A mathematical Model for Rogue Waves, using Saint-Venant Equations with Friction

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Abstract

We propose to contruct a temporary wave on the surface of the ocean, as a particular solution of the Saint-Venant equations with a source term involving the friction, whose shape is expected to mimic a rogue wave.

The phenomenon of Rogue Waves, or Freak Waves, is a transitory phenomenon which appears in the open ocean, under the shape of a gigantic and devastator wave, because it accumulates a significant quantity of energy. Many article are devoted to them in [6]. We propose here an hydraulic model, making use of the Saint-Venant equations. The model is limited to only one dimension of space, in the direction of the propagation, since the side movements near the wave are uniform. The validity of this model requires a large wavelength, particularly in deep ocean. The wave itself has a rather short wavelength, but the whole phenomenon involves two waves of large wavelength, and relatively low amplitude.

We denote by H the mean depth of the ocean, and by c_s the sonic velocity of waves inside the water (we have $c_s = 1647\ ms^{-1}$). The wavelength λ of the Saint-Venant waves must satisfy a condition of the form

$$\lambda \ \geq \ 2\ N\ \frac{H\ \sqrt{gH}}{c_s} \quad , \tag{0.1}$$

where g is the gravity constant and N the number of sonic interactions (back-and-forth) between the bottom and the surface of the ocean. This condition means that along a horizontal distance of a wavelength λ, there are at least N such sonic interactions . The use of the Saint-Venant model is as more appropriate as N is great. We usually require $N \geq 25$, which implies for example a wavelength greater than 21400 m for an ocean depth of $3700m$. We suppose here, for simplicity, that the propagation goes from West ($x < 0$) to East ($x > 0$).

1 The initial configuration of the wave

We denote by $q = q(x, t)$ the ocean depth at a point x at a time t, and $m = m(x, t)$ the corresponding flux. We have $m = q\ u$, where u is the water velocity. The relative velocity of the waves of the ocean surface is given by

$$c \;=\; \sqrt{gq} \qquad (= c(q)) \;.$$

We denote by $k > 0$ the friction coefficient of Strickler and we consider as a simplification purpose that the bottom is flat. In this configuration, the Saint-Venant model reads

$$q_t \;+\; m_x \;=\; 0 \;, \qquad (1.1)$$

$$m_t \;+\; 2u\, m_x \;+\; \left(c^2 - u^2\right)\, q_x \;+\; k\; |u|\, u \;=\; 0 \;. \qquad (1.2)$$

We shall use the representation of waves proposed in [3] when a source term (here $k\,|u|\,u$) is present. The different states of the same wave are described by a segment of a straight line

$$m \;=\; A\, q \;-\; B \;, \qquad (1.3)$$

in the phase plane, that is the plane (q, m) here. The parameter A is a constant which corresponds to the wave velocity, of the form $A = u_{ref} - c(q_{ref})$ or $A = u_{ref} + c(q_{ref})$ depending if the wave is travelling eastwards (sign $+$) or westwards (sign $-$) and for a given reference state $M_{ref} = (q_{ref}, m_{ref})$, with $m_{ref} = q_{ref} u_{ref}$, of course. The parameter B is also a constant and is determined by the reference state since $B = m_{ref} + A\; q_{ref}$. We denote by M_0 the state of the ocean far on the west side, and by M_* the state of the ocean far on the east side. In both cases, the distances are supposed to be larger than the reference wavelength λ proposed in (0.1), which allows to use the Saint-Venant model. We suppose that these states also corresponds to reference velocity equal to zero; this assumption is linked to the hypothesis of a flat bottom of the ocean. That way M_0 represents a state $M_0 = (q_0, 0)$ and M_* represents a state $M_* = (q_*, 0)$. We suppose

$$q_0 \;>\; q_* \;\;.$$

A difference $q_0 - q_*$ of a few decimeters is enough even for a wide depth of the ocean. The profile of the initial state is made of two branches which meet in a state $P = (q_P, m_P)$ for example at $x = 0$, which corresponds to the choice of the origin. From the state P to the state M_*, that is on the East side, the profile is decreasing and referred to the state M_*. The corresponding states are so situated on the straight line

$$m \;=\; c_* \;\left(q - q_*\right) \quad ,$$

with $c_* = c(q_*) = \sqrt{gq_*}$. As a matter of fact, we have $A = c_*$ and $B = c_* q_*$ for this part of the wave profile. The explicit formulation of the profile is obtained, following [1], by inverting the profile relation

$$\psi_E(q) \;=\; x \;, \qquad (1.4)$$

where the index E stands for East. The inverse profile $\psi_E(q)$ is determined by integrating

$$\psi'_E(q) \;=\; \frac{B^2 - c^2 q^2}{k\,|Aq - B|\,(Aq - B)} \quad , \quad \psi_E(q_P) = 0 \;\;.$$

We need to have $q_* \;<\; q_0 \;<\; q_P$, in order to ensure an increasing profile on the West side, then a decreasing one on the East side. We get that way a positive flux: $Aq - B = c_*\,(q - q_*) \;>\; 0$, and by writing $\xi = \frac{q}{q_*}$ we get

$$\psi'_E(q) \;=\; \frac{1 - \xi^3}{k\,(1 - \xi)^2} \;=\; \frac{1 + \xi + \xi^2}{k\,(1 - \xi)} \;=\; \frac{1}{k}\,\left(\frac{3}{1 - \xi} \;-2\; - \xi\right) \;,$$

thus

$$\psi_E(q) \;=\; -\frac{3q_*}{k}\, ln\left(\frac{q-q_*}{q_P-q_*}\right) \;+\; \frac{2}{k}\;(q_P-q) \;+\; \frac{1}{2kq_*}\;\left(q_P^2-q_*^2\right)\ .$$

By inverting this function $\psi_E(q)$ and using (1.4) we get q as a decreasing function of x which is equal to q_P when $x = 0$. For the West side of the profile, with the index W, the reference state $M_{ref} = (q_{ref}, m_{ref})$ must correspond to a depth satisfying

$$q_{ref} \;\geq\; q_P\;(\geq q_0)\ \ ,$$

in order to ensure an increasing profile. The reference velocity associated with the state M_{ref} that is

$$A_{ref} \;=\; \frac{m_{ref}}{q_{ref}} \;+\; c_{ref}\quad ,\quad with\ \ c_{ref} \;=\; c(q_{ref}) \;=\; \sqrt{g q_{ref}}\ \ ,$$

is also the velocity of the West side profile of the wave and is positive. The straight line representing this West profile in the phase plane has the form

$$m \;=\; m_{ref} \;+\; A_{ref}\;(q-q_{ref})\ \ ,$$

and passes by the state $M_0 = (q_0, 0)$. Hence

$$m_{ref} \;=\; A_{ref}\;(q_{ref}-q_0)\quad ,\quad B_{ref} \;=\; c_{ref}\;q_{ref}\ \ .$$

The invert profile is described by a function ψ_W satisfying

$$\psi_W'(q) \;=\; \frac{B_{ref}^2 - c_{ref}^2 q_{ref}^2}{k\;\left(A_{ref}q - B_{ref}\right)^2}\ \ ,\ \ \psi_W(q_P) \;=\; 0\ \ ,$$

with $q_0 \;\leq\; q_P \;\leq\; q_{ref}$. The initial West side profile of the wave is then obtained by inverting, for any $x < 0$,

$$\psi_W(q) \;=\; x\ . \tag{1.5}$$

We set

$$\xi \;=\; \frac{q}{q_{ref}}\quad ,\quad \xi_0 \;=\; \frac{q_0}{q_{ref}}\quad ,\quad F_{ref} \;=\; \frac{m_{ref}}{c_{ref}\;q_{ref}}\ ,\ (\ Froude\ number\ )\ ,$$

to obtain

$$\psi_W(q) \;=\; \frac{q_{ref}}{k\;\left(F_{ref}+1\right)^2}\;\int_{\frac{q_P}{q_{ref}}}^{\frac{q}{q_{ref}}}\left(-\xi - 2\xi_0 + \frac{\xi_0\,(\xi_0-4)}{\xi-\xi_0} + \frac{1-\xi_0^3}{(\xi-\xi_0)^2}\right)\,d\xi\ \ ,$$

that is

$$\psi_W(q) = K\left[\frac{q_P^2-q^2}{2\;q_{ref}} + 2\frac{q_0}{q_{ref}}\,(q_P-q) + \frac{q_0}{q_{ref}}\,(q_0-4q_{ref})\,ln\left(\frac{q-q_0}{q_P-q_0}\right) + \frac{\left(q_{ref}^3-q_0^3\right)(q-q_P)}{q_{ref}\,(q-q_0)\,(q_P-q_0)}\right]$$

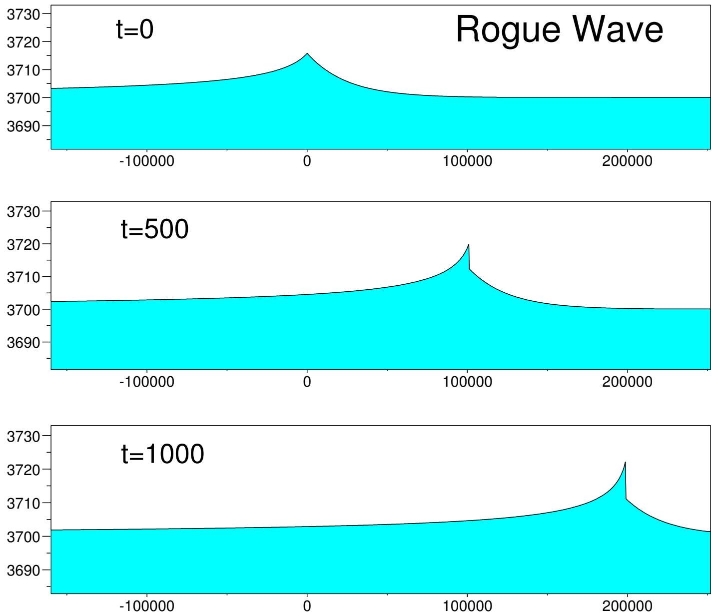
with

$$K \;=\; \frac{1}{k\;\left(F_{ref}+1\right)^2}\ \ .$$

The whole initial profile is now given by (1.5) for $x < 0$ and by (1.4) for $x > 0$. It corresponds to an increasing function $q(x, 0)$ for $x < 0$ and a decreasing one for $x > 0$, which is continuous at $x = 0$ where its value is $q = q_P$.

2 The propagation of the wave

The wave profile is expected to propagate eastwards, with the respective velocities A_{ref} and c_* which are different for each part West or East. Since the west profile moves lightly faster, the crest of the profile will move up, at the junction of the two parts. The left side of this crest corresponds to the continuity of the West profile, extropolated for depth values q going from q_P to the maximal value q_{ref}.



The right part of the crest corresponds to a discontinuity, that is a shock wave, whose location is imposed by the mass conservation. At any time t the water contained in the bump under the crest comes from the column of water of length $(A_{ref} - c_*)\,t$ shifted since the initial time. We denote by $q_l(t)$ and $q_r(t)$ the water depths on the left side (index l) and on the right side (index r) of the discontinuity, and by $x_0(t)$ its position. We always have

$$q_* \;\leq\; q_r(t) \;\leq\; q_P \;\leq\; q_l(t) \;\leq\; q_{ref}\ .$$

As the West profile is moving with the constant velocity A_{ref}, we get it by simply inverting for any time t, the relation

$$\psi_W(q) \;=\; x \;-\; A_{ref}t$$

for $x < x_0(t)$, with the function ψ_W defined above. By the same way, the East profile is obtained by inverting the relation

$$\psi_E(q) \;=\; x \;-\; c_* t$$

for $x > x_0(t)$, with the function ψ_E defined above. For a given time t, the depths $q_l t)$ and $q_r(t)$, and the shock position $x_0(t)$ are linked by three conditions, entailing three equations. The first equation says that $q_l(t)$ is the value of the West profile at $x = x_0(t)$, that is

$$\psi_W(q_l(t)) \;=\; x_0(t) \;-\; A_{ref}\; t \;.$$

The second equation says that $q_r(t)$ is the value of the East profile at $x = x_0(t)$, that is

$$\psi_E(q_r(t)) \;=\; x_0(t) \;-\; c_*\; t \;.$$

The third equation is given by the compatibility relation of Rankine-Hugoniot

$$(q_l(t) - q_r(t)) \sqrt{g \frac{q_r(t) + q_l(t)}{2 q_r(t) q_l(t)}} \;+\; \frac{A_{ref}\; q_0}{q_l(t)} \;-\; \frac{c_*\; q_*}{q_r(t)} \;=\; A_{ref} \;-\; c_* \;, \tag{2.1}$$

which ensures the mass conservation. A dichotomy method running on the parameter $x_0(t)$ allows the simultaneous determination of these three parameters. For practical purposes it is however easier to determine $x_0(t)$ by checking directly the mass conservation. Let us consider two points x_1 and x_2 such that

$$x_1 \;+\; A_{ref}\; t \;<\; x_0(t) \;<\; x_2 \;+\; c_*\; t \;.$$

If it is not the case in a first choice, one can increase x_2 or decrease x_1 sufficiently. We denote by M_0 the mass of water laying between the two points x_1 and x_2 at the initial time:

$$M_0 \;=\; \int_{x_1}^{x_2} q(x,0)\; dx \;=\; \int_{x_1}^{0} q_W(x,0)\; dx \;+\; \int_{0}^{x_2} q_E(x,0)\; dx \;,$$

where q_W and q_E are the respectives depths of the West and East profiles. The same mass M_0 has to be found at any time t between the two trajectories of equations

$$x_1'(t) \;=\; A_{ref} \;-\; \frac{q_{ref} c_{ref}}{q_W\,(x_1(t) - A_{ref} t)} \;,\; x_1(0) \;=\; x_1 \;, \tag{2.2}$$

and

$$x_2'(t) \;=\; c_* \left(1 - \frac{q_*}{q_E\,(x_2(t) - c_* t)}\right) \;,\; x_2(0) \;=\; x_2 \;, \tag{2.3}$$

which means that the relation $M(t) \;=\; M_0$ always occurs, with

$$M(t) \;=\; \int_{x_1(t)}^{x_2(t)} q(x,t)\; dx \;.$$

We remark that the mass $M(t)$ may be explicitely calculated from the relation

$$M(t) = \int_{q_1(t)}^{q_l(t)} q\; \psi_W'(q)\; dq \;+\; \int_{q_r(t)}^{q_2(t)} q\; \psi_E'(q)\; dq \;,$$

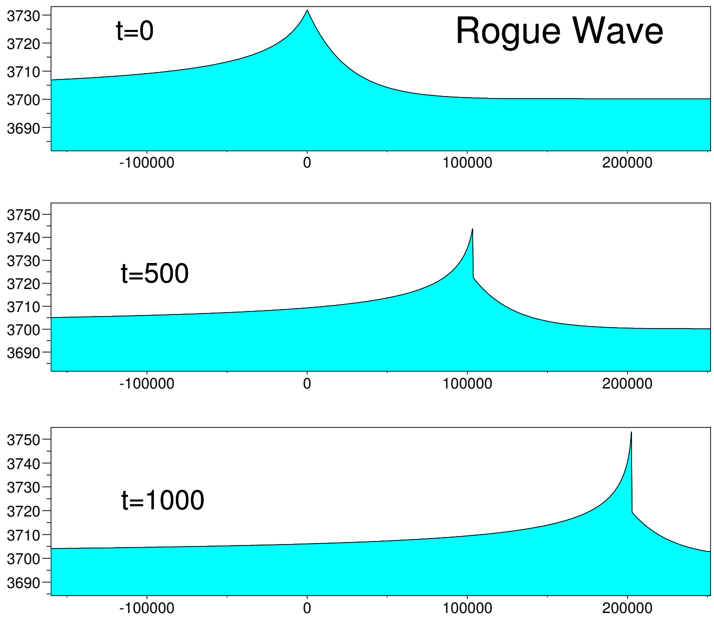
where $q_1(t) = q_W(x_1(t))$, $q_2(t) = q_E(x_2(t))$. This calculation involves only primitives of rationnal fractions. More simply, using the computing files giving q_W on the West side of a point x_0 and q_E on the East side of this point x_0, one can compute

$$F(x_0) \ = \ \int_{x_1(t)}^{x_0} q_W \ dx \ + \ \int_{x_0}^{x_2(t)} q_E \ dx \ ,$$

and next determine $x_0 = x_0(t)$ such that

$$F(x_0) \ = \ M_0$$

by noticing that F is an increasing function.

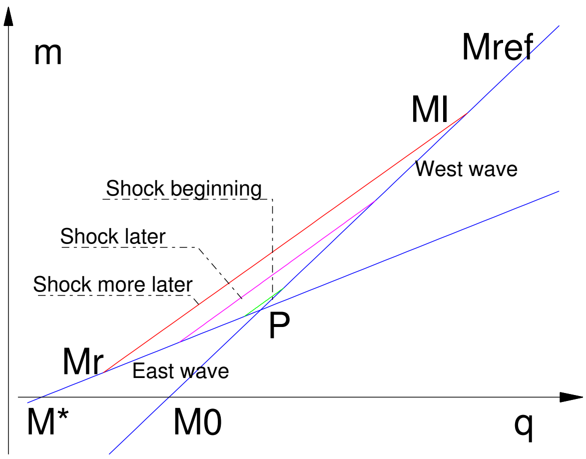


This last process has been used to compute the results shown on the figure above. For this example, we have taken $q_* = 3700\ m$ and $q_0 = 3700.2\ m$, then q_{ref} at its maximal value, that is $q_{ref} = 3731.6737\ m$ here. This choice leads to the value of q_P, which is $q_P = 3715.8087\ m$ here. The friction coefficient was taken equal to $k = 0.45$. We have to notice that the value of the friction coefficient is strongly linked to the wavelength of the profiles since the smallest frictions give the larger wavelengths. For terrestrial hydraulic flows (rivers or estuaries for example) the values of the friction coefficient is far smaller, of the order of 10^{-3} but corresponds to far less deep flows. It seems natural to consider that for larger depths this parameter has to be upgraded. At the time $t = 1000$, the shock amplitude $q_l - q_r$ reaches the value 11 m, between the two cells adjascent to the computed position of $x_0(t)$. The relative error on the mass is of order of 10^{-4} (mainly due to the meshsize), and the trajectories (2.2) and (2.3) were approached by a simple trapezoid formula, since the depth has a small variation along the trajectories $x_1(t)$ and $x_2(t)$ when they are chosen relatively far from the shock.

By upgrading lightly the value of q_0, one get more important shock amplitudes. For the next example, with $q_0 = 3700.8$ m, we obtain the values $q_{ref} = 3763.8773$, and $q_P = 3731.8248$. The computed shock amplitude $q_l - q_r$ is equal to 33.787 m. The wave crest culminates at more than 50 meters above the sea level.

3 Graphic interpretation

We propose a graphic interpretation in the phase plane (q, m) using the picture below.



The states M_* and M_0 are represented on the $q-$axis. The line passing through M_* corresponds to the East wave, of equation $m = c_*(q - q_*)$. The line passing through M_0 corresponds to the West wave, of equation $m = A_{ref}(q - q_0)$. These two lines meet at the point P. The shock wave is represented by the segment M_lM_r, with M_l sliding along the West wave line, from P to M_{ref} and M_r sliding along the East wave line, from P to M_*. This segment M_lM_r is not exactly a straight one, and its equation is obtained from the Rankine Hugoniot condition (2.1). The maximal amplitude is reached when the first event between M_l reaching M_{ref} or M_r reaching M_* occurs. After that time the wave is expected to collapse. The scaling of the picture has been strongly modified for readability, since in the real example all those lines are so close to one another that it is impossible to perceive any difference.

4 Conclusion

This study shows that the outbreak of these waves is due to a differential in the pressure field, resulting from two waves with different profiles, with close but different velocities and sufficiently large wavelengths. The water is pushed up from this pressure effect together with a friction effect. The later behaviour is not studied here: one expects that, when $q_l(t)$ reaches the la value q_{ref} (which occurs in a finite time), a backwards wave will appear, with a negative velocity, and will provoke strong perturbations on the West profile near the wave crest, which will collapse soon. However this backwards wave will have a shorter wavelength, probably too small to fit up with the use of the Saint-Venant model, as described by the condition (0.1).

Another remaining work is to fit out the different parameters in order to be in accordance with real world observations. We also outline the strong sensitivity of the difference $q_0 - q_*$ on the hight of the wave crest, and then on the shock amplitude. A little more important difference between q_0 and q_* causes noticeably more important shock amplitudes. We have only proposed empirical choices of the parameters, in order to get realistic results showing that this study may be a suitable way to understand the shaping of Rogue Waves. We emphasize the friction plays here a fundamental role since it allows a linear behaviuor of the West and the East waves.

Another idea to retain is a new example of the application of the notion of Source Waves after the Roll Waves in channels and rivers, the Tidal Bore waves in the estuaries, the surf waves on the shore near the beach, the hurricanes and tsunamis (see [5], [1]).

The authors thank Michel Olagnon from Ifremer-Brest for some useful answers by e-mail. Reading for example [2] or [4] is also very instructive for the description of the phenomenon of Rogue Waves and starting some bibliography research. It appears that a discussion on either the linear character or the non linear character of the waves is spreading. In this study we put together both characters since linearity is introduced through the source term, that is the friction term, with different parameters on the two sides of the wave, and the nonlinearity effect is present in the shockwave, linking the two sides of the wave.

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