E.Yu. Panov

Prolonged systems for a scalar conservation law and entropies of higher orders

Abstract

We give a matrix representation for prolonged systems corresponding to scalar conservation laws and describe entropies of such systems.

Let $f \in C^{n-1}(\mathbb{R})$. We denote $D_n f = Df(x) \in \mathbb{R}^n$ the column $(f, f', \dots, f^{(n-1)})^\top$ consisting of derivatives of f, and consider the $n \times n$-matrix $T_n(f) = T_n(f)(x)$, which is defined by the equality

$$D_n(fg) = T_n(f)D_n g \quad \forall g \in C^{n-1}(\mathbb{R}). \tag{1}$$

The coefficients of $T_n(f)$ are continuous functions, depending on derivatives $f^{(k)}(x)$, $k = 0, \dots, n-1$. For instance, if $n = 2, 3$ then

$$T_2(f) = \begin{pmatrix} f & 0 \\ f' & f \end{pmatrix}, \; T_3(f) = \begin{pmatrix} f & 0 & 0 \\ f' & f & 0 \\ f'' & 2f' & f \end{pmatrix}$$

respectively. In the general case, as it follows from the Leibnitz formula $(fg)^{(i-1)} = \sum_{j=1}^{i} C_{i-1}^{j-1} f^{(i-j)} g^{(j-1)}$,

$$T_n(f)_{ij} = C_{i-1}^{j-1} f^{(i-j)} \text{ for } 1 \le j \le i \le n, \quad T_n(f)_{ij} = 0 \text{ for } j > i$$

(in particular, the matrix $T_n(f)$ is triangular). Here $C_m^k = \dfrac{m!}{k!(m-k)!}$ are binomial coefficients.

Clearly, $T_n(\alpha f_1 + \beta f_2) = \alpha T_n(f_1) + \beta T_n(f_2)$, $\forall f_1, f_2 \in C^{n-1}(\mathbb{R})$, $\alpha, \beta \in \mathbb{R}$.

Further, by the obvious identity

$$T_n(f_1 f_2)Dg = D_n(f_1(f_2 g)) = T_n(f_1)D_n(f_2 g) = T_n(f_1)T_n(f_2)D_n g,$$

$T_n(f_1 f_2) = T_n(f_1)T_n(f_2)$ $\forall f_1, f_2 \in C^{n-1}(\mathbb{R})$.

Thus, the correspondence $f \to T_n(f)$ is a homomorphizm of algebras, so that it is a linear representation of the algebra $C^{n-1}(\mathbb{R})$ in the space of vector-functions $C(\mathbb{R}, \mathbb{R}^n)$. In particular, $\forall \eta(u) \in C^{n-1}(\mathbb{R})$ we have the equality $T_n(\eta(f)) = \eta(T_n(f))$, i.e. $D_n(\eta(f)g) = \eta(T_n(f))D_n g\ \forall g \in C^{n-1}(\mathbb{R})$. Here $\eta(T_n(f))$ is a function of the matrix $T_n(f)$ understood in the sense of functional calculus and, which is well-defined for $\eta(u) \in C^{n-1}(\mathbb{R})$.

By the construction for any fixed x the image of the representation $f \to T_n(f)(x)$ is a commutative n-dimensional matrix algebra X_n, consisting of triangular matrices $U_n = U_n(\bar{u})$, $\bar{u} = (u_1, \dots, u_n)$, where

$$U_n(\bar{u})_{ij} = C_{i-1}^{j-1} u_{i-j+1} \text{ for } 1 \le j \le i \le n, \quad U_n(\bar{u})_{ij} = 0 \text{ for } j > i.$$

This algebra is isomorphic a quotient algebra of the polynomial algebra with respect to the ideal generated by x^n. The isomorphism is realized by the map $f \to T_n(f)(0)$, so that for $f(x) = \sum_{i=1}^{n} u_i x^{i-1}/(i-1)!$ $T_n(f)(0) = U_n(\bar{u})$.

The following simple lemma will be needed for the sequel.

Lemma 1. *Let $\eta(u) \in C^n(\mathbb{R})$, $\bar{u} \in \mathbb{R}^n$, $U = U_n(\bar{u})$ and $\eta(U) = U_n(\bar{v})$, $\eta'(U) = U_n(\bar{w})$, where $\bar{v} = (v_1, \dots, v_n) \in \mathbb{R}^n$, $\bar{w} = (w_1, \dots, w_n) \in \mathbb{R}^n$ (i.e. $v_i = \eta(U)_{i1}$, $w_i = \eta'(U)_{i1}$). Then*

$$\frac{\partial v_i}{\partial u_j} = \begin{cases} C_{i-1}^{j-1} w_{i-j+1} & , \ j \le i \\ 0 & , \ j > i \end{cases}.$$

Proof. Since $d\eta(U) = \eta'(U)dU$ then

$$\frac{\partial v_i}{\partial u_j} = \sum_{k=1}^{i} \eta'(U)_{ik} \frac{\partial U_{k1}}{\partial u_j} = \sum_{k=1}^{i} \eta'(U)_{ik} \frac{\partial u_k}{\partial u_j},$$

which directly implies that $\frac{\partial v_i}{\partial u_j} = 0$ for $j > i$. If $j \le i$ then

$$\frac{\partial v_i}{\partial u_j} = \eta'(U)_{ij} = C_{i-1}^{j-1} w_{i-j+1}.$$

The proof is complete.

Now we consider a scalar first order quasilinear equation

$$u_t + \varphi(u)_x = 0, \tag{2}$$

$u = u(t,x)$, $(t,x) \in \Pi = \mathbb{R}_+ \times \mathbb{R}$, $\varphi(u) \in C^n(\mathbb{R})$. Differentiating this equation $n-1$ times over the variable x, we obtain the so-called prolonged system of conservation laws, which consists of n equations (together with the original one) depending on n unknown functions $u_i = u_x^{(i-1)}$, $i = 1, \ldots, n$. For instance, if $n = 3$ then the prolonged system has the form

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \dfrac{\partial}{\partial x}(\varphi(u_1)) = 0 \\ \dfrac{\partial u_2}{\partial t} + \dfrac{\partial}{\partial x}(\varphi'(u_1)u_2) = 0 \\ \dfrac{\partial u_3}{\partial t} + \dfrac{\partial}{\partial x}(\varphi'(u_1)u_3 + \varphi''(u_1)u_2^2) = 0 \end{cases}.$$

If we apply the map T_n to equality (2), where functions in Π are treated as functions of the variable x with parameter t then we obtain that

$$0 = T_n(u_t + \varphi(u)_x) = T_n(u)_t + T_n(\varphi(u))_x = T_n(u)_t + \varphi(T_n(u))_x. \quad (3)$$

Now we introduce the $n \times n$-matrix $U = U(t, x) = T_n(u)$. Clearly, $U = U_n(\bar{u})$, where $\bar{u} = (u_1, \ldots, u_n)$, $u_i = u_i(t, x) = u_x^{(i-1)}$, $i = 1, \ldots, n$.

Then equality (3) can be rewritten as the equation

$$U_t + \varphi(U)_x = 0 \quad (4)$$

similar to equation (2), but here the unknown function $U = U(t, x)$ takes its values in the algebra X_n. Since $U = U_n(\bar{u})$ the system (4) is a form of the prolonged system.

Our aim is to describe entropies of the prolonged system. Recall (see [1]) that the entropy of system (4) is a function $p(\bar{u}) \in C^1(\mathbb{R}^n)$ such that there exists a function $q(\bar{u}) \in C^1(\mathbb{R}^n)$ (called the corresponding entropy flux) satisfying the identity: $\forall \bar{u} \in \mathbb{R}^n$

$$dp(\bar{u}) \circ d\varphi(U) = dq(\bar{u}),\ U = U_n(\bar{u}). \quad (5)$$

Suppose $p(\bar{u})$ is an entropy of (4) with flux $q(\bar{u})$. Applying the operator dp to system (4) we derive the conservation law

$$p(\bar{u})_t + q(\bar{u})_x = 0. \quad (6)$$

In particular, for a solution $u(t, x) \in C^n(\Pi)$ of (2) the vector $\bar{u} \in \mathbb{R}^n$ with coordinates $u_i = u_x^{(i-1)}$, $i = 1, \ldots, n$ satisfies equality (6). Hence, this equality is a consequence of the original scalar equation (2) and it is natural to consider entropies of system (4) as entropies of order n for equation (2) (in contrast to "usual" entropies they depend not only on a solution u, but also on its derivatives up to order $n-1$).

Taking into account that $d\varphi(U) = \varphi'(U)dU$ we can rewrite equality (6) in the form

$$\sum_{i=j}^{n} C_{i-1}^{j-1} v_{i-j+1} \frac{\partial p(\bar{u})}{\partial u_i} = \frac{\partial q(\bar{u})}{\partial u_j}, \ v_k = \varphi'(U)_{k1}. \tag{7}$$

Suppose $1 \le j \le n$, $U = U_j(u_1, \dots, u_j)$ is a $j \times j$-matrix (i.e. the principal minor of the matrix U_n), and $\eta(u) \in C^j(\mathbb{R})$. Then the following statement holds.

Theorem 1 *A function $p(\bar{u}) = \eta(U_j)_{j1}$ is an entropy of system* (4) *with corresponding flux $q(\bar{u}) = \psi(U_j)_{j1}$, where the function $\psi(u) \in C^j(\mathbb{R})$ is defined, up to an additive constant, by the equality $\psi'(u) = \eta'(u)\varphi'(u)$.*

Proof. Let $U = U_n(\bar{u})$. Then

$$dq(\bar{u}) = d\psi(U_j)_{j1} = (\psi'(U_j)dU_j)_{j1} = (\eta'(U_j)\varphi'(U_j)dU_j)_{j1} = dp(\bar{u})d\varphi(U),$$

i.e. the identity (5) is satisfied. Thus, $p(\bar{u})$ is an entropy of (4) with flux $q(\bar{u})$.

Remark that the entropy $p(\bar{u})$ and the flux $q(\bar{u})$ in Theorem 1 naturally arise after differentiating of the "scalar" entropy pair $\eta(u)$, $\psi(u)$ $j-1$ times over x, so that

$$\frac{d^{j-1}\eta(u)}{dx^{j-1}} = p(\bar{u}), \quad \frac{d^{j-1}\psi(u)}{dx^{j-1}} = q(\bar{u}), \quad u_i = u_x^{(i-1)}, \ i = 1, \dots, j.$$

Now, suppose that the function $\varphi(u)$ is not linear on nondegenerate intervals, i.e. $\varphi''(u) \ne 0$ on a dense set in $\mathbb{R}$. We want to show that in this case any entropy $p(\bar{u}) \in C^2(\mathbb{R}^n)$ of system (4) is a sum of entropies indicated in Theorem 1:

$$p(\bar{u}) = \sum_{j=1}^{n} \eta_j(U_j)_{j1}, \quad \eta_j(u) \in C^{j+1}(\mathbb{R}). \tag{8}$$

For this, we have to analyse relation (7). Denote $\alpha_{ij} = C_{i-1}^{j-1} v_{i-j+1}$. By Lemma 1

$$\alpha_{ij} = \frac{\partial \beta_i}{\partial u_j}, \quad \text{where } \beta_i = \varphi(U_n)_{i1}.$$

Therefore, for $k, r = 1, \dots, n$

$$\sum_{i=k}^{n} \frac{\partial \alpha_{ik}}{\partial u_r} \frac{\partial p(\bar{u})}{\partial u_i} = \sum_{i=r}^{n} \frac{\partial \alpha_{ir}}{\partial u_k} \frac{\partial p(\bar{u})}{\partial u_i} = \sum_{i=k+r-1}^{n} \frac{\partial^2 \beta_i}{\partial u_k \partial u_r} \frac{\partial p(\bar{u})}{\partial u_i}.$$

We also take into account that by Lemma 1 $\dfrac{\partial^2\beta_i}{\partial u_k \partial u_r} = C_{i-1}^{k-1}\dfrac{\partial v_{i-k+1}}{\partial u_r} = 0$ for $i < k + r - 1$.

From this equality and (7) it follows the relation: for all $k, r = 1, \ldots, n$

$$\sum_{i=k}^{n} \alpha_{ik} \frac{\partial^2 p(\bar{u})}{\partial u_i \partial u_r} - \sum_{i=r}^{n} \alpha_{ir} \frac{\partial^2 p(\bar{u})}{\partial u_i \partial u_k} =$$

$$\frac{\partial}{\partial u_r} \sum_{i=k}^{n} \alpha_{ik} \frac{\partial p(\bar{u})}{\partial u_i} - \frac{\partial}{\partial u_k} \sum_{i=r}^{n} \alpha_{ir} \frac{\partial p(\bar{u})}{\partial u_i} = \frac{\partial^2 q(\bar{u})}{\partial u_r \partial u_k} - \frac{\partial^2 q(\bar{u})}{\partial u_k \partial u_r} = 0.$$

Since the first terms in the sums from the left side of this equality, corresponding to $i = k$ and $i = r$, coincides, we conclude that

$$\sum_{i=k+1}^{n} \alpha_{ik} \frac{\partial^2 p(\bar{u})}{\partial u_i \partial u_r} = \sum_{i=r+1}^{n} \alpha_{ir} \frac{\partial^2 p(\bar{u})}{\partial u_i \partial u_k}. \tag{9}$$

Using relations (9) we are ready to prove the following result.

Proposition 1. *Let $p(\bar{u}) \in C^2(\mathbb{R}^n)$ is an entropy of order n. Then for $k = 2, \ldots, n$, $r = 1, \ldots, n$*

$$(k-1)\frac{\partial^2 p(\bar{u})}{\partial u_k \partial u_r} = r\frac{\partial^2 p(\bar{u})}{\partial u_{k-1} \partial u_{r+1}}. \tag{10}$$

Here we agree that for $r = n$ the derivative from the right side of (10) *equals zero.*

Proof. We shall draw the proof by induction on $k + r = 2n, \ldots, 3$. The base of induction $k+r = 2n$ reduces to verification of the equality $\dfrac{\partial^2 p(\bar{u})}{\partial u_n^2} = 0$. This equality directly follows from (9) with $k = n-1$, $r = n$ and the fact that $\alpha_{nn-1} = (n-1)v_2 = (n-1)\varphi''(u_1)u_2 \neq 0$ on a dense set of $\bar{u} \in \mathbb{R}^n$. Moreover, applying (9) consequently for $k = n-1, \ldots, 1$ and $r = n$, we derive in the same way as above that

$$\frac{\partial^2 p(\bar{u})}{\partial u_k \partial u_n} = 0, \quad k = 2, \ldots, n. \tag{11}$$

Now suppose that (10) holds for $l + 1 \leq k + r \leq 2n$. Then for such values k, r

$$\frac{\partial^2 p(\bar{u})}{\partial u_k \partial u_r} = \frac{r}{k-1}\frac{\partial^2 p(\bar{u})}{\partial u_{k-1} \partial u_{r+1}} = \cdots$$

$$= \begin{cases} \text{const}\dfrac{\partial^2 p(\bar{u})}{\partial u_{k+r-n}\partial u_n} = 0 & \text{for} \quad k+r > n+1 \\ C_{k+r-2}^{k-1}\dfrac{\partial^2 p(\bar{u})}{\partial u_1 \partial u_{k+r-1}} & \text{for} \quad k+r \le n+1 \end{cases}. \tag{12}$$

In turn from (12) it follows that for $k_1 + r \ge l + 1$

$$\sum_{i=k_1}^{n} \alpha_{ik} \frac{\partial^2 p(\bar{u})}{\partial u_i \partial u_r} = \sum_{i=k_1}^{n+1-r} \alpha_{ik} C_{i+r-2}^{i-1} \frac{\partial^2 p(\bar{u})}{\partial u_1 \partial u_{i+r-1}} =$$
$$\sum_{j=k_1+r-1}^{n} \gamma_{krj} \frac{\partial^2 p(\bar{u})}{\partial u_1 \partial u_j}. \tag{13}$$

Here we make the change $j = i + r - 1$ and denote

$$\gamma_{krj} = \gamma_{krj}(\bar{u}) = \alpha_{ik} C_{i+r-2}^{i-1} = \frac{(j-1)!}{(k-1)!(r-1)!(j+1-k-r)!} v_{j+2-k-r}.$$

Now we are ready to prove that equality (10) holds for $k + r = l$. By (9) we have an equality

$$\sum_{i=k}^{n} \alpha_{ik-1} \frac{\partial^2 p(\bar{u})}{\partial u_i \partial u_r} = \sum_{i=r+1}^{n} \alpha_{ir} \frac{\partial^2 p(\bar{u})}{\partial u_i \partial u_{k-1}}. \tag{14}$$

Further, $k + r + 1 > l$ and, as it easily follows from (13) and the equality $\gamma_{k-1rj} = \gamma_{rk-1j}$,

$$\sum_{i=k+1}^{n} \alpha_{ik-1} \frac{\partial^2 p(\bar{u})}{\partial u_i \partial u_r} = \sum_{i=r+2}^{n} \alpha_{ir} \frac{\partial^2 p(\bar{u})}{\partial u_i \partial u_{k-1}} = \sum_{j=k+r}^{n} \gamma_{k-1rj} \frac{\partial^2 p(\bar{u})}{\partial u_1 \partial u_j}.$$

Therefore, (14) is reduced to coincidence of the first terms:

$$(k-1) v_2 \frac{\partial^2 p(\bar{u})}{\partial u_k \partial u_r} = r v_2 \frac{\partial^2 p(\bar{u})}{\partial u_{r+1} \partial u_{k-1}}$$

and since $v_2 = \varphi''(u_1) u_2 \neq 0$ on a dense set of the arguments we conclude that (10) holds. According to the mathematical induction method the proof is complete.

Corollary 1. *In the sense of distributions on $\mathbb{R}^n$ (in $\mathcal{D}'(\mathbb{R}^n)$)*

$$\frac{\partial^k p(\bar{u})}{\partial u_2^k} = \begin{cases} k! \dfrac{\partial^{k-1}}{\partial u_1^{k-1}} \dfrac{\partial p(\bar{u})}{\partial u_{k+1}} & \textit{for} \quad k \le n-1 \\ 0 & \textit{for} \quad k > n-1 \end{cases}. \tag{15}$$

Proof. If $k = 1$ then equality (15) is trivial. Next, if (15) holds for $k = r - 1 < n - 1$ then, taking into account (10), we see that

$$\frac{\partial^r p(\bar{u})}{\partial u_2^r} = \frac{\partial}{\partial u_2}\frac{\partial^{r-1} p(\bar{u})}{\partial u_2^{r-1}} = (r-1)!\frac{\partial}{\partial u_2}\left(\frac{\partial^{r-2}}{\partial u_1^{r-2}}\frac{\partial p(\bar{u})}{\partial u_r}\right) =$$
$$(r-1)!\frac{\partial^{r-2}}{\partial u_1^{r-2}}\frac{\partial^2 p(\bar{u})}{\partial u_2 \partial u_r} = r!\frac{\partial^{r-1}}{\partial u_1^{r-1}}\frac{\partial p(\bar{u})}{\partial u_{r+1}}.$$

At last, if $r = n$ then by (10) again the right hand side of the above equality is null (all the more this is true for $r > n$). According to the mathematical induction method the proof is complete.

Now we are ready to show that under our assumptions any entropy of order n has the form (8).

Theorem 2 *Suppose the function $\varphi(u)$ is not linear on nondegenerate intervals. Then an entropy $p(\bar{u}) \in C^2(\mathbb{R}^n)$ of system (4) has the form (8).*

Proof. Let $p(\bar{u})$ depend only on first r coordinates, i.e. it is an entropy of order r. We are going to prove that $p(\bar{u})$ has a representation like (8)

$$p(\bar{u}) = \sum_{j=1}^{r} \eta_j(U_j)_{j1}, \quad \eta_j(u) \in C^{j+1}(\mathbb{R}). \tag{16}$$

We shall draw the proof by induction on order r. If $r = 1$ (the base of induction) then $p(\bar{u}) = \eta(u_1) = \eta(U_1)$ and (16) is clear. Now, assume that (16) holds for entropies of order $r < n$ and show that it holds for entropies of order n. (with arbitrary $n > 1$). Remark firstly that from (11) it follows that $\dfrac{\partial p}{\partial u_n} = a(u_1)$, therefore $p(\bar{u}) = a(u_1)u_n + b(u_1, \ldots, u_{n-1})$, $b \in C^2(\mathbb{R}^{n-1})$. We show that $a(u) \in C^n(\mathbb{R})$. For this, remark that by Corollary 1 $p(\bar{u})$ is a polynomial of degree not more than $n - 1$ with respect to a variable u_2:

$$p(\bar{u}) = \sum_{k=0}^{n-1} q_k u_2^k, \ q_k = q_k(u_1, u_3, \ldots, u_n),$$

and also, as follows from the condition $p(\bar{u}) \in C^2(\mathbb{R}^n)$, the coefficients $q_k \in C^2(\mathbb{R}^{n-1})$. By Corollary 1 again, for $k = n - 1$

$$(n-1)!q_{n-1} = \frac{\partial^{n-1} p(\bar{u})}{\partial u_2^{n-1}} = (n-1)!\frac{\partial^{n-2}}{\partial u_1^{n-2}}\frac{\partial p(\bar{u})}{\partial u_n} = (n-1)!a^{(n-2)}(u_1)$$

in $\mathcal{D}'(\mathbb{R}^n)$. Thus, $a^{(n-2)}(u_1) = q_{n-1} \in C^2$ and $a(u) \in C^n(\mathbb{R})$, as required. Let $\eta_n(u) = \int a(u)du \in C^{n+1}(\mathbb{R})$, $p_n(\bar{u}) = \eta_n(U_n)_{n1}$. Then $p_n(\bar{u}) \in C^2(\mathbb{R}^n)$ and, evidently, $p_n(\bar{u}) = a(u_1)u_n + c(u_1, \ldots, u_{n-1})$, $c \in C^2(\mathbb{R}^{n-1})$. We see that $p(\bar{u}) - p_n(\bar{u}) = (b - c)(u_1, \ldots, u_{n-1}) \in C^2(\mathbb{R}^{n-1})$ is an entropy of order $n-1$. By the inductive assumption

$$p(\bar{u}) - p_n(\bar{u}) = \sum_{j=1}^{n-1} \eta_j(U_j)_{j1}, \quad \eta_j(u) \in C^{j+1}(\mathbb{R}),$$

and we conclude that

$$p(\bar{u}) = \sum_{j=1}^{n} \eta_j(U_j)_{j1}, \quad \eta_j(u) \in C^{j+1}(\mathbb{R}),$$

as was to be proved.

Remark. It is easy to verify that an entropy $p(\bar{u})$ of the form (8) is convex only in the case when

$$p(\bar{u}) = \eta(u_1) + \sum_{j=2}^{n} c_j u_j, \quad c_j = \text{const}$$

with a convex function $\eta(u)$. In particular, there are no strictly convex entropies of order $n > 1$.

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References

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