epsilon \to 0$ in (2.14)–(2.17), we deduce that

$$\begin{cases} \partial_t \rho + \frac{1}{r}\partial_r(r\rho u_1) + \frac{n}{r}\partial_\xi(\rho u_2) + \alpha\partial_\xi(\rho u_3) = 0, \\ \partial_t(\rho u_1) + \frac{1}{r}\partial_r(r\rho u_1^2) + \frac{n}{r}\partial_\xi(\rho u_1 u_2) + \alpha\partial_\xi(\rho u_1 u_3) - \frac{\rho u_2^2}{r} + a\partial_r \overline{\rho^\gamma} \\ \qquad = \mu\Big[\frac{1}{r}\partial_r(r\partial_r u_1) + (\alpha^2 + \frac{n^2}{r^2})\partial_\xi^2 u_1 - \frac{u_1}{r^2} - \frac{2n}{r^2}\partial_\xi u_2\Big], \\ \partial_t(\rho u_2) + \frac{1}{r}\partial_r(r\rho u_1 u_2) + \frac{n}{r}\partial_\xi(\rho u_2^2) + \alpha\partial_\xi(\rho u_2 u_3) + \frac{\rho u_1 u_2}{r} + a\frac{n}{r}\partial_\xi \overline{\rho^\gamma} \\ \qquad = \mu\Big[\frac{1}{r}\partial_r(r\partial_r u_2) + (\alpha^2 + \frac{n^2}{r^2})\partial_\xi^2 u_2 - \frac{u_2}{r^2} + \frac{2n}{r^2}\partial_\xi u_1\Big], \\ \partial_t(\rho u_3) + \frac{1}{r}\partial_r(r\rho u_1 u_3) + \frac{n}{r}\partial_\xi(\rho u_2 u_3) + \alpha\partial_\xi(\rho u_3^2) + a\alpha\partial_\xi \overline{\rho^\gamma} \\ \qquad = \mu\Big[\frac{1}{r}\partial_r(r\partial_r u_3) + (\alpha^2 + \frac{n^2}{r^2})\partial_\xi^2 u_3\Big] \end{cases} \tag{3.7}$$

in the sense of $\mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^+ \times [0,2\pi])$, and the weak limit $(\rho, \boldsymbol{u})$ is obviously periodic in ξ of period 2π. *Here and in what follows*, we denote by $\overline{f(\rho)}$ the weak limit of $f(\rho^\epsilon)$ (in the sense of distributions) as $\epsilon \to 0$.

Moreover, by the similar arguments to those in the proof of Lemma 4.4 in [4], we find that the weak limit $(\rho, \mathbf{u})$ solves $(3.7)_1$ in the sense of renormalized solutions, i.e.,

$$\begin{aligned}
&\partial_t b(\rho) + \frac{1}{r}\partial_r(rb(\rho)u_1) + \partial_\xi\big[\frac{n}{r}b(\rho)u_2 + \alpha b(\rho)u_3\big] \\
&\quad + \big(b'(\rho)\rho - b(\rho)\big)\Big[\frac{1}{r}\partial_r(ru_1) + \partial_\xi\big(\frac{n}{r}u_2 + \alpha u_3\big)\Big] = 0 \qquad (3.8)
\end{aligned}$$

holds in $\mathcal{D}'((0,T)\times\Omega)$ for any $b \in C^1(\mathbb{R})$ satisfying $|b'(z)z| + |b(z)| \le C$.

Thus, to show that the weak limit is indeed a finite energy 2π-periodic in-ξ weak solution of (1.1)–(1.4), we need first to prove that $\overline{\rho^\gamma} = \rho^\gamma$. To this end, we apply the same argument as the one in the derivation of (3.3) to (3.4), taking into account that the Riesz operator $(-\tilde{\Delta})^{-1}\partial_i\partial_j$ $(i,j=1,2)$ is bounded from L^p to L^p for any $1<p<\infty$, to deduce that

$$\begin{aligned}
&(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_r\big[r\partial_r u_1^\epsilon \partial_r(\frac{\phi}{r})\big] \rightharpoonup \overline{\rho^\theta}\partial_r\big[r\partial_r u_1 \partial_r(\frac{\phi}{r})\big], \\
&(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\big\{\partial_\xi^2 u_1^\epsilon \phi'(r)\big\} \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\big\{\partial_\xi^2 u_1^\epsilon \phi'(r)\big\}, \\
&(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_{r\xi}^2\big[u_3^\epsilon \partial_r\phi\big] \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_{r\xi}^2\big[u_3\partial_r\phi\big], \qquad (3.9)\\
&(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_\xi\big[r\partial_r u_3^\epsilon \partial_r(\frac{\phi}{r})\big] \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_\xi\big[r\partial_r u_3^\epsilon \partial_r(\frac{\phi}{r})\big], \\
&(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_r\big\{\frac{w_1^\epsilon}{r}\phi(r)\big\} \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_r\big\{\frac{w_1}{r}\phi(r)\big\} \\
&(\rho^\epsilon)^\theta \boldsymbol{u}^\epsilon \rightharpoonup \overline{\rho^\theta}\boldsymbol{u} \text{ in } L^2(0,T;,L^p_{\rm loc}(\Omega)), \qquad \forall\, p<\gamma/\theta,
\end{aligned}$$

where $\phi \in C_0^\infty(0,+\infty)$, and we have used the fact that $H^1_{loc}(\mathbb{R}^2) \hookrightarrow L^q_{loc}(\mathbb{R}^2)$ for any $1 \le q < \infty$.

Similarly we obtain that

$$\begin{aligned}
&(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_r\big\{\frac{n}{r}\partial_\xi(w_2^\epsilon\phi(r))\big\} \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_r\big\{\frac{n}{r}\partial_\xi(w_2\phi(r))\big\} \\
&(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_\xi\big\{\frac{n^2}{r^2}\partial_\xi^2(\phi(r)u_3^\epsilon)\big\} \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_\xi\big\{\frac{n^2}{r^2}\partial_\xi^2(\phi(r)u_3)\big\}
\end{aligned}$$

in the sense of $\mathcal{D}'((0,T)\times\Omega)$ for all $\theta < \gamma/2$.

From equations (2.15) and (2.17), we get

$$\partial_t(-\tilde{\Delta})^{-1}\{\tilde{\rm div}\,(\phi(r)\rho^\epsilon\mathbf{u}^\epsilon)\} \in L^\infty(0,T;L^1_{\rm loc}(\Omega)) + L^2(0,T;L^2_{\rm loc}(\Omega)).$$

Therefore, by the classical Lions-Aubin Lemma, one obtains

$$(-\tilde{\Delta})^{-1}\{\tilde{\rm div}\,(\phi(r)\rho^\epsilon\boldsymbol{u}^\epsilon)\} \to (-\tilde{\Delta})^{-1}\{\tilde{\rm div}\,(\phi(r)\rho\boldsymbol{u})\} \text{ in } L^q(0,T;L^p_{\rm loc}(\Omega)) \qquad (3.10)$$

for any $1<q<\infty,\ p<2\gamma$. Hence, (3.9) combined with (3.10) implies that

$$\begin{aligned}
&(\rho^\epsilon)^\theta\boldsymbol{u}^\epsilon(-\tilde{\Delta})^{-1}\{\tilde{\rm div}\,(\phi(r)\rho^\epsilon\tilde{\mathbf{u}}^\epsilon)\} \rightharpoonup \overline{\rho^\theta}\boldsymbol{u}(-\tilde{\Delta})^{-1}\{\tilde{\rm div}\,(\phi(r)\rho\tilde{\mathbf{u}})\}, \\
&(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\{\tilde{\rm div}\,(\phi(r)\rho^\epsilon\tilde{\mathbf{u}}^\epsilon)\} \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\{\tilde{\rm div}\,(\phi(r)\rho\tilde{\mathbf{u}})\}, \qquad (3.11)\\
&\qquad \text{weakly in } L^q(0,T;L^p_{\rm loc}(\Omega)) \text{ for any } q<2,\ p<\tfrac{2\gamma}{1+2\theta},
\end{aligned}$$

where we have used

$$\begin{aligned}
&(\rho^\epsilon)^\theta \rightharpoonup \overline{\rho^\theta} \quad \text{weak-}* \text{ in } L^\infty(0,T;\mathcal{L}^{\gamma/\theta}(\Omega)) \text{ and} \\
&(\rho^\epsilon)^\theta \rightharpoonup \overline{\rho^\theta} \quad \text{in } C^0([0,T];L^p_{\rm loc}(\Omega)-w) \text{ for any } 1<p\le\gamma/\theta,\ 0<\theta<\gamma/2, \qquad (3.12)
\end{aligned}$$

which follows from the (2.26)-(2.27) and Appendix C of [12] (also cf. $(3.4)_2$).

From (2.26) we can see that

$$(-\tilde{\Delta})^{-1}\partial_r\big[(\rho^\epsilon)^\gamma\phi'(r)\big] \rightharpoonup (-\tilde{\Delta})^{-1}\partial_r\big[\overline{\rho^\gamma}\phi'(r)\big] \text{ in } L^{(2\gamma-1)/\gamma}(0,T;W^{1,(2\gamma-1)/\gamma}(\Omega)). \tag{3.13}$$

On the other hand, the imbedding $L^p_{\rm loc}(\Omega) \hookrightarrow\hookrightarrow W^{-1,(2\gamma+1)/(\gamma-1)}_{\rm loc}(\Omega)$, for any $(4\gamma+2)/(4\gamma-1) < p < \min\{2,\gamma/(\gamma-1)\}$, together with (3.12) implies that

$$(\rho^\epsilon)^\theta \to \overline{\rho^\theta} \text{ in } C^0([0,T];W^{-1,(2\gamma+1)/(\gamma-1)}_{\rm loc}(\Omega)). \tag{3.14}$$

Thus, from (3.13), (3.14) and Sobolev's imbedding theorem ($W^{1,(2\gamma-1)/\gamma}_{\rm loc} \hookrightarrow L^{2(2\gamma-1)}_{\rm loc}$), it follows that

$$(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_r\big[(\rho^\epsilon)^\gamma\partial_r\phi\big] \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_r\big[\overline{\rho^\gamma}\partial_r\phi\big], \tag{3.15}$$

weakly in $L^{(2\gamma-1)/\gamma}(0,T;L^p_{\rm loc}(\Omega)$ with $p = \frac{2\gamma(2\gamma-1)}{\gamma+2\theta(2\gamma-1)}$ (> 1 for $\theta < \gamma/2$).

If we make use of (2.26), (2.29) and $W^{1,(\beta+\theta)/\beta}_{\rm loc} \hookrightarrow L^{2(\beta+\theta)/(\beta-\theta)}_{\rm loc}$, we find for any $K \subset\subset \Omega$ and $0 < \theta < \min\{\gamma-1,\gamma/2\}$ that

$$\begin{aligned}
&\epsilon^\lambda\|(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_r[(\rho^\epsilon)^\beta\partial_r\phi]\|_{L^1((0,T)\times K)}\\
&\quad\le C\epsilon^\lambda\|(\rho^\epsilon)^\theta\|_{L^{2(\beta+\theta)/(\beta+3\theta)}((0,T)\times K)}\|(-\tilde{\Delta})^{-1}\partial_r[(\rho^\epsilon)^\beta\partial_r\phi]\|_{w^{1,(\beta+\theta)/\beta}((0,T)\times K)}\\
&\quad\le C\epsilon^\lambda\|\rho^\epsilon\|^\theta_{L^{2\theta(\beta+\theta)/(\beta+3\theta)}((0,T)\times K)}\|(\rho^\epsilon)^\beta\|_{L^{(\beta+\theta)/\beta}((0,T)\times K)}\\
&\quad\le C\epsilon^{\lambda\theta/(\beta+\theta)}\|\rho^\epsilon\|^\theta_{L^\gamma((0,T)\times K)}\|\epsilon^{\lambda/(\beta+\theta)}\rho^\epsilon\|^\beta_{L^{\beta+\theta}((0,T)\times K)} \qquad\text{and}\\
&\epsilon^\lambda\|(\rho^\epsilon)^{\theta+\beta}\|_{L^1((0,T)\times K)}\\
&\quad\le C\epsilon^{\lambda(\gamma-1-\theta)/(\beta+\gamma-1)}\|\epsilon^{\lambda/(\beta+\gamma-1)}\rho^\epsilon\|^{\beta+\theta}_{L^{\beta+\gamma-1}((0,T)\times k)} \to 0, \quad\text{as } \epsilon\to 0.
\end{aligned}\tag{3.16}$$

From (3.1), (3.3), (3.4) and (3.5) we easily get

$$(-\tilde{\Delta})^{-1}\partial_r[r\rho^\epsilon(u_1^\epsilon)^2\partial_r(\frac{\phi}{r})] \rightharpoonup (-\tilde{\Delta})^{-1}\partial_r[r\rho(u_1)^2\partial_r(\frac{\phi}{r})], \tag{3.17}$$

weakly in $L^2(0,T;W^{1,p}_{\rm loc}(\Omega))$ for all $1 < p < 2\gamma/(\gamma+1)$. So, by virtue of (3.12), (3.13), (3.16) and Sobolev's imbedding theorem, we obtain

$$(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_r[r\rho^\epsilon(u_1^\epsilon)^2\partial_r(\frac{\phi}{r})] \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_r[r\rho(u_1)^2\partial_r(\frac{\phi}{r})], \tag{3.18}$$

weakly in $L^2(0,T;L^p_{\rm loc}(\Omega))$ for any $1 < p < 2\gamma/(2\theta+1)$ (recall here that $2\gamma/(2\theta+1) > 1$ for $\theta < \gamma/2$).

Analogously to (3.18), we can show that

$$\begin{aligned}
&(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_\xi[r\rho^\epsilon u_1^\epsilon u_3^\epsilon\partial_r(\frac{\phi}{r})] \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_\xi[r\rho u_1u_3\partial_r(\frac{\phi}{r})],\\
&(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_r[\rho^\epsilon(u_2^\epsilon)^2\frac{\phi(r)}{r}] \rightharpoonup \overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_r[\rho(u_2)^2\frac{\phi(r)}{r}]
\end{aligned}\tag{3.19}$$

weakly in $L^2(0,T;L^p_{\rm loc}(\Omega))$ for any $1 < p < 2\gamma/(2\theta+1)$.

Finally, we estimate the commutator in (2.28). Denote $\mathcal{R}_{ij}=(-\tilde{\Delta})^{-1}\partial_i\partial_j$ $(i,j=1,2)$. Let $\psi\in C_0^\infty(\mathbb{R}^+\times\Omega)$ be a $2\pi-$periodic function in the variable ξ, one has by symmetry of the Riesz operator that

$$
\begin{aligned}
&\int_{\mathbb{R}^+\times\Omega}\psi(t,r,\xi)\big\{(\rho^\epsilon)^\theta\mathcal{R}_{ij}(\rho^\epsilon\tilde{u_i^\epsilon}\tilde{u_j^\epsilon}\phi(r))-(\rho^\epsilon)^\theta\tilde{\mathbf{u}^\epsilon}\cdot\tilde{\nabla}(-\tilde{\Delta})^{-1}[\tilde{\operatorname{div}}\,(\rho^\epsilon\tilde{\mathbf{u}^\epsilon}\phi(r))]\big\}drd\xi dt\\
&=\int_{\mathbb{R}^+\times\Omega}\big\{\phi(r)\rho^\epsilon\tilde{u_i^\epsilon}\tilde{u_j^\epsilon}\mathcal{R}_{ij}((\rho^\epsilon)^\theta\psi(t,r,\xi))-\psi(\rho^\epsilon)^\theta\tilde{u_i^\epsilon}\mathcal{R}_{ij}(\phi(r)\rho^\epsilon\tilde{u_j^\epsilon})\big\}drd\xi dt\\
&=\int_{\mathbb{R}^+\times\Omega}\big\{\tilde{u_i^\epsilon}[\phi(r)\rho^\epsilon\tilde{u_j^\epsilon}\mathcal{R}_{ij}(\psi(t,r,\xi)(\rho^\epsilon)^\theta)-\psi(t,r,\xi)(\rho^\epsilon)^\theta\mathcal{R}_{ij}(\phi(r)\rho^\epsilon\tilde{u_j^\epsilon})]\big\}drd\xi dt,\qquad(3.20)
\end{aligned}
$$

where we have denoted $\tilde{\boldsymbol{u}}^\epsilon=(\tilde{u_1^\epsilon},\tilde{u_2^\epsilon})=(u_1^\epsilon,u_3^\epsilon)$. Similarly we can show that

$$
\begin{aligned}
&\int_{\mathbb{R}^+\times\Omega}\psi(t,r,\xi)\big\{(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_\xi[\tilde{\operatorname{div}}\,(\frac{n}{r}\rho^\epsilon u_2^\epsilon\tilde{\boldsymbol{u}}^\epsilon\phi(r))]-\frac{n}{r}(\rho^\epsilon)^\theta u_2^\epsilon\partial_\xi(-\tilde{\Delta})^{-1}[\tilde{\operatorname{div}}\,(\rho^\epsilon\tilde{\mathbf{u}}^\epsilon\phi(r))]\big\}drd\xi dt\\
&=\int_{\mathbb{R}^+\times\Omega}\big\{\frac{n}{r}\phi\rho^\epsilon u_2^\epsilon\tilde{\boldsymbol{u}}^\epsilon\cdot\tilde{\nabla}(-\tilde{\Delta})^{-1}(\partial_\xi(\rho^\epsilon)^\theta\psi)-\frac{n}{r}\psi(\rho^\epsilon)^\theta u_2^\epsilon\partial_\xi(-\tilde{\Delta})^{-1}\tilde{\operatorname{div}}\,(\phi\rho^\epsilon\tilde{\boldsymbol{u}}^\epsilon)\big\}drd\xi dt\\
&=\int_{\mathbb{R}^+\times\Omega}\big\{\frac{n}{r}u_2^\epsilon[\phi\rho^\epsilon\tilde{\boldsymbol{u}}^\epsilon\cdot\tilde{\nabla}(-\tilde{\Delta})^{-1}(\psi(\rho^\epsilon)^\theta)-\psi(\rho^\epsilon)^\theta\partial_\xi(-\tilde{\Delta})^{-1}\tilde{\operatorname{div}}\,(\phi\rho^\epsilon\tilde{\boldsymbol{u}}^\epsilon)]\big\}drd\xi dt.\qquad(3.21)
\end{aligned}
$$

On the other hand, from Corollary 4.1 of [5], $(3.4)_2$ and $(3.11)_2$ one gets

$$
\begin{aligned}
&\big[\phi(r)\rho^\epsilon\tilde{u_j^\epsilon}\mathcal{R}_{ij}(\psi(t,r,\xi)(\rho^\epsilon)^\theta)-\psi(t,r,\xi)(\rho^\epsilon)^\theta\mathcal{R}_{ij}(\phi(r)\rho^\epsilon\tilde{u_j^\epsilon})\big]\\
&\qquad\rightharpoonup\big[\phi(r)\rho\tilde{u_j}\mathcal{R}_{ij}(\psi(t,r,\xi)\overline{\rho^\theta})-\psi(t,r,\xi)\overline{\rho^\theta}\mathcal{R}_{ij}(\phi(r)\rho\tilde{u_j})\big],\\
&\big[\phi(r)\rho^\epsilon\tilde{\boldsymbol{u}}^\epsilon\cdot\tilde{\nabla}(-\tilde{\Delta})^{-1}(\psi(\rho^\epsilon)^\theta)-\psi(\rho^\epsilon)^\theta\partial_\xi(-\tilde{\Delta})^{-1}\tilde{\operatorname{div}}\,(\phi(r)\rho^\epsilon\tilde{\boldsymbol{u}}^\epsilon)\big]\\
&\qquad\rightharpoonup\big[\phi(r)\rho\tilde{\boldsymbol{u}}\cdot\tilde{\nabla}(-\tilde{\Delta})^{-1}(\psi\overline{\rho^\theta})-\psi\overline{\rho^\theta}\partial_\xi(-\tilde{\Delta})^{-1}\tilde{\operatorname{div}}\,(\phi(r)\rho\tilde{\boldsymbol{u}})\big]\qquad(3.22)
\end{aligned}
$$

for any $1<s<2\gamma/(\gamma+2\theta+1)$ with $\theta<(\gamma-1)/2$. Since $\mathcal{L}^s_{\mathrm{loc}}(\Omega)\hookrightarrow\hookrightarrow\mathcal{H}^{-1}_{\mathrm{loc}}(\Omega)$, the weak convergence in (3.22) is in fact strong convergence in $C^0([0,T];\mathcal{H}^{-1}_{\mathrm{loc}}(\Omega))$. Hence, combining (3.20), (3.21) with (3.1) and (3.22), we arrive at

$$
\begin{aligned}
&\lim_{\epsilon\to0}\int_{\mathbb{R}^+\times\Omega}\Big\{(\rho^\epsilon)^\theta(-\tilde{\Delta})^{-1}\partial_i[\operatorname{div}\,(\rho^\epsilon\tilde{u_i^\epsilon}\boldsymbol{u}^\epsilon\phi)]-(\rho^\epsilon)^\theta\mathbf{u}^\epsilon\cdot\nabla(-\tilde{\Delta})^{-1}[\tilde{\operatorname{div}}\,(\rho^\epsilon\tilde{\boldsymbol{u}^\epsilon}\phi)]\Big\}\psi drd\xi dt\\
&=\int_{\mathbb{R}^+\times\Omega}\Big\{\overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_i[\operatorname{div}\,(\rho\tilde{u}_i\boldsymbol{u}\phi)]-\overline{\rho^\theta}\boldsymbol{u}\cdot\nabla(-\tilde{\Delta})^{-1}[\tilde{\operatorname{div}}\,(\rho\tilde{\boldsymbol{u}}\phi)]\Big\}\psi drd\xi dt.\qquad(3.23)
\end{aligned}
$$

Now, taking $\epsilon\to0$ in (2.28) and utilizing (3.9)-(3.19) and (3.23), we conclude

$$
\begin{aligned}
&\phi(r)(a\overline{\rho^{\theta+\gamma}}-\mu\mathcal{Q})=\partial_t\{\overline{\rho^\theta}(-\tilde{\Delta})^{-1}\tilde{\operatorname{div}}\,(\phi(r)\rho\tilde{\mathbf{u}})\}\\
&+\operatorname{div}\{\overline{\rho^\theta}(-\tilde{\Delta})^{-1}\tilde{\operatorname{div}}\,(\phi(r)\rho\tilde{\boldsymbol{u}})\boldsymbol{u}\}+(-\tilde{\Delta})^{-1}\tilde{\operatorname{div}}\,(\phi(r)\rho\tilde{\boldsymbol{u}})\{\theta\overline{\rho^\theta}\frac{u_1}{r}+(\theta-1)\mathcal{Q}_1+(\theta-1)\mathcal{Q}\}\\
&-\overline{\rho^\theta}\boldsymbol{u}\cdot\nabla\{(-\tilde{\Delta})^{-1}\tilde{\operatorname{div}}\,(\phi(r)\rho\tilde{\boldsymbol{u}})\}+\overline{\rho^\theta}(-\tilde{\Delta})^{-1}\partial_i\{\operatorname{div}\,(\rho\tilde{u}_i\boldsymbol{u})\}\\
&-\overline{\rho^\theta}(-\tilde{\Delta})^{-1}\Big\{\partial_r\Big(r\rho(u_1)^2\partial_r[\frac{\phi}{r}]+\frac{\phi\rho(u_2)^2}{r}+a\overline{\rho^\gamma}\phi'(r)\Big)+\alpha\partial_\xi\big(r\rho u_1u_3\partial_r(\frac{\phi}{r})\big)\Big\}\\
&-\mu\overline{\rho^\theta}(-\tilde{\Delta})^{-1}\{\partial_r[\frac{n^2}{r^2}\partial_\xi^2(\phi u_1)]+\alpha\partial_\xi(\frac{n^2}{r^2}\partial_\xi^2(\phi u_3))\}\\
&+\mu\overline{\rho^\theta}(-\tilde{\Delta})^{-1}\Big\{\partial_r\Big(r\frac{\partial u_1}{\partial r}\partial_r(\frac{\phi}{r})+(\frac{u_1}{r^2}+\frac{2n}{r^2}\frac{\partial u_2}{\partial\xi})\phi\Big)+\alpha\partial_\xi\Big(r\partial_ru_3(\frac{\phi}{r})\\
&+\partial_r(\phi'(r)u_3)\Big)-\alpha^2\partial_\xi^2(\phi'(r)u_1)\Big\}\qquad(3.24)
\end{aligned}
$$

in the sense of distributions, where $0<\theta<(\gamma-1)/2$ and we have denoted by $\mathcal{Q}_1$ and $\mathcal{Q}$ the weak limit of $(\rho^\epsilon)^\theta w_1^\epsilon$ and $(\rho^\epsilon)^\theta\,\tilde{\operatorname{div}}\,\tilde{\boldsymbol{u}}^\epsilon$, respectively.

Since ρ^ϵ is a renormalized solution of (2.14), approximating s^θ by $b\in C^1(R)$ with $|b(s)|\le C$ and $|b'(s)s|\le C$, one finds by (2.24) that

$$\begin{aligned}\partial_t(\rho^\epsilon)^\theta+\frac{1}{r}\partial_r(r(\rho^\epsilon)^\theta u_1^\epsilon)+\frac{n}{r}\partial_\xi((\rho^\epsilon)^\theta u_2^\epsilon)+\alpha\partial_\xi((\rho^\epsilon)^\theta u_3^\epsilon)\\ +(\theta-1)(\rho^\epsilon)^\theta\big\{\frac{1}{r}\partial_r(ru_1^\epsilon)+\frac{n}{r}\partial_\xi u_2^\epsilon+\alpha\partial_\xi u_3^\epsilon\big\}=0.\end{aligned}$$

Thus, by applying (3.9) and (3.11) and letting $\epsilon\to 0$ in the above equation, we conclude

$$\begin{aligned}\partial_t\overline{\rho^\theta}+\frac{1}{r}\partial_r(r\overline{\rho^\theta}u_1)+\frac{n}{r}\partial_\xi(\overline{\rho^\theta}u_2)+\alpha\partial_\xi(\overline{\rho^\theta}u_3)\\ +(\theta-1)\mathcal{Q}_1+(\theta-1)\mathcal{Q}=0.\end{aligned}\tag{3.25}$$

Therefore, using (3.7) and (3.25), following the same procedure to the one used in [13, PP. 8-9], we deduce that

$$\phi(r)\overline{\rho^\theta}(a\overline{\rho^\gamma}-\mu\,\tilde{\operatorname{div}}\tilde{\boldsymbol{u}})=\text{R.H.S of (3.24)},$$

which combined with (3.24) gives

Lemma 3.1

$$\phi(r)(a\overline{\rho^{\gamma+\theta}}-\mu\mathcal{Q})=\phi(r)\overline{\rho^\theta}(a\overline{\rho^\gamma}-\mu\,\tilde{\operatorname{div}}\tilde{\boldsymbol{u}})$$

for all $0<\theta<\frac{\gamma-1}{2}$.

As a result of Lemma 3.1, one has

$$\mu\phi(r)(\mathcal{Q}-\overline{\rho^\theta}\,\tilde{\operatorname{div}}\tilde{\boldsymbol{u}})=a\phi(r)(\overline{\rho^{\gamma+\theta}}-\overline{\rho^\gamma}\,\overline{\rho^\theta}).\tag{3.26}$$

On the other hand, by convexity,

$$\overline{\rho^\gamma}\le\overline{\rho^{\gamma+\theta}}^{\frac{\gamma}{\gamma+\theta}},\qquad\overline{\rho^\theta}\le\overline{\rho^{\gamma+\theta}}^{\frac{\theta}{\gamma+\theta}},\tag{3.27}$$

which together with (3.25) yields

$$\mu\phi(r)(\mathcal{Q}-\overline{\rho^\theta}\,\tilde{\operatorname{div}}\tilde{\boldsymbol{u}})\ge 0.$$

To exclude possible concentration on the axis $r=0$, we will use the following estimate which is a similar version of the important Lemma 3.2 in [9].

Lemma 3.2 *Let* $0<\theta<\min\{1/2,(\gamma-1)/2\}$ *and* $\frac{1}{2}(1-\theta+\sqrt{1+6\theta+\theta^2})\le\gamma$. *Then,*

$$\rho^\theta-\overline{\rho^\theta}\in L^{2/\theta}(0,T;\mathcal{L}^{2/\theta}(\mathbb{R}^+\times(0,2\pi))).$$

Proof By Lemma 3.1 we know that

$$a(\overline{\rho^{\gamma+\theta}}-\overline{\rho^\gamma}\,\overline{\rho^\theta})=\mu(\mathcal{Q}-\overline{\rho^\theta}\tilde{\operatorname{div}}\tilde{\boldsymbol{u}})\tag{3.28}$$

By virtue of convexity, $\overline{\rho^{\gamma+\theta}}\ge\overline{\rho^\gamma}^{\frac{\gamma+\theta}{\gamma}}$, $\overline{\rho^\gamma}\ge\rho^\gamma$ and $\rho^\theta\ge\overline{\rho^\theta}$. Hence

$$\begin{aligned}\overline{\rho^{\gamma+\theta}}-\overline{\rho^\gamma}\,\overline{\rho^\theta}&\ge\overline{\rho^\gamma}^{\frac{\gamma+\theta}{\gamma}}-\overline{\rho^\gamma}\,\overline{\rho^\theta}\\ &=\overline{\rho^\gamma}(\overline{\rho^\gamma}^{\frac{\theta}{\gamma}}-\overline{\rho^\theta})\\ &\ge\rho^\gamma(\rho^\theta-\overline{\rho^\theta})\ge 0.\end{aligned}\tag{3.29}$$

On the other hand, it is easy to see that $(\rho^\epsilon)^\theta \mathrm{div}\, \boldsymbol{u}^\epsilon$ and $\overline{\rho^\theta}\mathrm{div}\, \boldsymbol{u}$ are bounded in $L^{2\gamma/(\gamma+2\theta)}(0,T;$ $\mathcal{L}^{2\gamma/(\gamma+2\theta)}_{\mathrm{loc}}(\Omega))$, which implies that $\rho^\gamma(\rho^\theta - \overline{\rho^\theta}) \in L^{2\gamma/(\gamma+2\theta)}(0,T;\mathcal{L}^{2\gamma/(\gamma+2\theta)}_{\mathrm{loc}}(\Omega))$. Thus,

$$
\begin{aligned}
(\rho^\theta - \overline{\rho^\theta})^{2/\theta} &= (\rho^\theta - \overline{\rho^\theta})^{\frac{2\gamma}{\gamma+2\theta}}(\rho^\theta - \overline{\rho^\theta})^{\frac{2}{\theta}-\frac{2\gamma}{\gamma+2\theta}} \\
&\le C(\rho^\theta - \overline{\rho^\theta})^{\frac{2\gamma}{\gamma+2\theta}}\rho^{2-\frac{2\gamma\theta}{\gamma+2\theta}} \\
&\le C(\rho^\theta - \overline{\rho^\theta})^{\frac{2\gamma}{\gamma+2\theta}}(1+\rho^{\frac{2\gamma^2}{\gamma+2\theta}}) \\
&\le C(1+\rho^\gamma + (\rho^\theta - \overline{\rho^\theta})^{\frac{2\gamma}{\gamma+2\theta}}\rho^{\frac{2\gamma^2}{\gamma+2\theta}}),
\end{aligned}
$$

which completes the proof. □

Now, with Lemma 3.2 and the discussions similar to those in [9], we can prove that

$$
\rho^\epsilon \to \rho \quad \text{strongly in } L^p_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}) \qquad \forall\, p < 2\gamma - 1. \tag{3.30}
$$

Thus, to show that $(\rho, \boldsymbol{u})$ is a weak solution of (2.1)-(2.7), we would prove that $(\rho, \boldsymbol{u})$ satisfies (2.1)-(2.4) in the sense of distributions, i.e., it need to prove (2.9)-(2.12).

By using (2.26) and (2.33), we see that for any test function $\phi(r,\xi) \in C^1_0(\Omega\times[0,T])$ with $\phi(t,0,\xi) = \partial_\xi\phi(t,0,\xi) = 0$, there holds for $i = 1,2$, as $h \to 0$,

$$
\begin{aligned}
&\int_0^T\int_0^h\int_0^{2\pi} |\partial_r u_i^\epsilon \phi_r| r dt dr d\xi \to 0, \quad \int_0^T\int_0^h\int_0^{2\pi} |\tfrac{w_i^\epsilon}{r}\phi_\xi| r dt dr d\xi \to 0,\\
&\int_0^T\int_0^h\int_0^{2\pi} |\partial_\xi u_i^\epsilon \phi_\xi| r dt dr d\xi \to 0, \quad \int_0^T\int_0^h\int_0^{2\pi} |\tfrac{w_i^\epsilon}{r}\phi| r dt dr d\xi \to 0.\\
&\int_0^T\int_0^h\int_0^{2\pi} |\rho^\epsilon u_i^\epsilon u_j^\epsilon \phi_r| r dt dr d\xi \to 0, \quad \int_0^T\int_0^h\int_0^{2\pi} |\rho^\epsilon u_i^\epsilon u_j^\epsilon \tfrac{\phi_\xi}{r}| r dt dr d\xi \to 0, \qquad j = 1,2,3.
\end{aligned} \tag{3.31}
$$

If we employ (2.33) again, we easily infer

$$
\int_0^t\int_0^h\int_0^{2\pi}\{a(\rho^\epsilon)^\gamma + \epsilon^\lambda(\rho^\epsilon)^\beta\}\phi_i r dr d\xi dt \le Ch^{1/3} \to 0, \qquad \text{as } h \to 0.
$$

where we denote that $\phi_1 = \phi_r$ and $\phi_2 = \phi_\xi$, therefore, utilizing (3.6) and (3.30), one gets analogously to (3.31) that

$$
\int_0^T\int_\epsilon^\infty\int_0^{2\pi}\{a(\rho^\epsilon)^\gamma + \epsilon^\lambda(\rho^\epsilon)^\beta\}\phi_i r dr d\xi dt \to \int_0^T\int_{\mathbb{R}^+}\int_0^{2\pi} a\rho^\gamma\phi_i r dr d\xi dt, \quad \text{as } \epsilon \to 0. \tag{3.32}
$$

Thus, by (3.30), (3.31), (3.32) we see that the equations (2.1)-(2.3) satisfied by $\rho, \rho u_1, \rho u_2$ respectively hold in the sense of distributions $\mathcal{D}'((0,T)\times\Omega)$. Hence, it remains to show that equation of (2.4) satisfied by ρu_3 holds in the sense of distributions $\mathcal{D}'((0,T)\times\Omega)$.

To show this, we first notice that by (2.33) for any text function $\phi(t,r,\xi) \in C^1([0,T]\times\Omega)$ with $\partial_r\phi(t,0,\xi) = \partial_\xi\phi(t,0,\xi) = 0$, then as $h \to 0$ we have

$$
\begin{aligned}
&\int_0^T\int_0^h\int_0^{2\pi} |\rho^\epsilon u_1^\epsilon u_3^\epsilon \phi_r| r dr d\xi dt \to 0, \quad \int_0^T\int_0^h\int_0^{2\pi} |\rho^\epsilon u_2^\epsilon u_3^\epsilon \tfrac{\phi_\xi}{r}| r dr d\xi dt \to 0,\\
&\int_0^T\int_0^h\int_0^{2\pi} |\partial_r u_3^\epsilon \phi_r| r dr d\xi dt \to 0, \quad \int_0^T\int_0^h\int_0^{2\pi} (|\partial_\xi u_3^\epsilon \phi_\xi| + |\tfrac{\partial_\xi u_3^\epsilon}{r^2}\phi_\xi|) r dr d\xi dt \to 0.
\end{aligned}
$$

Thus we need only to deal with the term $\rho^\epsilon(u_3^\epsilon)^2$, for this we shall use concentration compactness arguments similar to those of Lions for the stationary isothermal case and the estimate (2.33) to show no concentration on the axis $r = 0$.

To this end, notice that for any $T > 0$, $\rho^\epsilon(u_3^\epsilon)^2 r$ is uniformly bounded with respect to ϵ in $L^1((0,T)\times\mathbb{R}^+\times[0,2\pi])$ by virtue of (2.12). Therefore, $\rho^\epsilon(u_3^\epsilon)^2 rdrd\xi dt$ converges weakly in the sense of measures(on $(0,T)\times\mathbb{R}_0^+\times\mathbb{R}$) to a bounded non-negative Radon measure $\nu(t,r,\xi)$:

$$\rho^\epsilon(u_3^\epsilon)^2 rdrd\xi dt \rightharpoonup \nu \quad \text{ in the sense of measure.} \tag{3.34}$$

On the other hand, since ν is bounded, the set $\{(t,r,\xi)|\nu(\{t,r,\xi\}) > 0\}$ is at most countable (also see [2, p.13]). Hence, by the Lebesgue decomposition and the Radon-Nikodym theorem, there is a $f \in L^1$, an at most countable set J(possibly empty), distinct points $\{t_i, r_i, \xi_i\}_{i\in J} \in [0,T]\times\mathbb{R}_0^+\times[0,2\pi]$ and positive constants $\{c_i\}_{i\in J}$, such that

$$\nu = frdrd\xi dt + \sum_{i\in J} c_i\delta(t_i, r_i, \xi_i), \quad \sum_{i\in J} c_i < \infty. \tag{3.35}$$

Moreover, by virtue of (3.5), we easily see that

$$f = \rho(u_3)^2, \qquad r_i = 0. \tag{3.36}$$

Thus, for any $\varphi(t,r,\xi) \in C_0^1([0,T]\times\mathbb{R}_0^+\times[0,2\pi])$ with $\varphi_r(t,0,\xi) = 0$, test equation (2.4) with $r\varphi$ in the weak form of (2.17) and make use of (3.1), (3.4) and (3.34)–(3.36) to deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^+\times[0,2\pi]} \rho u_3\varphi rdrd\xi|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_{\mathbb{R}^+\times[0,2\pi]} \Big\{\rho u_3\varphi_t + \rho u_1 u_3\varphi_r + \rho\frac{nu_2}{r}u_3\varphi_\xi + \alpha\rho(u_3)^2\varphi_\xi\Big\} rdrd\xi dt \\
&= \int_{t_1}^{t_2}\int_{\mathbb{R}^+\times[0,2\pi]} \Big\{\alpha a(\rho)^\gamma\varphi_\xi - \mu\partial_r u_3\varphi_r - \mu\big(\frac{n^2}{r^2}\partial_\xi u_3\varphi_\xi + \alpha^2\partial_\xi u_3\varphi_\xi\big)\Big\} rdrd\xi dt \\
&\quad + \sum_{i\in J} c_i\varphi_\xi(t_i, 0, \xi)
\end{aligned} \tag{3.37}$$

for any $0 \le t_1 \le t_2 \le T$.

Now, (3.37) and the fact that $\varphi_\xi(t,0,\xi) = 0$ show that $c_i = 0$, or in the other words that $J = \emptyset$. Thus, we prove that $(\rho, \boldsymbol{u})$ is indeed a finite energy weak solutions of (2.1)-(2.7).

Proof of theorem 1.1: In order to complete the proof of Theorem 1.1 in the case of even n, it remains to prove that $(\varrho, \boldsymbol{U})$ of the form (1.3) satisfies (1.1) in the sense of distributions.

Now, let $\phi = \phi(\mathbf{x}, t)$ be a C^1 function such that, ϕ is a periodic function in x_3 with period $\frac{2\pi}{\alpha}$, and for $0 \le t_1 \le t \le t_2$, $\operatorname{supp}\phi(\mathbf{x}, t)$ is contained in a fixed compact set in $\mathbb{R}^2$ with respect to x_1, x_2. Denote $y_1 = r\cos\frac{(\xi+\eta)}{2n}$, $y_2 = r\sin\frac{(\xi+\eta)}{2n}$, $y_3 = \frac{(\xi-\eta)}{2\alpha}$, and for $0 \le \xi \le 2\pi$, let

$$\zeta(t,r,\xi) = \frac{1}{2n\alpha}\sum_{k=0}^{n-1}\int_{\xi-4\pi}^{\xi} \phi(r\cos\frac{\xi+\eta+4k\pi}{2n}, r\sin\frac{\xi+\eta+4k\pi}{2n}, \frac{\xi-\eta}{2\alpha})d\eta.$$

Then, (2.9) with the test function ζ becomes

$$\int_\Omega \rho\zeta rdrd\xi|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_\Omega \{\rho\zeta_t + \rho u_1\zeta_r + (\frac{\rho nu_2}{r} + \alpha\rho u_3)\zeta_\xi\} rdrd\xi dt = 0. \tag{3.38}$$

Observe that for t fixed,

$$\int_\Omega \rho\zeta rdrd\xi = \int_{\mathbb{R}^+}\int_0^{2\pi}\int_0^{2\pi/\alpha} \rho(r, n\theta+\alpha z)\phi(r,\theta,z) rdrd\theta dz = \int_{\mathbb{R}^2}\int_0^{2\pi/\alpha} \varrho\phi(\boldsymbol{x}, t)d\boldsymbol{x}.$$

Similarly, one has

$$\int_{t_1}^{t_2}\int_\Omega \rho\zeta_t r dr d\xi = \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_0^{2\pi/\alpha} \varrho\phi_t(\boldsymbol{x},t)dtd\boldsymbol{x},$$

$$\int_{t_1}^{t_2}\int_\Omega \rho(u_1\zeta_r + (\frac{n}{r}u_2 + \alpha u_3)\zeta_\xi)rdrd\xi =$$

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^+}\int_0^{2\pi}\int_0^{2\pi/\alpha} \{(\rho u_1)(r, n\theta+\alpha z, t)\phi_r(r\cos\theta, r\sin\theta, z)$$

$$+\frac{(\rho u_2)}{r}(r, n\theta+\alpha z, t)\phi_\theta(r\cos\theta, r\sin\theta, z) + (\rho u_3)(r, n\theta+\alpha z, t)\phi_z(r\cos\theta, r\sin\theta, z)\}rdrd\theta d\xi$$

$$= \int_{\mathbb{R}^2}\int_0^{2\pi/\alpha}(\varrho U_1\phi_{x_1} + \varrho U_2\phi_{x_2} + \varrho U_3\phi_{x_3})d\boldsymbol{x} = \int_{\mathbb{R}^2}\int_0^{2\pi/\alpha} \varrho \boldsymbol{U}\cdot\nabla_{\boldsymbol{x}}\phi d\boldsymbol{x}. \tag{3.39}$$

Substituting the above identities into (3.38), we see that the first equation in (1.1) for ϱ is satisfied in the sense of distributions.

Now, let ϕ be the same as above and define

$$\zeta^1(t,r,\xi) = \frac{1}{2n\alpha}\sum_{k=0}^{n-1}\int_{\xi-4\pi}^{\xi} \cos\frac{\xi+\eta+4k\pi}{2n}\phi(r\cos\frac{\xi+\eta+4k\pi}{2n}, r\sin\frac{\xi+\eta+4k\pi}{2n}, \frac{\xi-\eta}{2\alpha})d\eta$$

and

$$\zeta^2(t,r,\xi) = \frac{1}{2n\alpha}\sum_{k=0}^{n-1}\int_{\xi-4\pi}^{\xi} \sin\frac{\xi+\eta+4k\pi}{2n}\phi(r\cos\frac{\xi+\eta+4k\pi}{2n}, r\sin\frac{\xi+\eta+4k\pi}{2n}, \frac{\xi-\eta}{2\alpha})d\eta.$$

Recalling that $\phi(\boldsymbol{x})$ is periodic in x_3 of period $2\pi/\alpha$, one easily sees that $\partial_\xi\zeta^1(t,0,\xi) = \partial_\xi\zeta^2(t,0,\xi) = 0$. Moreover, for even n, $\zeta^1(t,0,\xi) = \zeta^2(t,0,\xi) = 0$ by a straightforward calculation. Thus, we apply (2.10) and (2.11) with ζ^1 and ζ^2 respectively, to arrive that

$$\int_\Omega \rho u_1\zeta^1 rdrd\xi|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_\Omega\{\rho u_1\zeta_t^1 + \rho u_1(u_1\zeta_r^1 + (\frac{nu_2}{r}+u_3)\zeta_\xi^1)\}rdrd\xi dt =$$

$$\int_{t_1}^{t_2}\int_\Omega\{a\rho^\gamma(\zeta_r^1 + \frac{\zeta^1}{r}) + \frac{\rho u_2^2}{r}\zeta^1 - \mu(\partial_r u_1\zeta_r^1 + (\frac{w_2}{r} + \alpha^2\partial_\xi u_1)\zeta_\xi^1 + \frac{w_1}{r}\zeta^1)\}rdrd\xi dt, \tag{3.40}$$

and

$$\int_\Omega \rho u_2\zeta^2 rdrd\xi|_{t_1}^{t_2} - \int_{t_1}^{t_2}\int_\Omega\{\rho u_2\zeta_t^2 + \rho u_2(u_1\zeta_r^2 + (\frac{nu_2}{r}+u_3)\zeta_\xi^2)\}rdrd\xi dt =$$

$$\int_{t_1}^{t_2}\int_\Omega\{a\rho^\gamma\zeta_\xi^2 - \frac{\rho u_1 u_2}{r}\zeta^2 - \mu(\partial_r u_2\zeta_r^2 + (\frac{w_1}{r} + \alpha^2\partial_\xi u_2)\zeta_\xi^2 - \frac{w_2}{r}\zeta^2)\}rdrd\xi dt. \tag{3.41}$$

For fixed t,

$$\begin{aligned}\int_\Omega(\rho u_1\zeta^1 - \rho u_2\zeta_2)rdrd\xi &= \int_0^\infty\int_0^{2\pi}\int_0^{2\pi/\alpha}\rho(u_1 cos\theta - u_2\sin\theta)rdrd\theta dz\\ &= \int_{\mathbb{R}^2}\int_0^{2\pi/\alpha}\varrho U_1\phi(\boldsymbol{x},t)d\boldsymbol{x}.\end{aligned}$$

and similarly,

$$\int_{t_1}^{t_2}\int_\Omega \rho(u_1\zeta_t^1 - u_2\zeta_t^2) r dr d\xi = \int_{\mathbb{R}^2}\int_0^{2\pi/\alpha} \varrho U_1 \phi_t(t,\boldsymbol{x}) dt d\boldsymbol{x}.$$

As in (3.39), we have by a straightforward calculation that

$$\int_{t_1}^{t_2}\{\rho u_1(u_1\zeta_r^1 + (\frac{nu_2}{r}+u_3)\zeta_\xi^1) - \rho u_2(u_1\zeta_r^2 + (\frac{nu_2}{r}+u_3)\zeta_\xi^2) - \frac{\rho u_2^2}{r}\zeta^1 - \frac{\rho u_1 u_2}{r}\zeta^2\} r dr d\xi dt$$
$$= \int_{\mathbb{R}^2}\int_0^{2\pi/\alpha} \varrho U_1\, \boldsymbol{U}\cdot\nabla_{\boldsymbol{x}}\phi(t,\boldsymbol{x}) dt d\boldsymbol{x}.$$

To deal with the last two terms in (3.40) and (3.41), noticing that

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega \{a(\rho)^\gamma(\zeta_r^1 + \frac{\zeta^1}{r})\} r dr d\xi dt &= \int_{t_1}^{t_2}\int_\Omega \frac{a(\rho)^\gamma}{r}(r\zeta^1)_r dr d\xi dt \\ &= \int_{t_1}^{t_2}\int_0^{2\pi}\int_0^{\frac{2\pi}{\alpha}} a(\rho(t,r,n\theta+\alpha z))^\gamma \cos\theta \phi_r r dt dr d\theta dz,\end{aligned} \tag{3.42}$$

we obtain

$$\int_{t_1}^{t_2}\int_\Omega a(\rho)^\gamma\{(\zeta_r^1 + \frac{\zeta^1}{r}) - \frac{n\zeta_\xi^2}{r}\} r dr d\xi dt = \int_{\mathbb{R}^2}\int_0^{2\pi/\alpha} a\varrho^\gamma \phi_{x_1}(t,\boldsymbol{x}) dt d\boldsymbol{x}.$$

Applying (3.42) again, we may write that

$$\int_{t_1}^{t_2}\int_\Omega \{(\partial_r u_1 \zeta^1 r + (\frac{w_2}{r} + \alpha^2\partial_\xi u_1)\zeta_\xi^1 + \frac{w_1}{r}\zeta^1) - (\partial_r u_2\zeta_r^2 + (\frac{w_1}{r}+\alpha^2\partial_\xi u_2)\zeta_\xi^2 - \frac{w_2}{r}\zeta^2)\} r dr d\xi dt$$
$$= \int_{t_1}^{t_2}\int_{\mathbb{R}^2}\int_0^{2\pi/\alpha} \nabla_{\boldsymbol{x}} U_1\cdot\nabla_{\boldsymbol{x}}\phi(t,\boldsymbol{x}) dt d\boldsymbol{x}.$$

Finally, putting all the above related identities together, we find that ϱU_1 satisfies the second equation in (1.1) in the sense of distributions.

If we use ζ^2 in (2.10) and ζ^1 in (2.11) and add these two equations together, then we can show that ϱU_2 satisfies the third equation in (1.1) in the sense of distributions. In the same manner, the last equation can be handled by applying ζ to (2.12) directly. We should point out here that for *even* n, the test function ζ satisfies $\partial_r\zeta(t,0,\xi)=0$ which is used in the derivation of the last equation of (1.1). Therefore, we see that $(\varrho, \boldsymbol{U})$ in the form of (1.3) satisfies (1.1) in the sense of distributions.

Finally, for $T, L, h > 0$, let $\varphi \in C_0^1(\mathbb{R}^3), \chi^h \in C_0^\infty(\mathbb{R})$ satisfy $\varphi(t,r,\xi) = 1$ when $(t,r,\xi) \in [0,T]\times[0,1]\times[-L,L]$, $\chi^h(r) = 0$ when $0 \le r \le h$ and $\chi^h(r) = 1$ when $r \ge 2h$. Then, taking $\phi = r^{1+\beta}\varphi(t,r,\xi)\chi^h(r)$ $(\beta > 0)$ in (2.32), we obtain (1.8) by the same arguments as used for (2.33). This completes the proof of Theorem 1.1.

4 Existence of the approximate solutions

In this section we prove the existence of solutions of (2.14)–(2.20) by adapting the ideas in [13, Theorem 7.2] and [4, 9, 11].

The approximate solutions will be constructed by means of a three-level approximate scheme based on a modified system of (2.14)–(2.20)

$$\partial_t\rho+\frac{1}{r}\partial_r(r\rho u_1)+\frac{n}{r}\partial_\xi(\rho u_2)+\alpha\partial_\xi(\rho u_3)=\delta\Big(\frac{1}{r}\partial_r(r\partial_r\rho)+\partial_\xi^2\rho\Big), \tag{4.1}$$

$$\begin{aligned}&\partial_t(\rho u_1)+\frac{1}{r}\partial_r(r\rho(u_1)^2)+\frac{n}{r}\partial_\xi(\rho u_1u_2)+\alpha\partial_\xi(\rho u_1u_3)-\frac{\rho(u_2)^2}{r}+\delta(\partial_ru_1\partial_r\rho+\partial_\xi u_1\partial_\xi\rho)\\&\quad+a\partial_r(\rho)^\gamma+\epsilon^\lambda\partial_r(\rho)^\beta=\mu\Big(\frac{1}{r}\partial_r(r\partial_ru_1)+(\alpha^2+\frac{n^2}{r^2})\partial_\xi^2u_1-\frac{u_1}{r^2}-\frac{2n}{r^2}\partial_\xi u_2\Big)\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\partial_t(\rho u_2)+\frac{1}{r}\partial_r(r\rho u_1u_2)+\frac{n}{r}\partial_\xi(\rho(u_2)^2)+\alpha\partial_\xi(\rho u_2u_3)+\frac{\rho u_1u_2}{r}+\delta(\partial_ru_2\partial_r\rho+\partial_\xi u_2\partial_\xi\rho)\\&\quad+\frac{n}{r}\partial_\xi(a(\rho)^\gamma+\epsilon^\lambda(\rho)^\beta)=\mu\Big(\frac{1}{r}\partial_r(r\partial_ru_2)+(\alpha^2+\frac{n^2}{r^2})\partial_\xi^2u_2-\frac{u_2}{r^2}+\frac{2n}{r^2}\partial_\xi u_1\Big)\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\partial_t(\rho u_3)+\frac{1}{r}\partial_r(r\rho u_1u_3)+\frac{n}{r}\partial_\xi(\rho u_2u_3)+\alpha\partial_\xi(\rho(u_3)^2)+\alpha\partial_\xi\big(a(\rho)^\gamma+\epsilon^\lambda(\rho)^\beta\big)\\&\quad+\delta(\partial_ru_3\partial_r\rho+\partial_\xi u_3\partial_\xi\rho)=\mu\Big(\frac{1}{r}\partial_r(r\partial_ru_3)+(\alpha^2+\frac{n^2}{r^2})\partial_\xi^2u_3\Big),\end{aligned} \tag{4.4}$$

where $\epsilon,\delta,\beta,\lambda>0$ are constants, ϵ and δ are small.

We will solve the problem (4.1)–(4.4) in the square domain $\Omega_R:=(\epsilon,R)\times\mathbb{R}$ in the first step, then the second step we let the artificial viscosity δ go to zero to obtain a solution of (2.14)–(2.20) on the domain Ω_R, and in the final step three, we prove the existence of solutions to (2.14)–(2.20) by passing to the limit $R\to\infty$. In the following of this section we denote that $C_R:=(\epsilon,R)\times[0,2\pi]$.

Step 1. The first level approximate solutions. We consider the system (4.1)–(4.4) in $\Omega_R:=(\epsilon,R)\times\mathbb{R}$, together with initial and boundary conditions:

$$\rho(0,r,\xi)=\rho_0,\qquad \rho\mathbf{u}(0,r,\xi)=\mathbf{m}_0,\qquad (r,\xi)\in\Omega_R, \tag{4.5}$$

$$\partial_r\rho=u_1=u_2=\partial_ru_3=0,\qquad \text{on } \{\epsilon,R\}\times\mathbb{R}, \tag{4.6}$$

$$\rho(t,r,\xi)\text{ and }\boldsymbol{u}(t,r,\xi)\text{ are periodic in }\xi\text{ of periodic }2\pi. \tag{4.7}$$

We obtain in this step that

Lemma 4.1 *Let $\beta>\max\{4,\gamma\}$. Assume that $(\rho_0,\boldsymbol{m}_0)$ is periodic in ξ with period 2π, and $\rho_0\in L^\gamma(C_R)\cap L^\beta(C_R)\cap L^\infty(C_R),\inf_{C_R}\rho_0>0$ and $\boldsymbol{m}_0/\sqrt{\rho_0}\in L^2(C_R)$. Then there exists a global weak solution $(\rho,\boldsymbol{u})$ of (4.1)–(4.7) with $\rho\geq0$ a.e., such that $\rho\in L^{\beta+1}(C_R)$ and*

$$\begin{aligned}&\sup_{t\in[0,T]}(\|\rho(t)\|_{L^\gamma(C_R)}+\epsilon^\lambda\|\rho(t)\|_{L^\beta(C_R)}+\|(\sqrt{\rho}\boldsymbol{u})(t)\|_{L^2(C_R)})\\&\qquad\int_0^T(\|\boldsymbol{u}\|_{L^2(C_R)^2}+\|\nabla\boldsymbol{u}\|^2_{L^2(C_R)})(t)dt\leq C,\end{aligned} \tag{4.8}$$

$$\delta\int_0^T\|\nabla\rho(t)\|^2_{L^2(C_R)}\leq C, \tag{4.9}$$

where the constant C does not depend on δ but on ϵ,R,β,ρ_0 and $\boldsymbol{m}_0$. Moreover, the energy inequality

$$\begin{aligned}&\frac{d}{dt}\int_{C_R}\Big(\frac{\rho|\boldsymbol{u}|^2}{2}+\frac{a}{\gamma-1}\rho^\gamma+\frac{\epsilon^\lambda}{\beta-1}\rho^\beta\Big)(t)rdrd\xi+\mu\int_{C_R}\Big\{|\partial_r\boldsymbol{u}|^2+\alpha^2|\partial_\xi\boldsymbol{u}|^2+\frac{n^2}{r^2}|\partial_\xi u_3|^2\\&\quad+\big(\frac{u_1}{r}+\frac{n}{r}\partial_\xi u_2\big)^2+\big(\frac{u_2}{r}-\frac{n}{r}\partial_\xi u_1\big)^2\Big\}rdrd\xi\leq0\end{aligned} \tag{4.10}$$

holds in $\mathcal{D}'(0,T)$.

Proof In fact, if we multiply (4.2), (4.3), (4.4) by ru_1, ru_2, ru_3 respectively and integrating the resulting equation, then integrating by parts, using the equation (4.1) and the boundary condition (4.7), we obtain (4.10). Thus, (4.8) follows from (4.10) and the (generalized) Poincaré inequality.

Following the same procedure as in the proof of Proposition 4.1 in [3] and the proposition in [4], we can obtain the existence of weak solutions to (4.1)–(4.7)and the estimate (4.9) by solving the equation (4.1) directly, then solving the equations (4.2)–(4.4) by a Faedo-Galerkin approximate (cf. [14] for the incompressible case). The proof of Lemma 4.1 is completed. □

Step 2. The vanishing artificial viscosity limit. In this step we let the artificial viscosity δ in (4.1)–(4.4) go to zero, accordingly, we obtain the weak solutions to (2.1)-(2.4) in the domain Ω_R with initial and boundary conditions:

$$\begin{aligned}
&\rho(0,r,\xi)=\rho_0, \qquad \rho\boldsymbol{u}(0,r,\xi)=\boldsymbol{u}_0, \qquad (r,\xi)\in\Omega_R,\\
&u_1=u_2=\partial_r u_3=0, \qquad \text{on } \{\epsilon,R\}\times\mathbb{R},\\
&\rho(t,r,\xi) \text{ and } \boldsymbol{u}(t,r,\xi) \text{ are periodic in } \xi \text{ of period } 2\pi.
\end{aligned} \tag{4.11}$$

Then, we have:

Lemma 4.2 *Let $\beta>\max\{4,\gamma\}$. Assume that $(\rho_0,\boldsymbol{m}_0)$ is periodic in ξ with period 2π, and $\rho_0\in L^\gamma(C_R)\cap L^\beta(C_R), \rho_0\ge 0$ a.e., and $\boldsymbol{m}_0/\sqrt{\rho_0}\in L^2(C_R)$. Then there exists a global weak solution $(\rho,\boldsymbol{u})$ of (2.1)-(2.4), (4.11) with $\rho\ge 0$ a.e., such that for any $T>0$, holds that*

$$\begin{aligned}
\sup_{0\le t\le T}\int_{C_R}\Big(\frac{\rho|\boldsymbol{u}|^2}{2}+\frac{a}{\gamma-1}\rho^\gamma+\frac{\epsilon^\lambda}{\beta-1}\rho^\beta\Big)(t)rdrd\xi+\mu\int_0^T\int_{C_R}\Big\{|\partial_r\boldsymbol{u}|^2+\alpha^2|\partial_\xi\boldsymbol{u}|^2\\
+\frac{n^2}{r^2}|\partial_\xi u_3|^2+\big(\frac{u_1}{r}+\frac{n}{r}\partial_\xi u_2\big)^2+\big(\frac{u_2}{r}-\frac{n}{r}\partial_\xi u_1\big)^2\Big\}rdrd\xi\le E_R(\rho_0,\boldsymbol{m}_0),
\end{aligned} \tag{4.12}$$

where

$$E_R(\rho_0,\boldsymbol{m}_0):=\int_{C_R}\Big(\frac{|\boldsymbol{m}_0|^2}{2\rho_0}+\frac{a}{\gamma-1}(\rho_0)^\gamma+\frac{\epsilon^\lambda}{\beta-1}\rho_0^\beta\Big)rdrd\xi.$$

Proof Let ρ_0^δ be a smooth function sequence satisfying

$$\begin{aligned}
&0<C_1(\delta)\le\rho_0^\delta\le C_2(\delta),\\
&\nabla\rho_0^\delta\cdot\mathbf{n}|_{\partial\Omega_R}=0, \qquad \text{on } \{\epsilon,R\}\times\mathbb{R},\\
&\rho_0^\delta(r,\xi) \text{ is periodic in } \xi \text{ of period } 2\pi,\\
&\rho_0^\delta\to\rho_0 \quad \text{in } L^\gamma(C_R)\cap L^\beta(C_R) \text{ as } \delta\to 0.
\end{aligned}$$

Take $\chi_1^\delta\in C_0^\infty(R)$ such that $\chi_1^\delta(r)=1$ when $\epsilon+2\delta\le r\le R-2\delta$, and $\chi_1^\delta(r)=0$ when $r\le\epsilon+\delta$ or $r\ge R-\delta$. Then we set

$$\boldsymbol{m}_0^\delta(r,\xi):=\chi_1^\delta(r)\times\begin{cases}[(\boldsymbol{m}_0/\sqrt{\rho_0})*j_\delta]\sqrt{\rho_0^\delta}, & \text{if } \rho_0(x)>0,\\ 0, & \text{if } \rho_0(x)=0.\end{cases}$$

Denote by $(\rho^\delta,\boldsymbol{u}^\delta)$ the solution of (4.1)–(4.7) with the initial data $(\rho_0^\delta,\boldsymbol{m}_0^\delta)$ obtained in Lemma 4.1. We first observe that as $\delta\to 0$,

$$\begin{aligned}
&\delta(\partial_r(r\partial_r\rho^\delta)+r\partial_\xi^2\rho^\delta)\to 0 \ \text{ in } L^2(0,T;H^{-1}(C_R)),\\
&\delta(\partial_r u_j^\delta\partial_r\rho^\delta+\partial_\xi u_j^\delta\partial_\xi\rho^\delta)\to 0 \ \text{ in } L^1((0,T)\times C_R), \quad j=1,2,3,
\end{aligned} \tag{4.13}$$

which follows directly from (4.8) and (4.9). In the same manner as in the derivation of (2.29) (with $\phi(r) \equiv 1$) we get

$$\int_0^T \int_{\Omega_R} \{(\rho^\delta)^{\gamma+\theta} + \epsilon^\lambda (\rho^\delta)^{\beta+\theta}\} r dr d\xi dt \le C, \qquad \theta = \gamma - 1, \tag{4.14}$$

where C is a positive constant independent of δ.

The estimates (4.8) and (4.9) imply that $(\rho^\delta, \boldsymbol{u}^\delta) \rightharpoonup (\rho, \boldsymbol{u})$ weakly or weak-$*$. Then, using the fact (4.13) and the estimates (4.8), (4.9) and (4.14), by the same arguments as in Section 3 to prove the precompactness that the weak limit $(\rho, \boldsymbol{u})$ just obtained is indeed a weak solution of (2.14)–(2.17), (4.11) on $[0, \infty) \times \Omega_R$. And the estimate (4.12) follows from the equation (4.10), the lower semicontinuity of weak convergence and the convergence of the $(\rho_0^\delta, \boldsymbol{u}_0^\delta)$.

Step 3. Passing the limit as $R \to +\infty$. In this step, we let $R \to +\infty$ in (2.14)–(2.17), (4.11) to obtain the solution of (2.14)–(2.20). The main result we obtain in this step is the following:

Theorem 4.3 *Let $\beta > \max\{4, \gamma\}$, and denote $\tilde{G}_\epsilon := [\epsilon, \infty) \times \mathbb{R}$ and $G_\epsilon := [\epsilon, \infty) \times [0, 2\pi]$. Suppose that $(\rho_0, \boldsymbol{m}_0)$ is periodic in ξ with period 2π, and $\rho_0 \in \mathcal{L}^\gamma(G_\epsilon) \cap \mathcal{L}^\beta(G_\epsilon) \cap \mathcal{L}^1(G_\epsilon)$, $\rho_0 \ge 0$ a.e., and $\boldsymbol{m}_0/\sqrt{\rho_0} \in \mathcal{L}^2(G_\epsilon)$. Then there exists a global weak solution $(\rho, \boldsymbol{u})$ of (2.14)–(2.20) in $\tilde{G}_\epsilon$ with initial data $(\rho_0^\epsilon, \boldsymbol{m}_0^\epsilon)$ replaced by $(\rho_0, \boldsymbol{u}_0)$, such that $\rho \ge 0$ a.e., and for any $T > 0$ there hold*

$$\sup_{0 \le t \le T} \int_{G_\epsilon} \Big(\frac{\rho |\boldsymbol{u}|^2}{2} + \frac{a}{\gamma-1}\rho^\gamma + \frac{\epsilon^\lambda}{\beta-1}\rho^\beta\Big)(t) r dr d\xi + \mu \int_0^T \int_{G_\epsilon} \Big\{ |\partial_r \boldsymbol{u}|^2 + \alpha^2 |\partial_\xi \boldsymbol{u}|^2$$
$$+ \frac{n^2}{r^2}|\partial_\xi u_3|^2 + \Big(\frac{u_1}{r} + \frac{n}{r}\partial_\xi u_2\Big)^2 + \Big(\frac{u_2}{r} - \frac{n}{r}\partial_\xi u_1\Big)^2 \Big\} r dr d\xi \le E_R(\rho_0, \boldsymbol{m}_0), \tag{4.15}$$

$$\sup_{t \in [0,T]} \int_{G_\epsilon} \rho(t, r, \xi) r dr d\xi \le \int_{G_\epsilon} \rho_0(r, \xi) r dr d\xi, \tag{4.16}$$

where

$$E_\epsilon(\rho_0, \boldsymbol{m}_0) := \int_{G_\epsilon} \Big(\frac{|\boldsymbol{m}_0|^2}{2\rho_0} + \frac{a}{\gamma-1}\rho_0^\gamma + \frac{\epsilon^\lambda}{\beta-1}\rho_0^\beta \Big) r dr d\xi.$$

Proof To prove the theorem, we first approximate the initial data $(\rho_0, \boldsymbol{u}_0)$ as follows:

$$\rho_0^R(r, \xi) := \rho_0(r, \xi), \qquad \boldsymbol{m}_0^R(r, \xi) := \boldsymbol{m}_0(r, \xi)\chi_1^R(r),$$

where $\chi_1^R \in C_0^\infty(R)$ satisfying $\chi_1^R(r) = 1$ when $\epsilon + 1/R \le r \le R - 1$, and $\chi_1^R(r) = 0$ when $r \le \epsilon + 1/(2R)$ or $r \ge R - 1/2$. Then, it is easy to see that as $R \to +\infty$,

$$\begin{aligned} &\rho_0^R \to \rho_0 \quad \text{in } \mathcal{L}^\gamma(G_\epsilon) \cap \mathcal{L}^\beta(G_\epsilon) \cap \mathcal{L}^1(G_\epsilon), \\ &\boldsymbol{m}_0^R/\sqrt{\rho_0^R} \to \boldsymbol{m}_0/\sqrt{\rho_0} \quad \text{in } \mathcal{L}^2(G_\epsilon). \end{aligned} \tag{4.17}$$

If we denote $(\rho^R, \boldsymbol{u}^R)$ the solution of (2.14)–(2.17), (4.11) with the initial data $(\rho_0^R, \boldsymbol{u}_0^R)$

obtained in Lemma 4.2, then extend $(\rho^R, \boldsymbol{u}^R)$ to the domain $(\epsilon, \infty) \times \mathbb{R}$ in the following way

$$\tilde{\rho}^R(t,r,\xi) := \begin{cases} \rho^R(t,r,\xi), & (r,\xi) \in \bar{\Omega}_R, \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{\boldsymbol{u}}^R(t,r,\xi) := \begin{cases} \boldsymbol{u}^R(t,r,\xi), & (r,\xi) \in \bar{\Omega}_R, \\ \boldsymbol{u}^R(t,r,R), & \epsilon < r < R, \quad \xi \geq R, \\ \boldsymbol{u}^R(t,r,-R), & \epsilon < r < R, \quad -\xi \geq R, \\ \boldsymbol{u}^R(t,R,z), & r \geq R, \quad |\xi| < R, \\ 0, & \text{otherwise}. \end{cases}$$

Then, by the estimate (4.12) and the relation (4.17) as well as the arguments in Lions' book [13, P.43], we obtain that $\hat{\boldsymbol{u}}^R \in L^2(0,T;\mathcal{H}^1_{\text{loc}}(\bar{G}_\epsilon)$ and

$$\|\hat{\rho}^R\|_{L^\infty(0,T;\mathcal{L}^\gamma(G_\epsilon)\cap\mathcal{L}^\beta(G_\epsilon))} + \|(\sqrt{\hat{\rho}^R}\hat{\boldsymbol{u}}^R)\|_{L^\infty(0,T;\mathcal{L}^2(G_\epsilon))} + \|\hat{\boldsymbol{u}}^R\|_{L^2(0,T;\mathcal{H}^1_{\text{loc}}(G_\epsilon))} \leq C$$

with C being independent of R. Hence, if we let $R \to \infty$, we get

$$\begin{aligned} &\hat{\rho}^R \rightharpoonup \rho \quad \text{weakl-}* \text{ in } L^\infty(0,T;\mathcal{L}^\gamma(G_\epsilon) \cap \mathcal{L}^\beta(G_\epsilon)), \\ &\hat{\boldsymbol{u}}^R \rightharpoonup \boldsymbol{u} \quad \text{weakly in } L^2(0,T;\mathcal{H}^1_{\text{loc}}(G_\epsilon)). \end{aligned}$$

On the other hand, using the estimate (4.12) again, by the same arguments to derive the estimate (2.29) (with $\phi(r) \equiv 1$), we can deduce that there exists a constant C which independent of R such that for all R large enough,

$$\int_0^T \int_K \{(\rho^R)^{\gamma+\theta} + \epsilon^\lambda (\rho^R)^{\beta+\theta}\} r dr d\xi \leq C, \quad \theta = \gamma - 1, \tag{4.18}$$

for any compact set $K \subset \mathbb{R}^+ \times \mathbb{R}$.

To complete the proof of the theorem, we take into account that in any compact set of $\tilde{G}_\epsilon$, for R large enough, there holds $(\hat{\rho}^R, \hat{\boldsymbol{u}}^R) = (\rho^R, \boldsymbol{u}^R)$, thus by the same proof of Lemma 2.3 in [4], we find that $(\hat{\rho}^R, \hat{\boldsymbol{u}}^R)$ satisfies (3.8) in $\mathcal{D}'((0,T) \times G_\epsilon)$ with $(\rho, \boldsymbol{u})$ replaced by $(\hat{\rho}^R, \hat{\boldsymbol{u}}^R)$. Then using estimates (4.12) and (4.18), following the same procedure as in the proof of precompactness in Section 3, we see the weak limit $(\rho, \boldsymbol{u})$ of $(\hat{\rho}^R, \hat{\boldsymbol{u}}^R)$ by taking $R \to 0$ in (2.14)–(2.17), (4.11) is indeed a weak solution of (2.14)–(2.20). Moreover, by the lower semicontinuity of weak convergence, estimates (4.12) and (4.17), we see that for any $l > 0$, there holds

$$\begin{aligned} &\sup_{0\leq t\leq T} \int_{\Omega_l} \Big(\frac{\rho|\boldsymbol{u}|^2}{2} + \frac{a}{\gamma-1}\rho^\gamma + \frac{\epsilon^\lambda}{\beta-1}\rho^\beta\Big)(t) r dr d\xi + \mu \int_0^T \int_{\Omega_l} \Big\{|\partial_r \boldsymbol{u}|^2 + \alpha^2 |\partial_\xi \boldsymbol{u}|^2 \\ &\qquad + \frac{n^2}{r^2}|\partial_\xi u_3|^2 + (\frac{u_1}{r} + \frac{n}{r}\partial_\xi u_2)^2 + (\frac{u_2}{r} - \frac{n}{r}\partial_\xi u_1)^2\Big\} r dr d\xi \leq \liminf_{R\to\infty} E_R(\rho_0^R, \boldsymbol{m}_0^R) \\ &\qquad \leq \int_{G_\epsilon} \Big(\frac{|\boldsymbol{m}_0|^2}{2\rho_0} + \frac{a}{\gamma-1}\rho_0^\gamma + \frac{\epsilon^\lambda}{\beta-1}\rho_0^\beta\Big) r dr d\xi, \end{aligned}$$

where $\Omega_l := (\epsilon, l) \times [0, 2\pi]$. Hence, (4.15) holds by Fatou's Lemma.

Finally, if we apply the Lemma C.1 of [12, Appendix C] and equation (2.14), we observe that $\rho \in C^0([0,T]; L^\gamma_{\text{loc}} - w)$. Thus, if we integrate (2.14) over $(0,t) \times \Omega_R$ and take into account the boundary condition (4.11), we infer that

$$\sup_{t\in[0,T]} \int_{\Omega_R} \rho^R r dr d\xi = \sup_{t\in[0,T]} \int_{\Omega_R} \rho^R r dr d\xi \le \int_{\Omega_R} \rho_0^R r dr d\xi \le \int_{G_\epsilon} \rho_0 r dr d\xi.$$

Consequently, with the help of (4.17) one has that for any $l > 0$

$$\sup_{t\in[0,T]} \int_{\Omega_l} \rho(t,r,\xi) r dr d\xi \le \lim \inf_{R\to\infty} \int_{\Omega_R} \rho^R r dr d\xi \le \int_{G_\epsilon} \rho_0 r dr d\xi,$$

which yields (4.16). Thus the proof of Theorem 4.3 is completed. □

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