Hyperbolicity and kinematic waves of a class of multi-population partial differential equations*

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Abstract: In this paper we study the fundamental mathematical aspects of a class of multi-population partial differential equations. We thoroughly discuss the hyperbolicity of the system. The admissible waves of the Riemann problem are also investigated in detail. We present some interesting results and interpret their physical meanings. Numerical examples are also given to support our conclusions.

1 Introduction

In dealing with hyperbolic conservation laws, the Lighthill-Whitham [1] and Richards [2] (LWR) traffic flow model has played an important role for better understanding of linear and nonlinear waves. This model reads:

$$\rho_t + (\rho v_e(\rho))_x = 0, \tag{1.1}$$

where ρ denotes the density, and the velocity is determined by the state equation $v = v_e(\rho)$, with $v_e'(\rho) < 0$. To interpret shock and rarefaction waves, model (1.1) was intensively studied in Whitham's masterpiece [3]. Actually, by the transformation $c = q'(\rho)$, where $q(\rho) = \rho v_e(\rho)$ is strictly concave, (1.1) becomes the following Burgers equation:

$$c_t + (\frac{1}{2}c^2)_x = 0. \tag{1.2}$$

It is well known that (1.2) is most critical for the study of hyperbolic differential equations, and we note that (1.1) and (1.2) are equivalent also in the distribution sense for linear function $v_e(\rho)$.

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Based on (1.2), or more likely on (1.1) in the physical sense, extensions could be made for the study of many properties of hyperbolic PDEs, which in return serve for application purpose; see e.g., [4, 5, 6]. To learn about more complicated kinematic waves, one may consider a continuum flow with heterogeneous (m) media, in which the fields of the velocity v_i of the i-th class is a function of all density $\{\rho_j\}_{j=1}^m$, governed by the state equation $v_i = v_i(\rho_1, \dots, \rho_m)$. Thus, by the mass conservation for each class, a multi-population system can be written as

$$(\rho_i)_t + (\rho_i v_i)_x = 0, \quad i = 1, \dots, m. \tag{1.3}$$

In general, (1.3) might model some flow phenomena [7, 8, 9, 10] at least at the level of an analogue. However, the proof or disproof of its being hyperbolic would be difficult for $m > 2$, and more would be the study of its wave propagation properties. This is because the eigen-polynomial might be implicit, and more likely would be the eigenvalues. For example, in [7, 9], the state equation is set to be

$$v_i = v_i(\rho), \quad v_i'(\rho) < 0, \quad i = 1, \dots, m, \tag{1.4}$$

where ρ is the total density, i.e., $\rho = \Sigma_{i=1}^m \rho_i$. Furthermore, in some problems the 'fluid' is considered compressible so that the total density is no greater than some ρ_{max}. That is, the solution vector is bounded in $\bar{D} \subset R^m$, where the corresponding open domain is given by

$$D = \{u | \rho_i > 0, i = 1, \dots, m; \Sigma_{i=1}^m \rho_i < \rho_{max}\}. \tag{1.5}$$

In addition, all velocities of (1.4) are bounded such that

$$v_i(0) = b_i, \quad v_i(\rho_{max}) = 0, \quad \forall\ i. \tag{1.6}$$

We note that system (1.3) and (1.4) is precisely an extension of scalar hyperbolic conservation equation (1.1), such that they are identical for $m = 1$, $\rho_1 \equiv \rho$, and $v_1 \equiv v_e$. For this reason and mostly for theoretical curiosity, the present paper firstly provides a detailed discussion on the hyperbolicity of this system. In [11], the hyperbolicity remains unknown and has been taken as a conjecture. In [9], this was shown by means of a symmetriser for the special case that $v_i = v_i(\rho)$ are linear functions, but the approach cannot be applied to the general case. Based on a thorough study of the hyperbolicity for the general case, we derive a clear mathematical structure of the system.

The discussion of the present paper is organized as follows. In Section 2, a concise mathematical expression of the eigen-polynomial is derived. By the intermediate value theorem and mathematical induction, m properly bounded real eigenvalues are implicitly ensured, and thus the system is hyperbolic (Section 2.1, Theorem 2.1). Further, classification of being hyperbolic is made over the entire solution domain $\bar{D}$; accordingly, the eigenvectors are solved, depending on the corresponding eigenvalues (Section 2.1, Lemma 2.1, Theorem 2.2 and 2.3). It is concluded that the system is strongly hyperbolic wherever the solution involves intersections by two or more velocity curves, and that the system is non-strictly hyperbolic somewhere at a boundary $\rho_i = 0$ ($\forall\ i$). Meanwhile, the system is strictly hyperbolic in other solution regions in general. The characteristic fields are defined in Section 2.2. Their continuity is guaranteed in the whole solution domain

$\bar{D}$ (Theorem 2.4), whereas a multiple eigenvalue is not differentiable in some $(m-1)$ or $(m-2)$ dimensional subsets in which the hyperbolicity of the system also degenerates.

In Section 3, the k-shock and k-rarefaction waves of the Riemann problem are thoroughly investigated. These waves are either 'compressive' or 'expansive' (Theorem 3.1 in Section 3.1, and Theorem 3.3 in Section 3.2), subject to the velocity fields. However, it is indicated that the compressive shock and expansive rarefaction should be regular. That is, all densities increase after passing through the k-shock, whereas they decrease after entering the k-rarefaction fan. Note here that the first $(k-1)$ classes travel slower than the k-wave, but the last $(m-k+1)$ flows are faster. The above descriptions are guaranteed for certain types of velocity fields (Theorem 3.2 in Section 3.1, and Theorem 3.4 in Section 3.2). In Section 3.3, we indicate that a contact merely arises from an eigenvalue that is identical to the velocities of at least two flows. For such a contact, all necessary Riemann invariants are obtained.

In Section 4, the fifth-order accurate WENO scheme is introduced for numerical approximations, based on Lax-Friedrichs flux-spitting (Section 4.1). All numerical examples are designed to examine the main conclusions in Section 3, namely to observe all waves of the Riemann problem (Section 4.2).

The final results are summarized in Section 5.

2 Hyperbolicity and characteristic fields

We rewrite the system (1.3) and (1.4) in the following conservation form:

$$u_t + f(u)_x = 0, \tag{2.1}$$

where the solution vector $u = (\rho_1, \dots, \rho_m)^T$, the flux $f(u) = (\rho_1 v_1(\rho), \dots, \rho_m v_m(\rho))^T$. The Jacobian of $f(u)$ is written as

$$f_u = \begin{pmatrix} v_1 + c_1 & c_1 & \cdots & c_1 & c_1 \\ c_2 & v_2 + c_2 & \cdots & c_2 & c_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m-1} & c_{m-1} & \cdots & v_{m-1} + c_{m-1} & c_{m-1} \\ c_m & c_m & \cdots & c_m & v_m + c_m \end{pmatrix}, \ c_i = \rho_i v_i'(\rho) \leq 0.$$

Here, we note that $c_i = 0$ if and only if $\rho_i = 0$. To concisely express the eigenpolynomial $P_m(\lambda) \equiv det(f_u - \lambda I)$, namely the determinant

$$\begin{vmatrix} c_1 + v_1 - \lambda & c_1 & \cdots & c_1 & c_1 \\ c_2 & c_2 + v_2 - \lambda & \cdots & c_2 & c_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m-1} & c_{m-1} & \cdots & c_{m-1} + v_{m-1} - \lambda & c_{m-1} \\ c_m & c_m & \cdots & c_m & c_m + v_m - \lambda \end{vmatrix}, \tag{2.2}$$

we proceed as follows. First, let $(v_i - \lambda)$ be taken out as a factor from the i-th row, $i = 1, \ldots, m$, so that (2.2) has the form

$$\prod_{i=1}^{m}(v_1 - \lambda) \begin{vmatrix} 1+K_1 & K_1 & \cdots & K_1 & K_1 \\ K_2 & 1+K_2 & \cdots & K_2 & K_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ K_{m-1} & K_{m-1} & \cdots & 1+K_{m-1} & K_{m-1} \\ K_m & K_m & \cdots & K_m & 1+K_m \end{vmatrix}, \; K_i = \frac{c_i}{v_i - \lambda}. \tag{2.3}$$

Then let the first $(m-1)$ rows be added to the last row, hence, all its elements are identical. We then have

$$P_m(\lambda) = \prod_{i=1}^{m}(v_i - \lambda)Q_m(\lambda), \quad Q_m(\lambda) = 1 + \sum_{i=1}^{m} K_i(\lambda). \tag{2.4}$$

Note that only row transformations are involved in the above arguments.

2.1 Hyperbolicities of the model equation

¿From (2.4), the hyperbolicities of system (2.1) are clearly discussed below.

Theorem 2.1 *The Jacobian f_u has m bounded real eigenvalues for $u \in \bar{D}$; thus system (2.1) is hyperbolic on $\bar{D}$.*

Proof Given $u \in \bar{D}$, we first deal with one special case in which no two v_i are equal and no ρ_i is zero. For this case we assume that

$$v_1 < v_2 < \cdots < v_{m-1} < v_m, \; \rho_i > 0, \; \forall i. \tag{2.5}$$

We can verify that, by (2.4)-(2.5),

$$sgn(P_m(v_i)) = (-1)^i, \quad i = 1, \ldots, m; \quad sgn(P_m(v_1 + \sum_{i=1}^{m} c_i)) = 1. \tag{2.6}$$

By the intermediate value theorem, (2.6) implies that the polynomial $P_m(\lambda)$ has m distinct eigenvalues $\{\lambda_i\}_{i=1}^m$ bounded such that

$$v_1 + \sum_{i=1}^{m} c_i < \lambda_1 < v_1 < \lambda_2 < v_2 \cdots < v_{i-1} < \lambda_i < v_i < \cdots < v_{m-1} < \lambda_m < v_m. \tag{2.7}$$

We then apply mathematical induction to other cases in which at least two v_i's are equal or at least one ρ_i (or c_i) is zero. The conclusion is obvious for $m = 1$. Assuming that this is true for all l, where $1 \leq l < m$, we prove it is also true for $l = m$.

Obviously, by (2.4) this case is equivalent to having an eigenvalue $\lambda = v_i$. We can always arrange the sequence of $\{v_i\}_{i=1}^m$ to reset $\lambda = v_{m-k} = \cdots = v_m$ $(0 \leq k < m)$. Here, v_m is not equal to any other v_i, and $\{v_i\}_{i=1}^m$ do not necessarily follow the sequence of (2.5). For the case we rewrite (2.4) to be

$$P_m(\lambda) = (v_m - \lambda)^k \tilde{P}_{m-k}(\lambda), \tag{2.8}$$

where

$$\tilde{P}_{m-k}(\lambda) = \prod_{i=1}^{m-k} (v_i - \lambda)\tilde{Q}_{m-k}(\lambda), \qquad \tilde{Q}_{m-k}(\lambda) = 1 + \sum_{i=1}^{m-k} \tilde{K}_i,$$

and

$$\tilde{K}_i = K_i,\ for\ i \leq m-k-1, \quad \tilde{K}_{m-k} = \frac{\tilde{c}_{m-k}}{v_{m-k} - \lambda}, \quad \tilde{c}_{m-k} = \sum_{i=m-k}^{m} c_i \leq 0.$$

For $k \geq 1$, we claim that all roots of $P_m(\lambda)$ are real, simply by applying either (2.7) or the assumption for l on the polynomial $\tilde{P}_{m-k}(\lambda)$ of (2.8). For $k = 0$, that $\lambda = v_m$ is an eigenvalue is equivalent to having $c_m = 0$. This means that

$$P_m(\lambda) = (v_m - \lambda)P_{m-1}(\lambda),$$

and we have the same conclusion. □

The above arguments lead to the following conclusion.

Corollary 2.1 *For $u \in \bar{D}$, some $v_j \in \{v_i\}_{i=1}^m$ is an eigenvalue of f_u, if and only if $\rho_j = 0$, or $\exists\ l \neq j$, s.t. $v_l = v_j$.*

Corollary 2.2 *For $u \in \bar{D}$, $P_m(\lambda)$ has a multiple root λ only if $\lambda \in \{v_i\}_{i=1}^m$.*

Further identification of system (2.1) must involve multiple eigenvalues, which could be some $\lambda = v_m$. With reference to (2.8), we start with the following lemma.

Lemma 2.1 *For $u \in \bar{D}$, suppose that $\lambda = v_m$ is a real root of $P_m(\lambda)$ satisfying $v_{m-k} = \cdots = v_m$ $(0 \leq k < m)$, and v_m is not equal to any other v_i. We have*
(i) if $\tilde{c}_{m-k} < 0$, then the multiplicity of λ is k $(k \geq 1)$;
(ii) if $\tilde{c}_{m-k} = 0$ and $\tilde{Q}_{m-k}(v_m) \neq 0$, then the multiplicity of λ is $(k+1)$ $(k \geq 0)$; and
(iii) if $\tilde{c}_{m-k} = 0$ and $\tilde{Q}_{m-k}(v_m) = 0$, then the multiplicity of λ is $(k+2)$ $(k \geq 0)$.

Proof Because $\tilde{P}_{m-k}(v_m) = \tilde{c}_{m-k} \neq 0$, (2.8) implies conclusion (i). For $\tilde{c}_{m-k} = 0$, we rewrite (2.8) to be

$$P_m(\lambda) = (v_m - \lambda)^{k+1} \prod_{i=1}^{m-k-1} (v_i - \lambda)(1 + \sum_{i=1}^{m-k-1} K_i) = (v_m - \lambda)^{k+1} P_{m-k-1}(\lambda).$$

We reach conclusion (ii) because $\tilde{Q}_{m-k}(v_m) \neq 0$ implies $P_{m-k-1}(v_m) \neq 0$, and $P_{m-k-1}(v_m) < \infty$. Finally, we reach conclusion (iii) because $\tilde{Q}_{m-k}(v_m) = 0$ implies $P_{m-k-1}(v_m) = 0$, and $\lambda = v_m$ is just a single root of $P_{m-k-1}(\lambda)$ by Corollary 2.2. □

With reference to Lemma 2.1 and by the theorem below, we investigate whether the multiplicity of each eigenvalue $\lambda = v_j$ coincides with the maximum number of its linearly independent eigenvectors. A positive answer ensures a complete set of linearly independent eigenvectors of f_u, thus system (2.1) is at least strongly hyperbolic; otherwise it is non-strictly hyperbolic.

Theorem 2.2 *For $u \in \bar{D}$, the Jacobian f_u has a complete set of linearly independent eigenvectors and thus system (2.1) is at least strongly hyperbolic if and only if $Q_m(v_i) \neq 0$, $i = 1, \dots, m$. Here $Q_m(\lambda)$ is defined in (2.4).*

Proof For any v_j, $Q_m(v_j) = 0$ implies that v_j is an eigenvalue. Therefore, we only need to prove that the multiplicity of an eigenvalue $\lambda = v_j$ coincides with the maximum number of its linearly independent eigenvectors if and only if $Q_m(v_j) \neq 0$. Here, we again set $j = m$ and follow Lemma 2.1. Hence, there are three cases for consideration.

Also note that $Q_m(\lambda) = \tilde{Q}_{m-k}(\lambda)$ by (2.4) and (2.8), and that we can track back to (2.2) (but not (2.3)) to continue row transformations on the corresponding matrix $f_u - \lambda I$. The common step for these transformations is to divide the first $(m-k-1)$ rows of (2.2), respectively by $(v_1 - \lambda), \dots, (v_{m-k-1} - \lambda)$, to have

$$f_u - \lambda I \sim \begin{pmatrix} 1+K_1 & \cdots & K_1 & \cdots & K_1 & K_1 \\ \vdots & \ddots & \vdots & & \vdots & \vdots \\ K_{m-k-1} & \cdots & 1+K_{m-k-1} & \cdots & K_{m-k-1} & K_{m-k-1} \\ c_{m-k} & \cdots & c_{m-k} & \cdots & c_{m-k} & c_{m-k} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\ c_m & \cdots & c_m & \cdots & c_m & c_m \end{pmatrix}. \tag{2.9}$$

For case (i) of Lemma 2.1, we sum up the last k rows of (2.9) to its $(m-k)$-th row, and all elements in the row become $\sum_{i=1}^{m-k} c_i = \tilde{c}_{m-k} \neq 0$. The further transformations follow so that we have

$$f_u - \lambda I \sim \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

By this, we have $rank(f_u - \lambda I) = m - k$, and k linearly independent eigenvectors,

$$p_{m_1} = (0, \cdots, 0, -1, 1, 0, \cdots, 0)^T, \dots, p_{m_k} = (0, \cdots, 0, -1, 0, 0, \cdots, 1)^T, \tag{2.10}$$

where the -1 is in the $(m-k)$-th position.

For case (ii) of Lemma 2.1, $\tilde{c}_{m-k} = 0$ implies $c_i = 0$, $i = m-k, \dots, m$. We sum up all of the first $(m-k-2)$ rows to the $(m-k-1)$-th row, thus the first $(m-k-1)$ elements in the row become $Q_m(v_m) \neq 0$. Then, similar row transformations give

$$f_u - \lambda I \sim \begin{pmatrix} 1 & \cdots & 0 & \bar{K}_1 & \cdots & \bar{K}_1 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & \bar{K}_{m-k-1} & \cdots & \bar{K}_{m-k-1} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\bar{K}_i = K_i(v_m)/Q_m(v_m)$, $i = 1, \cdots, m-k-1$. Accordingly, $rank(f_u - \lambda I) = m-k-1$, and we have $k+1$ linearly independent eigenvectors,

$$p_{m_1} = (-\bar{K}_1, \cdots, -\bar{K}_{m-k-1}, 1, 0, \cdots, 0)^T, \ldots, p_{m_{k+1}} = (-\bar{K}_1, \cdots, -\bar{K}_{m-k-1}, 0, 0, \cdots, 1)^T. \tag{2.11}$$

In the above two cases, $Q_m(v_m) = \infty$ ($\neq 0$) and the multiplicity of eigenvalue $\lambda = v_m$ is truly equal to the number of the solved linearly independent eigenvectors. Therefore, we only need to verify that the two numbers are not equal for case (iii) of Lemma 2.1. We sum up the last $(m-k-2)$ rows of (2.9) to its $(m-k-1)$-th row, which yields, by $Q_m(v_m) = 0$,

$$f_u - \lambda I \sim \begin{pmatrix} 1+K_1 & \cdots & K_1 & K_1 & K_1 & \cdots & K_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{m-k-2} & \cdots & 1+K_{m-k-2} & K_{m-k-2} & K_{m-k-2} & \cdots & K_{m-k-2} \\ 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{2.12}$$

As $Q_m(v_m) = 0$ also implies that at least one element of the set $\{K_i\}_{i=1}^{m-k-1}$ is not zero, we assume that $K_{m-k-1} \neq 0$ as well. We then sum up the first $(m-k-3)$ rows of (2.12) to its $(m-k-2)$-th row, thus the first $(m-k-2)$ elements in the row become $-K_{m-k-1}$. Let the row be divided by $-K_{m-k-1}$, then the further transformations give

$$f_u - \lambda I \sim \begin{pmatrix} 1 & \cdots & 0 & \alpha_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 1 & \alpha_{m-k-2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\alpha_i = K_i/K_{m-k-1}$, $i = 1, \cdots, m-k-2$. It is easy to see that $rank(f_u - v_m) = m-k-1$, and we can only solve for at most $k+1$ linearly independent eigenvectors, i.e.,

$$p_{m_1} = (0, \cdots, 0, 0, -1, 1, \cdots, 0)^T, \ldots, p_{m_k} = (0, \cdots, 0, 0, -1, 0, \cdots, 1)^T,$$

and

$$p_{m_{k+1}} = (K_1, \cdots, K_{m-k-2}, K_{m-k-1}, 0, 0, \cdots, 0)^T. \tag{2.13}$$

In this case the multiplicity of $\lambda = v_m$ is $(k+2)$. This completes the proof. $\square$

Finally, some sufficient conditions for system (2.1) to be strictly hyperbolic can be derived directly from the above. In a stronger sense, (2.5) guarantees m distinct eigenvalues for $u \in D$, and the corresponding eigenvectors can be solved similarly. This is concluded by the following theorem.

Theorem 2.3 *Suppose that the inequalities of (2.5) hold for all $u \in D$. Then, system (2.1) is strictly hyperbolic in D with m distinct eigenvalues, being in the sequence of inequality (2.7). Moreover, the m linearly independent eigenvectors are*

$$p_i(\lambda_i) = (K_1(\lambda_i), \cdots, K_m(\lambda_i))^T, \quad i = 1, \dots, m, \tag{2.14}$$

where $K_j(\lambda_i)$ is given by (2.3).

2.2 General descriptions on characteristic fields

Based on the previous discussion, we define the characteristic fields of system (2.1), on which some heuristic comments and conclusions are also presented. Primarily, we define the velocity fields $\tilde{s} = \{\tilde{v}_i(\rho)\}_{i=1}^m$ as follows.

Given u and denote by $s_0 = \{v_i(\rho)\}_{i=1}^m$, then (i) set $\tilde{v}_1(\rho) = min_{v_j \in s_0}\{v_j(\rho)\}$ and $s_1 = s_0 - \{\tilde{v}_1(\rho)\}$; and (ii) for $i = 2, \dots, m$, set $\tilde{v}_i(\rho) = min_{v_j \in s_{i-1}}\{v_j(\rho)\}$ and $s_i = s_{i-1} - \{\tilde{v}_i(\rho)\}$. The definition directly gives the following lemma.

Lemma 2.2 *The function set $\tilde{s} = \{\tilde{v}_i(\rho)\}_{i=1}^m$ is in the sequence of inequalities:*

$$\tilde{v}_1(\rho) \leq \cdots \leq \tilde{v}_{i-1}(\rho) \leq \tilde{v}_i(\rho) \leq \cdots \leq \tilde{v}_m(\rho), \quad \forall \rho; \tag{2.15}$$

and each function $\tilde{v}_i(\rho)$ is continuous and strictly decreasing.

The proof of the continuity of $\tilde{v}_i(\rho)$ is trivial. Take $m = 3$, for example, the continuity is evident since

$$\tilde{v}_1(\rho) = min(v_1, v_2, v_3), \ \tilde{v}_3(\rho) = max(v_1, v_2, v_3), \ \tilde{v}_2(\rho) = \sum_{i=1}^{3} v_i - \tilde{v}_1 - \tilde{v}_3.$$

We then define m characteristic fields simply by numbering the m eigenvalues from the smaller to the larger for a fixed $u \in \bar{D}$, which implies that

$$\lambda_1(u) \leq \cdots \leq \lambda_{i-1}(u) \leq \lambda_i(u) \leq \cdots \leq \lambda_m(u). \tag{2.16}$$

By the definition, we also have

Lemma 2.3 *The function sets $\{\lambda_i(u)\}_{i=1}^m$ and $\tilde{s} = \{\tilde{v}_i(\rho)\}_{i=1}^m$ form the following interlaced sequence of inequalities:*

$$\lambda_1(u) \leq \tilde{v}_1(\rho); \quad \tilde{v}_{i-1}(\rho) \leq \lambda_i(u) \leq \tilde{v}_i(\rho), \ for \ i = 2, \dots, m, \quad u \in \tilde{D}. \tag{2.17}$$

Proof For a certain u, we can always arrange the sequence of $\{v_i(\rho)\}_{i=1}^m$ such that $v_i(\rho) = \tilde{v}_i(\rho)$ for all i. Thus (2.17) is implied from the proof of Theorem 2.1. □

We have several comments on the cases when equality holds in (2.15)-(2.17), which means an eigenvalue acquired by the intersection of some curves of $\{v_i(\rho)\}_{i=1}^m$, or at a boundary $\rho_i = 0$. For the former case, the intersection is denoted by $\rho = \rho_I$ in Fig.1(a) ($m = 2$), where the curves $\tilde{v}_1(\rho)$ and $\tilde{v}_2(\rho)$ are respectively the lower and upper parts of $v_1(\rho) \cup v_2(\rho)$.

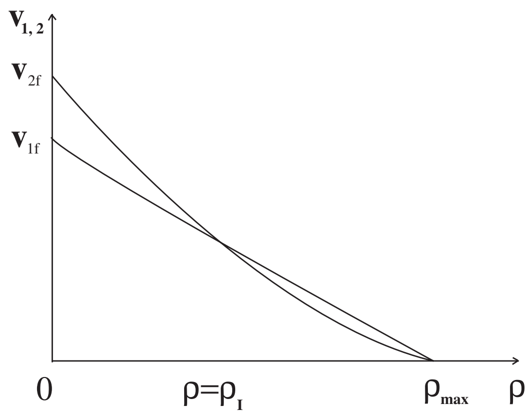

Fig.1(a) Intersection by velocity curves.

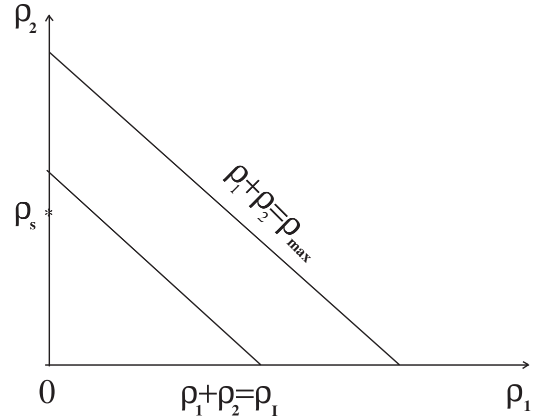

Fig.1(b) Involved $(m-1)$d subset on $\bar{D}$.

If all of these intersections are isolated, that is they are finite or infinitely denumerable, then such a representative eigenvalue $\lambda = v_m(\rho_I)$ (see the proof of Theorem 2.1) only involves an $m-1$ dimensional subset of $\bar{D}$, as shown by $\rho_1+\rho_2 = \rho_I$ in Fig.1(b) $(m = 2)$. If the intersections form a continuous section of a curve, then such $\lambda = v_m(\rho_I)$ involves an m-dimensional domain. The former case should be trivial in affecting the strict hyperbolicity of the system in D. However, $\lambda = v_m(\rho_I)$ is not differentiable when involving such an single intersection. This is also shown by Fig.1, where $\tilde{v}_1(\rho)$ and $\tilde{v}_2(\rho)$ are generally not smooth at ρ_I, so $\lambda_{1,2}(u)$ must not be differentiable in $\rho_1 + \rho_2 = \rho_I$. For the latter case, the eigenvalue $\lambda = v_m(\rho)$ is differentiable in the involved open domain, where the hyperbolicity is essentially different. See the further discussion in Section 3.3.

For the case at a boundary $\rho_i = 0$, the eigenvalue that arises is not differentiable on some subset that is just $(m-2)$ dimensional. It is in this subset where $Q_m(v_i) = 0$ and thus system (2.1) is non-strictly hyperbolic (Theorem 2.2). This argument is illustrated by the case of $m = 2$ below.

Assume that $v_1(\rho) < v_2(\rho)$ $(\rho \neq \rho_{max})$, it is easily shown that

$$\lambda_1 = 0.5(v_1 + c_1 + v_2 + c_2 - \sqrt{(v_1 + c_1 - v_2 - c_2)^2 - 4c_1c_2}),$$

$$\lambda_2 = 0.5(v_1 + c_1 + v_2 + c_2 + \sqrt{(v_1 + c_1 - v_2 - c_2)^2 - 4c_1c_2}).$$

At the boundary $\rho_1 = 0$ (also $c_1 = 0$),

$$\lambda_1(u) = \begin{cases} v_1(\rho), & \text{if } v_1 \leq v_2 + c_2, \\ v_2 + c_2, & \text{otherwise,} \end{cases} \qquad \lambda_2(u) = v_1 + v_2 + c_2 - \lambda_1. \tag{2.18}$$

Clearly, $\lambda_1(u)$ and $\lambda_2(u)$ are not partially differentiable at $(0, \rho_s)$ that is intersected by $v_1 = v_2 + c_2$, shown in Fig.1(b). At this same point, we have $\lambda_1(u) = \lambda_2(u) = v_1(\rho)$, and the corresponding eigen-matrix reads

$$f_u - \lambda I = \begin{pmatrix} v_1 + c_1 - \lambda & \rho_1 + c_1 \\ \rho_2 + c_2 & v_2 + c_2 - \lambda \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c_2 & 0 \end{pmatrix}.$$

For $c_2 \neq 0$, no two linearly independent eigenvectors exist, and (2.1) is thus non-strictly hyperbolic.

In addition, for $\rho_i = 0$, whether $v_i = \lambda_i$ or $v_i = \lambda_{i-1}$ depends on data u at the boundary $\rho_i = 0$. Again, this is easily shown by (2.18). In general, we remark that it is just in the critical subset $\{u \mid v_i(\rho) = \lambda_i(u) = \lambda_{i-1}(\rho)\}$ where $\lambda_i(u)$ is not differentiable and where system (2.1) is non-strictly hyperbolic.

¿From the above, we argue that the loss of the differentiability of some $\lambda_i(u)$ corresponds to the transitions at which the strict hyperoblicity of system (2.1) fails. However, $\lambda_i(u)$ is always differentiable in an open domain in which it is never identical to any velocity, and thus system (2.1) is strictly hyperbolic. The latter statement is obvious because such an eigenvalue is always derived from $Q_m(\lambda) = 0$. See the discussion on Theorem 2.1 and Lemma 3.2 for this argument.

Finally, it is our opinion that the non-strict hyperbolicity is inherent in the system. At a whole boundary $\rho_i = 0$, system (2.1) actually reduces to an $(m-1) \times (m-1)$ system because its i-th equation becomes an identity. This reduction means that any discussion must be made separately, as is done in (2.12)-(2.13). More significantly, the handling is more complex for characteristic decomposition (if it is possible); i.e., the decomposition can only be proceeded on the reduced system.

By the following theorem, however, all states of $\bar{D}$ can be connected continuously by the eigenvalues, and thus by their corresponding eigenvectors. This continuity may explain the successful numerical simulations in Section 4.2, even when they involve non-differential states.

Theorem 2.4 *All functions of* $\{\lambda_i(u)\}_{i=1}^m$ *are continuous on* $\bar{D}$.

Proof We denote by $\Lambda(u) \equiv (\lambda_1(u), \ldots, \lambda_m(u))$. Given $u \in \bar{D}$ and the increment Δu, $u + \Delta u \in \bar{D}$, suppose that $\Lambda(u + \Delta u)$ does not converge on $\Lambda(u)$ as $\Delta u \to 0$. That is, to have $\Delta u_n \to 0$, such that the sequence $\{\Lambda(u + \Delta u_n)\}$ along with all its subsequences does not converge on $\Lambda(u)$. As $\{\lambda_i(u + \Delta u_n)\}$ are all bounded (see (2.7) and (2.17)), hence $\{\Lambda(u + \Delta u_n)\}$ must have one convergent subsequence $\{\Lambda(u + \Delta u_{n_k})\}$, *i.e.*,

$$\lim_{k\to\infty} \{\Lambda(u + \Delta u_{n_k})\} = \tilde{\Lambda} \equiv (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m), \qquad \tilde{\Lambda} \neq \Lambda(u).$$

First, by (2.17), we have

$$\lambda_1(u + \Delta u_{n_k}) \leq \tilde{v}_1(\rho + \Delta \rho_{n_k}),\ \tilde{v}_{i-1}(\rho + \Delta \rho_{n_k}) \leq \lambda_i(u + \Delta u_{n_k}) \leq \tilde{v}_i(\rho + \Delta \rho_{n_k}); \quad (2.19)$$

then we re-denote by $P_m(\lambda, u)$ the eigenpolynomial (2.4), which gives

$$P_m(\lambda_i(u + \Delta u_{n_k}), u + \Delta u_{n_k}) = 0. \quad (2.20)$$

Note that $\tilde{\lambda}_i = \lim_{k\to\infty} \lambda_i(u + \Delta u_{n_k})$, and that $P_m(\lambda, u)$ and all functions $\tilde{v}_i(\rho)$ are continuous (Lemma 2.2). Hence, as $k \to \infty$, (2.19)-(2.20) become

$$\tilde{\lambda}_1 \leq \tilde{v}_1(\rho), \qquad \tilde{v}_{i-1}(\rho) \leq \tilde{\lambda}_i \leq \tilde{v}_i(\rho); \quad (2.21)$$

$$P_m(\tilde{\lambda}_i, u) = 0, \quad i = 1, \ldots, m. \quad (2.22)$$

Eq. (2.22) indicates that all $\{\tilde{\lambda}_i\}_{i=1}^m$ are m eigenvalues that correspond to u. By comparison between (2.21) and (2.17), we must have $\tilde{\lambda}_i = \lambda_i(u)$, $i = 1, \ldots, m$, namely $\tilde{\Lambda} = \Lambda(u)$. But this is a contradiction. □

We now explain the physical meanings of the characteristic fields in the motion, assuming that (2.5), (2.7) and $u \in D$.

First, for $i \leq k$, $\lambda_i < v_k$ indicates that the k-th flow from some $x = x_0$ are influenced by $\{\lambda_i\}_{i=1}^{k}$ from the downstream. Meanwhile, they are influenced by $\{\lambda_i\}_{i=k+1}^{m}$ from the upstream because $\lambda_i > v_k$ for $i \geq k+1$: see Fig.2, where l_k denotes the trajectory of the k-th flow, and $\{\lambda_i\}_{i=1}^{m}$ denotes the propagations of the m characteristic fields. Note that all trajectories can be approximated as straight lines by means of local linearization.

Next, we say that the last $m-1$ characteristic fields all arise from velocity differences among the flow media. Precisely, the k-th field ($k \geq 2$) arises from the $(k-1)$-th flow being overtaken by the k-th. This is true by reduction to absurdity. If there is no overtaking between the two flows, namely we reassume $v_{k-1} \equiv v_k$ in D, then the k-th characteristic field becomes $\lambda_k(u) = v_k(\rho)$: see the proof of Theorem 2.1. For the case, ρ_{k-1}, ρ_k and $\lambda_k(u)$ disappear after merging the $(k-1)$-th and k-th equations of (2.1), and (2.1) reduces to an $(m-1) \times (m-1)$ system with new variable $\tilde{\rho}_{k-1} \equiv \rho_{k-1} + \rho_k$. Also see Section 4.3, where the system remains being $m \times m$ but the wave by $\lambda_k = v_k$ is explained as a contact across which $\rho_{k-1} + \rho_k$ is a Riemann invariant.

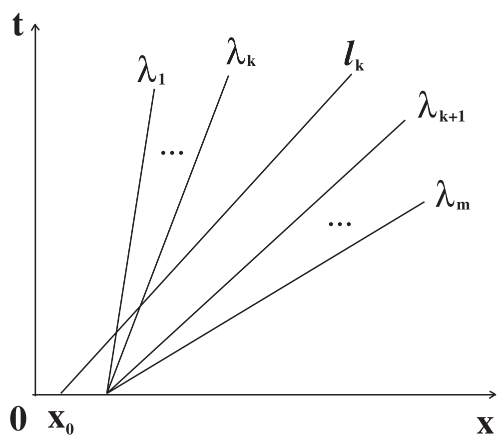

Fig.2(a) l_k is met by k propagations ahead.

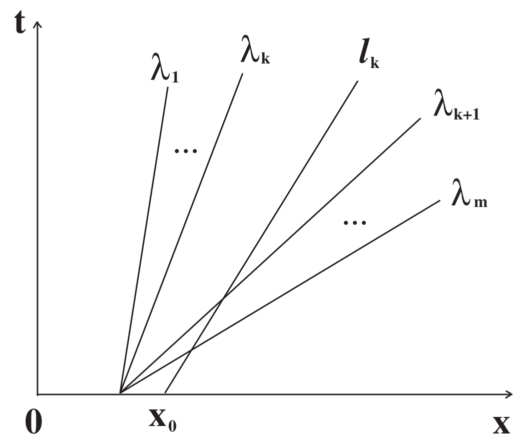

Fig.2(b) l_k is met by $m-k$ propagations behind.

Finally, $\lambda_1(u) < v_1(\rho)$ is the fundamental characteristic field. Its influence flows from the downstream. This has nothing to do with "overtaking" because it never disappears even when we reset all velocities to be identical and thus (2.1) reduces to the scalar case of (1.1).

Incidentally, for $\rho = \rho_{max}$, we have $v_1(\rho_{max}) = \ldots = v_m(\rho_{max}) = 0$ and $\lambda_2 = \ldots = \lambda_m = 0$. This means that the whole flow is stagnant. Hence, "overtaking" is impossible. For $\rho_i = 0$, which means the absence of the i-th flow, the eigenvalue $\lambda = v_i$ is a multiple root, and its multiplicity equals the reduction in the number of equations of the system.

3 Kinematic waves and solution properities

We are mainly concerned with a strictly hyperbolic admissible system (2.1)−a system that consists of either nonlinear or linearly degenerate fields. This is achieved by assuming (2.5) and (2.7) for $u \in D$; thus the strict hyperbolicity is ensured by Theorem 2.3. Of course, the boundary ∂D is excluded from this discussion in Section 3.1 and Section 3.2. In addition, the investigation of the strong hyperbolicity is also briefed in Section 3.3, where $\rho = \rho_{jam}$ can be included. The related Riemann problem is given by

$$u(x,0) = \begin{cases} u_l, & \text{if } x < 0, \\ u_r, & \text{if } x > 0. \end{cases} \tag{3.1}$$

It is well known that local admissible solutions of strictly hyperbolic systems (2.1) and (3.1) are existent, namely for sufficiently small $u_r - u_l$. That is, the initial data (3.1) break into m simple waves in the sense of Lax, corresponding to the defined m characteristic fields. Moreover, these waves divide the x-t upper plane into $m+1$ constant regions of all solution variables. For detailed accounts of strictly hyperbolic systems, see [12, 13, 14] and [15, 16, 3].

Global admissible solutions of the same problem are definitely a great concern. Indeed, for considerable functions $v_i(\rho)$, this existence is strongly supported by many of our conclusions, especially by the numerical results in Section 4.2. However, the main difficulty for the strict proof remains the same– all $\lambda_i(u)$ are implicit for $m > 2$.

Therefore, it is important to investigate how to connect a wave and how solution variables change in or across the wave. These changes are regular, as is shown in the discussion. In Section 3.1 and 3.2, only shocks and rarefactions are considered, because Section 3.3 shows that a contact is possible only by reassuming, such that some $\lambda = v_m$ holds continuously in a domain $D_c \subseteq D$. Such a contact is also investigated.

For all of these discussions, the left and right constant states of the studied k-wave ($k = 1, \ldots, m$) are represented by the superscripts "$-$" and "$+$", respectively.

3.1 Shock waves

Let s_k (or simply s) be the speed of the $k-$shock that arises from λ_k; the Rankine-Hugoniot conditions read

$$s = \frac{\rho_i^+ v_i(\rho^+) - \rho_i^- v_i(\rho^-)}{\rho_i^+ - \rho_i^-} = \frac{\sum_{i=1}^m \rho_i^+ v_i(\rho^+) - \sum_{i=1}^m \rho_i^- v_i(\rho^-)}{\rho^+ - \rho^-}, \quad \forall i. \tag{3.2}$$

These equalities of (3.2) also imply that

$$v_i^+ - s = \rho_i^- \frac{v_i^- - v_i^+}{\rho_i^+ - \rho_i^-}, \quad v_i^- - s = \rho_i^+ \frac{v_i^- - v_i^+}{\rho_i^+ - \rho_i^-}, \quad \frac{v_i^- - s}{v_i^+ - s} = \frac{\rho_i^+}{\rho_i^-} > 0, \quad v_i^\pm \equiv v_i(\rho^\pm). \tag{3.3}$$

For a valid shock, the Lax entropy conditions must be satisfied, i.e.,

$$\lambda_k^- > s > \lambda_k^+, \quad \lambda_{k+1}^+ > s > \lambda_{k-1}^-, \quad \lambda^\pm \equiv \lambda(u^\pm). \tag{3.4}$$

Here note that some invalid expressions in the above and in the following, namely λ_{k+1} for $k = m$, and λ_{k-1} for $k = 1$. They should be removed automatically.

Theorem 3.1 *For s satisfying (3.2), $\exists k \in \{i\}_{i=1}^m$, such that $v_{k-1}^\pm < s < v_k^\pm$. Moreover, we must have either*
(i). $\rho^- < \rho^+$, $\rho_i^- > \rho_i^+$ for $i < k$, and $\rho_i^- < \rho_i^+$ for $i \geq k$; or
(ii). $\rho^- > \rho^+$, $\rho_i^- < \rho_i^+$ for $i < k$, and $\rho_i^- > \rho_i^+$ for $i \geq k$.

Proof We change the third equality of (3.3) to

$$\rho_i^+ = \frac{v_i^- - s}{v_i^+ - s}\rho_i^- > 0, \tag{3.5}$$

then summation over i yields

$$\rho^+ = \sum_{i=1}^{m} \frac{v_i^- - s}{v_i^+ - s}\rho_i^- = \sum_{i=1}^{m}(1 + \frac{v_i^- - v_i^+}{v_i^+ - s})\rho_i^- = \sum_{i=1}^{m} \frac{v_i^- - v_i^+}{v_i^+ - s}\rho_i^- + \rho^-,$$

which is equivalent to

$$1 + \frac{1}{\rho^+ - \rho^-}\sum_{i=1}^{m} \frac{v_i^+ - v_i^-}{v_i^+ - s}\rho_i^- = 0.$$

It is easy to verify that the left hand side of the above would be positive or ∞ if $s \geq v_i^+$ for all i. This suggests some k, such that $v_{k-1}^+ < s < v_k^+$. Thus we have, by (3.5),

$$v_{k-1}^{\pm} < s < v_k^{\pm}. \tag{3.6}$$

Suppose that $\rho^- < \rho^+$, namely $v_i^- > v_i^+$ $(\forall i)$, then we have (i) by (3.3), (3.6), and (2.5). Similarly, we derive (ii) if $\rho^- > \rho^+$. Finally, $\rho^- = \rho^+$ is possible only at $\rho = \rho_{max}$, such that $s = v_i = 0$ for all i. □

In the proof, we see that the second inequality of (3.4) is implied in the first, provided that the Rankine-Hugoniot conditions of (3.2) hold. Therefore, it is unnecessary in the consideration for a valid shock.

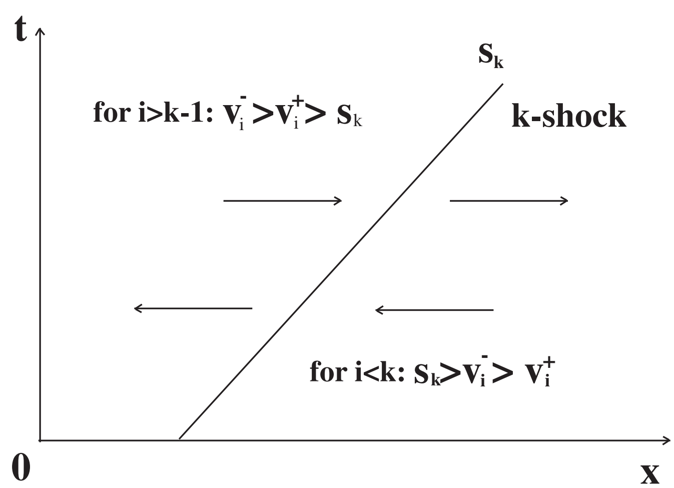

Fig.3 Changes in density and velocity of all flows across the compressive k-shock.

Importantly, we show that the k-shock which is characterized by $\rho^- < \rho^+$ is truly compressive for all flow media in the following sense. By setting the shock as stationary, a certain flow increases in density and decreases in velocity after it passes through the shock. Actually, for the i-th flow of $i \geq k$, $v_i^{\pm} - s > 0$ (see (3.6) and (2.5)), so it travels across the k-shock from the flow direction, such that $\rho_i^- < \rho_i^+$ and $v_i^- - s > v_i^+ - s$ (Theorem

3.1.(i)). For the i-th flow of $i \leq k-1$, we have $s - v_i^{\pm} > 0$. This suggests that the flow crosses the k-shock from the inverse direction, such that $\rho_i^- > \rho_i^+$ and $s - v_i^- > s - v_i^+$. See Fig.3 for the illustration.

In general, we argue that these compressive shocks occur frequently. This is absolutely true for the scalar case ($m = 1$) with strictly concave flux $f(\rho) = \rho v(\rho)$. Here, the strict concavity $f''(\rho) < 0$ also means the genuine nonlinearity of the waves [15, 3]. For the cases $m > 1$, it is difficult to define a similar 'concavity' of the flux, namely to define its properties sufficient and necessary to guarantee that all shocks are compressive. However, we can see the high frequency of these shocks in our problem, by its analogy to the scalar case, especially by the numerical simulations (Section 4.2). In contrast, the k-shock that is characterized by $\rho^- > \rho^+$ (Theorem 3.1.(ii)) must be expansive in the same sense; it is unlikely to occur but cannot be completely excluded.

As an important evidence of the argument, a range of velocity functions of certain types are verified to ensure that all shocks are compressive. For this proof and more, two Lemmas are derived from our previous discussions.

Lemma 3.1 *For $u \in D$, $\lambda(u)$ is an eigenvalue if and only if*

$$Q(\lambda, u) = 0, \tag{3.7}$$

where we denote by $Q(\lambda, u) \equiv Q_m(\lambda)$, which is given by (2.4).

Lemma 3.2 *For $u \in D$, the function $G(\lambda, u)$ is strictly decreasing in λ in the $m+1$ open intervals divided by $\lambda \neq v_k(\rho)$, $k \in \{i\}_{i=1}^m$. Moreover, a certain $\lambda_k(u)$ that is determined by (3.7) is differentiable in D.*

Proof We rewrite

$$Q(\lambda, u) = 1 + \sum_{i=1}^{m} \frac{c_i(u)}{v_i(\rho) - \lambda}. \tag{3.8}$$

Recall that $c_i(u) = \rho_i v_i'(\rho) < 0$, so it is obvious that

$$\frac{\partial Q(\lambda, u)}{\partial \lambda} = \sum_{i=1}^{m} \frac{c_i}{(v_i - \lambda)^2} < 0, \quad \lambda \neq v_k(\rho), \quad k = 1, \ldots, m. \tag{3.9}$$

Note that $\lambda_k(u) \in (v_{k-1}(\rho), v_k(\rho))$ in D, and it is valid to write, by (3.9),

$$\frac{\partial \lambda_k}{\partial \rho_i} = -\frac{\partial Q}{\partial \rho_i} / \frac{\partial Q}{\partial \lambda_k}; \tag{3.10}$$

and the differentiability of $\lambda_k(u)$ is ensured by the implicit function theorem. $\square$

We now define all velocity fields from a concave function $v(\rho)$, namely we have

$$v_i(\rho) = b_i v(\rho), \quad 0 < b_i < b_{i+1}, \quad \forall i; \quad v'(\rho) < 0, \quad v''(\rho) \leq 0, \quad \forall \rho \in [0, \rho_{max}], \tag{3.11}$$

Compared with (1.6), this definition implies that $v(0) = 1$ and $v(\rho_{max}) = 0$.

Theorem 3.2 *Suppose that the velocities are given by (3.11), and the two states u^- and u^+ of D satisfy (3.2). Then, u^- and u^+ form a valid k-shock if and only if $\rho^- < \rho^+$.*

Proof By Theorem 3.1, there exists $k \in \{i\}_{i=1}^m$, $v_{k-1}^{\pm} < s < v_k^{\pm}$ (see (3.6)). We only need to prove that $\rho^- < \rho^+$ if and only if we have the first inequality of (3.4). As mentioned, the second inequality of (3.4) is self-evident in the proof.

Suppose that $\rho^- < \rho^+$. By Theorem 3.1, we then have

$$\frac{\rho_i^+}{\rho_i^-} \begin{cases} < 1, & \text{if } i < k, \\ > 1, & \text{if } i \geq k. \end{cases} \tag{3.12}$$

Note that s, $\lambda_k^+ \in (v_{k-1}^+, v_k^+)$, in which $Q(\lambda, u^+)$ is strictly decreasing in λ (Lemma 3.2). Therefore, we prove $Q(s, u^+) < 0$, and thus $s > \lambda_k^+$ is implied by $Q(s, u^+) < 0 \equiv Q(\lambda_k^+, u^+)$ (Lemma 3.1). For this estimation of $Q(s, u^+)$, we replace all denominators of (3.8) with the first equality of (3.3) to have

$$Q(s, u^+) = 1 + \sum_{i=1}^m [v_i'(\rho^+)\frac{\rho_i^+ - \rho_i^-}{v_i^- - v_i^+}]\frac{\rho_i^+}{\rho_i^-} < 1 + \sum_{i=1}^m v_i'(\rho^+)\frac{\rho_i^+ - \rho_i^-}{v_i^- - v_i^+}.$$

The above inequality is achieved by (3.12) because the terms in $[\cdot]$ are positive for $i < k$ and negative for $i \geq k$. The substitution of v_i by (3.11) makes further estimation,

$$Q(s, u^+) < 1 - v'(\rho^+)\frac{\rho^+ - \rho^-}{v^+ - v^-} = 1 - \frac{v'(\rho^+)}{v'(\tilde{\rho})} \leq 0, \quad \rho^- < \tilde{\rho} < \rho^+. \tag{3.13}$$

We can similarly prove that $Q(s, u^-) > 0$ to have $s < \lambda_k^-$.

In a similar manner, it can be verified that all of the inequalities above will be inverse if $\rho^- > \rho^+$, so is the first inequality of the Lax entropy conditions (3.4). □

Note that the estimation of (3.12) can be altered to be closer, replaced by

$$\frac{\rho_i^+}{\rho_i^-} \begin{cases} < v_i^-/v_i^+, & \text{if } i < k, \\ > v_i^-/v_i^+, & \text{if } i \geq k. \end{cases} \tag{3.14}$$

This is due to the last inequality of (3.3), but only for $s > 0$. Accordingly, (3.13) is altered to be

$$Q(s, u^+) < 1 - v'(\rho^+)\frac{\rho^+ - \rho^-}{v^+ - v^-} \cdot \frac{v^-}{v^+} < 1 - \frac{v'(\rho^+)/v(\rho^+)}{v'(\tilde{\rho})/v(\tilde{\rho})} \leq 0, \quad \rho^- < \tilde{\rho} < \rho^+.$$

The last inequality above is acquired by reassuming $v(\rho)$ of (3.11), namely that $v'(\rho)/v(\rho)$ is decreasing or $v'' \leq (v')^2/v$. The new assumption is weaker but so will be the new conclusion (for $s > 0$). This indicates that the proof of $Q(s, u^+) < 0$ is very difficult for weaker settings of $\{v_i(\rho)\}_{i=1}^m$. However, we stress that this cannot exclude the compressive shocks that are widely observed in the numerical simulations (Section 4.2).

3.2 Rarefaction waves

The k-rarefaction wave arises from the following inequality:

$$\lambda_k^- < \lambda_k^+. \tag{3.15}$$

Wave solutions are well-known for their self-similarity, say $u(x,t) = u(\theta)$ and $\theta = x/t$. By the substitution, system (2.1) becomes

$$(f_u(u(\theta)) - \theta I)u'(\theta) = 0, \quad u'(\theta) \equiv (\rho_1'(\theta), \ldots, \rho_m'(\theta))^T. \tag{3.16}$$

In the rarefaction fan $\theta \in [\lambda_k^-, \lambda_k^+]$, $u'(\theta) \neq 0$, so by (3.16) $(\theta, u'(\theta))$ is an eigen-pair. That is, $u'(\theta) \mid\mid p_k(\theta)$, namely

$$\frac{\rho_1'(\theta)}{K_1(\theta)} = \cdots = \frac{\rho_i'(\theta)}{K_i(\theta)} = \cdots = \frac{\rho_m'(\theta)}{K_m(\theta)} = \frac{\rho'(\theta)}{-1}, \; u'(\theta) \neq 0, \tag{3.17}$$

where $p_k(\theta)$ and $K_i(\theta)$ are given by (2.14) and (2.4), respectively. The last equality of (3.17) is by Lemma 3.1, namely by

$$Q(\theta, u(\theta)) = 0, \quad or \quad \sum_{i=1}^{m} K_i(\theta) = -1, \quad v_{k-1}(\rho(\theta)) < \theta < v_k(\rho(\theta)). \tag{3.18}$$

It is obvious that (3.17) and (3.18) define $m+1$ nontrivial smooth curves $\rho = \rho(\theta)$ and $\rho_i = \rho_i(\theta)$, $i = 1, \ldots, m$; and the k rarefaction fan including the two states must be linked by these curves such that (3.15) holds. We describe the main features of these smooth curves namely rarefaction curves.

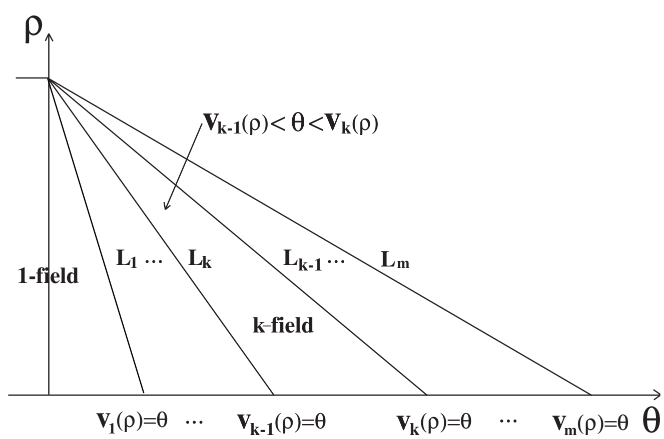

Fig.4 Division of solution fields of rarefaction curves by solving (3.17)-(3.18).

For fixed k, the curves are confined to $v_{k-1}(\rho(\theta)) < \theta < v_k(\rho(\theta))$, which is called the k-th solution field of (3.17)-(3.18). See Fig.4, we have m solution fields such that all curves $\rho = \rho(\theta)$ of these m fields in the ρ-θ coordinate plane are completely separated by m curves L_i: $v_i(\rho) = \theta$, $i = 1, \ldots, m$. In the k-th (each) field, we note that $v_i(\rho(\theta)) - \theta$ or $K_i(\theta)$ keep the sign unchanged for certain i. This property is stressed. Hence, we can give the following lemma.

Lemma 3.3 *For solutions of (3.17)-(3.18) in the k-th field, we have*

$$v_i(\rho(\theta)) < \theta \; or \; K_i(\theta) < 0, \; for \; i < k; \quad v_i(\rho(\theta)) > \theta \; or \; K_i(\theta) > 0, \; for \; i \geq k. \tag{3.19}$$

Combining the above lemma and (3.17), we directly have

Theorem 3.3 *In the k-th field of (3.17)-(3.18), we have either*
(i). $\rho'(\theta) < 0$, $\rho_i'(\theta) > 0$ *for* $i < k$, $\rho_i'(\theta) < 0$ *for* $i \geq k$; *or*
(ii). $\rho'(\theta) > 0$, *then* $\rho_i'(\theta) < 0$ *for* $i < k$, *and* $\rho_i'(\theta) > 0$ *for* $i \geq k$.

It is interesting (but not surprising) to note that Theorem 3.3 is parallel to Theorem 3.1, as are many of the following conclusions and comments. Likewise, we say that the k-rarefaction that is characterized by $\rho'(\theta) < 0$ (Theorem 3.3.(i)) is expansive in the same sense; that is, all media accelerate and increase their densities after they enter the rarefaction fan. Furthermore, these expansive rarefaction waves should be regular in our problem. Compare this to the corresponding discussion for shocks. Meanwhile, these compressive rarefactions that are characterized by $\rho'(\theta) > 0$ (Theorem 3.3.(ii)) are infrequent, subject to the velocity fields.

Parallel to Theorem 3.2, we prove that the monotonicity of each k curve is certain, i.e., $\rho'(\theta) < 0$, provided that the velocity fields are given by (3.11). This is ascribed to the following lemma.

Lemma 3.4 *The velocities are given by (3.11). Then the solutions of (3.17)-(3.18) in the k-th field are monotone such that* $\rho'(\theta) < 0$.

Proof Suppose that the conclusion is not true, then there exist θ_1 and θ_2, $\theta_1 < \theta_2$, such that $\rho'(\theta) > 0$ for $\theta \in [\theta_1, \theta_2]$. As $\lim_{\theta \to \theta_2} (v_k(\rho(\theta)) - \theta_2) = v_k(\rho(\theta_2)) - \theta_2 > 0$, θ_1 can be sufficiently close to θ_2 such that $v_k(\rho(\theta)) - \theta_2 > 0$, $\forall \theta \in [\theta_1, \theta_2]$. Generally, for $\theta \in [\theta_1, \theta_2)$, we can have,

$$v_i(\rho(\theta)) - \theta_2 < v_i(\rho(\theta)) - \theta < 0, \ i < k; \ \ v_i(\rho(\theta)) - \theta > v_i(\rho(\theta)) - \theta_2 > 0, \ i \geq k, \tag{3.20}$$

By (3.17), we always have

$$\frac{\rho_i'(\theta)}{\rho_i(\theta)} = \frac{-(v_i(\rho(\theta)))'}{v_i(\rho(\theta)) - \theta} < \frac{-(v_i(\rho(\theta)) - \theta_2)'}{v_i(\rho(\theta)) - \theta_2}, \quad \forall i. \tag{3.21}$$

Note that $-(v_i(\rho(\theta)))' = -v_i'(\rho)\rho'(\theta) > 0$ in the above. By the formula $(\ln|\varphi|)' = \varphi'/\varphi$, (3.21) changes to

$$(\rho_i(\theta)|v_i(\rho(\theta)) - \theta_2|)' < 0,$$

which gives

$$\rho_i(\theta_2)|v_i(\rho(\theta_2)) - \theta_2| < \rho_i(\theta_1)|v_i(\rho(\theta_1)) - \theta_2|, \ \forall i.$$

More conveniently, the inequality above is rewritten as

$$\frac{\rho_i(\theta_2)}{\rho_i(\theta_1)} < \frac{v_i(\rho(\theta_1)) - \theta_2}{v_i(\rho(\theta_2)) - \theta_2}, \quad \forall i. \tag{3.22}$$

The signs in $|\cdot|$ above are decided by (3.17)-(3.18), of which θ is replaced by θ_1. In addition, (3.22) can be written as

$$\frac{\rho_i(\theta_2) - \rho_i(\theta_1)}{\rho_i(\theta_1)} < \frac{v_i(\rho(\theta_1)) - v_i(\rho(\theta_2))}{v_i(\rho(\theta_2)) - \theta_2}, \quad \forall i. \tag{3.23}$$

Furthermore we have, by Theorem 3.3 and the assumption $\rho'(\theta) > 0$ on $[\theta_1, \theta_2]$,

$$\rho(\theta_2) > \rho(\theta_1), \quad \frac{\rho_i(\theta_2)}{\rho_i(\theta_1)} \begin{cases} < 1, & \text{if } i < k, \\ > 1, & \text{if } i \geq k, \end{cases} \quad v_i(\rho(\theta_1)) > v_i(\rho(\theta_2)), \ \forall i. \tag{3.24}$$

Based on (3.8), (3.11), (3.23) and (3.24), and also noting that $v_i'(\rho(\theta_2)) < 0$, we have

$$Q(\theta_2, u(\theta_2)) = 1 + \sum_{i=1}^{m} \frac{\rho_i(\theta_2) v_i'(\rho(\theta_2))}{v_i(\rho(\theta_2)) - \theta_2} < 1 + \sum_{i=1}^{m} \frac{\rho_i(\theta_2)}{\rho_i(\theta_1)} \frac{(\rho_i(\theta_2) - \rho_i(\theta_1)) v_i'(\rho(\theta_2))}{v_i(\rho(\theta_1)) - v_i(\rho(\theta_2))}$$

$$< 1 + \sum_{i=1}^{m} \frac{(\rho_i(\theta_2) - \rho_i(\theta_1)) v_i'(\rho(\theta_2))}{v_i(\rho(\theta_1)) - v_i(\rho(\theta_2))} = 1 + \frac{(\rho(\theta_2) - \rho(\theta_1)) v'(\rho(\theta_2))}{v(\rho(\theta_1)) - v(\rho(\theta_2))} < 1 - \frac{v'(\rho(\theta_2))}{v'(\tilde{\rho})} \leq 0, \tag{3.25}$$

where $\rho(\theta_1) < \tilde{\rho} < \rho(\theta_2)$. However, this contradicts with the fact that $Q(\theta_2, u(\theta_2)) = 0$ from Lemma 3.1. This completes the proof. □

By Lemma 3.4, (3.15) is equivalent to having $\rho^- > \rho^+$ (and the description by Theorem 3.3(i)) for the two states at the same k curves of (3.17)-(3.18), provided that the velocities are given by (3.11). Therefore the following theorem is self-evident.

Theorem 3.4 *Suppose that the velocities are given by (3.11). Then, for two states u^- and u^+ that are in the same certain curves of (3.17)-(3.18) in the k-th field, the k-rarefaction wave is formed if and only if $\rho^- > \rho^+$, and is expansive as described by Theorem 3.3(i).*

The conclusions by Lemma 3.4 and Theorem 3.4 are true for $\theta \in [\lambda_k^-, \lambda_k^+] \subset (0, \lambda_k^+]$, provided that $v''(\rho) \leq 0$ of (3.11) is altered to be $v'' \leq (v')^2/v$. Following the same steps that prove Lemma 3.4, the conclusion is reached through the replacement of (3.24) by

$$\frac{\rho_i(\theta_2)}{\rho_i(\theta_1)} < \frac{v_i(\rho(\theta_1))}{v_i(\rho(\theta_2))}, \quad \text{if } i < k, \quad \frac{\rho_i(\theta_2)}{\rho_i(\theta_1)} > \frac{v_i(\rho(\theta_1))}{v_i(\rho(\theta_2))}, \quad \text{if } i \geq k.$$

The first part of the above is implied by (3.22) ($\theta_2 > 0$); the second part can be similarly obtained such that in (3.21) θ_2 is replaced by θ_1 and the inequality is inverse. Compare this to (3.14) and the associated comments.

Finally, the discussion on shock curves is similar to that of rarefaction curves, though for simplicity they are not mentioned in Section 3.1. See [15] for a detailed account of this issue.

3.3 Heuristic comments on nonlinearity and linearly degeneration

A linearly degenerate field of the eigen-pair (λ, p) corresponds to

$$\nabla_u \lambda \cdot p \equiv 0, \quad \nabla_u = (\partial_{\rho_1}, \cdots, \partial_{\rho_1}), \tag{3.26}$$

from which a contact discontinuity arises. If $\nabla_u \lambda \cdot p \neq 0$, then the characteristic field is genuinely nonlinear [15]. By (3.10), we proceed with the following,

$$(-\frac{\partial Q}{\partial \lambda_k}) \nabla \lambda_k \cdot p_k = (-\frac{\partial Q}{\partial \lambda_k}) \sum_{i=1}^{m} \frac{\partial \lambda_k}{\partial \rho_i} K_i = \sum_{i=1}^{m} \frac{\partial Q}{\partial \rho_i} K_i = \sum_{i=1}^{m} \frac{\rho_i (v_i')^2}{(v_i - \lambda_k)^2} - \sum_{i=1}^{m} \frac{\rho_i v_i''}{v_i - \lambda_k}. \tag{3.27}$$

Above, by Lemma 3.2 note that $-\partial Q/\partial \lambda_k > 0$.

Several comments are necessary on the nonlinearity of the characteristic fields. We first have two conclusions as follows.

1. If all $v_i(\rho)$ are linear functions such that $v_i''(\rho) = 0$, then $\nabla_u \lambda_k \cdot p_k > 0$, and thus all fields λ_k are nonlinear globally in D.
2. If $v_i''(\rho) < 0$, which includes those given by (3.11), then $\nabla_u \lambda_1 \cdot p_1 > 0$, and thus λ_1 field is nonlinear globally in D.

Secondly, whether $\nabla_u \lambda_k \cdot p_k > 0$ in D and for all k is generally unclear, subject to the velocity fields. This investigation is also difficult mainly due to the implicit $\lambda_k(u)$. By (3.27), however, it seems that $\nabla_u \lambda_k \cdot p_k \leq 0$ is unlikely in D for most cases. Even that is true, the set $\{u \mid \nabla_u \lambda_k \cdot p_k = 0\}$ is only $m-1$ dimensional, and not so "large" in D. In this sense, we can say that all fields λ_k are essentially nonlinear.

Finally, for the above two statements, λ_1 generates the main wave to connect the two initial states u_l and u_r given by (3.1). As observed in numerical simulations, this means that 1-wave is always sharp for large $u_r - u_l$, whereas other waves are very thin in connecting, so that $\nabla_u \lambda_k \cdot p_k = 0$ $(k \geq 2)$ might be avoided in the k-wave. Also see Section 4.2, where these 'standard' compressive shock and expansive rarefaction waves are always observed.

These comments are merely heuristic and might be conducive to future studies.

However, a contact discontinuity is possible if the assumptions are released such that some $\lambda = v_m(\rho)$ is an eigenvalue in some sub-domain say $D_c \subseteq D \cup \{u \mid \Sigma_{i=1}^m \rho_i = \rho_{max}\}$. This is to recall the discussions in Section 2 (Lemma 2.1, Theorem 2.3 and their proofs). We assume that

$$v_m(\rho) = v_{m-1}(\rho) = \cdots = v_{m-k}(\rho), \quad \rho \in [\rho_c, \rho_{max}], \tag{3.28}$$

where $v_m(\rho)$ is not necessarily the largest velocity and not identical to any other velocity in the same interval. This corresponds to the k multiple eigenvalue,

$$\lambda_m = \cdots = \lambda_{m-k+1} = v_m, \quad u \in D_c.$$

The k linearly independent eigenvectors that are given by (2.10) are redenoted as

$$p_{m_1} = (0, \cdots, 0, -1, 1, 0, \cdots, 0)^T, \ldots, p_{m_k} = (0, \cdots, 0, -1, 0, 0, \cdots, 1)^T.$$

For $u \in D_c$, (3.26) is obviously satisfied by all eigen-pairs (λ_m, p_{m_j}), i.e.,

$$\nabla_u \lambda_m \cdot p_{m_j} \equiv 0, \quad u \in D_c, \quad j = 1, \ldots, k.$$

We show that the wave arising from λ_m is actually a contact discontinuity.

Note that system (2.1) is strongly hyperbolic on D_c: namely, m linearly independent eigenvectors are guaranteed. Hence, by generalization, $m-k$ Riemann invariants of the wave (casually named the m_k-wave here) are defined such that

$$l_i \cdot du = 0, \quad i = 1, \ldots, m-k.$$

Here l_i are left eigenvectors of λ_i; and by scaling $\{l_i\}_{i=1}^m$ are bi-orthonormal to the set of right eigenvectors $\{p_i\}_{i=1}^m$, which implies that

$$l_i \cdot p_{m_j} = 0, \quad i = 1, \cdots, m-k, \; j = 1, \ldots, k.$$

Therefore, we have

$$l_i \cdot (du - p_{m_j}) = 0, \quad i = 1, \cdots, m-k, \; j = 1, \ldots, k. \tag{3.29}$$

Clearly, (3.29) suggests that $du \,||\, S$, where $S = \{\alpha_1 p_{m_1} + \cdots + \alpha_k p_{m_k} \mid \forall \alpha_i, \; i = 1, \ldots, k\}$, and $\alpha_1 p_{m_1} + \cdots + \alpha_k p_{m_k} = (0, \cdots, 0, -\Sigma_{j=1}^k \alpha_j, \alpha_1, \cdots, \alpha_k)$. That is,

$$\frac{d\rho_1}{0} = \cdots = \frac{d\rho_{m-k-1}}{0} = \frac{d\rho_{m-k}}{-\Sigma_{j=1}^k \alpha_j} = \frac{d\rho_{m-k+1}}{\alpha_1} = \cdots = \frac{d\rho_m}{\alpha_k},$$

by which these $m-k$ Riemann invariants of the m_k-wave are clearly $\rho_1, \ldots, \rho_{m-k-1}$ and $\Sigma_{j=m-k}^m \rho_j$. As a result, $\rho = \Sigma_{j=m}^m \rho_j$ is also a Riemann invariant, as are all $v_i(\rho)$ and $\lambda_m(u) = v_i(\rho)$. This clearly indicates that the m_k-wave is a contact discontinity. Across the wave we have

$$\rho_1^- = \rho_1^+, \ldots, \rho_{m-k-1}^- = \rho_{m-k-1}^+, \; \Sigma_{j=m-k}^m \rho_j^- = \Sigma_{j=m-k}^m \rho_j^+,$$

$$\rho^- = \rho^+, \; v_i^- = v_i^+, \; \forall i, \quad \lambda_m^- = \lambda_m^+.$$

The Rankine-Hugoniot conditions of (3.2) can be easily verified.

The interpretation of such a contact is similar to what is discussed at the end of Section 2.2. Let $k = m-1$, for example, then (3.28) means that all flow media are the same in their velocity as the total density is larger than some critical value ρ_c. Moreover, it is easy to see that the contact is formed if and only if, initially, $u_l = u^- \in D_c$ or $u_r = u^+ \in D_c$. Suppose that $u_r = u^+ \in D_c$, for example, the contact is needed to separate the original state $u_r = u^+$ from the state u^- formed behind, such that generally $\rho_i^+ \neq \rho_i^-$ but $\rho^- = \rho^+$. Note that we only have two waves by the assumption. Also compare this argument with Fig.10 in Section 4.2.

4 Numerical Approximations

To confirm the conclusions that are clear (or unclear) in the previous discussions, numerical implementation is made. High resolution schemes are adopted. However, here the main difficulty is the characteristic decomposition: e.g., see [17] and [18]. Analytically it is impossible for $m > 2$ because in this case all $\lambda_i(u)$ are implicit. Though it could be made for $m = 2$, the discussion must be separated at each boundary $\rho_i = 0$; thus the handing will be very complex. See the comments in Section 2.2 and around Theorem 2.4.

Due to this difficulty, Lax-Friedrichs flux-splitting is applied as an alternative. Because this compromise involves considerable numerical viscosity, the fifth order accurate WENO (Weighted Essentially Non-Oscillatory) reconstruction is used, coupled with the third order accurate TVD Runge-Kutta time discretization. The scheme is only briefly described. For a detailed account of the ENO and WENO methods, see [19, 20, 21].

4.1 Flux-splitting WENO schemes

The solution procedure consists of three major blocks: the flux-splitting, the WENO reconstruction, and the TVD Runge-Kutta time discretization. For the first step, the flux vector $f(u)$ of (2.1) is split into two parts as follows:

$$f(u) = f^+(u) + f^-(u), \quad f^+(u) = \frac{1}{2}(f(u) + \alpha u), \quad f^-(u) = \frac{1}{2}(f(u) - \alpha u).$$

In the above, the constant α is given by

$$\alpha_0 = \max_u \max_{1 \le j \le m} |\lambda_j(u)|,$$

or greater, where the first maximum is taken over all u involved. In each of our examples, α is well evaluated by (2.8), such that $\alpha - \alpha_0$ is small enough. By such α, $f^+(u)$ and $f^-(u)$ are guaranteed to be non-negative and non-positive, respectively, i.e.,

$$f_u^+(u) = f_u(u) + \alpha I \ge 0, \quad f_u^-(u) = f_u(u) - \alpha I \le 0.$$

For the discretization of (2.1), $(f^+(u))_x$ and $(f^-(u))_x$ are approximated by numerical fluxes separately, such that

$$\frac{du_i}{dt} + \frac{1}{\Delta x}(\hat{f}^+_{i+1/2} - \hat{f}^+_{i-1/2}) + \frac{1}{\Delta x}(\hat{f}^-_{i+1/2} - \hat{f}^-_{i-1/2}) = 0. \tag{4.1}$$

These fluxes $\hat{f}^{\pm}_{i+1/2}$ are acquired by the WENO reconstruction below.

Given all of the discrete values v_j of a function $v(x)$ in I_j, we denote by $v_{i\pm1/2}$ the approximate boundary values of $v(x)$ on a fixed cell I_i. Then, $v_{i\pm1/2}$ are obtained by applying v_i and its neighboring $r+s$ point values. Here, r is the number of the cells in the left side of I_i, and $k = r+s+1$ cells are thus involved. Moreover, we have k approximations of $v(x_{i\pm1/2})$, denoted by

$$v^{(r)}_{i+1/2} = \sum_{j=0}^{k-1} c_{rj} v_{i-r+j}, \quad v^{(r)}_{i-1/2} = \sum_{j=0}^{k-1} \tilde{c}_{rj} v_{i-r+j}, \quad r = 0, \dots, m-1. \tag{4.2}$$

In (4.2), $\tilde{c}_{rj} = c_{r-1,j}$; and for $k=3$, c_{rj} are given by

$$c_{-1,0} = 11/6,\ c_{-1,1} = -7/6,\ c_{-1,2} = 1/3; \quad c_{00} = 1/3,\ c_{01} = 5/6,\ c_{02} = -1/6;$$

$$c_{10} = -1/6, \quad c_{11} = 5/6, \quad c_{12} = 1/3; \quad c_{20} = 1/3, \quad c_{21} = -7/6, \quad c_{22} = 11/6.$$

For certainty, $v_{i\pm1/2}$ are made to be the following weighted averages of (4.2):

$$v^-_{i+1/2} = \sum_{r=0}^{k-1} \omega_r v^{(r)}_{i+1/2}, \quad v^+_{i-1/2} = \sum_{r=0}^{k-1} \tilde{\omega}_r v^{(r)}_{i-1/2}, \tag{4.3}$$

with all weights given by

$$\omega_r = \frac{\alpha_r}{\sum_{r=0}^{k-1} \alpha_s}, \quad \alpha_r = \frac{d_r}{(\varepsilon + \beta_r)^2}; \quad \tilde{\omega} = \frac{\tilde{\alpha}_r}{\sum_{r=0}^{k-1} \tilde{\alpha}_s}, \quad \tilde{\alpha}_r = \frac{\tilde{d}_r}{(\varepsilon + \beta_r)^2}, \quad \tilde{d}_r = d_{k-1-r}.$$

In the above, $\varepsilon = 10^{-12}$; and for our application of $k = 3$, these constants are $d_0 = 0.3$, $d_1 = 0.6$, $d_2 = 0.1$, and

$$\beta_0 = \frac{13}{12}(v_{j-2} - 2v_{j-1} + v_j)^2 + \frac{1}{4}(v_{j-2} - 4v_{j-1} + 3v_j)^2;$$

$$\beta_1 = \frac{13}{12}(v_{j-1} - 2v_j + v_{j+1})^2 + \frac{1}{4}(v_{j-1} - v_j)^2;$$

$$\beta_2 = \frac{13}{12}(v_j - 2v_{j+1} + v_{j+2})^2 + \frac{1}{4}(3v_j - 4v_{j+1} + v_{j+2})^2.$$

Now, the $\hat{f}^{\pm}_{i+1/2}$ that are applied in scheme (4.1) are determined by the following procedure:

1. let $v_i = f_i^+$ and compute $v^-_{i+1/2}$ by (4.3), then set $f^+_{i+1/2} = v^-_{i+1/2}$; and
2. let $v_i = f_i^-$ and compute $v^+_{i+1/2}$ by (4.3), then set $f^-_{i+1/2} = v^+_{i+1/2}$.

The reconstruction guarantees the $(2k-1)$-th order accuracy.

Finally, for the TVD-Runge-Kutta time discretization, scheme (4.1) can be rewritten as the following ODEs:

$$u_t = L(u).$$

Given the initial values $u(x, 0) \equiv u_0(x)$ and the division $\{u^n\}_{n=0}^N$ in time direction, then for $n = 0, \ldots, N$, we follow the steps below.

(1) Set $u^{(0)} = u^n$.

(2) For $j = 1, \ldots, K$, compute the values of intermediate functions as

$$u^{(j)} = \sum_{l=0}^{j-1}(\alpha_{jl}u^{(l)} + \Delta t \beta_{jl} L(u^{(l)})).$$

(3) Set $u^{n+1} = u^{(K)}$.

This approximation achieves the K-th order accuracy in time direction. For our application of $K = 3$, the coefficients are

$$\alpha_{10} = 1,\ \alpha_{20} = 3/4,\ \alpha_{21} = 1/4,\ \alpha_{30} = 1/3,\ \alpha_{31} = 0,\ \alpha_{32} = 2/3;$$

$$\beta_{10} = 1,\ \beta_{20} = 0,\ \beta_{21} = 1/4,\ \beta_{30} = 0,\ \beta_{31} = 0,\ \beta_{32} = 2/3.$$

4.2 Resolution of waves in the Riemann Problem

To compare with the analytical results, numerical examples are given by the Riemann problem (3.1), with the change of the interface from $x = 0$ to some $x = x_0$. To capture these compressive shocks and expansive rarefactions, moreover, the velocity fields of (3.11) are applied with $\rho_{max} = 1$, $m = 3$, $(b_1, b_2, b_3) = (0.6, 0.8, 1)$, and

$$v(\rho) = 1 - \rho^\mu, \tag{4.4}$$

except that $v''(\rho) > 0$ for $\mu < 1$. The computational range is $(0, 1)$, with division $\Delta x = 1/800$; the temporal increment $\Delta t = 0.6\Delta x/\alpha$. In all examples, the types of all m waves are also identified in comparison with analytical results in Section 3.

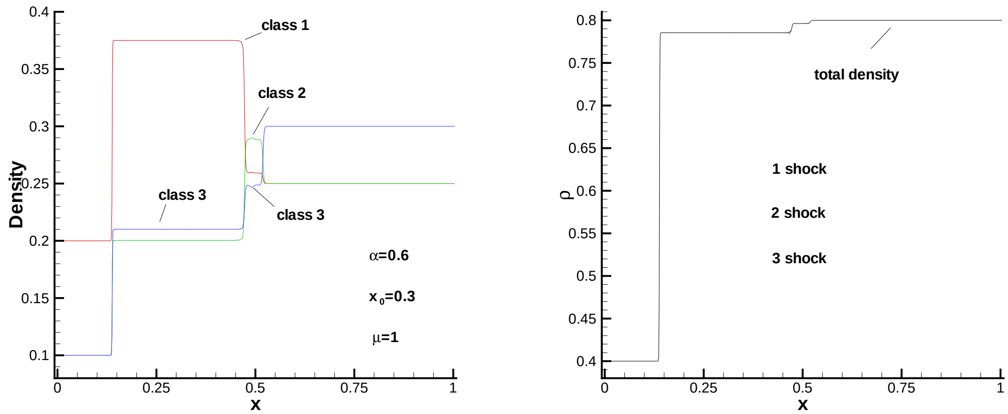

Fig.5 Changes in all densities at $t = 1.2$, with $u_l = (0.2, 0.1, 0.1)$, and $u_r = (0.25, 0.25, 0.3)$.

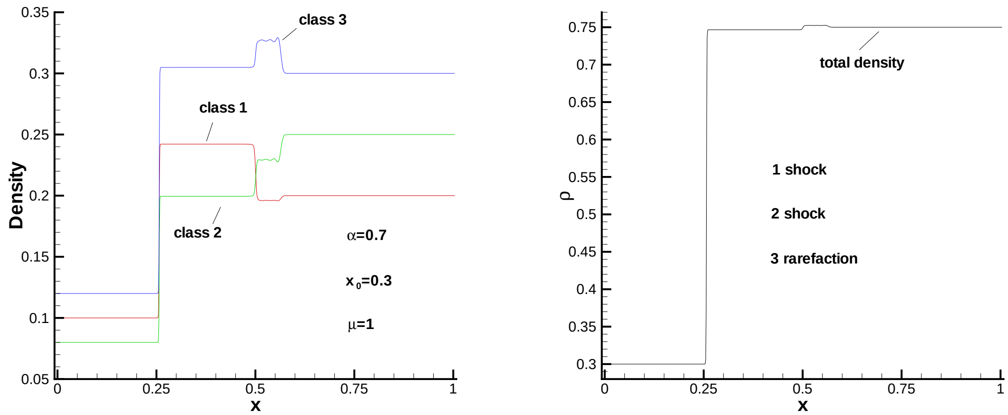

Fig.6 Changes in all densities at $t = 1.2$, with $u_l = (0.1, 0.08, 0.12)$, and $u_r = (0.2, 0.25, 0.3)$.

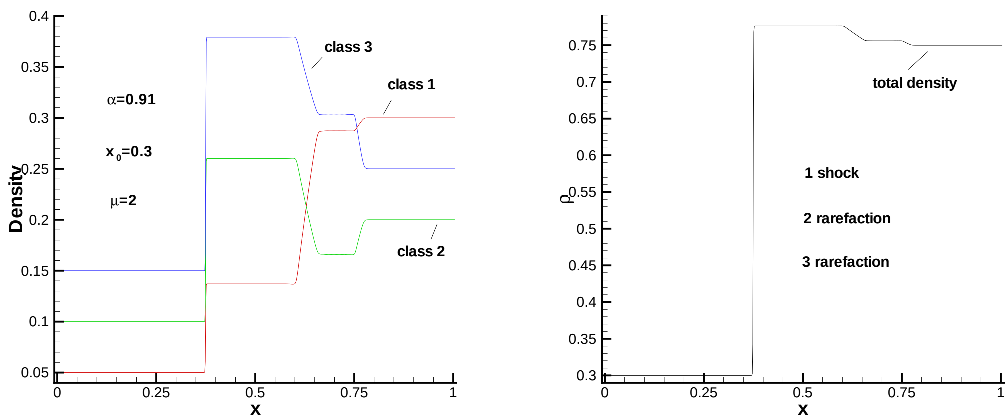

Fig.7 Changes in all densities at $t = 1.2$, with $u_l = (0.05, 0.1, 0.15)$, $u_r = (0.3, 0.2, 0.25)$.

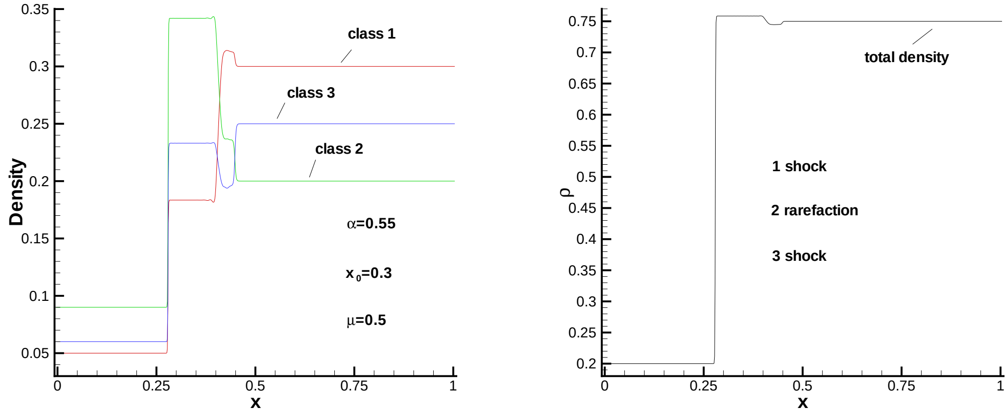

Fig.8 Changes in all densities at $t = 1.2$, with $u_l = (0.05, 0.09, 0.06)$, and $u_r = (0.3, 0.2, 0.25)$.

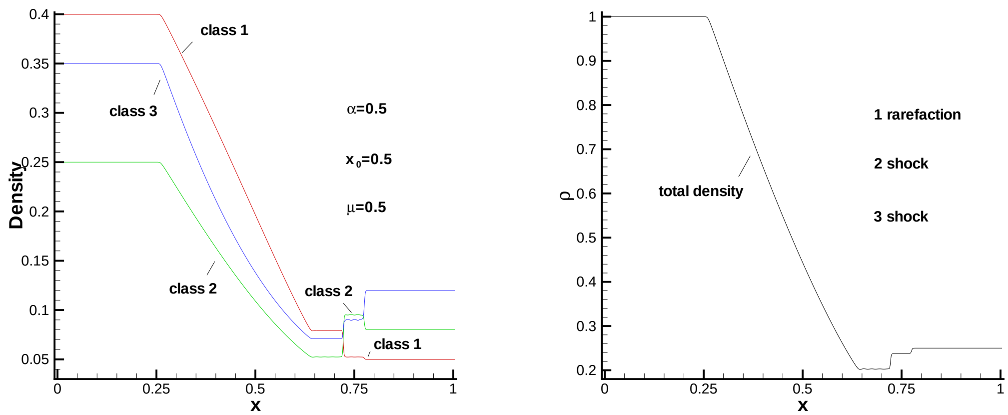

Fig.9 Changes in all densities at $t = 0.6$, with $u_l = (0.4, 0.25, 0.35)$, and $u_r = (0.05, 0.08, 0.12)$.

In Figs.5-9, the density of all different classes are shown on the left side; whereas the total density is shown on the right side. All waves follow the descriptions given by Theorem 3.2.(i) or Theorem 3.4.(i); i.e., all shocks are compressive and all rarefactions are expansive. Note that these are also true even with $\mu < 1$ of (4.4) (Figs.8-9), which means the convexity $v_i''(\rho) > 0$, $\forall i$.

To simulate a contact, (4.4) is altered to be

$$v_i(\rho) = \begin{cases} v(\rho), & \text{if} \quad \rho > \rho_c, \\ b_i + \rho(1 - b_i - \rho_c)/\rho_c, & \text{otherwise}, \end{cases} \quad i = 1, 2, 3; \tag{4.5}$$

see the discussion in Section 3.3. The numerical result is shown in Fig.10. Note that the initial state $u_r = (0.25, 0.25, 0.3)$ is in the region of D_c. On the left side, the number of the waves reduces to two, because $\lambda_2 = \lambda_3$ for $u \in D_c$. On the right side, the second wave disappears because the total density is a Riemann invariant for this contact. See our comments in the end of Section 3.3.

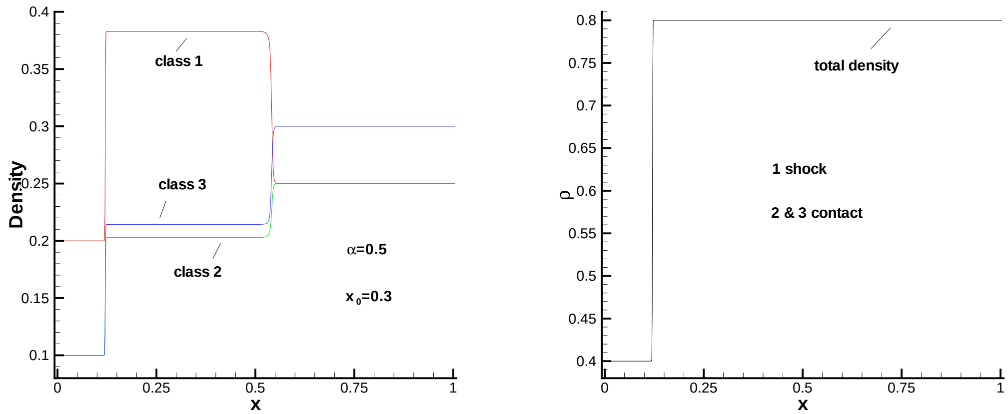

Fig.10 Changes in all densities at $t = 1.2$, with $\rho_c = 0.5$, $u_l = (0.2, 0.1, 0.1)$, $u_r = (0.25, 0.25, 0.3)$.

5 Conclusions

We have proven that system (2.1) is hyperbolic, because it has m real eigenvalues. More precisely, it is non-strictly hyperbolic somewhere at a boundary $\rho_i = 0$, and it is only $(m-2)$ dimensional. Moreover, it is strongly hyperbolic at the intersections by at least two velocity curves, and it is just $m-1$ dimensional. The system is strictly hyperbolic in other solution regions at large.

Some important properties of the characteristic fields are also discussed. In particular, it is confirmed that the last $(m-1)$ fields are due to overtaking. Precisely, the k-field is due to the $(k-1)$-th flow being overtaken by the k-th, provided that $\rho_i \neq 0$, $\forall i$.

All waves are investigated. In general, a characteristic field is essentially genuinely nonlinear, and the involved wave is characterized by the so called compressive shock or expansive rarefaction. That is, the density of each class increases after it passes through a shock but decreases after entering a rarefaction fan. A contact is formed such that the propagation of a certain family of characteristics coincides with at least two flows. For this case all Riemann invariants of the wave are easily acquired and the meanings in the motion are clearly understood.

All of the main results are verified numerically. All of the examples adopt the Lax-Friedrichs flux-splitting WENO scheme. Their declared compressive shocks and expansive rarefactions are always observed, along with the described contact.

In summary, this paper presents a comprehensive study and provides many important results for the discussed system. These are of much significance for studies of hyperbolic equations, and probably also for applications.

What is left unclear is the existence of global solution of the Riemann problem. A strict proof for this will be more challenging, based on the conclusions in this paper, and subject to assumptions on the velocity fields. Besides, system 2.1 can be further extended say through combination with those factors considered in [4, 5, 6], so that more interesting and complicated waves could be presented.

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References

[1] M. J. Lighthill and G. B. Whitham. On kinematic waves: II. a theory of traffic flow on long crowded roads. *Proc. Roy. Soc.London, Ser.*, A 229:317–345, 1955.

[2] P. I. Richards. Shock waves on the highway. *Operations Research*, 4:42–51, 1956.

[3] G. B. Whitham. Linear and nonlinear waves. *Springer-Verlay*, 1974.

[4] P. Zhang and R. X. Liu. Hyperbolic conservation laws with space-dependent flux: I. Characteristics theory and riemann problem. *J. Comput. Appl. Math.*, 156:1–21, 2003.

[5] P. Zhang and R. X. Liu. Hyperbolic conservation laws with space-dependent flux: II. General study of numerical fluxes. *J. Comput. Appl. Math.*, 176:105–129, 2005.

[6] P. Zhang and R. X. Liu. Generalization of Runge-Kutta discontinuous Galerkin method to LWR traffic flow model with inhomogeneous road conditions. *Numer. Meth. Partial Diff. Equ.*, 21(1):80–88, 2005.

[7] G. C. K. Wong and S. C. Wong. A multi-class traffic flow model-an extension of lwr model with heterogeneous drivers. *Transpn. Res. A*, 36:827–841, 2002.

[8] P. Bagnerini and M. Rascle. A multiclass homogenized hyperbolic model of traffic flow. *SIAM Journal on Mathematical Analysis*, 35:949–937, 2003.

[9] S. Benzoni-Gavage and R. M. Colombo. An n-populations model for traffic flow. *European Journal of Applied Mathematics*, 14:587–612, 2003.

[10] A. Klar J. M. Greenberg and M. Rascle. Congestion on multi-lane highways. *SIAM Journal on Applied Mathematics*, 63:818–833, 2003.

[11] M. P. Zhang, C.-W. Shu, C. K. Wong and S. C. Wong. A weighted essentially non-oscillatory numerical scheme for a multi-class Lighthill-Whitham-Richards traffic flow model. *J. Comput. Phys.*, 191:639–659, 2003.

[12] P. D. Lax. Hyperbolic systems of conservation Laws II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[13] P. D. Lax. Shock waves and entropy. *Contributions to Nonlinear Functional Anal. Proc. Symposium at the University of Wisconsin*, page 603, 1971.

[14] P. D. Lax. Hyperbolic systems of conservation laws and the mathematical theory of shock waves. *SIAM, Philadelphia*, 1973.

[15] I. Smoller. Shock waves and reaction-diffusion equations. *Springer-Verlag, New York*, 1994.

[16] E. F. Toro. Riemann solvers and numerical methods for fluid dynamics. *NY: John Wiley and Sons*, 1999.

[17] A. Harten, P. D. Lax and B. Van Leer. On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. *SIAM Review*, 25(1):35–61, 1983.

[18] P. L. Roe. Approximate Riemann solvers, parameter vectors, and difference schemes. *J. Comp. Phys.*, 43:357–372, 1981.

[19] A. Harten, B. Engquist, S. Osher and S. Chakravarthy. Uniformly high order essentially non-oscillatory schemes III. *J. Comput. Phys.*, 71:231–303, 1987.

[20] S. Osher X.-D. Liu and T. Chan. Weighted essentially nonoscillatory schemes. *J. Comput. Phys.*, 115:200–212, 1994.

[21] C.-W. Shu. Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws. Technical Report ICASE Report No. 1997-65.