Upwinding of source term at interfaces for Euler equations with high friction

Haythem Ounaissa, Benoît Perthame, Francois Bouchut
Emails: ounaissa@dma.ens.fr, perthame@dma.ens.fr, fbouchut@dma.ens.fr
Département de Mathématiques et Applications, CNRS UMR 8553
École Normale Supérieure
45, rue d'Ulm, 75230 Paris Cedex 05, France

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Abstract

We consider Euler equations with a friction term that describe an isentropic gas flow in a porous domain. More precisely, we consider the transition between low and high friction regions. In the high friction region the system is reduced to a parabolic equation, the *porous media equation*. In this paper we present a hyperbolic approach based on a finite volume technique to compute numerical solutions for the system in both regimes. The *Upwind Source at Interfaces* (USI) scheme we propose satisfies the following properties. Firstly it preserves the nonnegativity of gas density. Secondly and this is the motivation, the scheme is asymptotically consistent with the limit model (porous media equation) when the friction coefficient goes to infinity. We show analytically and through numerical results, that the above properties are satisfied. We shall also compare results given with the use of *USI*, hyperbolic-parabolic coupling and classical centered sources schemes.

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Contents

1 Introduction

The 2×2 Euler system describes an isentropic gas flow at a time $t \geq 0$ and at a point $x \in \mathbb{R}$ through the gas density $\varrho(t,x) \geq 0$ and its velocity $u(t,x) \in \mathbb{R}$ by the hyperbolic equations

$$\begin{cases} \dfrac{\partial}{\partial t}\varrho + \dfrac{\partial}{\partial x}(\varrho u) = 0, \qquad t \geq 0,\ x \in \mathbb{R}, \\[2ex] \dfrac{\partial}{\partial t}(\varrho u) + \dfrac{\partial}{\partial x}\Big(\varrho u^2 + p(\varrho)\Big) = -\alpha \varrho u, \end{cases} \tag{1}$$

where α is the friction coefficient. We consider only polytropic gases, hence the pressure is given by the equation of state

$$p(\varrho) = \kappa \varrho^{\gamma}, \qquad 1 < \gamma \leq 3, \qquad \kappa > 0. \tag{2}$$

A classical approach for solving systems of conservation laws consists in using finite volume technique which requires to compute fluxes at the control volumes interfaces, and the overall stability of the method requires some upwinding in the interpolation of the fluxes.

In this paper we restrict our study to the one dimensional case and consider a heterogeneous domain composed mainly of two areas. The first is transparent, i.e., the friction coefficient vanishes, whereas the second is porous and characterized by a very large friction coefficient $\alpha(x)$ ($\alpha \gg 1$). It was proved in [18] that in this area, the system is reduced in an appropriate time scale to a parabolic equation called *porous media equation*

$$\frac{\partial}{\partial \tau}\varrho - \frac{\partial^2}{\partial x^2}p(\varrho) = 0, \qquad \tau = \frac{t}{\alpha}, \quad t \geq 0, \quad x \geq x_0, \tag{3}$$

where x_0 is the interface separating the two regions.

Several approaches to compute such a transition low to large friction can be proposed. At first, one can try to use a classical solver with centered friction term, but it is in practice computationally too expensive, we will show that the mesh size should be smaller than $1/\alpha$! A second approach is to couple the hyperbolic homogeneous scheme to parabolic scheme in the regions where they apply; this has the drawback not to capture the transition but it copes with extreme cases. A third approach, this is our contribution in this paper, consists in designing a hyperbolic method that copes with the two regimes, in particular it preserves *Darcy steady states*. Being given a finite volume solver for the homogeneous problem with a certain consistency prorperty (32) below, we show that the source term can be discretized at interfaces and upwinded so as to be consistent with both regimes. We prove that this numerical scheme not only preserves non-negativity of the gas density, but also it is asymptotically consistent with the limit system when α takes very large values.We compare via numerical tests results given by the three approaches.

Thus, our main point is to derive a finite volume scheme which incorporates an appropriate discretization of the source term $\alpha\varrho u$. It is known, and widely used, since several years that an accurate method to achieve this is to upwind the source at interface. Generally this method follows from the fact of balancing the source term so as to preserve steady states. It was introduced independently by several authors Roe[23], LeRoux and coauthors [6], [9], and now is well understood in various contexts [2], [24], [5], [11]. For shallow water system and when focusing on steady states of a lake at rest, such a balancing can be achieved with a unique method whatever is the hyperbolic solver, and with nonlinear stability properties, see [1]. Here we will follow the spirit of this construction for the problem of transition hyperbolic/parabolic. Notice however an important difference: there is no balancing here because non trivial steady states do not exist for a fixed friction term. Our guide line is to preserve the steady states of the limiting porous media equation.

Hyperbolic balance laws with stiff source terms often lead to parabolic asymptotics. Of course, their numerical treatment requires specific schemes. The most famous cases arise in kinetic theory as in Rooseland approximation of neutron or radiative transfer, see [8, 12, 4, 10]. The idea to dicretize the source at interfaces already appears here.

This paper begins with the diffusive limit of the system (1) where we show the relation between high friction and porous media equation and the *hyperbolic-parabolic coupling* approach (section 2). In section 3, we present a hyperbolic approach namely the so-called *USI scheme* used to compute solutions for (1). And finally, in section 4, we present numerical results to compare various approaches.

2 Diffusive limit

In this section we recall the relation between high friction and porous media equation. We also recall the general framework of finite volume schemes and introduce the hyperbolic-parabolic coupling method that will serve later for comparison between various possible approaches.

2.1 Parabolic rescaling

We recall the theorem proved by Marcati and Milani in [18] which we summarize as

Theorem 2.1 *With the equation of state (2), consider for all $\varepsilon > 0$ the system of equations*

$$\begin{cases} \dfrac{\partial}{\partial \tau}\varrho^\varepsilon + \partial_x(\varrho^\varepsilon v^\varepsilon) = 0, \qquad \tau = \varepsilon t, \quad t \geq 0,\ x \in \mathbb{R}, \\ \varepsilon^2 \dfrac{\partial}{\partial \tau}(\varrho^\varepsilon v^\varepsilon) + \dfrac{\partial}{\partial x}\Big(\varepsilon^2 \varrho^\varepsilon (v^\varepsilon)^2 + p(\varrho^\varepsilon)\Big) = -c\varrho^\varepsilon v^\varepsilon. \\ \varrho^\varepsilon(0,x) = \varrho_0(x) \geq 0, \quad v^\varepsilon(0,x) = v_0(x), \quad p^\varepsilon = p(\varrho^\varepsilon), \quad v^\varepsilon = \dfrac{u^\varepsilon}{\varepsilon}. \end{cases} \tag{4}$$

with $c > 0$. Then there exist limit functions ϱ and v such that as $\varepsilon \to 0$, $\varrho^\varepsilon \to \varrho$ in L^p_{loc}, $v^\varepsilon \to v$ in L^2 weak and $\sqrt{\varepsilon} v^\varepsilon \to 0$ in L^p_{loc}, for all $p \in]1, +\infty[$. Moreover, ϱ satisfies, in the sense of distribution, Darcy's law

$$\frac{\partial}{\partial x} p(\varrho) = -c\varrho v. \tag{5}$$

As a consequence, ϱ is a weak solution of the porous media equation

$$\frac{\partial}{\partial \tau}\varrho - \frac{\partial^2}{\partial x^2}\frac{p(\varrho)}{c} = 0, \qquad \tau \geq 0,\ x \geq x_0. \tag{6}$$

Notice that, as usual for compressible flows, one of the difficulties in this result is to deal with vacuum. Now we consider the system (1), according to physics, when the friction becomes very high the flow velocity tends to zero, this result is justified by the following

Theorem 2.2 *Consider the system (1), when $\alpha \to \infty$, then $u \to 0$ in L^p strongly in the sets $\varrho \geq \varrho_{min} > 0$.*

Proof The gas dynamics system admits a convex entropy, namely the physical energy given by

$$E = \varrho u^2/2 + \varrho e(\varrho), \quad e'(\varrho) = \frac{p(\varrho)}{\varrho^2},$$

where e represents the internal energy. The entropy flux associated is

$$G = \Big(\varrho u^2/2 + \varrho e(\varrho) + p(\varrho)\Big)u.$$

It follows that the system (1) satisfies the following entropy inequality

$$\partial_t E(\varrho, u) + \partial_x G(\varrho, u) \leq -\alpha \varrho u^2, \tag{7}$$

which implies that

$$\int_0^\infty \int_{x\in\mathbb{R}} \alpha \varrho u^2 dx dt \leq \int_{x\in\mathbb{R}} E^0(x) dx.$$

On the other hand, we know from [17] that u is bounded in L^∞ therefore the result follows.□

When $\alpha \to \infty$, u can be written (at least formally) in smooth regions

$$u = \frac{v}{\alpha} + o(\frac{1}{\alpha}),$$

and with this notation and $\tau = t/\alpha$, for α large, the system (4) is another version of (1), we refer to [18] for more details.

2.2 Numerical Scheme

For a later purpose, we consider a standard finite volume scheme for

$$\frac{\partial}{\partial t}\varrho - \frac{\partial}{\partial x}\Big(b(x)\frac{\partial}{\partial x}F(\varrho)\Big) = 0, \tag{8}$$

with $F \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ a non decreasing function such that $F(0) = 0$ and $b \in L^\infty$ with $b \geq \underline{b} > 0$. We approximate the solution of (8) by discrete values $\varrho_i^n, i \in \mathbb{Z}, n \in \mathbb{N}$. In order to do so, we consider a grid of points $x_{i+1/2}$, $i \in \mathbb{Z}$,

$$... < x_{-1/2} < x_{1/2} < x_{3/2} ...$$

We define also cells and their lengths

$$C_i =]x_{i-1/2}, x_{i+1/2}[, \quad h_i = x_{i+1/2} - x_{i-1/2}, \quad h_{i+1/2} = \frac{h_i + h_{i+1}}{2}, \quad h = \sup_{i\in\mathbb{Z}} h_i.$$

Here, we will always consider grids which are regular enough. An explicit, three points, finite volume scheme for (8) is

$$\varrho_i^{n+1} - \varrho_i^n + \frac{\Delta t}{h_i}(F_{i+1/2}^n - F_{i-1/2}^n) = 0, \quad \forall n \in \mathbb{N}, \forall i \in \mathbb{Z} \tag{9}$$

where $F_{i+1/2}^n$ is given by

$$F_{i+1/2}^n = \frac{b_{i+1/2}}{h_{i+1/2}}\mathcal{F}(\varrho_i^n, \varrho_{i+1}^n), \quad b_{i+1/2} \approx \frac{1}{h_{i+1/2}}\int_{x_i}^{x_{i+1}} b(x) dx. \tag{10}$$

where the flux function $\mathcal{F}$ is defined as

$$\mathcal{F}(u, v) = F(u) - F(v), \quad \forall u, v \in \mathbb{R}_+. \tag{11}$$

We request that the above scheme satisfies two basic properties: it is consistent with (8) and it preserves nonnegativity of gas density under a CFL condition .

2.2.1 Consistency

Definition 2.3 *We say that the scheme (9)-(10)-(11) is consistent with (8) if the numerical flux function $\mathcal{F}$ (11) satisfies*

$$\lim_{u\to u_0, v\to u_0} \frac{\mathcal{F}(u,v)}{u-v} = F'(u_0). \tag{12}$$

So when the function F is C^1 the scheme presented above is consistent.

2.2.2 Nonnegativity of ϱ

The scheme (9)-(10)-(11) keeps the gas density positive thanks to the following proposition

Proposition 2.4 *Assume $\varrho_i^0 \geq 0, \forall i \in \mathbb{Z}$, and the CFL condition*

$$2\Delta t \max_{j\in\mathbb{Z}}(b_{j+1/2}) \sup_{\varrho\in\mathbb{R}} F'(\varrho) \leq h_i \min_{j\in\mathbb{Z}}(h_{j+1/2}), \quad \forall i \in \mathbb{Z}. \tag{13}$$

Then we have $\varrho_i^n \geq 0\ \forall i \in \mathbb{Z}, \forall n \in \mathbb{N}$.

This CFL condition is restrictive, but it is well adapted to our purposes: the coupling with the hyperbolic regions and the transition regime.
Proof We know from (9)-(10)-(11) that

$$\varrho_i^{n+1} = \varrho_i^n - \sigma_i\Big[\frac{b_{i+1/2}}{h_{i+1/2}}[F(\varrho_i^n) - F(\varrho_{i+1}^n)] - \frac{b_{i-1/2}}{h_{i-1/2}}[F(\varrho_{i-1}^n) - F(\varrho_i^n)]\Big],$$

with $\sigma_i = \Delta t/h_i$. As F is C^1, the above equality may be written as follows

$$\varrho_i^{n+1} = \varrho_i^n - \sigma_i\Big[\frac{b_{i+1/2}}{h_{i+1/2}}F'(\xi_{i+1/2})(\varrho_i^n - \varrho_{i+1}^n) - \frac{b_{i-1/2}}{h_{i-1/2}}F'(\xi_{i-1/2})(\varrho_{i-1}^n - \varrho_i^n)\Big],$$

for some $\xi_{i-1/2}$ and $\xi_{i+1/2}$. This implies that ϱ_i^{n+1} is a convex combination of ϱ_i^n, ϱ_{i-1}^n and ϱ_{i+1}^n

$$\begin{aligned}\varrho_i^{n+1} &= \Big(1 - \sigma_i\frac{b_{i+1/2}}{h_{i+1/2}}F'(\xi_{i+1/2}) - \sigma_i\frac{b_{i-1/2}}{h_{i-1/2}}F'(\xi_{i-1/2})\Big)\varrho_i^n + \sigma_i\frac{b_{i+1/2}}{h_{i+1/2}}F'(\xi_{i+1/2})\varrho_{i+1}^n \\ &+ \sigma_i\frac{b_{i-1/2}}{h_{i-1/2}}F'(\xi_{i-1/2})\varrho_{i-1}^n,\end{aligned}$$

the two last terms of the right hand side are nonnegative since F' is nonnegative. Then we check easily that the first term is also nonnegative whenever the CFL condition (13) is satisfied. □

2.2.3 Interface flux for hyperbolic-parabolic coupling

We now present the hyperbolic-parabolic coupling method between the hyperbolic and parabolic region of the domain. Then, the crucial point is how to compute a flux at the interface (separating the hyperbolic and parabolic region of the domain) that ensures density conservation. To do so, we denote by $x_0 = x_{i_0+1/2}$ the interface point, for $x < x_{i_0+1/2}$ we consider the transparent region described by (1) with $\alpha = 0$. For $x > x_{i_0+1/2}$ we consider (8). We construct an artificial velocity u_{i_0+1}

$$u_{i_0+1}^n = -\kappa \frac{\text{minmod}\Big((\partial_x \varrho^\gamma)_{i_0}^n, (\partial_x \varrho^\gamma)_{i_0+1}^n\Big)}{\alpha \varrho_{i_0+1}^n}, \tag{14}$$

and

$$\text{minmod}(x, y) = \begin{cases} \min(x, y) \text{ if } x, y \geq 0, \\ \max(x, y) \text{ if } x, y \leq 0, \\ 0 \text{ otherwise.} \end{cases}$$

Then we compute the flux at the interface using a solver for the homogeneous problem. To summarize, on the left transparent domain we use a classical hyperbolic scheme (see §3.1), and on the right (porous domain) we use (9)-(10). This construction is motivated by *Darcy Law*, as follows. For very large values of the friction, ϱ and u adjusts so as to satisfy

$$\partial_x p(\varrho) = -\alpha \varrho u,$$

and considering a discrete version we obtain (14).

3 USI Scheme

In this section, we present the defects of a classical hyperbolic approach to compute solutions of (1) when the friction term is centered. To overcome these defects, we propose the Upwind Source at Interfaces (USI) scheme and we show that it satisfies some stability and consistency properties. First, we prove that this scheme is consistent with (1), that it preserves nonnegativity of gas density and finally that it is asymptotically consistent with "Porous media" equation (8) with $b = 1/\alpha, F(\varrho) = p(\varrho)$.

3.1 Finite volume formalism

We consider again the system (1). The natural semi-implicit finite volume three points source centered scheme is the following

$$\begin{cases} \varrho_i^{n+1} - \varrho_i^n + \sigma_i \, (A_{i+1/2}^{\varrho,n} - A_{i-1/2}^{\varrho,n}) = 0, \\ \\ q_i^{n+1} - q_i^n + \sigma_i \, (A_{i+1/2}^{q,n} - A_{i-1/2}^{q,n}) = -\alpha_i q_i^{n+1}, \qquad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N}, \end{cases} \tag{15}$$

with the notation in §2.2 and the following definitions; we define $\sigma_i = \Delta t/h_i$ for some time step Δt which is chosen small enough using a CFL condition. Also the principle of finite volume methods is to use approximation in L^1 sense, namely we have in mind

$$\varrho_i^n \approx \frac{1}{h_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \varrho(n\Delta t, x)dx, \quad q_i^n \approx \frac{1}{h_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \varrho u(n\Delta t, x)dx, \quad \alpha_i \approx \frac{1}{h_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \alpha(x)dx.$$

The finite volume method is very classical and efficient, see [16, 7]. Now we shall see that the above scheme is not well adapted for the system (1) with α very large. It was shown in the first section that $u \to 0$ when $\alpha \to \infty$. We can prove the same result in the discrete case. Indeed, if the scheme satisfies some in-cell entropy inequalities, one has, setting $E_i^n = \varrho_i^n (u_i^n)^2/2 + \varrho_i^n e_i^n$

$$E_i^{n+1} - E_i^n + \sigma_i (A_{i+1/2}^{E,n} - A_{i-1/2}^{E,n}) \le -\alpha \Delta t \varrho_i^n (u_i^n)^2, \quad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N},$$

therefore

$$\alpha \Delta t \sum_n \sum_i h_i \varrho_i^n (u_i^n)^2 < \infty.$$

Thus, we have $u_i^n = O(1/\sqrt{\alpha})$ apart from vacuum i.e $\varrho_i^n > 0$. But from (15) we indeed expect that $u_i^n \sim 1/\alpha$. Now, up to an extraction, let us denote by r_i^n the limit of ϱ_i^n when $\alpha \to \infty$. The second equation of (15) becomes

$$\sigma_i \Big(\mathcal{A}^q(r_i^n, 0, r_{i+1}^n, 0) - \mathcal{A}^q(r_{i-1}^n, 0, r_i^n, 0) \Big) = -\lim_{\alpha \to \infty} \alpha \varrho_i^{n+1} u_i^{n+1}, \quad r_i^n = \lim_{\alpha \to \infty} \varrho_i^n, \quad n \in \mathbb{N}, \quad \forall i \in \mathbb{Z},$$

assuming the flux $\mathcal{A}$ is regular enough and $(r_i^n)_{n \in \mathbb{N}, i \in \mathbb{Z}}$ are L^∞ bounded, we conclude that the quantity $\alpha_i u_i^n$ is L^∞ bounded when $\varrho_i^n > 0$. Now let us analyze the behavior of the numerical flux when $\alpha \to \infty$. In fact given a solver $\mathcal{A}$ for the homogeneous system (regular enough), numerical fluxes are given by

$$A_{i+1/2}^n = \mathcal{A}(\varrho_i^n, u_i^n, \varrho_{i+1}^n, u_{i+1}^n), \quad \forall n \in \mathbb{N}, \quad \forall i \in \mathbb{Z},$$

when passing to the limit as $\alpha \to \infty$ we obtain

$$A_{i+1/2}^{n,\varrho} = \mathcal{A}^\varrho(r_i^n, 0, r_{i+1}^n, 0) + O(\frac{1}{\alpha}), \quad \forall n \in \mathbb{N}, \quad \forall i \in \mathbb{Z}.$$

For a uniform grid of size $h \to 0$, this does not go to zero in general, which means that such schemes are not asymptotically consistent with "porous media equation" (6). These kinds of schemes may give quite good results provided that we consider a mesh size h smaller than $1/\alpha$, but this solution is computationally too expensive (in terms of computation time and memory especially in the 2D case) as we consider very large values of α.

3.2 Upwinding the source at interfaces

We first recall the formalism of USI finite volume scheme and then we present our specific reconstruction at interfaces. A general introduction and theoretical aspects can be found in [3, 21, 13, 14].

We denote by U_i^n the cell-centered vector of discrete unknowns: $U_i^n = (\varrho_i^n, \varrho_i^n u_i^n)^t$. A USI finite volume scheme for (1) is the following

$$\frac{h_i}{\Delta t}(U_i^{n+1} - U_i^n) + A_{i+1/2}^n - A_{i-1/2}^n = S_i^n, \tag{16}$$

with sources given by

$$S_i^n = S_{i+1/2,-}^n + S_{i-1/2,+}^n \equiv \begin{pmatrix} 0 \\ p(\varrho_{i+1/2,-}^n) - p(\varrho_i^n) + p(\varrho_i^n) - p(\varrho_{i-1/2,+}^n) \end{pmatrix}.$$

and numerical fluxes are computed such that

$$A_{i+1/2}^n = \mathcal{A}(U_{i+1/2,-}^n, U_{i+1/2,+}^n), \tag{17}$$

$\mathcal{A}$ satisfies $\mathcal{A}(U,U) = A(U)$, A and $U_{i+1/2,\pm}$ are given by

$$A(U) = \begin{pmatrix} \varrho u \\ \varrho u^2 + p(\varrho) \end{pmatrix}, \quad U_{i+1/2,-}^n = \begin{pmatrix} \varrho_{i+1/2,-}^n \\ \varrho_{i+1/2,-}^n u_i^n \end{pmatrix}, \quad U_{i+1/2,+}^n = \begin{pmatrix} \varrho_{i+1/2,+}^n \\ \varrho_{i+1/2,+}^n u_{i+1}^n \end{pmatrix}. \tag{18}$$

The new reconstructed variables are

$$\begin{cases} \kappa(\varrho_{i+1/2,-}^n)^\gamma = \Big(\kappa(\varrho_i^n)^\gamma - \alpha_i(\varrho_i^n u_i^n)_+ h_{i+1/2}\Big)_+, \\ \kappa(\varrho_{i+1/2,+}^n)^\gamma = \Big(\kappa(\varrho_{i+1}^n)^\gamma + \alpha_{i+1}(\varrho_{i+1}^n u_{i+1}^n)_- h_{i+1/2}\Big)_+, \end{cases} \tag{19}$$

where

$$(\varrho_i^n u_i^n)_+ = \max(0, \varrho_i^n u_i^n), \quad (\varrho_i^n u_i^n)_- = \min(0, \varrho_i^n u_i^n).$$

The motivation of this reconstruction is that when $\alpha \to \infty$, ϱ satisfies formally Darcy's law

$$\partial_x \Big(\kappa \underline{\varrho}^\gamma\Big) = -\underline{\varrho} v, \quad \underline{\varrho} = \lim_{\alpha \to \infty} \varrho, \quad v = \lim_{\alpha \to \infty} \alpha u. \tag{20}$$

When integrating the above relation between x_i and $x_{i+1/2,-}$, then between $x_{i+1/2,+}$ and x_{i+1} we obtain

$$\begin{cases} \kappa \underline{\varrho}^{\gamma}_{i+1/2,-} = \kappa \underline{\varrho}^{\gamma}_{i} - \underline{\varrho}_i v_i \dfrac{h_i}{2}, \\ \kappa \underline{\varrho}^{\gamma}_{i+1/2,+} = \kappa \underline{\varrho}^{\gamma}_{i+1} + \underline{\varrho}_{i+1} v_{i+1} \dfrac{h_{i+1}}{2}. \end{cases} \tag{21}$$

Notice that there is a difference between formulas (19) and (21). In fact $\varrho^n_{i+1/2,\pm}$ must be nonnegative, hence we take the positive part of $\kappa(\varrho^n_i)^\gamma \pm \alpha_i(\varrho^n_i u^n_i)_\pm h_{i\pm1/2}$. Moreover, $\varrho^n_{i+1/2,\pm}$ should satisfy $\varrho^n_{i+1/2,-} \le \varrho^n_i$ and $\varrho^n_{i+1/2,+} \le \varrho^n_{i+1}, \forall i \in \mathbb{Z}, \forall n \ge 0$ which is sufficient to ensure nonnegativity of gas density at the next time step, this will be seen later in §3.2.2. And finally for a reason of asymptotic consistency of the scheme, we replace h_i and h_{i+1} by $h_{i+1/2}$ (see §3.3).

3.2.1 Hyperbolic consistency

Of course we wish that this method is firstly consistent with the hyperbolic system (1). In fact this property is sufficient to ensure that when $h \to 0$, if the method converges then numerical solutions converge to the solution of (1) (consistency in the sense of Lax-wendroff). We show here that the numerical scheme presented in the previous subsection satisfies this theoretical property.

Let us start by rewriting the scheme as

$$h_i \frac{d}{dt} U_i(t) + \underline{A}_{i+1/2}(t) - \underline{A}_{i-1/2}(t) = \underline{S}_{i+1/2-} + \underline{S}_{i-1/2+}, \tag{22}$$

$$\underline{S}_{i+1/2,-} = \underline{S}^-(U_i, U_{i+1}, \alpha_i, \alpha_{i+1}, h_{i+1/2}) = \underline{A}_{i+1/2} - A_{i+1/2} + S_{i+1/2,-}$$

$$\underline{S}_{i-1/2,+} = \underline{S}^+(U_{i-1}, U_i, \alpha_{i-1}, \alpha_i, h_{i-1/2}) = -\underline{A}_{i-1/2} + A_{i-1/2} - S_{i-1/2,+}$$

where

$$\underline{A}_{i+1/2} = \mathcal{A}(U_i, U_{i+1}), \quad A_{i+1/2} = \mathcal{A}(U_{i+1/2,-}, U_{i+1/2,+}).$$

To prove consistency of the scheme with system (1), we apply criterion in [21], i.e, we have to check the followings

i) $\mathcal{A}(U,U) = A(U)$,

ii) $\underline{S}^-(U,V,\alpha,\beta,0) = \underline{S}^+(U,V,\alpha,\beta,0) = 0$,

iii) $\lim_{h\to0}\Big(\underline{S}^-(U,U,\alpha,\alpha,h) + \underline{S}^+(U,U,\alpha,\alpha,h)\Big)/h = (0, -\alpha\varrho u)^t$,

where $U = (\varrho, u)^t$.

Theorem 3.1 *Consider a flux function $\mathcal{A}$ consistent with the exact flux i.e $\mathcal{A}(U,U) = A(U)$. Then the USI scheme satisfies i), ii) and iii).*

Proof. The points *i)* and *ii)* are trivial. Concerning *iii)* we use the definition of $\varrho_{i+1/2,\pm}$ and the consistency of $\mathcal{A}$ with the exact flux A. For h small enough, $\kappa(\varrho_{i+1/2,\pm})^\gamma = \kappa(\varrho_i)^\gamma \pm \alpha_i(\varrho_i u_i)_\pm h_{i+1/2}$, then

$$(\underline{S}^- + \underline{S}^+)(U, U, \alpha, \alpha, h) = \Big(0, -\alpha(\varrho u)_+ - \alpha(\varrho u)_-\Big)^t h,$$

which proves *iii)* and the theorem.□

3.2.2 Positivity of ϱ

As a weak stability condition, the finite volume scheme has to ensure the nonnegativity of gas density. We prove in this section that this property is satisfied by the *USI* scheme in both discrete and semi-discrete version.

Semi discrete stability

Proposition 3.2 *Consider a solver $\mathcal{A}$ for the homogeneous problem that preserves nonnegativity of $\varrho_i(t)$ then the finite volume scheme keeps $\varrho_i(t)$ nonnegative.*

Proof. The statement that $\mathcal{A}$ preserves the nonnegativity of $\varrho_i(t)$ means that whenever $\varrho_i(t)$ vanishes, the following inequality

$$\mathcal{A}^\varrho(U_i, U_{i+1}) - \mathcal{A}^\varrho(U_{i-1}, U_i) \leq 0,$$

holds for all choices of the other arguments. Similarly, in our case, we have to check

$$\mathcal{A}^\varrho(U_{i+1/2,+}, U_{i+1/2,-}) - \mathcal{A}^\varrho(U_{i-1,+}, U_{i-1/2,-}) \leq 0,$$

whenever $\varrho_i = 0$. Notice that our reconstruction of $\varrho_{i+1/2,\pm}$ (19) ensures that $\varrho_{i+1/2,-} = \varrho_{i+1/2,+} = 0$. whenever ϱ_i vanishes, which concludes the proof.□

Fully discrete stability

In order to preserve the positivity of ϱ_i, the CFL condition that needs to be used is not more restrictive than that of the homogeneous problem.

Definition 3.3 *We say that a solver $\mathcal{A}$ preserves the nonnegativity of ϱ by interface with a numerical speed $\sigma(U_i, U_{i+1}) \geq 0$ under the CFL condition*

$$\sigma(U_i^n, U_{i+1}^n)\Delta t \leq \min(h_i, h_{i+1}), \tag{23}$$

if we have

$$\begin{aligned} &\varrho_i^n - \frac{\Delta t}{h_i}\Big(\mathcal{A}^\varrho(U_i^n, U_{i+1}^n) - \varrho_i^n u_i^n\Big) \geq 0, \\ &\varrho_{i+1}^n - \frac{\Delta t}{h_{i+1}}\Big(\varrho_{i+1}^n u_{i+1}^n - \mathcal{A}^\varrho(U_i^n, U_{i+1}^n)\Big) \geq 0. \end{aligned} \tag{24}$$

Proposition 3.4 *Assume that the solver $\mathcal{A}$ for the homogeneous problem preserves the nonnegativity of ϱ by interface, then the USI scheme also preserves the nonnegativity of ϱ by interface,*

$$\begin{aligned} &\varrho_i^n - \frac{\Delta t}{h_i}\Big(\mathcal{A}^\varrho(U^n_{i+1/2,-}, U^n_{i+1/2,+}) - \varrho_i^n u_i^n\Big) \geq 0,\\ &\varrho_{i+1}^n - \frac{\Delta t}{h_{i+1}}\Big(\varrho_{i+1}^n u_{i+1}^n - \mathcal{A}^\varrho(U^n_{i+1/2,-}, U^n_{i+1/2,+})\Big) \geq 0, \end{aligned} \tag{25}$$

under the CFL condition

$$\sigma(U^n_{i+1/2,-}, U^n_{i+1/2,+})\Delta t \leq \min(h_i, h_{i+1}). \tag{26}$$

Proof. Taking into account the CFL condition (26), the followings inequalities

$$\begin{aligned} &\varrho^n_{i+1/2,-} - \frac{\Delta t}{h_i}\Big(\mathcal{A}^\varrho(U^n_{i+1/2,-}, U^n_{i+1/2,+}) - \varrho^n_{i+1/2,-} u_i^n\Big) \geq 0,\\ &\varrho^n_{i+1/2,+} - \frac{\Delta t}{h_{i+1}}\Big(\varrho^n_{i+1/2,+} u_{i+1}^n - \mathcal{A}^\varrho(U^n_{i+1/2,-}, U^n_{i+1/2,+})\Big) \geq 0, \end{aligned}$$

hold. Moreover, our construction (19) ensures that $\varrho^n_{i+1/2,-} \leq \varrho^n_i$ and $\varrho^n_{i+1/2,+} \leq \varrho^n_{i+1}$, and as $1 + u_i^n \Delta t/h_i \geq 0$ and $1 - u_{i+1}^n \Delta t/h_{i+1} \geq 0$, the inequalities (25) hold, which concludes the proof.□

3.3 Asymptotic Consistency with porous media equation

We show in this paragraph that the *USI scheme* is asymptotically consistent with (6). This means that the asymptotic expansion when $\alpha \to \infty$ of the mass flux computed with *USI scheme* is a given consistent numerical flux to (6). Thus, the numerical scheme preserves Darcy's equilibrium for large values of the friction α.
From now $\mathcal{A}$ denotes a C^1 numerical flux consistent with the exact flux A i.e $\mathcal{A}(\varrho, u, \varrho, u) = A(\varrho, u) \equiv (\varrho u, \varrho u^2 + p(\varrho))$. Then, up to extraction we assume

$$\lim_{\alpha\to\infty} \varrho_i^n = r_i^n, \quad \lim_{\alpha\to\infty} u_i^n = 0, \quad \forall i \in \mathbb{Z}, \quad \forall n \in \mathbb{N}, \tag{27}$$

and we finally assume that α is constant.

Theorem 3.5 *Assume that $\mathcal{A}$ satisfies (32) below, (27) and the following asymptotic expansion of ϱ_i and u_i when $\alpha \to \infty$*

$$\varrho_i = r_i + \frac{r_i^{(1)}}{\alpha} + O(\frac{1}{\alpha^2}), \quad u_i = \frac{v_i^{(1)}}{\alpha} + \frac{v_i^{(2)}}{\alpha^2} + O(\frac{1}{\alpha^3}). \tag{28}$$

Then, as long as $r_i > 0$, we have

$$\mathcal{A}^\varrho(\varrho_{i+1/2,-}, u_i, \varrho_{i+1/2,+}, u_{i+1}) = \frac{\kappa}{\alpha h_{i+1/2}}(\varrho_i^\gamma - \varrho_{i+1}^\gamma) + O(\frac{h}{\alpha}) + O(\frac{1}{\alpha^2}). \tag{29}$$

Remark 3.6 *Theorem 3.5 expresses that for large values of the friction ($\alpha \to \infty$) we have*

$$\alpha \mathcal{A}^{\varrho}(\varrho_{i+1/2,-}, u_i, \varrho_{i+1/2,+}, u_{i+1}) = \underline{F}_{i+1/2} + O(h) + O(\frac{1}{\alpha}),$$

where $\underline{F}_{i+1/2}$ is a consistent flux for

$$\frac{\partial}{\partial \tau}\varrho - \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial x}(p(\varrho))\Big) = 0, \quad \tau \geq 0.$$

This means that the USI scheme is asymptotically consistent with porous media equation. In fact the main point is that the asymptotic expansion of $\mathcal{A}^{\varrho}_{i+1/2}$ does not contain terms in $O(h)$ which means that the mesh size does not depend on the friction. This makes the difference with the source centered scheme.

Proof. The proof is divided into two parts. For simplicity we shall write ϱ_i and u_i instead of ϱ_i^n resp u_i^n.

First part. Our aim in this paragraph is to select solvers for the homogeneous system that ensure

$$r_{i+1/2,-} = r_{i+1/2,+}, \quad \forall i \in \mathbb{Z}.$$

Consider the second discrete equation involving the momentum flux

$$\begin{aligned}\frac{h_i}{\Delta t}(\varrho_i^{n+1}u_i^{n+1} - \varrho_i u_i) \quad &+ \quad \Big(\mathcal{A}^{\varrho u}(\varrho_{i+1/2,-}, u_i, \varrho_{i+1/,+}, u_{i+1}) - \mathcal{A}^{\varrho u}(\varrho_{i-1/2,-}, u_{i-1}, \varrho_{i-1/2,+}, u_i)\Big) \\ &= \quad \kappa(\varrho_{i+1/2,-})^{\gamma} - \kappa(\varrho_{i-1/2,+})^{\gamma} \quad \forall i \in \mathbb{Z},\end{aligned}$$

When passing to the limit as $\alpha \to \infty$ we obtain for all i

$$\mathcal{A}^{\varrho u}(r_{i+1/2,-}, 0, r_{i+1/2,+}, 0) - \mathcal{A}^{\varrho u}(r_{i-1/2,-}, 0, r_{i-1/2,+}, 0) = \kappa(r_{i+1/2,-})^{\gamma} - \kappa(r_{i-1/2,+})^{\gamma}, \quad (30)$$

with $r_{i+1/2,\pm}$ are given by

$$\begin{cases} \kappa r_{i+1/2,-}^{\gamma} = \Big(\kappa r_i^{\gamma} - r_i(v_i^{(1)})_+ h_{i+1/2}\Big)_+, \\ \kappa r_{i+1/2,+}^{\gamma} = \Big(\kappa r_{i+1}^{\gamma} + r_{i+1}(v_{i+1}^{(1)})_- h_{i+1/2}\Big)_+, \end{cases} \quad (31)$$

If $r_{i+1/2,-} = r_{i+1/2,+}$ for all i then, by consistency of $\mathcal{A}$, (30) holds. The aim of this first part is to find a condition on $\mathcal{A}$ that ensures uniqueness of these solutions to (30).

Proposition 3.7 *Assume that unique solutions of (30) are ($r_{i+1/2,-} = r_{i+1/2,+}$) for all i. Then for all R strictly positive, $\mathcal{A}$ satisfies*

- *If $\mathcal{A}^{\varrho u}(r, 0, R, 0) = \mathcal{A}^{\varrho u}(r, 0, r, 0)$ then $r = R$.*

- *If* $\mathcal{A}^{\varrho u}(r,0,R,0)=\mathcal{A}^{\varrho u}(R,0,R,0)$ *then* $r=R$.

Proof We may rewrite equation (30) as

$$\mathcal{A}^{\varrho u}(r_{i+1/2,-},0,r_{i+1/2,+},0)-\kappa(r_{i+1/2,-})^{\gamma}=\mathcal{A}^{\varrho u}(r_{i-1/2,-},0,r_{i-1/2,+},0)-\kappa(r_{i-1/2,+})^{\gamma},\quad \forall i\in\mathbb{Z}.$$

We choose $r_{i-1/2,-}=r_{i-1/2,+}$. By consistency of $\mathcal{A}$ with the exact flux we have

$$\mathcal{A}^{\varrho u}(r_{i-1/2,-},0,r_{i-1/2,+},0)=\mathcal{A}^{\varrho u}(r_{i-1/2,+},0,r_{i-1/2,+},0)\Longrightarrow r_{i-1/2,-}=r_{i-1/2,+}\quad \forall i\in\mathbb{Z}.$$

Similarly by choosing $r_{i+1/2,-}=r_{i+1/2,+}$

$$\mathcal{A}^{\varrho u}(r_{i+1/2,-},0,r_{i+1/2,+},0)=\mathcal{A}^{\varrho u}(r_{i+1/2,-},0,r_{i+1/2,-},0)\Longrightarrow r_{i+1/2,-}=r_{i+1/2,+}\quad \forall i\in\mathbb{Z}.$$

which achieves the proof.□
Proposition 3.7 expresses a necessary condition on the function $\mathcal{A}$ to ensure that unique solutions of (30) are $(r_{i+1/2,-})_{i\in\mathbb{Z}}=(r_{i+1/2,+})_{i\in\mathbb{Z}}$. A sufficient one is given by the following

Proposition 3.8 *Assume* $\mathcal{A}$ *satisfies*

$$\mathcal{A}^{\varrho u}(r,0,R,0)=\frac{\kappa}{2}(r^{\gamma}+R^{\gamma})\quad \forall r,R\in\mathbb{R}_{+};\tag{32}$$

and $(r_{i+1/2,\pm})_{i\in\mathbb{Z}}$ *satisfy (30). Then for all* i *we have* $r_{i+1/2,-}=r_{i+1/2,+}$.

Proof Taking into account (32) we rewrite (30) as

$$(r_{i+1/2,-})^{\gamma}-(r_{i+1/2,+})^{\gamma}=-\Big((r_{i-1/2,-})^{\gamma}-(r_{i-1/2,+})^{\gamma}\Big),\quad \forall i\in\mathbb{Z},$$

which means that there exists a constant C such that

$$(r_{i+1/2,-})^{\gamma}-(r_{i+1/2,+})^{\gamma}=(-1)^{i}C\quad \forall i\in\mathbb{Z},$$

besides, from mass conservation we deduce that

$$\lim_{i\to\infty} r_i=\lim_{i\to\infty} r_{i+1/2,-}=\lim_{i\to\infty} r_{i+1/2,+}=0,$$

thus, $r_{i+1/2,-}=r_{i+1/,+}$, for all i in $\mathbb{Z}$, which concludes the proof of the proposition.□
Second part. Now we compute the asymptotic expansion of the mass flux. From now, $(\partial_q\mathcal{A}^{\varrho})_{i+1/2,-}$ denotes the partial derivative of $\mathcal{A}^{\varrho}$ with respect to the q^{ith} variable at $(\varrho_{i+1/2,-},0,\varrho_{i+1/2,-},0)$. First we start by the asymptotic expansion of $\varrho_{i+1/2,+}-\varrho_{i+1/2,-}$ when $\alpha\to\infty$. Indeed, using the construction of $\varrho_{i+1/2,\pm}$ we obtain

$$\kappa(\varrho_{i+1/2,+})^{\gamma}-\kappa(\varrho_{i+1/2,-})^{\gamma}=\kappa(\varrho_{i+1})^{\gamma}-\kappa(\varrho_i)^{\gamma}+\alpha\varrho_i(u_i)_{+}h_{i+1/2}+\alpha\varrho_{i+1}(u_{i+1})_{-}h_{i+1/2},$$

when passing to the limit when $\alpha \to \infty$ we obtain an equality that relates $v_i^{(1)}$ to $v_{i+1}^{(1)}$

$$\kappa(r_{i+1})^\gamma - \kappa(r_i)^\gamma + r_i(v_i^{(1)})_+ h_{i+1/2} + r_{i+1}(v_{i+1}^{(1)})_- h_{i+1/2} = 0. \tag{33}$$

Then taking into account the above relation, a first order asymptotic expansion of $(\varrho_{i+1/2,+})^\gamma - (\varrho_{i+1/2,-})^\gamma$ is the following

$$\begin{aligned}
(\varrho_{i+1/2,+})^\gamma - (\varrho_{i+1/2,-})^\gamma \quad &= \quad \frac{1}{\alpha}\Big(\gamma(r_{i+1}^{\gamma-1} r_{i+1}^{(1)} - r_i^{\gamma-1} r_i^{(1)}) + \frac{h_{i+1/2}}{\kappa}(r_i v_i^{(2)} + r_i^{(1)} v_i^{(2)})\ \mathbf{1}_{\{\mathbb{R}_+^*\}}(u_i) \\
&+ \quad \frac{h_{i+1/2}}{\kappa}(r_{i+1} v_{i+1}^{(2)} + r_{i+1}^{(1)} v_{i+1}^{(2)})\ \mathbf{1}_{\{\mathbb{R}_-^*\}}(u_{i+1})\Big) + O(\frac{1}{\alpha^2}).
\end{aligned}$$

We introduce c_i and $b_{i+1/2}$ such that

$$c_i = \frac{h_{i+1/2}}{\gamma\kappa}(r_i v_i^{(2)} + r_i^{(1)} v_i^{(2)}), \quad b_{i+1/2} = r_{i+1}^{\gamma-1} r_{i+1}^{(1)} - r_i^{\gamma-1} r_i^{(1)},$$

it follows

$$\varrho_{i+1/2,+} - \varrho_{i+1/2,-} = \frac{1}{\alpha(r_{i+1/2,-})^{\gamma-1}}\Big(b_{i+1/2} + \ \mathbf{1}_{\{\mathbb{R}_+^*\}}(u_i)c_i + \ \mathbf{1}_{\{\mathbb{R}_-^*\}}(u_{i+1})c_{i+1}\Big) + O(\frac{1}{\alpha^2}).$$

Now we perform a first order asymptotic expansion of $A^\varrho_{i+1/2}$ at the point$(\varrho_{i+1/2,-}, 0, \varrho_{i+1/2,-}, 0)$

$$\begin{aligned}
\mathcal{A}^\varrho(\varrho_{i+1/2,-}, u_i, \varrho_{i+1/2,+}, u_{i+1}) \quad &= \quad \varrho_i u_i + (\varrho_{i+1/2,-} - \varrho_i)u_i + (\varrho_{i+1/2,+} - \varrho_{i+1/2,-})(\partial_3 \mathcal{A}^\varrho)_{i+1/2,-} \\
&+ \quad (u_{i+1} - u_i)(\partial_4 \mathcal{A}^\varrho)_{i+1/2,-} + O((\varrho_{i+1/2,+} - \varrho_{i+1/2,-})^2) \\
&+ \quad O((u_i)^2) + O((u_{i+1})^2),
\end{aligned}$$

and from relation (33) we deduce

$$r_i(v_i^{(1)})_+ = \frac{\kappa}{h_{i+1/2}}(r_i^\gamma - r_{i+1}^\gamma) - r_{i+1}(v_{i+1}^{(1)})_-.$$

Then we divide by α and we use (28)

$$\varrho_i u_i = \frac{\kappa}{\alpha h_{i+1/2}}(\varrho_i^\gamma - \varrho_{i+1}^\gamma) + \frac{d_{i+1/2}}{\alpha} + O(\frac{1}{\alpha^2}),$$

where

$$d_{i+1/2} = r_i(v_i^{(1)})_- - r_{i+1}(v_{i+1}^{(1)})_-.$$

It follows that

$$\begin{aligned}
\mathcal{A}^\varrho(\varrho_{i+1/2,-}, u_i, \varrho_{i+1/2,+}, u_{i+1}) \quad &= \quad \frac{\kappa}{\alpha h_{i+1/2}}\Big(\varrho_i^\gamma - \varrho_{i+1}^\gamma\Big) + \frac{d_{i+1/2}}{\alpha} + (\varrho_{i+1/2,-} - \varrho_i)u_i \\
&+ \quad \frac{(\partial_3 \mathcal{A}^\varrho)_{i+1/2,-}}{\alpha r_{i+1/2,-}^{\gamma-1}}\Big(b_{i+1/2} + \ \mathbf{1}_{\{\mathbb{R}_+^*\}}(u_i)c_i + \ \mathbf{1}_{\{\mathbb{R}_-^*\}}(u_{i+1})c_{i+1}\Big) \\
&+ \quad \frac{(v_{i+1}^{(1)} - v_i^{(1)})}{\alpha}(\partial_4 \mathcal{A}^\varrho)_{i+1/2,-} + O(\frac{1}{\alpha^2}).
\end{aligned}$$

Notice that $(\varrho_{i+1/2,-} - \varrho_i)$, $(v_{i+1}^{(1)} - v_i^{(1)})$, $b_{i+1/2}$, c_i and $d_{i+1/2}$ are $O(h)$ for all i, therefore we conclude that

$$\mathcal{A}^{\varrho}(\varrho_{i+1/2,-}, u_i, \varrho_{i+1/2,+}, u_{i+1}) = \frac{\kappa}{\alpha h_{i+1/2}}\Big(\varrho_i^{\gamma} - \varrho_{i+1}^{\gamma}\Big) + O(\frac{h}{\alpha}) + O(\frac{1}{\alpha^2}),$$

and thus Theorem 3.5 is proved.□

Remark 3.9 *The crucial point in the proof is the property (32) that implies that $(\varrho_{i+1/2,-})_{i\in\mathbb{Z}}$ and $(\varrho_{i+1/2,+})_{i\in\mathbb{Z}}$ have the same limits when $\alpha \to \infty$. This equality holds when using kinetic and Lax-Friedrichs scheme since they satisfy property (32). However, it is a restrictive property and it does not hold for Godunov scheme for instance.*

4 Numerical results

We conclude this paper with numerical examples that illustrate the results stated in the previous sections. In particular we hilight the defects of just centering the source and we compare these results to those given by the *USI scheme* and *coupled scheme*.
All numerical tests are performed with a kinetic solver for the homogeneous problem. This solver is based on the kinetic theory developed in [19] and has the advantage to satisfy our sufficient asymptotic consistency condition (32), to keep the gas density nonnegative, to verify a discrete in-cell entropy inequality and to be able to compute problems with shocks or vacuum.

We present a non stationary test case. The flow domain consists of two heterogeneous subdomains: transparent part and porous one

$$\alpha(x) = \begin{cases} 0, & \text{for} \quad x \le x_0, \\ \alpha_0 \gg 1, & \text{for} \quad x > x_0. \end{cases}$$

where x_0 is the interface coordinate. The initial conditions are

$$u(0,x) = 0, \quad \varrho(0,x) = \begin{cases} \varrho_l, & \text{for} \quad x \le x_0, \\ \varrho_r, & \text{for} \quad x > x_0. \end{cases}$$

where $\varrho_l > \varrho_r$. Note that this case corresponds to a Riemann problem for the homogeneous system. We consider a domain which length $L = 1m$ and $x_0 = L/4$. For all numerical tests, we use a CFL number 0.4.
Figure 1 illustrates the convergence speed of the solution given by a source centered hyperbolic approach. Note that the *reference* solution is reached by a mesh $h = 1/8000$. Which confirms the theoretical analysis presented in the previous section, indeed $h \le 1/\alpha$.
In figure 2 we show also the convergence speed of the solution given by the coupled scheme. The

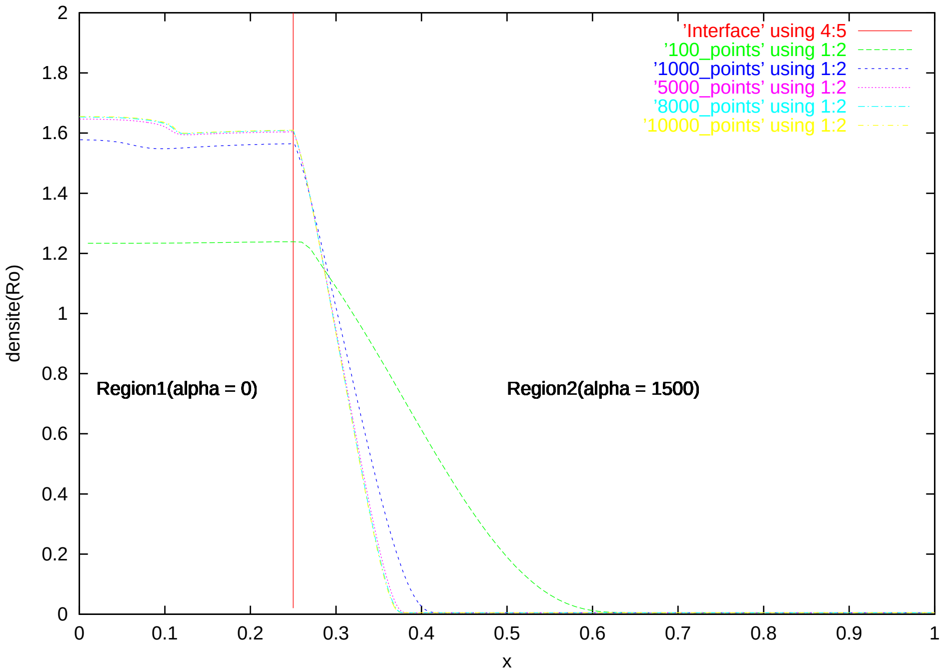

Figure 1: Gas density at tf=2 using a source centered hyperbolic scheme.

reference solution is given by the use of **2000 points**, and quite good solutions may be reached using 200 points.

Figure 3 illustrates the convergence speed of the solution given using the hyperbolic USI approach. Note that the *reference* solution is reached by the use of **2000 points**.
We also checked that the three approaches converge toward the same *reference solution*, this is shown in figure 4.
In figure 5 we compare solutions computed using 100 points with three approaches to the reference solution. The source centered hyperbolic approach is the less accurate one, this result is confirmed by figure 6 where we show the density error.

In order to illustrate the asymptotic consistency of the *USI scheme* with the *porous media equation*, we compared solutions given by *USI* and *hyperbolic-parabolic coupling* approach when we change the friction values. It is confirmed by figure 7 and 8 that for a fixed mesh size (we choose 1/400), when the friction takes very large values, solutions given by both approaches are very close.

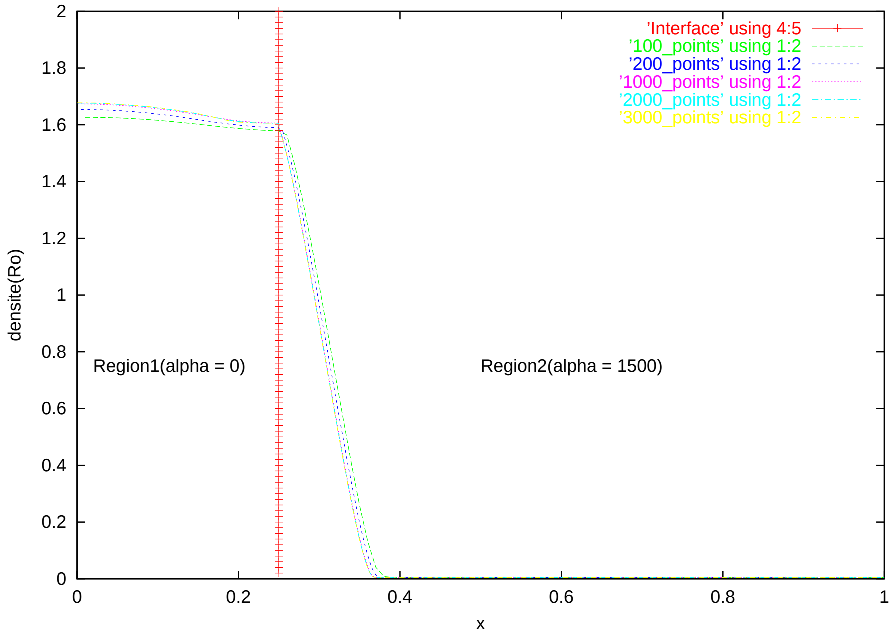

Figure 2: Gas density at tf=2 using the hyperbolic-parabolic coupling approach.

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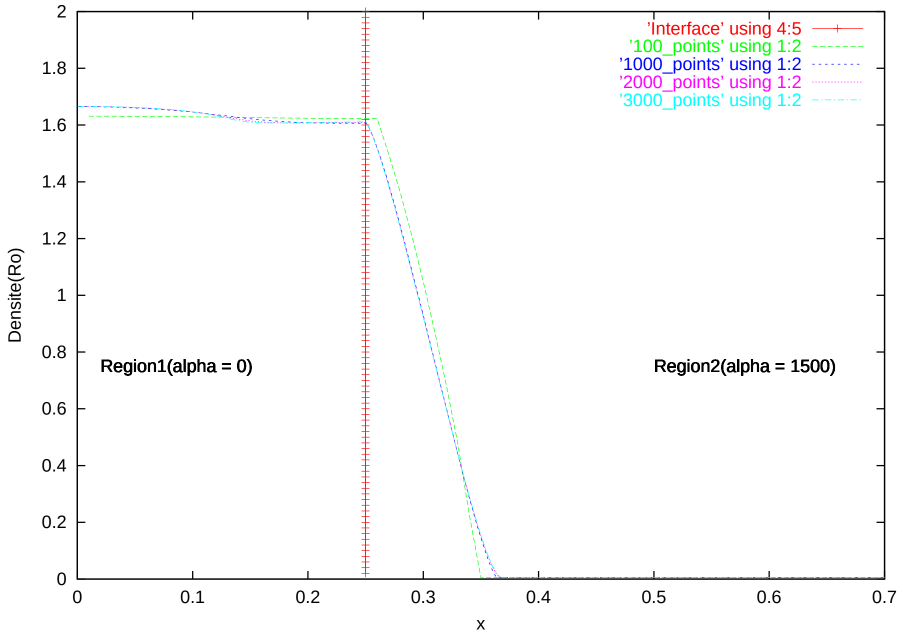

Figure 3: Gas density at tf=2 using the USI scheme

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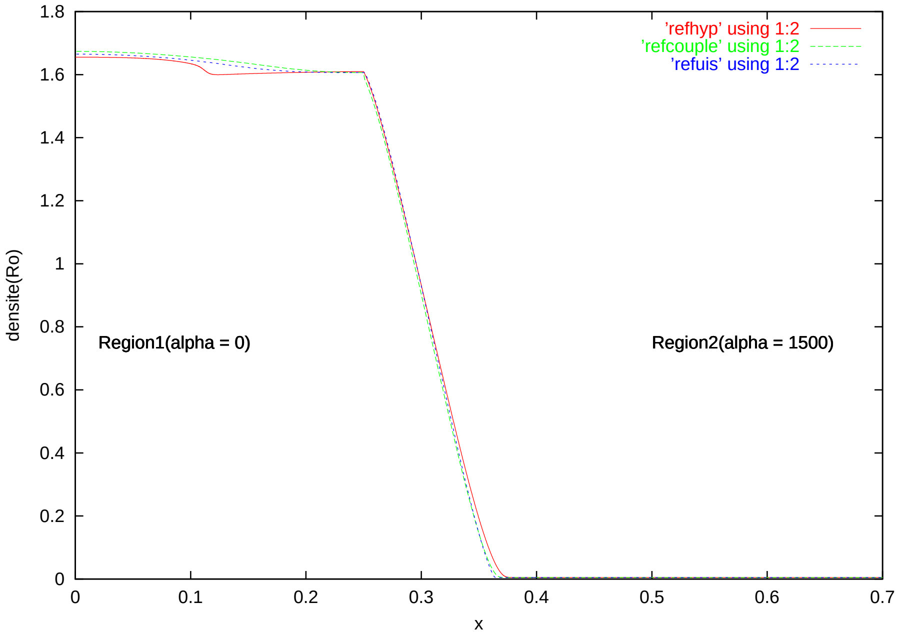

Figure 4: Reference solution given by three approaches at tf=2.

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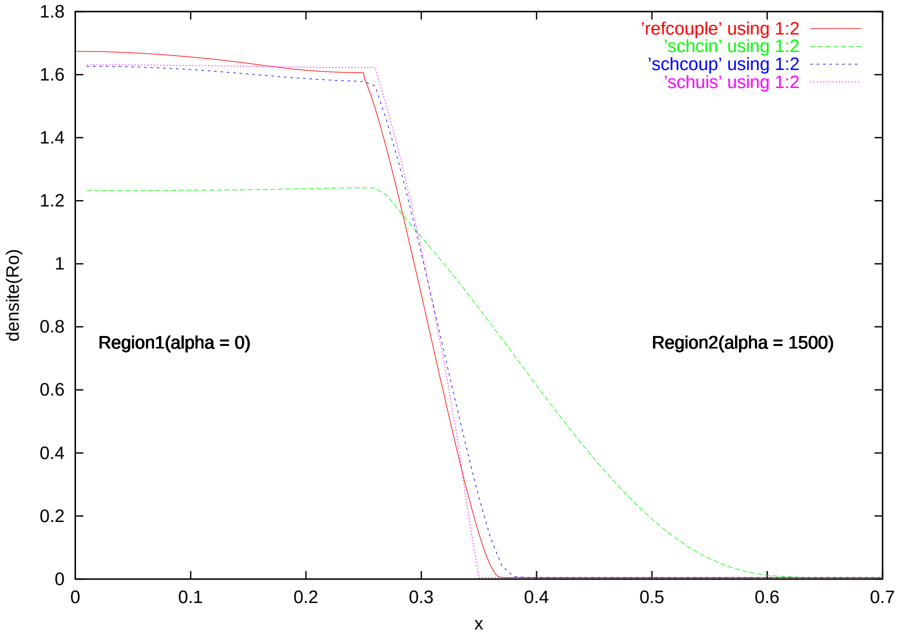

Figure 5: Comparison of solutions given using three approaches to the reference solution, tf=2, 100 points.

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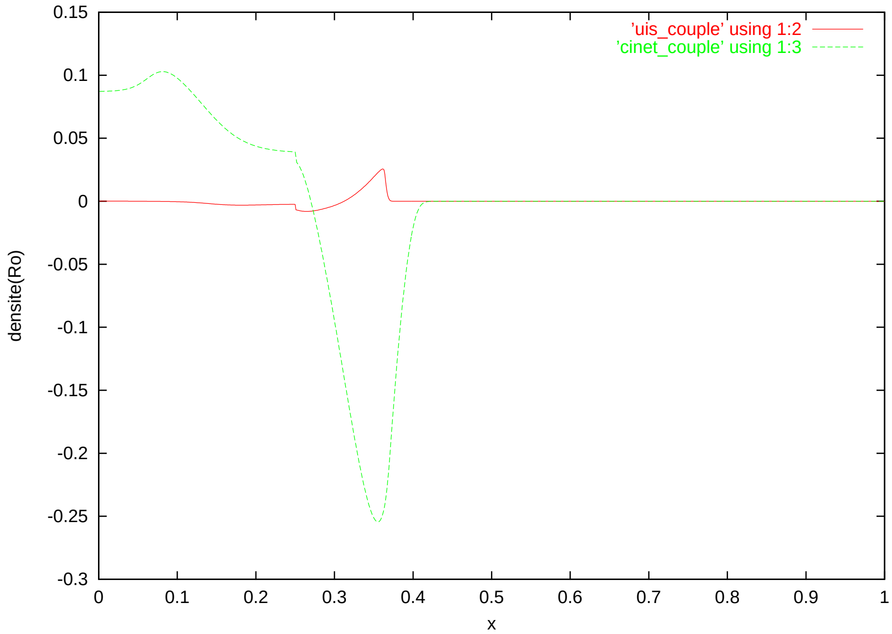

Figure 6: Density error: source centered hyperbolic scheme-coupled scheme, USI scheme - coupled scheme, tf=2, 1000 points.

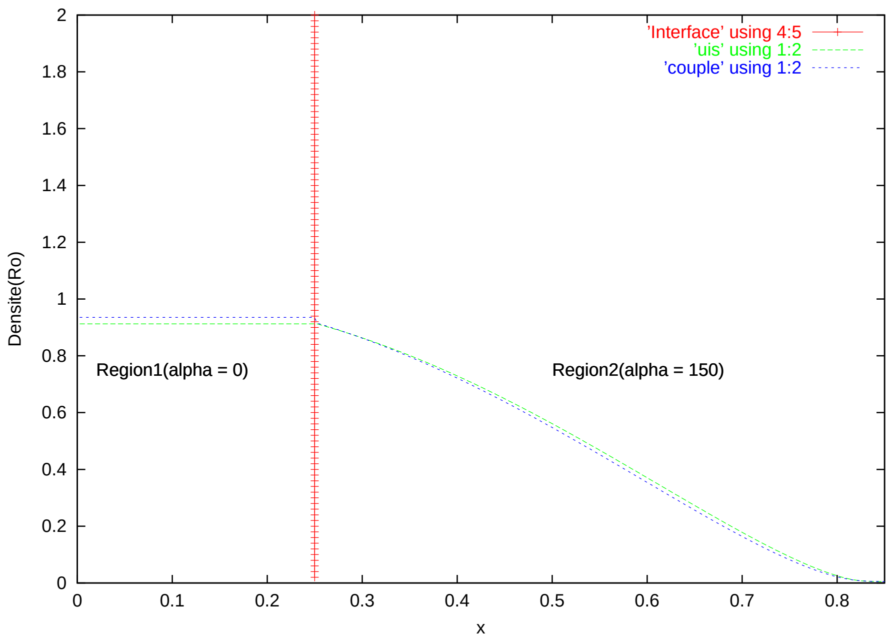

Figure 7: Solutions computed with USI and the coupled scheme: $\alpha = 150$, tf=6, 400 points.

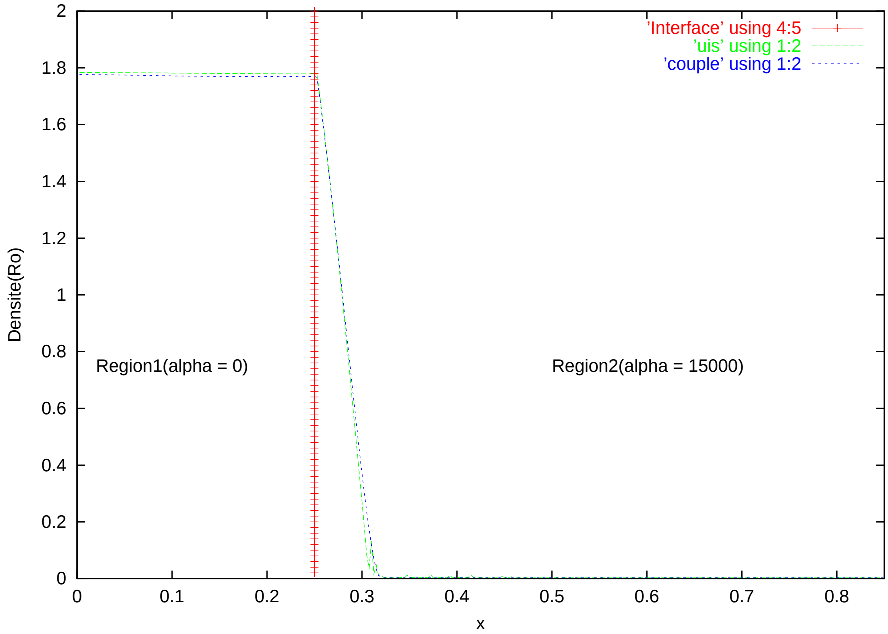

Figure 8: Solutions computed with USI and the coupled scheme: $\alpha = 15000$, tf=6, 400 points.