H^1–PERTURBATIONS OF SMOOTH SOLUTIONS FOR A WEAKLY DISSIPATIVE HYPERELASTIC-ROD WAVE EQUATION

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ABSTRACT. We consider a weakly dissipative hyperelastic-rod wave equation (or weakly dissipative Camassa–Holm equation) describing nonlinear dispersive dissipative waves in compressible hyperelastic rods. By fixed a smooth solution, we establish the existence of a strongly continuous semigroup of global weak solutions for any initial perturbation from $H^1(\mathbb{R})$. In particular, the supersonic solitary shock waves [8] are included in the analysis.

1. Introduction and Statement of Main Results

Consider the equation

$$\partial_t u - \partial_{txx}^3 u + 3u\partial_x u + \delta\partial_{xx}^2 u = \gamma\big(2\partial_x u\partial_{xx}^2 u + u\partial_{xxx}^3 u\big), \quad t > 0,\ x \in \mathbb{R}. \tag{1.1}$$

In the case $\gamma = 1$, $\delta = 0$ it is known as the *Camassa-Holm equation* and describes unidirectional shallow water waves above a flat bottom: u represents the fluid velocity [1, 12]. The Camassa–Holm equation possesses a bi-Hamiltonian structure (and thus an infinite number of conservation laws) [11, 1] and is completely integrable [1]. From a mathematical point of view the Camassa–Holm equation is well studied, see [3] for a complete list of references. In particular, we recall that existence and uniqueness results for global weak solutions have been proved by Constantin and Escher [4], Constantin and Molinet [5], and Xin and Zhang [17, 18], see also Danchin [9, 10].

When $\delta = 0$, it is termed *hyperelastic-rod wave equation* and describes the finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods. The constant $\gamma > 0$ depends on the material constants and the prestress of the rod [6, 7, 8].

The additional weakly dissipative term $\delta\partial_{xx}^2 u$ is introduced in [15]. We coin (1.1) the *weakly dissipative hyperelastic-rod wave equation.*

In [3] the authors consider the case $\delta = 0$ and prove the global existence and wellposedness of solutions belonging to $L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$. On the other hand in [8] it is showed that for $\delta = 0$ and any constants $0 < \gamma < 3$, $c > 0$ there exists a $\zeta \in \mathbb{R}$

Date: October 18, 2005.

2000 *Mathematics Subject Classification.* 35G25, 35L05, 35A05.

Key words and phrases. Shallow water equation, integrable equation, hyperbolic equation, weak solution, existence, uniqueness, stability.

The research of K. H. Karlsen is supported by an Outstanding Young Investigators Award from the Research Council of Norway. The current address of G. M. Coclite is Department of Mathematics, University of Bari, Via E. Orabona 4, 70125 Bari, Italy.

such that the following peakon like function is a traveling wave solution of (1.1)

$$U(t,x) = \frac{c}{2}\Big(1-\frac{1}{\gamma}\Big) + \frac{c}{2}\Big(\frac{3}{\gamma}-1\Big)e^{-|x-ct-\zeta|/\sqrt{\gamma}}, \tag{1.2}$$

called *supersonic solitary shock wave*. It is clear that the analysis in [3] does not cover this kind of solutions (that do not belong to $L^\infty(\mathbb{R}_+;H^1(\mathbb{R}))$!).

In this paper we extend the result of [3] to cover also (1.2). Roughly speaking the idea is to look at (1.2) as a $L^\infty(\mathbb{R}_+;H^1(\mathbb{R}))-$perturbation of a constant state. Indeed we can decompose U in the following way

$$U = U_1 + U_2, \quad U_1 := \frac{c}{2}\Big(1-\frac{1}{\gamma}\Big), \quad U_2(t,x) := \frac{c}{2}\Big(\frac{3}{\gamma}-1\Big)e^{-|x-ct-\zeta|/\sqrt{\gamma}},$$

where U_1 is a classical solution to (1.1) and U_2 is a perturbation that lies in the space $L^\infty(\mathbb{R}_+;H^1(\mathbb{R}))$.

To be more precise: let $\varphi = \varphi(t,x)$ be a solution of (1.1) such that

$$\varphi \in C^3([0,\infty)\times\mathbb{R}), \qquad \varphi,\, \partial_t\varphi,\, \partial_x\varphi,\, \partial_{tx}^2\varphi,\, \partial_{xx}^2\varphi,\, \partial_x^3\varphi \in L^\infty(\mathbb{R}_+\times\mathbb{R}), \tag{1.3}$$

(this is the case if φ is periodic or constant) and

$$v_0 \in H^1(\mathbb{R}), \qquad \gamma > 0, \qquad \delta \in \mathbb{R}. \tag{1.4}$$

We want to study the wellposedness of the Cauchy problem

$$\begin{cases} \partial_t u - \partial_{txx}^3 u + 3u\partial_x u + \delta\partial_{xx}^2 u = \gamma\big(2\partial_x u \partial_{xx}^2 u + u\partial_{xxx}^3 u\big), & t>0,\ x\in\mathbb{R},\\ u(0,x) = \varphi(0,x) + v_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.5}$$

Observe that, at least formally, (1.5) is equivalent to the elliptic-hyperbolic system

$$\begin{cases} \partial_t u + \gamma u\partial_x u + \partial_x P = 0, & t>0,\ x\in\mathbb{R},\\ -\partial_{xx}^2 P + P = \dfrac{3-\gamma}{2}u^2 + \dfrac{\gamma}{2}\big(\partial_x u\big)^2 + \delta\partial_x u, & t>0,\ x\in\mathbb{R},\\ u(0,x) = \varphi(0,x) + v_0(x), & x\in\mathbb{R}. \end{cases} \tag{1.6}$$

Motivated by this, we shall use the following definition of weak solution. Moreover, in the same spirit of [3, Definition 1.1] we define the admissible perturbations.

Definition 1.1. *We call $u : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ a weak solution of the Cauchy problem* (1.5) *if*

(i) $u \in C([0,\infty)\times\mathbb{R})$*;*
(ii) $u - \varphi \in L^\infty\big((0,T);H^1(\mathbb{R})\big)$, $T>0$*;*
(iii) u *satisfies* (1.6) *in the sense of distributions;*
(iv) $u(0,x) = \varphi(0,x) + v_0(x)$*, for every* $x\in\mathbb{R}$*.*

If, in addition, for each $T>0$ there exists a positive constant K_T depending only on $\|v_0\|_{H^1(\mathbb{R})}$, φ, γ, T, such that

$$\partial_x\big(u(t,x) - \varphi(t,x)\big) \le \frac{4}{\gamma t} + K_T, \qquad (t,x)\in(0,T)\times\mathbb{R}, \tag{1.7}$$

then we say that $u-\varphi$ is an admissible perturbation of (1.1)*.*

Our results are collected in the following theorem:

Theorem 1.1. *There exists a strongly continuous semigroup of solutions associated to the Cauchy problem* (1.5)*. More precisely, there exists a map*

$$S:[0,\infty)\times(0,\infty)\times\mathbb{R}\times H^1(\mathbb{R})\longrightarrow C([0,\infty)\times\mathbb{R}),\quad (t,\gamma,\delta,v_0)\longmapsto S_t(\gamma,\delta,v_0)(\cdot),$$

with the following properties:

(*i*) *for each* $v_0\in H^1(\mathbb{R})$, $\gamma>0$, $\delta\in\mathbb{R}$ *the map* $u(t,x)=S_t(\gamma,\delta,v_0)(x)$ *is a weak solution of* (1.5) *and* $u-\varphi$ *is an admissible perturbation of* (1.1)*;*

(*ii*) *it is stable with respect to the initial condition and the coefficient in the following sense, if*

$$v_{0,n}\longrightarrow v_0 \text{ in } H^1(\mathbb{R}),\qquad \gamma_n\longrightarrow\gamma,\ \delta_n\longrightarrow\delta \text{ in } \mathbb{R}, \tag{1.8}$$

then

$$S(\gamma_n,\delta_n,v_{0,n})-\varphi\longrightarrow S(\gamma,\delta,v_0)-\varphi \text{ in } L^\infty([0,T];H^1(\mathbb{R})), \tag{1.9}$$

for every $\{v_{0,n}\}_{n\in\mathbb{N}}\subset H^1(\mathbb{R})$, $\{\gamma_n\}_{n\in\mathbb{N}}\subset(0,\infty)$, $\{\delta_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$, $v_0\in H^1(\mathbb{R})$, $\gamma>0$, $\delta\in\mathbb{R}$, $T>0$.

Moreover, the following statements hold:

(*iii*) *the estimate* (1.7) *is valid with*

$$\begin{aligned}K_T:=&\frac{2}{\sqrt{\gamma}}\bigg(\Big(\frac{\gamma}{\sqrt{2}}\|\partial^2_{xx}\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}+|\gamma-3|\sqrt{2}\|\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}\Big)e^{\rho T}\|v_0\|_{H^1(\mathbb{R})}\\&+\frac{\max\{|3-\gamma|,2\gamma\}+|3-\gamma|}{4}e^{2\rho T}\|v_0\|^2_{H^1(\mathbb{R})}\\&+\frac{5\gamma}{2}\|\partial_x\varphi\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})}+2\frac{\delta^2}{\gamma}\bigg)^{1/2},\end{aligned}\tag{1.10}$$

$$\rho:=\frac{3+\gamma}{2}\|\partial_x\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}+\frac{\gamma}{2}\|\partial^3_{xxx}\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})}+|\delta|, \tag{1.11}$$

for $T>0$*;*

(*iv*) *there results*

$$\partial_x S(\gamma,\delta,v_0)\in L^p_{\mathrm{loc}}(\mathbb{R}_+\times\mathbb{R}), \tag{1.12}$$

with $1\le p<3$.

Our argument is based on the analysis of the evolution of the perturbation

$$v:=u-\varphi.$$

From (1.5) we get the following equation for v

$$\begin{cases}\partial_t v-\partial^3_{txx}v+3v\partial_x v+3\varphi\partial_x v+3v\partial_x\varphi+\delta\partial^2_{xx}v\\ \qquad=\gamma\big(2\partial_x v\partial^2_{xx}v+v\partial^3_{xxx}v+2\partial_x v\partial^2_{xx}\varphi+2\partial_x\varphi\partial^2_{xx}v+v\partial^3_{xxx}\varphi+\varphi\partial^3_{xxx}v\big),\\ v(0,\cdot)=v_0,\end{cases}\tag{1.13}$$

that is formally equivalent to the elliptic-hyperbolic system

$$\begin{cases}\partial_t v+\gamma v\partial_x v+\gamma v\partial_x\varphi+\gamma\varphi\partial_x v+\partial_x P=0,\\ -\partial^2_{xx}P+P=\dfrac{3-\gamma}{2}v^2+\dfrac{\gamma}{2}\big(\partial_x v\big)^2+(3-\gamma)\varphi v+\gamma\partial_x\varphi\partial_x v+\delta\partial_x v,\\ v(0,\cdot)=v_0.\end{cases}\tag{1.14}$$

Since the argument is very similar to the one in [3] we simply sketch it.

2. Viscous Approximations: Existence and A Priori Estimates

We prove existence of a weak solution to the Cauchy problem (1.13) (and equivalently to (1.5)) by proving compactness of a sequence of smooth solutions $\{v_\varepsilon\}_{\varepsilon>0}$ solving the following viscous problems (see [2]):

$$\begin{cases} \partial_t v_\varepsilon + \gamma v_\varepsilon \partial_x v_\varepsilon + \gamma v_\varepsilon \partial_x \varphi + \gamma \varphi \partial_x v_\varepsilon + \partial_x P_\varepsilon = \varepsilon \partial_{xx}^2 v_\varepsilon, \\ -\partial_{xx}^2 P_\varepsilon + P_\varepsilon = \dfrac{3-\gamma}{2} v_\varepsilon^2 + \dfrac{\gamma}{2}\big(\partial_x v_\varepsilon\big)^2 + (3-\gamma)\varphi v_\varepsilon + \gamma \partial_x \varphi \partial_x v_\varepsilon + \delta \partial_x v_\varepsilon, \\ v_\varepsilon(0,\cdot) = v_{\varepsilon,0}, \end{cases} \tag{2.1}$$

that is equivalent to the following fourth order one

$$\begin{cases} \partial_t v_\varepsilon - \partial_{txx}^3 v_\varepsilon + 3 v_\varepsilon \partial_x v_\varepsilon + 3\varphi \partial_x v_\varepsilon + 3 v_\varepsilon \partial_x \varphi + \delta \partial_{xx}^2 v_\varepsilon \\ \qquad = \gamma\big(2\partial_x v_\varepsilon \partial_{xx}^2 v_\varepsilon + v_\varepsilon \partial_{xxx}^3 v_\varepsilon + 2\partial_x v_\varepsilon \partial_{xx}^2 \varphi + 2\partial_x \varphi \partial_{xx}^2 v_\varepsilon\big) \\ \quad + \gamma\big(\varphi \partial_{xxx}^3 v_\epsilon + v_\varepsilon \partial_{xxx}^3 \varphi\big) + \varepsilon \partial_{xx}^2 v_\varepsilon - \varepsilon \partial_{xxxx}^4 v_\varepsilon, \\ v_\varepsilon(0,\cdot) = v_{\varepsilon,0}. \end{cases} \tag{2.2}$$

Formally, sending $\varepsilon \to 0$ in (2.2), (2.1) yields (1.13), (1.14), respectively.

We shall assume that

$$v_{\varepsilon,0} \in H^2(\mathbb{R}), \quad \|v_{\varepsilon,0}\|_{H^1(\mathbb{R})} \le \|v_0\|_{H^1(\mathbb{R})}, \varepsilon > 0, \quad \text{and} \quad v_{\varepsilon,0} \longrightarrow v_0 \text{in} H^1(\mathbb{R}). \tag{2.3}$$

The starting point of our analysis is the following wellposedness result for (2.1) (see [2, Theorem 2.3]).

Lemma 2.1. *Assume* (1.3), (1.4) *and* (2.3), *let* $\varepsilon > 0$. *There exists a unique smooth solution* $v_\varepsilon \in C\big([0,\infty); H^2(\mathbb{R})\big)$ *to the Cauchy problem* (2.1).

The next step in our analysis is to derive the following a priori estimates:

Lemma 2.2. *Assume* (1.3), (1.4) *and* (2.3), *and let* $\varepsilon > 0$. *Then the following estimates hold:*

j) (Energy Conservation) for each $t \ge 0$

$$\big\|v_\varepsilon(t,\cdot)\big\|_{H^1(\mathbb{R})}^2 + 2\varepsilon \int_0^t \big\|\partial_x v_\varepsilon(\tau,\cdot)\big\|_{H^1(\mathbb{R})}^2 \, d\tau \le e^{2\rho t}\|v_0\|_{H^1(\mathbb{R})}^2; \tag{2.4}$$

jj) (Oleinik type Estimate) for any $0 < t < T$ *and* $x \in \mathbb{R}$,

$$\partial_x v_\varepsilon(t,x) \le \frac{4}{\gamma t} + K_T, \tag{2.5}$$

where K_T *is defined in* (1.10)*;*

jj) (Higher Integrability Estimate) for every $0 \le \alpha < 1$, $T > 0$, *and* $a, b \in \mathbb{R}$, $a < b$, *there exists a positive constant* C_T *depending only on* $\|v_0\|_{H^1(\mathbb{R})}$, φ, α, T, a *and* b, *but independent on* ε, *such that*

$$\int_0^T \int_a^b \big|\partial_x v_\varepsilon(t,x)\big|^{2+\alpha} \, dt dx \le C_T. \tag{2.6}$$

Remark 2.1. *Due to* [13, Theorem 8.5], (2.3) *and* (2.4), *we have for each* $t \ge 0$

$$\|v_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}} \, \|v_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})} \le \frac{e^{\rho t}}{\sqrt{2}} \|v_0\|_{H^1(\mathbb{R})}. \tag{2.7}$$

Proof of Lemma 2.2. We begin with j). Multiplying (2.2) by v_ε, integrating on $\mathbb{R}$, and integrating by parts we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(v_\varepsilon^2+(\partial_x v_\varepsilon)^2\right)dx+\varepsilon\int_{\mathbb{R}}\left((\partial_x v_\varepsilon)^2+(\partial_{xx}^2 v_\varepsilon)^2\right)dx\le\rho\int_{\mathbb{R}}\left(v_\varepsilon^2+(\partial_x v_\varepsilon)^2\right)dx,$$

where ρ is defined in (1.11). Hence (2.4) is consequence of (2.3) and the Gronwall Lemma.

We continue by proving jj). Introduce the notation

$$q_\varepsilon := \partial_x v_\varepsilon.$$

From (2.1) we get the following equation for q_ε

$$\begin{aligned}\partial_t q_\varepsilon+\frac{\gamma}{2}q_\varepsilon^2+\gamma v_\varepsilon\partial_x q_\varepsilon+\gamma v_\varepsilon\partial_{xx}^2\varphi+\gamma\partial_x\varphi q_\varepsilon-\delta q_\varepsilon\\+\gamma\varphi\partial_x q_\varepsilon+\frac{\gamma-3}{2}v_\varepsilon^2+(\gamma-3)\varphi v_\varepsilon+P_\varepsilon-\varepsilon\partial_{xx}^2 q_\varepsilon=0.\end{aligned}\tag{2.8}$$

Using the fact that $e^{-|x|}/2$ is the Green's function of the operator $1-\partial_{xx}^2$

$$\begin{aligned}P_\varepsilon(t,x)=\frac{1}{2}\int_{\mathbb{R}}e^{-|x-y|}\Big(\frac{3-\gamma}{2}v_\varepsilon^2(t,y)+\frac{\gamma}{2}\big(\partial_x v_\varepsilon(t,y)\big)^2\\+(3-\gamma)\varphi(t,y)v_\varepsilon(t,y)+\gamma\partial_x\varphi(t,y)\partial_x v_\varepsilon(t,y)\Big)\,dy.\end{aligned}\tag{2.9}$$

It follows from (2.4) and (2.7) (see [3, Proof of Lemma 3.1]) that

$$\left\|\gamma v_\varepsilon\partial_{xx}^2\varphi+\frac{\gamma-3}{2}v_\varepsilon^2+(3-\gamma)\varphi v_\varepsilon+P_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}\le L_T,\tag{2.10}$$

for some constant $L_T>0$. Then, from (2.8),

$$\partial_t q_\varepsilon+\frac{\gamma}{2}q_\varepsilon^2+\gamma v_\varepsilon\partial_x q_\varepsilon+\gamma\partial_x\varphi q_\varepsilon-\delta q_\varepsilon+\gamma\varphi\partial_x q_\varepsilon-\varepsilon\partial_{xx}^2 q_\varepsilon\le L_T.$$

Since

$$\frac{\gamma}{2}\xi^2+(\gamma\partial_x\varphi-\delta)\xi\ge\frac{\gamma}{4}\xi^2-\frac{(\gamma\partial_x\varphi-\delta)^2}{\gamma},\qquad \xi\in\mathbb{R},$$

we conclude

$$\begin{aligned}\partial_t q_\varepsilon+\frac{\gamma}{4}q_\varepsilon^2+\gamma v_\varepsilon\partial_x q_\varepsilon+\gamma\varphi\partial_x q_\varepsilon-\varepsilon\partial_{xx}^2 q_\varepsilon\\\le L_T+2\gamma\|\partial_x\varphi\|^2_{L^\infty(\mathbb{R}_+\times\mathbb{R})}+2\frac{\delta^2}{\gamma}=:\widetilde{L}_T.\end{aligned}\tag{2.11}$$

Employing the comparison principle for parabolic equations, we get

$$q_\varepsilon(t,x)\le h(t),\qquad 0\le t\le T,\ x\in\mathbb{R}\tag{2.12}$$

where h solves

$$\frac{dh}{dt}+\frac{\gamma}{4}h^2=\widetilde{L}_T,\qquad h(0)=\|\partial_x v_{\varepsilon,0}\|_{L^\infty(\mathbb{R})}.\tag{2.13}$$

Since the map

$$H(t):=\frac{4}{\gamma t}+\sqrt{\frac{4\widetilde{L}_T}{\gamma}},\qquad t>0,$$

is a super-solution of (2.13) in the interval $[0,T]$. Due to the comparison principle for ordinary differential equations, we get $h(t)\le H(t)$ for all $0<t\le T$. Therefore, (2.5) is proved.

Finally, we consider jjj). The argument is very similar to the one of [3, Lemma 4.1]. Pick a cut-off function $\chi \in C^\infty(\mathbb{R})$ such that

$$0 \le \chi \le 1, \qquad \chi(x) = \begin{cases} 1, & \text{if } x \in [a,b], \\ 0, & \text{if } x \in (-\infty, a-1] \cup [b+1, \infty), \end{cases}$$

consider the map $\theta(\xi) := \xi\big(|\xi|+1\big)^\alpha$, $\xi \in \mathbb{R}$, then multiply (2.8) by $\chi\theta'(q_\varepsilon)$, integrate over $(0,T)\times\mathbb{R}$ and use (2.4). $\square$

3. Compactness

Lemma 3.1. *The family $\{P_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^\infty([0,T);W^{1,\infty}(\mathbb{R}))$ and $L^\infty([0,T);H^1(\mathbb{R}))$ for each $T>0$.*

Proof. The argument is the same of [3, Lemma 5.1]: use the integral representation of P_ε (2.9) and then employ (2.7). $\square$

Lemma 3.2. *There exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to zero and a function $v \in L^\infty([0,T);H^1(\mathbb{R})) \cap H^1([0,T]\times\mathbb{R})$, for each $T \ge 0$, such that*

$$v_{\varepsilon_j} \rightharpoonup v \quad \text{weakly in } H^1_{\mathrm{loc}}([0,T]\times\mathbb{R}), \text{ for each } T \ge 0, \tag{3.1}$$

$$v_{\varepsilon_j} \to v \quad \text{strongly in } L^\infty_{\mathrm{loc}}([0,\infty)\times\mathbb{R}). \tag{3.2}$$

Proof. Fix $T>0$. Observe that, from (2.1),

$$\partial_t v_\varepsilon = \varepsilon\partial^2_{xx}v_\varepsilon - \gamma v_\varepsilon\partial_x v_\varepsilon - \gamma v_\varepsilon\partial_x\varphi - \gamma\varphi\partial_x v_\varepsilon - \partial_x P_\varepsilon,$$

hence, by (2.7), (2.4), Lemma 3.1, and the Hölder inequality, $\{v_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $H^1([0,T]\times\mathbb{R}) \cap L^\infty([0,T);H^1(\mathbb{R}))$, and (3.1) follows. Finally, since $H^1(\mathbb{R}) \subset\subset L^\infty_{\mathrm{loc}}(\mathbb{R}) \subset L^2_{\mathrm{loc}}(\mathbb{R})$, (3.2) is consequence of [16, Theorem 5]. $\square$

Lemma 3.3. *The family $\{P_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $W^{1,1}_{\mathrm{loc}}([0,T)\times\mathbb{R})$ for any $T>0$. In particular, there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to zero and a function $P \in L^\infty([0,T);W^{1,\infty}(\mathbb{R}))$ such that for each $1 \le p < \infty$*

$$P_{\varepsilon_j} \to P \quad \text{strongly in } L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}). \tag{3.3}$$

Proof. The argument is analogous to the one of [3, Lemma 5.3]. Using the integral representation (2.9) of P_ε and then employing (2.7) we get the uniform boundedness of $\big\{\partial_t P_\varepsilon\big\}_{\varepsilon>0}$ in $L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$. Then, due to Lemma 3.1, $\{P_\varepsilon\}_{\varepsilon>0}$ is bounded in $W^{1,1}_{\mathrm{loc}}([0,T)\times\mathbb{R})$. Finally, using again Lemma 3.1, we have the existence of a pointwise converging subsequence that is uniformly bounded in $L^\infty([0,T)\times\mathbb{R})$. Clearly, this implies (3.3). $\square$

Lemma 3.4. *There exist a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ tending to zero and two functions $q \in L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$, $\overline{q^2} \in L^r_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$ such that*

$$q_{\varepsilon_j} \rightharpoonup q \quad \text{in } L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}), \qquad q_{\varepsilon_j} \overset{\star}{\rightharpoonup} q \quad \text{in } L^\infty_{\mathrm{loc}}([0,\infty);L^2(\mathbb{R})), \tag{3.4}$$

$$q^2_{\varepsilon_j} \rightharpoonup \overline{q^2} \quad \text{in } L^r_{\mathrm{loc}}([0,\infty)\times\mathbb{R}), \tag{3.5}$$

for each $1<p<3$ and $1<r<3/2$. Moreover,

$$q^2(t,x) \le \overline{q^2}(t,x) \quad \text{for almost every } (t,x) \in [0,\infty)\times\mathbb{R} \tag{3.6}$$

and

$$\partial_x v = q \quad \text{in the sense of distributions on } [0,\infty)\times\mathbb{R}. \tag{3.7}$$

Proof. Formulas (3.4) and (3.5) are direct consequences of Lemma 2.1 and (2.6). Inequality (3.6) is true thanks to the weak convergence in (3.5). Finally, (3.7) is a consequence of the definition of q_ε, Lemma 3.2, and (3.4). □

In the following, for notational convenience, we replace the sequences $\{v_{\varepsilon_j}\}_{j\in\mathbb{N}}$, $\{q_{\varepsilon_j}\}_{j\in\mathbb{N}}$, $\{P_{\varepsilon_j}\}_{j\in\mathbb{N}}$ by $\{v_\varepsilon\}_{\varepsilon>0}$, $\{q_\varepsilon\}_{\varepsilon>0}$, $\{P_\varepsilon\}_{\varepsilon>0}$, respectively.

In view of (3.4), we conclude that for any $\eta\in C^1(\mathbb{R})$ with η' bounded, Lipschitz continuous on $\mathbb{R}$ and any $1\le p<3$ we have

$$\eta(q_\varepsilon)\rightharpoonup\overline{\eta(q)}\quad\text{in } L^p_{\mathrm{loc}}([0,\infty)\times\mathbb{R}),\qquad \eta(q_\varepsilon)\overset{\star}{\rightharpoonup}\overline{\eta(q)}\quad\text{in } L^\infty_{\mathrm{loc}}([0,\infty);L^2(\mathbb{R})). \tag{3.8}$$

Multiplying the equation in (2.8) by $\eta'(q_\varepsilon)$, we get

$$\begin{aligned}\partial_t\eta(q_\varepsilon)&+\gamma\partial_x\big((v_\epsilon+\varphi)\eta(q_\varepsilon)\big)-\varepsilon\partial^2_{xx}\eta(q_\varepsilon)-\varepsilon\eta''(q_\varepsilon)(\partial_x\eta(q_\varepsilon))^2-\gamma q_\varepsilon\eta(q_\varepsilon)\\&+\gamma v_\varepsilon\partial^2_{xx}\varphi\eta'(q_\varepsilon)+\gamma\partial_x\varphi\big(q_\varepsilon\eta'(q_\varepsilon)-\eta(q_\varepsilon)\big)-\delta q_\varepsilon\eta'(q_\varepsilon)\\&+\frac{\gamma-3}{2}v_\varepsilon^2\eta'(q_\varepsilon)+\frac{\gamma}{2}q_\varepsilon^2\eta'(q_\varepsilon)+(\gamma-3)\varphi v_\varepsilon\eta'(q_\varepsilon)+P_\varepsilon\eta'(q_\varepsilon)=0.\end{aligned}\tag{3.9}$$

Lemma 3.5. *For any convex $\eta\in C^1(\mathbb{R})$ with η' bounded, Lipschitz continuous on $\mathbb{R}$, we have*

$$\begin{aligned}\partial_t\overline{\eta(q)}&+\gamma\partial_x\big((v+\varphi)\overline{\eta(q)}\big)-\gamma\overline{q\eta(q)}\\&+\gamma v\partial^2_{xx}\varphi\overline{\eta'(q)}+\gamma\partial_x\varphi\big(\overline{q\eta'(q)}-\overline{\eta(q)}\big)-\delta\overline{q\eta'(q)}\\&+\frac{\gamma-3}{2}v^2\overline{\eta'(q)}+\frac{\gamma}{2}\overline{q^2\eta'(q)}+(\gamma-3)\varphi v\overline{\eta'(q)}+P\overline{\eta'(q)}\le0,\end{aligned}\tag{3.10}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$. Here $\overline{q\eta(q)}$, $\overline{q^2\eta'(q)}$ and $\overline{\eta'(q)q}$ denote the weak limits of $q_\varepsilon\eta(q_\varepsilon)$, $q_\varepsilon^2\eta'(q_\varepsilon)$ and $\eta'(q_\varepsilon)q_\varepsilon$ in $L^r_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$, $1<r<3/2$, respectively.

Proof. In (3.9), by convexity of η, (3.2), (3.4), and (3.5), sending $\varepsilon\to0$ yields (3.10). □

Remark 3.1. *From* (3.4) *and* (3.5)*, it is clear that*

$$q=q_++q_-=\overline{q_+}+\overline{q_-},\qquad q^2=(q_+)^2+(q_-)^2,\qquad \overline{q^2}=\overline{(q_+)^2}+\overline{(q_-)^2},$$

almost everywhere in $[0,\infty)\times\mathbb{R}$, where $\xi_+:=\xi\chi_{[0,+\infty)}(\xi)$, $\xi_-:=\xi\chi_{(-\infty,0]}(\xi)$, $\xi\in\mathbb{R}$. Moreover, by (2.5) *and* (3.4)*,*

$$q_\varepsilon(t,x),q(t,x)\le\frac{4}{\gamma t}+K_T,\qquad 0<t<T,\ x\in\mathbb{R}.\tag{3.11}$$

Lemma 3.6. *There holds*

$$\partial_tq+\gamma\partial_x\big((v+\varphi)q\big)-\frac{\gamma}{2}\overline{q^2}+\gamma v\partial^2_{xx}\varphi+\frac{\gamma-3}{2}v^2+(\gamma-3)\varphi v+P-\delta q=0,\tag{3.12}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$.

Proof. Using (3.2), (3.3), (3.4), and (3.5), the result (3.12) follows by $\varepsilon\to0$ in (2.8). □

The next lemma contains a renormalized formulation of (3.12).

Lemma 3.7 ([3, Lemma 5.8])**.** *For any $\eta \in C^1(\mathbb{R})$ with $\eta' \in L^\infty(\mathbb{R})$,*

$$\begin{aligned}(3.13)\quad \partial_t \overline{\eta(q)} + \gamma\partial_x\big((v+\varphi)\overline{\eta(q)}\big) - \gamma\overline{q\eta(q)} - \gamma\Big(\frac{\overline{q^2}}{2} - q^2\Big)\eta'(q) \\ + \gamma v\partial_{xx}^2\varphi\overline{\eta'(q)} + \gamma\partial_x\varphi\big(\overline{q\eta'(q)} - \overline{\eta(q)}\big) - \delta\overline{q\eta(q)} \\ + \frac{\gamma-3}{2}v^2\overline{\eta'(q)} + (\gamma-3)\varphi v\overline{\eta'(q)} + P\overline{\eta'(q)} = 0,\end{aligned}$$

in the sense of distributions on $[0,\infty)\times\mathbb{R}$.

Following [3, Section 6] and [17], we improve the weak convergence of q_ε in (3.4) to strong convergence (and then we have an existence result for (1.5)). The idea is to derive a "transport equation" for the evolution of the defect measure $\big(\overline{q^2} - q^2\big)(t,\cdot) \geq 0$, so that if it is zero initially then it will continue to be zero at all later times $t > 0$. The proof is complicated by the fact that we do not have a uniform bound on q_ε from below but merely (3.11) and that in (2.6) we have only $\alpha < 1$.

Lemma 3.8. *Assume* (1.3) *and* (2.3)*. Then for each $t \geq 0$*

$$\begin{aligned}(3.14)\quad \int_{\mathbb{R}} \Big(\overline{(q_+)^2}(t,x) - (q_+)^2(t,x)\Big)\,dx \\ \leq 2e^{\lambda t}\int_0^t\int_{\mathbb{R}} e^{-\lambda s}S(s,x)\left[\overline{q_+}(s,x) - q_+(s,x)\right]dsdx,\end{aligned}$$

where

$$\lambda := \gamma\|\partial_x\varphi\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} + 2|\delta|,$$

$$S(s,x) := -v(s,x)\partial_{xx}^2\varphi(s,x) + \frac{3-\gamma}{2}v^2(s,x) + (3-\gamma)\varphi(s,x)v(s,x) - P(s,x).$$

Proof. Let $T > 0$, $R > K_T$ (see (2.5)). Subtract (3.13) from (3.10) using the renormalization

$$\eta_R^+(\xi) := \begin{cases} R\xi - R^2/2, & \text{if } \xi > R,\\ \xi^2/2, & \text{if } 0 \leq \xi \leq R,\\ 0, & \text{if } \xi < 0.\end{cases}$$

Arguing as in [3, Lemma 6.4] we get

$$\frac{d}{dt}\int_{\mathbb{R}}\Big(\overline{(q_+)^2} - (q_+)^2\Big)\,dx \leq \lambda\int_{\mathbb{R}}\Big(\overline{(q_+)^2} - (q_+)^2\Big)\,dx + 2\int_{\mathbb{R}} S(t,x)\left[\overline{q_+} - q_+\right]dx,$$

for $4/(\gamma(R-K_T)) < t < T$. First we have to apply the Gronwall Lemma to the previous inequality on the interval $(4/(\gamma(R-K_T)),T)$. Then sending $R\to\infty$ and using (see [3, Lemma 6.2])

$$(3.15)\qquad \lim_{t\to 0+}\int_{\mathbb{R}}\Big(\overline{\eta_R^+(q)}(t,x) - \eta_R^+(q(t,x))\Big)\,dx = 0, \qquad R > 0.$$

□

Lemma 3.9. *For any $t \geq 0$ and any $R > 0$,*

$$\begin{aligned}(3.16)\quad \int_{\mathbb{R}}\Big[\overline{\eta_R^-(q)}(t,x) - \eta_R^-(q)(t,x)\Big]\,dx \\ \leq \frac{Re^{\lambda t}}{2}\int_0^t\int_{\mathbb{R}} e^{-\lambda s}\left(\gamma R - 2\gamma\partial_x\varphi + 2\delta\right)\overline{(R+q)\chi_{(-\infty,-R)}(q)}\,dsdx\end{aligned}$$

$$
\begin{aligned}
&- \frac{Re^{\lambda t}}{2}\int_0^t\int_{\mathbb{R}} e^{-\lambda s}\left(\gamma R - 2\gamma\partial_x\varphi + 2\delta\right)(R+q)\chi_{(-\infty,-R)}(q)\,dsdx\\
&+ \frac{\gamma Re^{\lambda t}}{2}\int_0^t\int_{\mathbb{R}} e^{-\lambda s}\left(\overline{(q_+)^2} - (q_+)^2\right) dsdx\\
&+ \gamma Re^{\lambda t}\int_0^t\int_{\mathbb{R}} e^{-\lambda s}\left(\overline{\eta_R^-(q)} - \eta_R^-(q)\right) dsdx\\
&+ e^{\lambda t}\int_0^t\int_{\mathbb{R}} e^{-\lambda s} S(s,x)\left[\overline{(\eta_R^-)'(q)} - (\eta_R^-)'(q)\right] dsdx.
\end{aligned}
$$

Proof. The argument is very similar to the one of [3, Lemma 6.3]. We begin by subtracting (3.13) from (3.10), using the renormalization

$$
\eta_R^-(\xi) := \begin{cases} 0, & \text{if } \xi > 0,\\ \xi^2/2, & \text{if } -R \le \xi \le 0,\\ -R\xi - R^2/2, & \text{if } \xi < -R.\end{cases}
$$

Then we integrate on $\mathbb{R}$ and use the Gronwall Lemma and (see [3, Lemma 6.2])

$$
\lim_{t\to 0+}\int_{\mathbb{R}}\left(\overline{\eta_R^-(q)}(t,x) - \eta_R^-(q(t,x))\right) dx = 0, \qquad R>0. \tag{3.17}
$$

□

Lemma 3.10. *There holds* $\overline{q^2} = q^2$ *almost everywhere in* $[0,\infty)\times\mathbb{R}$.

Proof. We follow the argument of [3, Lemma 6.6]. We add (3.14) and (3.16). Using the concavity of $\xi \mapsto (R+\xi)\chi_{(-\infty,-R)}(\xi)$, the Gronwall Lemma, (3.15), (3.17) and

$$
\lim_{t\to 0+}\int_{\mathbb{R}} q^2(t,x)\,dx = \lim_{t\to 0+}\int_{\mathbb{R}}\overline{q^2}(t,x)\,dx = \int_{\mathbb{R}}\left(\partial_x v_0\right)^2 dx,
$$

we conclude that

$$
\int_{\mathbb{R}}\left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_R^-(q)} - \eta_R^-(q)\right]\right)(t,x)\,dx = 0, \qquad \text{for each } 0<t<T.
$$

By the Fatou Lemma, Remark 3.1, and (3.6), sending $R\to\infty$ yields

$$
0 \le \int_{\mathbb{R}}\left(\overline{q^2} - q^2\right)(t,x)\,dx \le 0, \qquad 0<t<T,
$$

and, since the argument holds for each $T>0$, we are done. □

4. Proof of Theorem 1.1

In this last section we prove Theorem 1.1. The first step consists in the proof of the existence of solutions for (1.5).

Lemma 4.1. *Assume* (1.3) *and* (2.3). *Then there exists an admissible weak solution of* (1.5), *satisfying* (*iii*) *and* (*iv*) *of Theorem 1.1.*

Proof. The conditions (*i*), (*ii*), (*iv*) of Definition 1.1 are satisfied, due to (2.3), (2.4) and Lemma 3.2. We have to verify (*ii*). Due to Lemma 3.10, we have

$$
q_\varepsilon \;\to\; q \qquad \text{strongly in } L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}). \tag{4.1}
$$

Clearly (3.2), (3.3), and (4.1) imply that v is a distributional solution of (1.14). Therefore $u := v+\varphi$ is a weak solution of (1.5) and v is an admissible perturbation of (1.1). Finally, (iii) and (iv) of Theorem 1.1 are consequences of (2.5) and (2.6), respectively. □

The second step is the existence of the semigroup.

Lemma 4.2. *There exists a strongly continuous semigroup of solutions associated with the Cauchy problem* (1.5)

$$S:\ [0,\infty)\times(0,\infty)\times\mathbb{R}\times H^1(\mathbb{R})\longrightarrow C([0,\infty)\times\mathbb{R}),$$

namely, for each $v_0\in H^1(\mathbb{R})$, $\gamma>0$, $\delta\in\mathbb{R}$ the map $u(t,x)=S_t(\gamma,\delta,v_0)(x)$ is an admissible weak solution and $u-\varphi$ and admissible perturbation of (1.5). *Moreover, (iii) and (iv) of Theorem 1.1 are satisfied.*

Clearly, this lemma is a direct consequence of the following one and of the ones in the previous sections.

Lemma 4.3. *Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\{\mu_n\}_{n\in\mathbb{N}}\subset(0,\infty)$ and v, $w\in L^\infty([0,T];H^1(\mathbb{R}))\cap H^1([0,T]\times\mathbb{R})$, for each $T\geq 0$, be such that ε_n, $\mu_n\to 0$ and*

$$v_{\varepsilon_n}\to v,\quad v_{\mu_n}\to v,\quad \textit{strongly in } L^\infty([0,T];H^1(\mathbb{R})),\quad T>0,$$

then

$$v=w.$$

Proof. Let $t>0$. From [2, Theorem 3.1], we have that

$$\|v_\varepsilon(t,\cdot)-v_\mu(t,\cdot)\|_{H^1(\mathbb{R})}\leq A(t,\varepsilon+\mu)\|v_{0,\varepsilon}-v_{0,\mu}\|_{H^1(\mathbb{R})}+B(t,\varepsilon+\mu)|\varepsilon-\mu|,$$

with

$$A(t,\varepsilon+\mu)=\mathcal{O}\big(e^{t/(\varepsilon+\mu)}\big),\qquad B(t,\varepsilon+\mu)=\mathcal{O}\big(e^{t/(\varepsilon+\mu)}\big),$$

for each ε, $\mu>0$. Hence

$$\|v_{\varepsilon_n}(t,\cdot)-v_{\mu_n}(t,\cdot)\|_{H^1(\mathbb{R})}\leq c_1e^{t/(\varepsilon+\mu)}\big(|\varepsilon_n-\mu_n|+\|v_{0,\varepsilon_n}-v_{0,\mu_n}\|_{H^1(\mathbb{R})}\big),\quad n\in\mathbb{N},$$

for some constant $c_1>0$. Choosing suitable subsequences as in [3, Lemma 7.2] we get $v=w$. □

The third and last step is the stability of the semigroup.

Lemma 4.4. *The semigroup S defined on $[0,\infty)\times(0,\infty)\times\mathbb{R}\times H^1(\mathbb{R})$ satisfies the stability property (ii) of Theorem 1.1.*

Proof. Fix $\varepsilon>0$ and denote by S^ε the semigroup associated to the viscous problem (2.1) and $\widetilde{S}:=S-\varphi$. Choose $\{v_{0,n}\}_{n\in\mathbb{N}}\subset H^1(\mathbb{R})$, $\{\gamma_n\}_{n\in\mathbb{N}}\subset(0,\infty)$, $\{\delta_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$, $v_0\in H^1(\mathbb{R})$, $\gamma>0$, $\delta\in\mathbb{R}$ satisfying (1.8). The initial data satisfy $v_{0,\varepsilon,n}$, $v_{0,\varepsilon}\in H^2(\mathbb{R})$ and (2.3). Finally, write

$$v_{\varepsilon,n}:=S^\varepsilon(\gamma_n,\delta_n,v_{0,n}),\qquad v_n:=S(\gamma_n,\delta_n,v_{0,n}),\qquad v:=S(\gamma,\delta,v_0).$$

Let $t>0$. Due to Lemmas 3.2 and 4.1,

$$\|v_n(t,\cdot)-v(t,\cdot)\|_{H^1(\mathbb{R})}=\lim_{\varepsilon\to 0}\|v_{\varepsilon,n}(t,\cdot)-v_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}.$$

Using [2, Theorem 3.1], we have that

$$\|v_{\varepsilon,n}(t,\cdot)-v_\varepsilon(t,\cdot)\|_{H^1(\mathbb{R})}\leq A(t,\varepsilon)\|v_{0,n}-v_0\|_{H^1(\mathbb{R})}+B(t,\varepsilon)\big(|\gamma_n-\gamma|+|\delta_n-\delta|\big),$$

with

$$A(t,\varepsilon)=\mathcal{O}(e^{T/\varepsilon}),\qquad B(t,\varepsilon)=\mathcal{O}(e^{T/\varepsilon}),\quad t\in[0,T].$$

Now, using the same argument as in [3, Lemma 8.1] we prove the claim. □

Proof of Theorem 1.1. It is direct consequence of Lemmas 4.2 and 4.4. □

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