On infinite dimensional systems of conservation laws of Keyfitz-Kranzer type

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Abstract

We prove existence and uniqueness of strong generalized entropy solution to the Cauchy problem for an infinite dimensional system of Keyfitz-Kranzer type, in which the unknown vector takes its value in an arbitrary Banach space.

We study the Cauchy problem for an equation

$$u_t + (\varphi(\|u\|)u)_x = 0 \tag{1}$$

with initial condition

$$u(0,x) = u_0(x). \tag{2}$$

Here the unknown vector $u = u(t,x)$ is defined in a half-plane $\Pi = \mathbb{R}_+ \times \mathbb{R}$, $\mathbb{R}_+ = (0,+\infty)$ and takes its values in some real Banach space X equipped with norm $\|\cdot\|$. We suppose that the function $\varphi(r) \in C(\mathbb{R}_+)$ and

$$r\varphi(r) \to 0 \quad \text{as } r \to 0+\,. \tag{3}$$

The initial function $u_0(x) \in L^\infty(\mathbb{R}, X)$, i.e. it is an essentially bounded strongly measurable function on $\mathbb{R}$.

Remark that in the case when $X = L^p(\mathbb{R})$ equation (1) can be written as the following integral-differential equation

$$\frac{\partial}{\partial t}u(t,x,\lambda) + \frac{\partial}{\partial x}\left[\varphi\left(\int |u(t,x,\mu)|^p d\mu\right) u(t,x,\lambda)\right] = 0, \ p \ge 1.$$

In the case of finite dimensional X equation (1) is reduced to the known Keyfitz-Kranzer system (see [1]). This case was completely investigated in [7], where in particular existence and uniqueness of a strong generalized entropy solution (strong g.e.s.) of problem (1), (2) were proved. In the present paper these results are extended for the case of an arbitrary Banach space X.

Let $f(r) = r\varphi(|r|) \in C(\mathbb{R})$, for $r = 0$ we set $f(r) = 0$ in accordance with (3). By analogy with finite-dimensional case we define a notion of a strong g.e.s. of problem (1), (2).

Definition. A function $u = u(t,x) \in L^\infty(\Pi, X)$ is called a strong g.e.s. of (1), (2) if:

1) $u(t,x)$ satisfies (1) in the sense of distributions (in $\mathcal{D}'(\Pi, X)$);

2) the function $r = \|u(t,x)\|$ is a g.e.s. of the Cauchy problem for the scalar equation

$$r_t + f(r)_x = 0, \quad r(0,x) = \|u_0(x)\| \tag{4}$$

in the sense of S.N. Kruzhkov [2];

3) initial condition (2) is satisfied in the following strict form

$$\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0 \ \text{ in } L^1_{loc}(\Pi, X).$$

If $u(t,x)$ is a strong g.e.s. of the problem (1), (2) then the function $r(t,x) = \|u(t,x)\|$ is uniquely determined by condition 2), as the unique g.e.s. of scalar problem (4). It is essential here that the spatial variable x is single, for the multidimensional equation $r_t + \operatorname{div}_x f(r) = 0$, $x \in \mathbb{R}^n$, $n > 1$ with only continuous flux vector $f(r) \in C(\mathbb{R}, \mathbb{R}^n)$ a g.e.s. of the Cauchy problem can be nonunique, see [3, 4, 8].

Thus, for the construction of a strong g.e.s. of problem (1), (2) we have only to find the value $v = u/r$ (here for $r = 0$ the value of v can be chosen arbitrarily, for instance we can set $v = 0$). From condition 1) it follows that the vector $v = v(t,x)$ must satisfy in $\mathcal{D}'(\Pi, X)$ the linear equation

$$(rv)_t + (f(r)v)_x = 0, \ v = v(t,x) \in L^\infty(\Pi, X) \tag{5}$$

and the initial condition $rv(0,x) = u_0(x)$. In the scalar case $v \in \mathbb{R}$ the theory of generalized solutions (g.s.) of the Cauchy problem for equation of the kind (5) was developed in papers [5, 6, 7], where existence and uniqueness (for the product rv) of g.s. were proved together with the following important property:

any continuous function of a finite set of g.s. to problem (5) *is also a g.s. to this problem with corresponding initial data.*

Going to investigation of infinite-dimensional case $v \in X$, let us consider the Cauchy problem for a general linear transport equation

$$(Av)_t + (Bv)_x = 0, \tag{6}$$

with initial condition

$$v(0,x) = v_0(x) \in L^\infty(\mathbb{R}, X). \tag{7}$$

Suppose, as in [5, 6, 7], that the coefficients $A, B \in L^\infty(\Pi)$ satisfy the following conditions:

$$\operatorname*{ess\,lim}_{t\to 0+} A(t,x) = A(0,x) \ \text{ in } L^1_{loc}(\mathbb{R}), \quad A(0,x)\in L^\infty(\mathbb{R}); \tag{8}$$

$$\forall \varepsilon > 0 \quad |B| \le N(\varepsilon)\cdot(A+\varepsilon) \ \text{ a.e. on } \Pi\ , \ \varepsilon N(\varepsilon) \underset{\varepsilon\to 0+}{\to} 0; \tag{9}$$

$$A_t + B_x = 0 \quad \text{in } \mathcal{D}'(\Pi). \tag{10}$$

As was shown in [7], coefficients $A = r$, $B = f(r)$ satisfy conditions (8)-(10), moreover one can take in (9) $N(\varepsilon) = \omega(\varepsilon)/\varepsilon$, where $\omega(\sigma)$ is the modulus of continuity of $f(u)$ on the segment $[-R, R]$, $R = \|r\|_\infty$. From condition (9) it easily follows that $A \ge 0$ a.e. on Π and $B = 0$ a.e. on the set, where $A = 0$.

The notion of a g.s. of problem (6), (7) is defined in the same way as in [5, 7]:

Definition 2. A function $v = v(t,x) \in L^\infty(\Pi, X)$ is called a g.s. of Cauchy problem (6), (7) if for any test function $h = h(t,x)$ from the space $C_0^\infty(\bar\Pi)$ with $\bar\Pi = [0,+\infty)\times\mathbb{R}$

$$\int_\Pi [Avh_t + Bvh_x]dtdx + \int_{\mathbb{R}} A(0,x)v_0(x)h(0,x)dx = 0. \tag{11}$$

Since the function v is supposed to be bounded and strongly measurable the integrals in (11) are well-defined. It is clear that a vector $v = v(t,x) \in L^\infty(\Pi, X)$ is a g.s. of problem (6), (7) if and only if the scalar functions $< x', v >$ are g.s. of this problem for all linear continuous functionals $x' \in X'$, here X' being a conjugate space to X. Remark also that values of the functions $v(t,x)$ and $v_0(x)$ on the sets, where respectively $A(t,x) = 0$, $A_0(x) = 0$ do not matter and we can consider these functions as elements of the spaces $L^\infty(\cdot, X)$ with respect to weighted measures $Adtdx$ and A_0dx.

To extend results of the papers [5, 6, 7] to the general case $v \in X$ we shall need the following technical lemma, which was proved in [7]:

Lemma 1. *Let $\alpha_t + \beta_x \le 0$ in $\mathcal{D}'(\Pi)$, where $\alpha = \alpha(t,x)$, $\beta = \beta(t,x) \in L^\infty(\Pi)$; $\operatorname*{ess\,lim}_{t\to 0+} \alpha(t,x) = \alpha(0,x)$ in $L^1_{loc}(\mathbb{R})$; $\forall \varepsilon > 0$ $|\beta(t,x)| \le N(\varepsilon)(\alpha(t,x)+\varepsilon)$ a.e. on Π (in particular this condition implies that $\alpha \ge 0$ a.e. on Π), $N(\varepsilon) \ge 1$. Then for a.e. $t > 0$ the following estimate holds:*

$$\int \alpha(t,x)e^{-|x|}dx \le e^t \cdot \inf_{\varepsilon>0}\left(\int \alpha(0,x)e^{-|x|/N(\varepsilon)}dx + 2\varepsilon N(\varepsilon)\right).$$

Now, we can prove the following

Theorem 1.

1) There exists a g.s. $v = v(t,x) \in L^\infty(\Pi, X)$ *of problem* (6), (7);

2) $\operatorname{ess\,lim}_{t\to 0+} A(t,x)v(t,x) = A_0(x)v_0(x)$ *in* $L^1_{loc}(\mathbb{R}, X)$*;*

3) for any continuous function $p(u) \in C(X)$*, which is bounded on bounded subsets of* X*, the composition* $p(v(t,x))$ *is a scalar g.s. of* (6), (7) *with initial data* $p(v_0(x))$*;*

4) if $A(0,x)v_0(x) = 0$ *a.e. on* $\mathbb{R}$ *then* $A(t,x)v(t,x) = 0$ *a.e. on* Π *(uniqueness).*

Proof. Remark firstly that for the finite-dimensional space X problem (6), (7) reduces to the scalar problem for the corresponding coordinate functions and in this case assertions 1)-4) have been already proved in [5, 6, 7] (see for instance Propositions 4-6 in [7]). In the general case to prove existence of g.s. we apply the technique of finite-dimensional approximations.

Thus, let $v_{0n} = v_{0n}(x)$, $n \in \mathbb{N}$ be a sequence of simple functions such that $v_{0n} \underset{n\to\infty}{\to} v_0$ in $L^1_{loc}(\mathbb{R}, X)$. Recall that a simple function is a measurable function, which takes only a finite number of values in X. Since $v_0(x)$ is strongly measurable the approximated sequence v_{0n} really exists, and in addition we can suppose that $\|v_{0n}\|_\infty \le M = \|v_0\|_\infty$ $\forall n \in \mathbb{N}$. Let $x_{nk} \in X$, $k = 1, \dots, m_n$ be values of the functions v_{0n}, and $X_n \subset X$ be a finite-dimensional linear space generated by the vectors x_{nk}, $k = 1, \dots, m_n$. As was mentioned above, there exists a unique g.s. $v_n = v_n(t,x) \in L^\infty(\Pi, X_n)$ of problem (6), (7) with initial functions v_{0n}. Applying property 3 to this g.s. with $p(u) = \max(\|u\| - M, 0)$, we obtain the g.s. $p(v_n)$ of scalar problem (6), (7) with initial function $p(v_{0n}) = 0$ and by the uniqueness property 4) $Ap(v_n) = 0$ a.e. on Π. The latter means (after appropriate definition of v_n on the set, where $A(t,x) = 0$) that $\|v_n\|_\infty \le M$ $\forall n \in \mathbb{N}$. For $k, l \in \mathbb{N}$ the difference $v_k - v_l$ takes its values in the finite-dimensional subspace $X_k + X_l$ and by property 3) with $p(u) = \|u\|$ the function $\theta = \|v_k - v_l\|$ is a g.s. of scalar problem (6), (7) with initial data $\theta_0 = \|v_{0k} - v_{0l}\|$.

Further, observe that in view of (9) $\forall \varepsilon > 0$

$$
\begin{gathered}
|B|\theta \le N(\varepsilon)(A + \varepsilon)\theta \le N(\varepsilon)(A\theta + \varepsilon\theta) \le \\
N(\varepsilon)(A\theta + 2M\varepsilon) \le \bar{N}(\varepsilon)(A\theta + \varepsilon) \quad \text{a.e. on } \Pi,
\end{gathered}
$$

where $\bar{N}(\varepsilon) = \max(2M, 1)N(\varepsilon) + 1$.

We see that the functions $\alpha(t,x) = A\theta = A\|v_k - v_l\|$, $\beta(t,x) = B\theta = B\|v_k - v_l\|$ satisfy the conditions of Lemma 1 with $N(\varepsilon) = \bar{N}(\varepsilon)$, $\alpha(0,x) = A(0,x)\theta_0(x) = A(0,x)\|v_{0k} - v_{0l}\|$. By Lemma 1 for a.e. $t > 0$ and all $k, l \in \mathbb{N}$

$$\int A(t,x)\|v_k(t,x) - v_l(t,x)\|e^{-|x|}dx \le e^t\omega_{kl}, \tag{12}$$

where

$$\omega_{kl} = \inf_{\varepsilon>0}\left(\int A(0,x)(v_{0k}(x) - v_{0l}(x))e^{-|x|/\bar{N}(\varepsilon)}dx + 2\varepsilon\bar{N}(\varepsilon)\right).$$

By the construction the sequence v_{0k} converges as $k \to \infty$ to the function v_0 in $L^1_{loc}(\mathbb{R}, X)$ and it is bounded in $L^\infty(\mathbb{R}, X)$: $\|v_{0k}\|_\infty \le M$. From this and the condition $\varepsilon\bar{N}(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} 0$ it easily follows (see the proof of Proposition 9 in [6]) that $\lim\limits_{k,l\to\infty} \omega_{kl} = 0$ and (12) implies that for any $T > 0$ the sequence $v_k(t,x)$ is fundamental in the spaces $L^1([0,T]\times\mathbb{R}, X)$ equipped with measure $e^{-|x|}A(t,x)dtdx$. By the Cauchy criterion this sequence converges in the indicated spaces to some function $v = v(t,x)$. Besides, we can assume that $v_k(t,x) = v(t,x) = 0$ on the set, where $A = 0$. Then $v_k(t,x) \to v(t,x)$ as $k \to \infty$ in the space $L^1_{loc}(\Pi, X)$ as well. In particular, the function $v(t,x)$ is strongly measurable and bounded, clearly $\|v\|_\infty \le M$. From relation (11) with $v = v_k$, $v_0 = v_{0k}$ it follows in the limit as $k \to \infty$ that the limit function $v \in L^\infty(\Pi, X)$ satisfies this relation with initial data v_0. Thus, v is a g.s. of problem (6), (7). Existence of g.s. is proved.

Further, passing to the limit in estimate (12) as $l \to \infty$ we obtain that $\forall t \in E$, where $E \subset \mathbb{R}_+$ is some set of full measure

$$\int A(t,x)\|v_k(t,x) - v(t,x)\|e^{-|x|}dx \le e^t\omega_k, \tag{13}$$

$$\omega_k = \inf_{\varepsilon>0}\left(\int A(0,x)(v_{0k}(x) - v_0(x))e^{-|x|/\bar{N}(\varepsilon)}dx + 2\varepsilon\bar{N}(\varepsilon)\right).$$

Obviously, $\omega_k \to 0$ as $k \to \infty$ and from (13) it follows that as $k \to \infty$ $A(t,\cdot)v_k(t,\cdot) \to A(t,\cdot)v(t,\cdot)$ in the space $L^1(\mathbb{R}, X)$ (equipped with the measure $e^{-|x|}dx$) uniformly with respect to $t \in [0,T] \cap E$, $\forall T > 0$. Since for the finite-dimensional solutions $v_k(t,x)$ property 2) is satisfied, and it can be written in the form $\operatorname*{ess\,lim}_{t\to 0+} A(t,\cdot)v_k(t,\cdot) = A_0 v_{0k}$ in $L^1(\mathbb{R}, X)$, then due to uniform convergence the limit function is essentially continuous at $t = 0$:

$$\operatorname*{ess\,lim}_{t\to 0+} A(t,x)v(t,x) = A_0(x)v_0(x) \text{ in } L^1(\mathbb{R}, X),$$

i.e. condition 2) is satisfied.

After a possible extraction of a subsequence we can assume that $v_k(t,x) \to v(t,x)$, $v_{0k}(x) \to v_0(x)$ in X as $k \to \infty$ a.e. on Π and on $\mathbb{R}$ respectively. Let a function $p(u)$ be continuous on X and be bounded on bounded subsets of X. Then the functions $p(v_k(t,x))$, $k \in \mathbb{N}$ and $p(v(t,x))$ are bounded and $p(v_k) \to p(v)$ as $k \to \infty$ a.e. on Π. By the Lebesgue dominated convergence theorem we see that $p(v_k) \to p(v)$ in $L^1_{loc}(\Pi, X)$ and, similarly, $p(v_{0k}) \to p(v_0)$ in $L^1_{loc}(\mathbb{R}, X)$. As was already mentioned, $p(v_k)$ is a scalar g.s. of problem (6), (7) with initial data $p(v_{0k})$. Passing to the limit in the corresponding equality (11) as $k \to \infty$ we derive that (11) holds for the limit function $p(v)$ and the initial function $p(v_0)$. Thus, $p(v)$ is a g.s. of problem (6), (7) with initial data $p(v_0)$.

It only remains to prove the uniqueness of g.s. Let $A(0,x)v_0(x) = 0$ a.e. on $\mathbb{R}$ and $v = v(t,x)$ be the corresponding g.s. of (6), (7). By strong measurability the set of essential values of $v(t,x)$ is separable (see for example [9]). Thus, we can change X to the closed linear hull of the essential image of $v(t,x)$ and without loss of generality assume that X is a separable Banach space. Then the conjugate space X' is weakly separable (see [9]) and we can choose a countable weakly dense set $S \subset X'$. For any functional $x' \in S$ the function $< x', v(t,x) >$ is a g.s. of (6), (7) with zero initial data. By the known uniqueness of scalar solutions we have $< x', A(t,x)v(t,x) >= 0$ a.e. on Π. Since S is countable then the set of full measure, on which the latter equality holds, can be chosen common for all $x' \in S$. Then on this set $A(t,x)v(t,x) = 0$ (in view of density of $S \subset X'$) that is $A(t,x)v(t,x) = 0$ a.e. on Π. The proof is complete.

From Theorem 1 it easily follows our main result:

Theorem 2. *There exists a unique strong g.e.s. of problem* (1), (2).

Proof. Let $r = r(t,x)$ be the g.e.s. of scalar problem (4) with initial function $r_0 = \|u_0(x)\|$. Notice that by the comparison principle (see [2, 3, 4, 8]) $r \geq 0$. By Theorem 1 we can find a g.s. $v = v(t,x)$ of problem (5), (7) with initial data $v_0(x) = \begin{cases} u_0(x)/r_0(x) & , \ r_0(x) > 0, \\ 0 & , \ r_0(x) = 0. \end{cases}$

Then $r_0\|v_0\| \equiv r_0$ and by statement 3) of Theorem 1 $\|v(t,x)\|$ is a g.s. of scalar equation (5) with initial function, which is equals 1. Since constant function are evidently g.s. of (5) (in view of condition (10)) then by uniqueness property 4) $r\|v\| = r$ a.e. on Π. We set $u = rv$. Then $u \in L^\infty(\Pi, X)$ and $\|u\| = r\|v\| = r$ is a g.e.s. of problem (4). Further,

$$u_t + (\varphi(\|u\|)u)_x = (rv)_t + (f(r)v)_x = 0 \ \text{ in } \mathcal{D}'(\Pi, X),$$

because v satisfies equation (5) in the sense of distributions. Finally, as it follows from the definition of g.e.s. of problem (4) and statement 2) of Theorem 1,

$$\operatorname*{ess\,lim}_{t\to 0} u(t,\cdot) = u_0 \ \text{ in } L^1_{loc}(\Pi, X).$$

By Definition 1 $u = u(t,x)$ is a strong g.e.s. of original problem (1), (2).

To prove uniqueness of a strong g.e.s. suppose that $u_1 = u_1(t,x)$, $u_2 = u_2(t,x)$ are strong g.e.s. of problem (1), (2). Then $\|u_1\| = \|u_2\| = r$ in view of uniqueness of a g.e.s. to problem (4). But then $v_i = u_i/r$, $i = 1,2$ are g.s. of problem (5), (7) with the same initial function u_0/r_0. By uniqueness of this g.s. $u_1 - u_2 = r(v_1 - v_2) = 0$ a.e. on Π, that was to be prove. The proof is complete.

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