A NOTE ON THE SHOCK WAVE FORMATION PROCESS FOR SCALAR CONSERVATION LAW

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Abstract. In the paper we consider the same problem as in [4]. We propose simpler procedure for describing the shock wave formation process in the case of scalar conservation law. We cleared up most of the questions appearing when using the weak asymptotic method. The procedure proposed here can be generalized on examining the problem of singularity formation for a class of system of conservation law considered in [6].

1. Introduction

In the paper, we describe the shock wave formation process for the equation

$$u_t + (f(u))_x = 0, \quad t \in \mathbf{R}^+, \quad x \in \mathbf{R}, \tag{1}$$

where $f \in C^3(\mathbf{R})$ and $f''(x) > 0$, with the initial condition

$$u|_{t=0} = \hat{u}_0(x), \tag{2}$$

for $\hat{u}_0(x)$ such that (below, u_0^0 and U are constants):

$$\hat{u}_0(x) = \begin{cases} U, \ x < a_2 \\ u_0(x), \ a_2 \le x \le a_1 \\ u_0^0, \ x > a_1 \end{cases}$$

and $f'(u_0(x)) = -Kx + b$, for the constants K and b determined by the equations $f'(u_0^0) = -Ka_1 + b$ and $f'(U) = -Ka_1 + b$.

It is well known that in some moment characteristics of equation (1) originating from the points a_1 and a_2 will intersect (we say a_1 and a_2 interact). We will denote that moment with t^*. The equations of characteristics are:

$$\varphi_{10}(t) = f'(U)t + a_2 \tag{3}$$

$$\varphi_{20}(t) = f'(u_0^0)t + a_1, \tag{4}$$

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and the moment of the interaction is

$$t^* = \frac{a_1 - a_2}{f'(U) - f'(u_0^0)}.$$

From that moment we do not have classical solution any more since the shock wave is formed. From the choice of the initial condition it follows that the formed shock wave will remain unchanged for every $t > t^*$. Therefore, the points a_1 and a_2 continue to move according to Rankine-Hugoniot condition. Our aim is to find smooth trajectories approximating generalized characteristics ([1], p.204.) of problem (1), (2).

To achieve this we will use the weak asymptotic method. We remind on the basic definitions of the method (see [4, 2]).

Definition 1. By $O_{\mathcal{D}'}(\varepsilon^\alpha) \subset \mathcal{D}'(\mathbf{R})$ we denote the family of distributions depending on $\varepsilon \in (0,1)$ and $t \in \mathbf{R}^+$ such that for any test function $\eta(x) \in C_0^1(\mathbf{R})$, the estimate

$$\langle O_{\mathcal{D}'}(\varepsilon^\alpha), \eta(x)\rangle = O(\varepsilon^\alpha)$$

holds, where the estimate on the right-hand side is understood in the usual sense and locally uniform in t, i.e., $|O(\varepsilon^\alpha)| \leq C_T \varepsilon^\alpha$ for $t \in [0,T]$.

Definition 2. The function $u_\varepsilon = u_\varepsilon(x,t)$ is called a weak asymptotic solution of problem (1), (2) if

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\partial f(u_\varepsilon)}{\partial x} = O_{\mathcal{D}'}(\varepsilon), \qquad u_\varepsilon\Big|_{t=0} - u\Big|_{t=0} = O_{\mathcal{D}'}(\varepsilon).$$

We search the weak asymptotic solution of problem (1), (2). We will need the following theorem (proved in [4]).

Theorem 3. *Let $\theta_{i\varepsilon}(x) = \omega_i(x/\varepsilon)$, $i = 1, 2$, where $\lim_{z\to+\infty} \omega_i(z) = 1$, $\lim_{z\to-\infty} \omega_i(z) = 0$ and $\frac{d\omega(z)}{dz} \in \mathcal{S}(\mathbf{R})$ where $\mathcal{S}(\mathbf{R})$ is the space of quickly decreasing functions. For the bounded functions a, b, c depending on $(x,t) \in \mathbf{R}^+ \times \mathbf{R}$ we have*

$$\begin{aligned}(5)\quad & f\left(a + b\theta_{1\varepsilon}(\varphi_1 - x) + c\theta_{2\varepsilon}(\varphi_2 - x)\right) = \\ & f(a) + \theta_{1\varepsilon}(\varphi_1 - x)\left(f(a+b+c)B_1 + f(a+b)B_2 - f(a+c)B_1 - f(a)B_2\right) + \\ & \theta_{2\varepsilon}(\varphi_2 - x)\left(f(a+b+c)B_2 - f(a+b)B_2 + f(a+c)B_1 - f(a)B_1\right) + \mathcal{O}_{\mathcal{D}'}(\varepsilon),\end{aligned}$$

where for $\rho \in \mathbf{R}$ we have

$$B_1(\rho) = \int \dot\omega_1(z)\omega_2(z+\rho)dz \text{ and } B_2(\rho) = \int \dot\omega_2(z)\omega_1(z-\rho)dz, \tag{6}$$

and

$$B_1(\rho) + B_2(\rho) = 1.$$

2. The main result

We introduce the main theorem of the paper. The functions $\theta_{i\varepsilon}$, $i = 1, 2$, appearing there represent weak approximations of the Heaviside function.

Theorem 4. *The weak asymptotic solution of problem (1), (2) we have in the form*

$$\begin{aligned}(7)\quad u_\varepsilon(x,t) = u_0^0 + (u_1(x,t,\varepsilon) - u_0^0)\theta_{1\varepsilon}(\varphi_1(t,\varepsilon) - x) + \\ (U - u_1(x,t,\varepsilon))\theta_{2\varepsilon}(\varphi_2(t,\varepsilon) - x).\end{aligned}$$

The functions $\varphi_i(t,\varepsilon)$ are given by:

$$\varphi_i(t,\varepsilon) = \int_0^t \left[(B_2(\rho) - B_1(\rho))(-Ka_i + b) + cB_1\right] dt + a_i, \quad i = 1,2.$$

where B_1 and B_2 are given by (6), ρ by (12) and c by (18).
The function u_1 is given by (9).

Remark 5. In the sequel we will use the following notations (as usual $x \in \mathbf{R}$, $t \in \mathbf{R}^+$):

$$\begin{aligned}
&u_1 = u_1(x,t,\varepsilon), \quad B_i = B_i(\rho), \quad \varphi_i = \varphi_i(t,\varepsilon),\\
&\theta_{i\varepsilon} = \theta_{i\varepsilon}(\varphi_i - x), \quad \delta_{i\varepsilon} = -\frac{d}{dx}\theta_{i\varepsilon}(\varphi_i - x),\ i = 1,2,\\
&\tau = \frac{f'(U)t + a_2 - f'(u_0^0)t - a_1}{\varepsilon}, \quad \rho = \frac{\varphi_2(t,\varepsilon) - \varphi_1(t,\varepsilon)}{\varepsilon}.
\end{aligned}$$

The function τ is called "fast variable" and, thanks to small parameter ε, it can be considered independent on so called "slow variable" t.

Proof: Substituting assumed solution into (1), using (5) and definition of the weak asymptotic solution we obtain:

$$\begin{aligned}
&\left[\frac{\partial u_1}{\partial t} + B_2 f'(u_1)\frac{\partial u_1}{\partial x} + B_1 f'(U + u_0^0 - u_1)\frac{\partial u_1}{\partial x}\right]\theta_{1\varepsilon} +\\
&\left[-\frac{\partial u_1}{\partial t} - B_2 f'(u_1)\frac{\partial u_1}{\partial x} - B_1 f'(U + u_0^0 - u_1)\frac{\partial u_1}{\partial x}\right]\theta_{2\varepsilon} +\\
&\left((u_1 - u_0^0)\varphi_{1t} - B_2\left(f(u_1) - f(u_0^0)\right) - B_1\left(f(U) - f(U + u_0^0 - u_1)\right)\right)\delta_{1\varepsilon} +\\
&\left((U - u_1)\varphi_{2t} - B_2\left(f(U) - f(u_1)\right) - B_1\left(f(U + u_0^0 - u_1) - f(u_0^0)\right)\right)\delta_{2\varepsilon} = \mathcal{O}_{\mathcal{D}'}(\varepsilon).
\end{aligned}$$

In order to pass to limit when $\varepsilon \to 0$ in the previous expression, the coefficients multiplying $\delta_{i\varepsilon} = -\frac{d}{dx}\theta_{i\varepsilon}$ have to be continuous in x when $\varepsilon \to 0$ for every $t \in \mathbf{R}^+$. Roughly speaking, we have the following situation $f_\varepsilon(x)\delta_\varepsilon(x)$ where δ_ε represents weak approximation of the Dirac distribution $\delta(x)$. If the function $f_\varepsilon \to f \notin C(\{0\})$, $\varepsilon \to 0$, then, letting ε to zero we would have the product $f(x)\delta(x) := f(0)\delta(x)$ and $f(0)$ is not uniquely defined. To overcome this problem we can use an approach from [5]. There, they assign appropriate value at the discontinuity point of a step function. Considering such an object as an element of a measure space one can write $f(x)\delta(x) := f(0)\delta(x)$ (since they assigned needed value at $x = 0$). Still, that approach is useless in describing passage of the solution of our problem from the continuous to discontinuous state.

Therefore, we have to combine coefficients in some way, to obtain the products of the form $f_\varepsilon(x)\delta_\varepsilon(x)$ where $f_\varepsilon \to f \in C(\{0\})$, $\varepsilon \to 0$ for every $t \in \mathbf{R}^+$. In [4] the special ansatz is used to obtain such situation.

For an unknown constant c we add and subtract the term $cB_1\frac{\partial u_1}{\partial x}$ in the coefficient multiplying $(\theta_{1\varepsilon} - \theta_{2\varepsilon})$ and then we rewrite the last expression in the following form:

(8)

$$\left(\frac{\partial u_1}{\partial t}+[(B_2-B_1)f'(u_1)+cB_1]\frac{\partial u_1}{\partial x}\right)(\theta_{1\varepsilon}-\theta_{2\varepsilon})+$$
$$B_1[\frac{d}{dx}\left(f(U+u_0^0-u_1)+f(u_1)-cu_1\right)](\theta_{1\varepsilon}-\theta_{2\varepsilon})+$$
$$\left((u_1-u_0^0)\varphi_{1t}-B_2\left(f(u_1)-f(u_0^0)\right)-B_1\left(f(U)-f(U+u_0^0-u_1)\right)\right)\delta_{1\varepsilon}+$$
$$\left((U-u_1)\varphi_{2t}-B_2\left(f(U)-f(u_1)\right)-B_1\left(f(U+u_0^0-u_1)-f(u_0^0)\right)\right)\delta_{2\varepsilon}=\mathcal{O}_{\mathcal{D}'}(\varepsilon).$$

We put

$$\frac{\partial u_1}{\partial t}+[(B_2-B_1)f'(u_1)+cB_1]\frac{\partial u_1}{\partial x}=0,\quad u_1(x,0,\varepsilon)=u_0(x),\quad x\in[a_2,a_1]. \tag{9}$$

Out of the segment $[a_2,a_1]$ initial function is constant and we define the solution u_1 of problem (9) to be equal to U on the left-hand side of the characteristic emanating from a_2 and to be equal to u_0^0 on the right-hand side of the characteristic emanating from a_1 (see figure 1).

dj1.lpSystem of characteristics for $u_1$fig1

For the functions φ_1 and φ_2 as the characteristics emanating from a_1 and a_2 respectively, we have

$$\varphi_{1t}=(B_2-B_1)(-Ka_1+b)+cB_1, \tag{10}$$

$$\varphi_{2t}=(B_2-B_1)(-Ka_2+b)+cB_1. \tag{11}$$

Since before the interaction $\rho\to-\infty$ (and consequently $B_1=\mathcal{O}(\varepsilon^N)$, $N\in\mathbf{N}$) we see that the expressions for φ_{it}, $i=1,2$, for $t<t^*$ coincides up to $\mathcal{O}(\varepsilon^N)$, $N\in\mathbf{N}$, with the expressions for standard characteristics (3), (4).

Now, we apply standard procedure (see [2, 3, 4])). Subtracting (10) from (11) and passing from the "slow" variable t to the "fast" variable τ we obtain the Cauchy problem:

$$\rho_\tau=1-2B_1(\rho),\quad \frac{\rho}{\tau}\Big|_{\tau\to-\infty}=1. \tag{12}$$

From the standard theory of ODE we see that $\rho\to\rho_0$ as $\tau\to+\infty$ where ρ_0 is constant such that $B_1(\rho_0)=B_2(\rho_0)=1/2$. That means that after the interaction we have $\varphi_1=\varphi_2+\mathcal{O}(\varepsilon)$ (see definitions of ρ and τ). Now, we can prove global resoluteness of Cauchy problem (9).

Problem (9) is globally solvable if characteristics emanating from the interval $[a_2,a_1]$ do not intersect. To prove that we will use the inverse function theorem. We will prove that for every t we have $\frac{\partial x}{\partial x_0}>0$ which means that for every $x=x(x_0,t)$, $x_0\in[a_2,a_1]$, we have unique $x_0=x_0(x,t)$ and we can write $u_1(x(x_0,t),t)=u_0(x_0)$.

The appropriate equation of characteristics reads (we use $f'(u_0(z))=-Kz+b$):

$$\dot{x}=(B_2-B_1)(-Kx_0+b)+cB_1,\quad x(0)=x_0.$$

As in [4] we will solve this problem with perturbed initial data:

$$\dot{x}=(B_2-B_1)(-Kx_0+b)+cB_1,\quad x(0)=x_0+\varepsilon Ax_0. \tag{13}$$

Differentiating (13) in x_0 and integrating from 0 to t we obtain:

$$\frac{\partial x}{\partial x_0}=1+\varepsilon A-K\int_0^t(B_2-B_1)dt' \tag{14}$$

for arbitrary fixed positive constant A. It is clear that this perturbation changes the solution of (1), (2) for $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ since initial condition (2) is continuous.

For $t \in [0, t^*]$ we have

$$\frac{\partial x}{\partial x_0} = 1 + A\varepsilon - K\int_0^t dt + K\int_0^t 2B_1 dt \geq$$
$$1 + A\varepsilon - K\int_0^{t*} dt + K\int_0^{t*} 2B_1 dt = A\varepsilon + K\int_0^{t*} 2B_1 dt > 0.$$

So, everything is correct for $t \leq t^*$.

To see what is happening for $t > t^*$, initially we estimate ρ_τ when $\tau \to \infty$. From equation (12) we have (we use Taylor expansion):

$$\rho_\tau = 1 - 2B_1(\rho) = -2(\rho - \rho_0)B_1'(\tilde\rho),$$

for some $\tilde\rho$ belonging to the interval with ends in ρ and ρ_0. From here we see:

$$\rho - \rho_0 = (\rho(\tau_0) - \rho_0)\exp(\int_{\tau_0}^{\tau} -2B_1'(\tilde\rho)d\tau') = (\rho(\tau_0) - \rho_0)\exp((\tau - \tau_0)2B_1'(\tilde\rho_1))$$

for some fixed $\rho_0 \in \mathbf{R}$ and $\tilde\rho_1 \in (\rho(\tau_0), \rho(\tau)) \subset [\rho(\tau_0), \rho_0]$. We remind that $B_1'(\tilde\rho_1)) \geq c > 0$, for some constant c, since B_1 is increasing function and $\tilde\rho_1$ belongs to the compact interval $[\rho(\tau_0), \rho_0]$, letting $\tau \to \infty$ we conclude that for any $N \in \mathbf{N}$

$$\rho - \rho_0 = \mathcal{O}(1/\tau^N), \quad \tau \to \infty.$$

From here we have $\rho_\tau = \mathcal{O}(1/\tau^N), \quad \tau \to \infty$, since:

$$(15) \quad \lim_{\tau\to\infty} \frac{\rho_\tau}{\rho - \rho_0} = \lim_{\tau\to\infty} \frac{1 - 2B_1(\rho)}{\rho - rho_0} = \lim_{\tau\to\infty} -2B_1'(\rho) = -2B_1(\rho_0) = \text{const.} < 0$$

This, in turn, means that for every $N \in \mathbf{N}$ and $t > t^*$ we have

$$\rho_\tau = \mathcal{O}(\varepsilon^N), \quad \varepsilon \to \infty.$$

Now we can prove resoluteness of problem (14) for $t > t^*$. We differentiate equation from (14) in ε. We obtain

$$\left(\frac{\partial x}{\partial x_0}\right)_\varepsilon' = \int_0^t 2KB_1'(\rho)\rho_\tau\tau_\varepsilon dt'.$$

For $t < t^*$ we have $B_1'(\rho) = \mathcal{O}(\varepsilon^N)$, $N \in \mathbf{N}$, and for $t > t^*$ we have $B_1' > 0$ (since then $\rho \to \rho_0 \neq \pm\infty$ and B_1 is increasing function). Also, for $t > t^*$ from (15) we see that for every $N \in \mathbf{N}$ we have $\rho_\tau = \mathcal{O}(\varepsilon^N)$, $\varepsilon \to 0$. From here:

$$\left(\frac{\partial x}{\partial x_0}\right)_\varepsilon' = A + \int_0^{t^*} 2KB_1'(\rho)\rho_\tau\tau_\varepsilon dt' + \int_{t^*}^t 2KB_1'(\rho)\rho_\tau\tau_\varepsilon dt' =$$
$$A + \mathcal{O}(\varepsilon^N) + \mathcal{O}(\varepsilon^N) > 0,$$

for ε small enough. This means that the function $\frac{\partial x}{\partial x_0}$ increases in ε for $t > t^*$ and it reaches its minimal value when $\varepsilon \to 0$. Letting $\varepsilon \to 0$ in (14) and taking into account that for $t > t^*$ we have $B_1, B_2 \to 1/2$, and for $t < t^*$ we have $B_1 \to 0$ we obtain

$$\frac{\partial x}{\partial x_0}|_{\varepsilon\to 0} = 1 - K\int_0^{t^*} dt' = 0.$$

So, from the previous we infer that $\frac{\partial x}{\partial x_0} > 0$ for each $t \in \mathbf{R}^+$.

Now, we have to obtain the constant c. We multiply (8) by $\eta \in C_0^1(\mathbf{R})$, integrate over $\mathbf{R}$ with respect to x and use (9).

$$\int B_1[\frac{d}{dx}\left(f(U+u_0^0-u_1)+f(u_1)-cu_1\right)]\left(\theta_{1\varepsilon}-\theta_{2\varepsilon}\right)\eta(x)dx+$$
$$\left((u_1-u_0^0)\varphi_{1t}-B_2\left(f(u_1)-f(u_0^0)\right)-B_1\left(f(U)-f(U+u_0^0-u_1)\right)\right)\delta_{1\varepsilon}+$$
$$\left((U-u_1)\varphi_{2t}-B_2\left(-f(u_1)+f(U)\right)-B_1\left(-f(u_0^0)+f(U+u_0^0-u_1)\right)\right)\delta_{2\varepsilon}=\mathcal{O}(\varepsilon).$$

We apply partial integration on the first integral in the previous expression to obtain:

$$\int B_1[f(U+u_0^0-u_1)+f(u_1)-cu_1]\left(\theta_{1\varepsilon}-\theta_{2\varepsilon}\right)\eta'(x)dx+ \tag{16}$$
$$\int\left((u_1-u_0^0)\varphi_{1t}-B_2\left(f(u_1)-f(u_0^0)\right)+B_1\left(f(u_1)+f(u_0^0)-cu_1\right)\right)\eta(x)\delta_{1\varepsilon}dx+$$
$$\int\left((U-u_1)\varphi_{2t}-B_2\left(-f(u_1)+f(U)\right)+B_1\left(-f(u_1)-f(U)+cu_1\right)\right)\eta(x)\delta_{2\varepsilon}dx=\mathcal{O}(\varepsilon).$$

From the properties of the functions B_i, $i=1,2$, we see that the coefficients multiplying $\delta_{i\varepsilon}$ are continuous in the limit as $\varepsilon\to 0$ and we can pass to limit when $\varepsilon\to 0$. Using the property $f(x)\delta(x-a)=f(a)$ we obtain the following expression from (16):

$$\begin{aligned}&\int B_1[f(U+u_0^0-u_1)+f(u_1)-cu_1]\left(\theta_{1\varepsilon}-\theta_{2\varepsilon}\right)\eta'(x)dx-\\&B_1\left(2f(u_0^0)-cu_0^0\right)\eta(\varphi_1)-B_1\left(-2f(U)+cU\right)\eta(\varphi_2)=\mathcal{O}(\varepsilon).\end{aligned} \tag{17}$$

We have

$$\begin{aligned}&\int B_1[f(U+u_0^0-u_1)+f(u_1)-cu_1]\left(\theta_{1\varepsilon}-\theta_{2\varepsilon}\right)\eta'(x)dx=\\&-\varepsilon\rho B_1\int[f(U+u_0^0-u_1)+f(u_1)-cu_1]\frac{\theta_{1\varepsilon}-\theta_{2\varepsilon}}{\varphi_{1\varepsilon}-\varphi_{2\varepsilon}}\eta'(x)dx=\mathcal{O}(\varepsilon)\end{aligned}$$

since

$$\rho B_1\int[f(U+u_0^0-u_1)+f(u_1)-cu_1]\frac{\theta_{1\varepsilon}-\theta_{2\varepsilon}}{\varphi_{1\varepsilon}-\varphi_{2\varepsilon}}\eta'(x)dx<\infty.$$

Therefore, (17) become

$$B_1\left(2f(u_0^0)-cu_0^0\right)\eta(\varphi_1)+B_1\left(-2f(U)+cU\right)\eta(\varphi_2)=\mathcal{O}(\varepsilon).$$

Rewrite this expression in the following manner:

$$B_1\left(2(f(u_0^0)-f(U))-c(u_0^0-U)\right)\eta(\varphi_1)+B_1\left(-2f(U)+cU\right)\left(\eta(\varphi_2)-\eta(\varphi_1)\right)=$$
$$B_1\left(2(f(u_0^0)-f(U))-c(u_0^0-U)\right)\eta(\varphi_1)+\varepsilon\rho B_1(\rho)\left(-2f(U)+cU\right)\frac{\eta(\varphi_2)-\eta(\varphi_1)}{\varphi_2-\varphi_1}=$$
$$B_1\left(2(f(u_0^0)-f(U))-c(u_0^0-U)\right)\eta(\varphi_1)+\mathcal{O}(\varepsilon).$$

From here, we see that the last relation is satisfied for

$$c=2\frac{f(U)-f(u_0^0)}{U-u_0^0}. \tag{18}$$

The theorem is proved. □

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