On A Lemma Of Diperna And Chen

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Abstract

This note gives an explicit lower bound of the function $\varphi(t)$ in the lemma of DiPerna and Chen, and some remarks.

1 Lemma of DiPerna and Chen

R.DiPerna in [5] firstly studied the global existence theory of the Cauchy problem of isentropic gas dynamics systems

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (\dfrac{m^2}{\rho} + p(\rho))_x = 0, \\ \rho(x,0) = \rho_0(x),\ m(x,0) = m_0(x),\ \rho_0(x) \geq 0, \end{cases} \tag{1}$$

where $p(\rho) = \frac{\rho^\gamma}{\gamma},\ \gamma = 1 + \frac{2}{2n+1},\ n \geq 2$ is an integer. But there is a gap in his proof of the global existence theorem of the corresponding Cauchy problem of the viscosity systems

$$\begin{cases} \rho_t + m_x = \varepsilon\rho_{xx}, \\ m_t + (\dfrac{m^2}{\rho} + p(\rho))_x = \varepsilon m_{xx}, \\ \rho(x,0) = \rho_0(x),\ m(x,0) = m_0(x),\ \rho_0(x) \geq \delta. \end{cases} \tag{2}$$

In order to get a lower bound of the density, he built his Lemma 4.1 in an incorrect form. Chen [2] again formed it in a corrected form as follows:

Lemma of DiPerna-Chen: If $\varphi(t)$ is a nonnegative measurable function satisfying

$$\varphi(t) - \varphi(s) \geq -c_1(t-s)^{\frac{1}{2}},\ c_1 > 0,\ \text{for } s < t < T < +\infty, \tag{3}$$

$$\int_0^T \varphi^{-\alpha}(s)ds \leq c_2,\ c_2 > 0 \text{ for some } \alpha \in [2, \infty), \tag{4}$$

and in addition

$$\varphi(0) \geq \delta > 0, \tag{5}$$

then there exists a positive constants $c_3 = c_3(c_1, c_2, T, \alpha, \delta)$ such that

$$\varphi(t) \geq c_3 > 0,\ \forall\ t \in [0, T]. \tag{6}$$

By this lemma, Chen got an implicit lower bound of the density of the viscosity systems (2). But if we want to get a global solution from the continuation of a local solution of (2), it is useful to have an explicit lower bound of $\rho(x,t)$ in (2). This needs to get an explicit c_3 of the above lemma. We will do it in this note. By the way, there have been some works on the explicit lower bound of $\rho(x,t)$. For example in [11], Lu got such an estimate when $\gamma \in [1,2)$ by standard maximum principle method of parabolic equation.

2 The explicit lower bound

Now we will get an explicit c_3 of the Lemma of DiPerna-Chen. For simplicity, we only treat the special case $\alpha = 3$, and this is enough for the global existence theory. Without loss of generality, we may suppose $\delta \leq 1$. Then we have the following lemma.

Lemma: If $\varphi(t)$ is a nonnegative measurable function satisfying

$$\varphi(t) - \varphi(s) \geq -c_1(t-s)^{\frac{1}{2}},\ c_1 > 0,\ \text{for } s < t < T < +\infty, \tag{3}$$

$$\int_0^T \varphi^{-3}(s)ds \leq c_2,\ c_2 > 0, \tag{4'}$$

$$\varphi(0) \geq \delta > 0, \tag{5}$$

then the following holds

$$\varphi(t) \geq \min\left(\frac{\delta}{2},\ \frac{\delta^2}{8c_1^2c_2(1+c_1^2t)}\right) \text{ for } t \in [0,T]. \tag{7}$$

Proof: From (3), we have

$$\varphi(s) \leq \varphi(t) + c_1(t-s)^{\frac{1}{2}},$$

which, together with (4'), implies

$$I = \int_0^t \frac{ds}{[\varphi(t) + c_1(t-s)^{\frac{1}{2}}]^3} \leq c_2.$$

Put $(t-s)^{\frac{1}{2}} = x$, $\varphi(t) = k$, we have

$$\begin{aligned} I &= 2\int_0^{t^{\frac{1}{2}}} \frac{xdx}{(k+c_1x)^3} = \frac{2}{c_1}\left[\int_0^{t^{\frac{1}{2}}} \frac{1}{(c_1x+k)^2}dx - k\int_0^{t^{\frac{1}{2}}} \frac{dx}{(c_1x+k)^3}\right] \\ &= \frac{2}{c_1^2}\left[\frac{1}{k} - \frac{1}{c_1t^{\frac{1}{2}}+k} - \frac{1}{2k} + \frac{k}{2(c_1t^{\frac{1}{2}}+k)^2}\right] = \frac{1}{c_1^2}\left[\frac{1}{\sqrt{k}} - \frac{\sqrt{k}}{c_1t^{\frac{1}{2}}+k}\right]^2. \end{aligned}$$

Hence we have

$$\frac{1}{\sqrt{k}} - \frac{\sqrt{k}}{c_1t^{\frac{1}{2}}+k} \leq c_1\sqrt{c_2} = a,$$

or

$$(\sqrt{k})^3 + c_1t^{\frac{1}{2}}\sqrt{k} - \frac{c_1t^{\frac{1}{2}}}{a} \geq 0.$$

If $k > 1$, then $k \geq \frac{\delta}{2}$. If $k \leq 1$, then

$$\sqrt{k} \geq \frac{c_1t^{\frac{1}{2}}}{a(1+c_1t^{\frac{1}{2}})}.$$

This means

$$k \geq \frac{c_1^2t}{2a^2(1+c_1^2t)} \geq \frac{t}{2c_2(1+c_1^2t)}. \tag{8}$$

On the other hand, from (3), we have

$$\varphi(t) - \varphi(0) \geq -c_1t^{\frac{1}{2}}.$$

Taking a positive constant t_1 such that

$$\delta - c_1 t_1^{\frac{1}{2}} = \frac{\delta}{2},$$

i.e. $t_1 = \frac{\delta^2}{4c_1^2}$. Then if $t \le t_1$, we have $\varphi(t) \ge \frac{\delta}{2}$. If $t > t_1$, from (8), we obtain

$$\varphi(t) > \frac{t_1}{2c_2(1+c_1^2 t)}.$$

Lastly, we get

$$\varphi(t) \ge \min\{\frac{\delta}{2}, \ \frac{t_1}{2c_2(1+c_1^2 t)}\} = \min\{\frac{\delta}{2}, \ \frac{\delta^2}{8c_1^2 c_2(1+c_1^2 t)}\}.$$

3 Remarks

Remark 1. In [4] we ever treated a nonhomogeneous viscosity p-system. We got the global existence theorem by successive continuation from the local existence. In that case the non-zero lower bound of the density is easy to get. With the aid of the explicit lower bound above (Lemma in the above section) we can treat the nonhomogeneous Euler system of gas dynamics with artificial viscosity and general $p(\rho)$,

$$\begin{cases} \rho_t + m_x = \varepsilon \rho_{xx}, \\ m_t + (\dfrac{m^2}{\rho} + p(\rho))_x = \varepsilon m_{xx} + \alpha m, \ \alpha \le 0, \\ \rho(x,0) = \rho_0(x) \ge \delta, \ m(x,0) = m_0(x), \\ \lim\limits_{x \to \pm\infty} \rho_0(x) = \bar{\rho}, \quad \lim\limits_{x \to \pm\infty} m_0(x) = \bar{m} = 0. \end{cases}$$

If $\bar{m} \ne 0$, we can take $\tilde{m}_0(x) = 1_{(-N,N)}(x) m_0(x) \star G^\delta(x)$ instead $m_0(x)$, where

$$1_{(-N,N)}(x) = \begin{cases} 1, \ |x| \le N, \\ 0, \ |x| > N, \end{cases}$$

$G^\delta(x)$ is a modifier.

Remark 2. For the case without viscosity, when the initial data is away from the vacuum, the global existence of BV solution with large BV initial data was first obtained by Nishida [11] by Glimm scheme for $\gamma = 1$ in 1968. When a vacuum occurs, the global existence of L^∞ entropy solution for the isentropic flow ($\gamma > 1$) with large L^∞ initial data was solved by DiPerna [5] for $\gamma = 1 + \frac{2}{2n+1}, n \geq 2$ integer, Ding, Chen, Luo [3] for $1 < \gamma \leq \frac{5}{3}$ and Lions, Perthame, Tadmor and Souganidis [8], [9] for $\gamma > \frac{5}{3}$. The existence problem for $\gamma = 1$ when the L^∞ initial data admits vacuum was solved by Huang and Wang [6] in 2002 by introducing complex entropies.

Later in 2005, LeFloch and Shelukhin [7] claimed that they obtained a similar result for $\gamma = 1$ by a different approach. However, it seems that their proof is not complete. The reason is in the following. In Lemma 3.5 of [7] which is an essential step for [7], the authors tried to show that for any given entropy-entropy flux (η, q), the sequence

$$\begin{aligned} \theta^\varepsilon = \ & \eta_t^\varepsilon + q_x^\varepsilon = \varepsilon \eta_{xx}^\varepsilon - \varepsilon[\eta_{\rho\rho}^\varepsilon \rho_x^2 + \eta_{mm}^\varepsilon m_x^2 + 2\eta_{\rho m}^\varepsilon \rho_x m_x] \\ & + \varepsilon_1 u_x (q_m^\varepsilon + \eta_\rho^\varepsilon) - 2\varepsilon_1 \frac{\rho_x \eta_m^\varepsilon}{\rho} \end{aligned} \tag{9}$$

is compact in $W_{loc}^{-1,2}(R_+^2)$. The authors proved Lemma 3.5 by Lemma 3.3,

$$\|\frac{\varepsilon \rho_x^2}{\rho} + \varepsilon \rho u_x^2\|_{L_{loc}^1} \leq C \quad \text{uniformly in } \varepsilon \tag{10}$$

and $\rho \geq 2\varepsilon_1$. To prove Lemma 3.5, they claimed that each term of $\theta^\varepsilon - \varepsilon \eta_{xx}^\varepsilon$ is bounded in L_{loc}^1 provided $\varepsilon_1 = \varepsilon$. For instance, they claimed

$$2\varepsilon_1 |u_x| = \frac{2\varepsilon_1 \rho^{\frac{1}{2}} |u_x|}{\rho^{\frac{1}{2}}} \leq \sqrt{2\varepsilon} \rho^{\frac{1}{2}} |u_x|. \tag{11}$$

However if $q_m^\varepsilon, \eta_\rho^\varepsilon$ has singularity near vacuum, the estimate (11) can not imply $\varepsilon_1 u_x (q_m^\varepsilon + \eta_\rho^\varepsilon)$ is bounded in L_{loc}^1 from (10) provided $\varepsilon_1 = \varepsilon$. Unfortunately the derivatives of entropy usually have singularities near vacuum. They treated the key term

$$\varepsilon[\eta_{\rho\rho}^\varepsilon \rho_x^2 + \eta_{mm}^\varepsilon m_x^2 + 2\eta_{\rho m}^\varepsilon \rho_x m_x] \tag{12}$$

in a similar incomplete way. In fact, it is not clear whether the term (12) is L_{loc}^1 bounded or not even for the special weak entropy η constructed by them (see Theorem 4.4 in [7]).

Remark 3. People may consider the Lax-Friedrichs difference scheme for a system of conservation laws

$$\begin{cases} u_t + f(u)_x = g(u), \\ u(x,0) = u_0(x). \end{cases} \tag{13}$$

This is

$$\begin{cases} \dfrac{u_n^{k+1} - \frac{u_{n-1}^k + u_{n+1}^k}{2}}{h} + \dfrac{f(u_{n+1}^k) - f(u_{n-1}^k)}{2l} = g(u_{n+1}^k), \\ u_n^0 = u_0(nl). \end{cases} \tag{14}$$

In many cases if we take $h/l = c$, the solutions of (14) approaches to the generalized solutions of (13) (see [1], [3]). But if $l^2/h \to 2\varepsilon > 0$, Oleinik in "Discontinuous solutions of nonlinear differential equations, USP. Mat. Nauk (N.S), 12, (1957), 3-73" proved that for a scalar equation it approaches to the generalized solutions of the parabolic equation

$$\begin{cases} u_t + f(u)_x = g(u) + \varepsilon u_{xx}, \\ u(x,0) = u_0(x). \end{cases} \tag{15}$$

Is it true for systems of conservation laws (13)? Especially is it true for the systems (1)?

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