The Aw-Rascle traffic flow model with phase transitions

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Abstract

We introduce a new model of traffic flow with phase transitions. The model is obtained by coupling together the classical Lighthill-Whitham-Richards (LWR) equation with the 2×2 system described in [A. Aw and M. Rascle, *Resurrection of "second order" models of traffic flow*, SIAM J. Appl. Math., 60 (2000), pp. 916–938]. We describe the solutions of the Riemann problem, and we compare the results with the ones obtained using the LWR model and the biphasic model described in [R.M. Colombo, *Hyperbolic phase transitions in traffic flow*, SIAM J. Appl. Math., 63 (2002), pp. 708–721]

Key words: Hyperbolic Conservation Laws, Riemann Problem, Phase Transitions, Continuum Traffic Models

1 Introduction

Any reasonable model for traffic flow should satisfy the following principles:

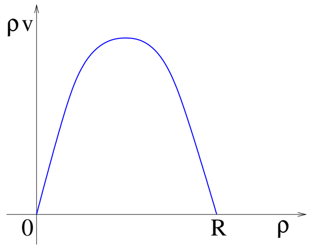
1. Drivers react to what happens in front of them, so no information travels faster than the cars.
2. Density and velocity must remain non-negative and bounded.

One of the first models introduced to describe traffic flow is the well known Lighthill-Whitham [1] and Richards [2] (LWR) model, which reads

$$\partial_t \rho + \partial_x [\rho v(\rho)] = 0, \tag{1.1}$$

where $\rho \in [0, R]$ is the mean traffic density, and $v(\rho)$, the mean traffic velocity, is a given non-increasing function, non-negative for ρ between 0 and the positive maximal density R, which corresponds to a traffic jam. This simple model expresses conservation of the number of cars, and relies on the assumption that the car speed depends only on the density. Nevertheless, the corresponding fundamental diagram in the $(\rho, \rho v)$-plane, see Figure 1, left, does not qualitatively match experimental data as reported in Figure 1, right. These experimental data, whose behavior is similar to what observed in several areas of the world, suggest that a good traffic flow model should exhibit two qualitative different behaviors:

- for low densities, the flow is *free* and essentially analogous to the LWR model;
- at high densities the flow is *congested* and has one more degree of freedom (it covers a 2-dimensional domain).



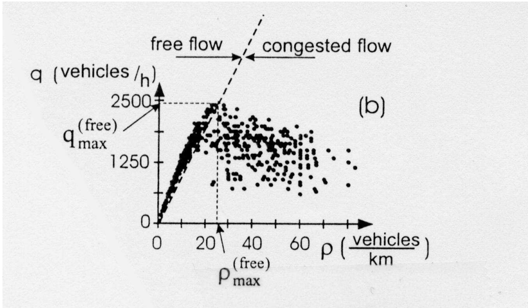


Figure 1: Left: standard flow for the LWR model. Right: experimental data, taken from [3]

A first prototype of 2×2 models was proposed by Payne [4] and Whitham [5]. The main drawback of this model is that it does not satisfy principles 1. and 2. above, as pointed out by Daganzo [6]. Later, Aw and Rascle [7] have corrected the Payne-Whitham model by replacing the space derivative of the "pressure" in the momentum equation with the convective derivative $\partial_t + v\partial_x$. Another 2×2 model has been introduced by Colombo [8], and used in the construction of the first traffic flow model with phase transition [9].

In this paper we describe and study in details a new traffic flow model with phase transition obtained combining the Aw-Rascle model with the LWR equation (the model has been introduced in [10]). Section 2 is devoted to the description of the model. In Section 3 we construct the solutions to the Riemann problem and in Section 4 we compare the results with the ones obtained using the LWR model and the biphasic model described in [9]. Several examples are considered, and numerical integrations are provided. Some results of existence and well-posedness for the solutions of the Cauchy and Initial-Boundary Value Problems are collected in the Appendix.

2 Description of the model

In analogy with the model introduced by Colombo in [9], the model presented here is obtained combining the classical LWR model describing the free flow with the 2×2 model introduced by Aw and Rascle [7] (in this work referred to as the AR model) to describe the congested phase.

In the following R is the maximal possible car density, V is the maximal speed allowed and V_{ref} a given reference velocity. The model is the following:

$$\begin{array}{ll} \text{Free flow:} & \text{Congested flow:} \\ (\rho, v) \in \Omega_f & (\rho, v) \in \Omega_c \\ \partial_t \rho + \partial_x[\rho v] = 0 & \left\{ \begin{array}{l} \partial_t \rho + \partial_x[\rho v] = 0 \\ \partial_t\big[\rho(v + p(\rho))\big] + \partial_x\big[\rho v(v + p(\rho))\big] = 0 \end{array} \right. \\ v = v_f(\rho) & p(\rho) = V_{ref} \ln(\rho/R) \end{array} \tag{2.2}$$

The sets Ω_f and Ω_c denote the free and the congested phases respectively. In Ω_f there is only one independent variable, the car density ρ, and the velocity v_f is a function that satisfies the same properties usually required in the LWR model. Here we choose the simplest standard linear function

$$v_f(\rho) = \left(1 - \frac{\rho}{R}\right) V.$$

In Ω_c the variables are the car density ρ and the car speed v or, equivalently, the conservative variables ρ and $y := \rho v + \rho p(\rho)$; see [7]. The "pressure" function p is assumed increasing and plays the role of an *anticipation factor*, taking in account drivers' reactions to the state of traffic in front of them. Here we take $p(\rho) = V_{ref} \ln(\rho/R)$ as in [11, 12], because this choice allows to define a unique Riemann solver without any further assumption on the parameters R and V. However, under suitable assumptions, the model allows more general pressures (see Remark 3.2 below).

As pointed out in [13], no evidence suggests that queues form spontaneously in free flow traffic for no apparent reason. Hence it is reasonable to assume that if the initial data are entirely in the free (resp. congested) phase, then the solution will remain in the free (resp. congested) phase for all time. Thus we are led to take Ω_f (resp. Ω_c) to be an invariant set for (2.2), left (resp., right). The resulting domain is given by

$$\Omega_f = \left\{(\rho, v) \in [0, R_f] \times [V_f, V] \colon v = v_f(\rho)\right\} ,$$

$$\Omega_c = \left\{(\rho, v) \in [0, R] \times [0, V_c] \colon p(r) \le v + p(\rho) \le p(R)\right\} ,$$

where V_f and V_c are the threshold speeds, i.e. above V_f the flow is free and below V_c the flow is congested. We assume they are strictly positive and $V > V_f > V_c$. The parameter $r \in]0, R]$ depends on the environmental conditions and determines the width of the congested region. The maximal free-flow density R_f must satisfy $V_f + p(R_f) = p(R)$ (that is $V_f + V_{ref} \ln(R_f/R) = 0$ with our choice of the pressure). In order to get this condition, we are led to assume $V_{ref} < V$. It is easy to check that the *capacity drop* in the passage from the free phase to the congested phase [3] is then automatically satisfied. In order to resume, we have the following order relation between the speed parameters:

$$V > V_{ref} > V_f > V_c.$$

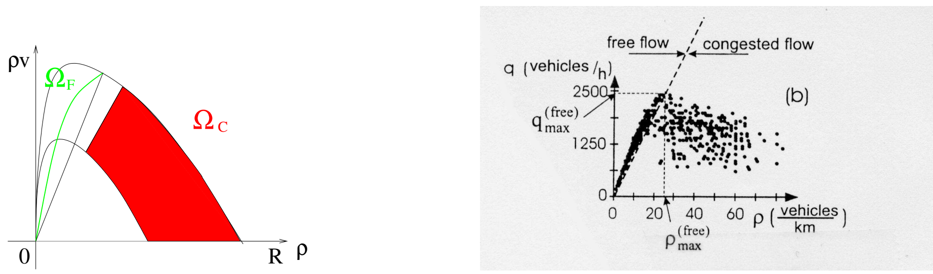


Figure 2: Left: invariant domain for (2.2). Right: experimental data, taken from [3].

Figure 2 shows that the shape of the invariant domain is in good agreement with experimental data.

We recall at this point the main features of the two models used in (2.2). In the free phase the characteristic speed is $\lambda(\rho) = V(1 - 2\rho/R)$, while the informations on the Aw-Rascle system are collected in the following table (see [7] for a more detailed study of the model):

$$\begin{array}{ll}
r_1(\rho, v) = \begin{bmatrix} 1 \\ -p'(\rho) \end{bmatrix} & r_2(\rho, v) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\lambda_1(\rho, v) = v - \rho\, p'(\rho) & \lambda_2(\rho, v) = v \\
\nabla\lambda_1 \cdot r_1 = -2\, p'(\rho) - \rho\, p''(\rho) & \nabla\lambda_2 \cdot r_2 = 0 \\
\mathcal{L}_1(\rho; \rho_o, v_o) = v_o + p(\rho_o) - p(\rho) & \mathcal{L}_2(\rho; \rho_o, v_o) = v_o \\
w_1 = v & w_2 = v + p(\rho)
\end{array} \tag{2.3}$$

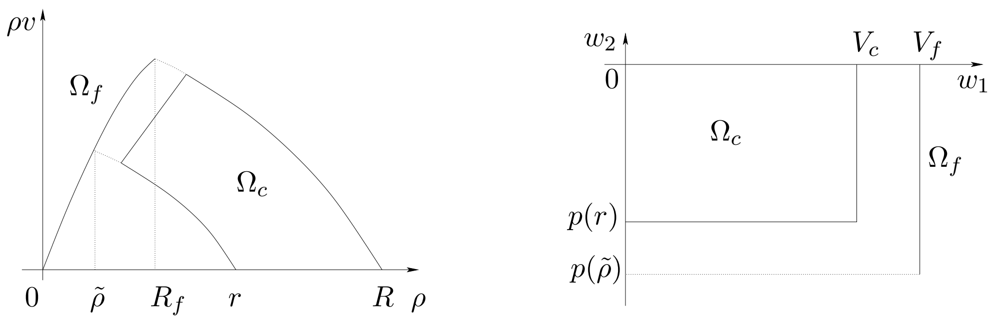


Figure 3: Notation used in the paper.

where r_i is the i-th right eigenvector, λ_i the corresponding eigenvalue and $\mathcal{L}_i$ is the i-Lax curve. Shock and rarefaction curves coincide, hence the system belongs to the Temple class [14].

Using Riemann coordinates (w_1, w_2), $\Omega_c = [0, V_c] \times [p(r), p(R)]$. For $(\rho, v) \in \Omega_f$, we extend the corresponding Riemann coordinates (w_1, w_2) as in [9]: Let $\tilde u = (\tilde\rho, v_f(\tilde\rho))$ be the point in Ω_f implicitly defined by $v_f(\tilde\rho) + p(\tilde\rho) = p(r)$. We define

$$w_1 = V_f \qquad\qquad w_2 = \begin{cases} v_f(\rho) + p(\rho) & \text{if } \rho \geq \tilde\rho \\ v_f(\rho) + p(\tilde\rho) & \text{if } \rho < \tilde\rho \end{cases} \tag{2.4}$$

so that, in Riemann coordinates, $\Omega_f = \{V_f\} \times [p(\tilde\rho), p(R)]$ (Figure 3).

3 The Riemann problem

This section is devoted to the description of the Riemann problem for (2.2), i.e. the Cauchy problem (in conservative variables)

$$\begin{cases} \partial_t \rho + \partial_x \left[\rho \cdot v_f(\rho)\right] = 0 & (\rho, y) \in \Omega_f \\ \begin{cases} \partial_t \rho + \partial_x \left[\rho \cdot v\right] = 0 \\ \partial_t y + \partial_x \left[y \cdot v\right] = 0 \end{cases} & (\rho, y) \in \Omega_c \\ (\rho, y)(0, x) = \begin{cases} (\rho^l, y^l) & \text{if } x < 0 \\ (\rho^r, y^r) & \text{if } x > 0\,. \end{cases} \end{cases} \tag{3.5}$$

However, the description will be carried out in the (ρ, v) variables or in the Riemann coordinates. The construction follows closely the one in [9, 15]. For every $(\rho^l, y^l), (\rho^r, y^r)$ in $\Omega_f \cup \Omega_c$, we define a unique self-similar admissible solution to (3.5) as defined in [9].

We consider several different cases:

(A) The data in (3.5) are in the same phase, i.e. they are either both in Ω_f or both in Ω_c. Then the solution is the standard Lax solution to (2.2), left (resp. right), and no phase boundary is present.

(B) $(w_1^l, w_2^l) \in \Omega_c$ and $(w_1^r, w_2^r) \in \Omega_f$ (as in Figure 4). We consider the points $(w_1^c, w_2^c) \in \Omega_c$ and $(w_1^m, w_2^m) \in \Omega_f$ implicitly defined by $V_c + p(\rho_c) = w_1^l$ and $v_f(\rho_m) + p(\rho_m) = w_1^l$ respectively. Then the solution is made of a rarefaction from (w_1^l, w_2^l) to (w_1^c, w_2^c), a phase transition from (w_1^c, w_2^c) to (w_1^m, w_2^m) and a Lax wave from (w_1^m, w_2^m) to (w_1^r, w_2^r).

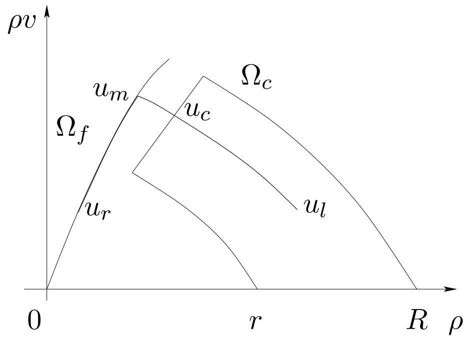


Figure 4: Case **(B)**

(C) $(w_1^l, w_2^l) \in \Omega_f$ and $(w_1^r, w_2^r) \in \Omega_c$ with $w_2^l \in [p(r), p(R)]$ (as in Figure 5, left). Consider the point $(w_1^m, w_2^m) \in \Omega_c$ implicitly defined by $v_m = v_r$, $v_m + p(\rho_m) = w_1^l$. Then the solution is made of a shock-like phase transition between (w_1^l, w_2^l) and (w_1^m, w_2^m) followed by a 2-Lax wave.

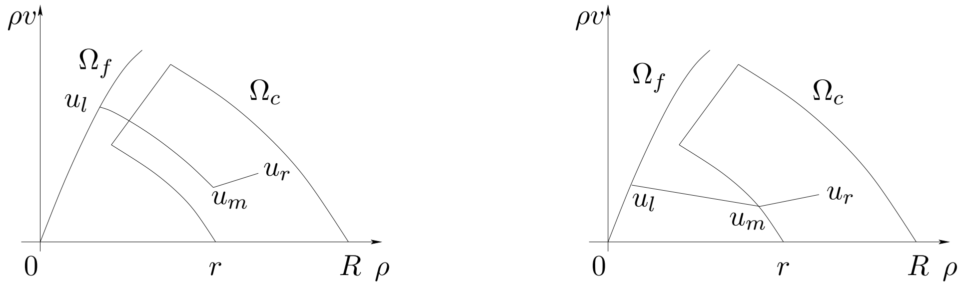


Figure 5: Left: case **(C)**. Right: case **(D)**

(D) $(w_1^l, w_2^l) \in \Omega_f$ with $w_2^l < p(r)$ and $(w_1^r, w_2^r) \in \Omega_c$ (see Figure 5, right). Due to the concavity of the curve $\rho\, v = \rho\, \big(p(r) - p(\rho)\big)$, this case is much simpler than the corresponding case in [9]. Let $(w_1^m, w_2^m) \in \Omega_c$ be the point in Ω_c implicitly defined by $v_m = v_r$, $v_m + p(\rho_m) = p(r)$. The solution to (3.5) consists of a phase boundary joining (w_1^l, w_2^l) with (w_1^m, w_2^m) followed by a 2-Lax wave in Ω_c between the states (w_1^m, w_2^m) and (w_1^r, w_2^r).

From the point of view of traffic flow, it is natural to pass to the Initial Boundary Value Problem (IBVP)

$$\left\{\begin{array}{lll} \partial_t \rho + \partial_x \left[\rho \cdot v_f(\rho)\right] = 0 & (\rho, y) \in \Omega_f\,, \quad t \geq 0\,, & x \geq 0\,, \\ \left\{\begin{array}{l} \partial_t \rho + \partial_x \left[\rho \cdot v\right] = 0 \\ \partial_t y + \partial_x \left[y \cdot v\right] = 0 \end{array}\right. & (\rho, y) \in \Omega_c\,, \quad t \geq 0\,, & x \geq 0\,, \\ (\rho, y)(0, x) = (\bar\rho, \bar y)(x) & & x \geq 0\,, \\ (\rho v)(t, 0) = \tilde f(t) & \qquad\qquad\quad\;\; t \geq 0\,. & \end{array}\right. \tag{3.6}$$

Problem (3.6) describes a road starting at $x = 0$ where the inflow $\tilde f$ is prescribed.

The definition of solution to (3.6) used here has been introduced in [16], see also [17, Definition NC]. According to it, a solution to (3.6) is a weak entropic solution to the Cauchy Problem for the conservation law where $x > 0$, that attains the boundary data in the sense of the limit:

$$\lim_{x \to 0+} \rho(t, x) \cdot v\left(\rho(t, x), y(t, x)\right) = \tilde f(t) \qquad \text{for a.e. } t \geq 0\,.$$

Under suitable conditions, see (3.8) in Proposition 3.1 below, the Riemann problem with boundary (3.7) is *non characteristic*, i.e. the condition prescribed along the boundary is attained by the trace of the solution on the boundary. Nevertheless, due to the possible presence of phase boundaries, the number of waves entering the domain cannot be *a priori* established.

The starting point for the study of (3.6) is the solution to the Riemann problem with boundary, namely

$$\begin{cases} \partial_t \rho + \partial_x \left[\rho \cdot v_f(\rho)\right] = 0 & (\rho, y) \in \Omega_f\,, \quad t \geq 0\,, \quad x \geq 0 \\ \left\{ \begin{array}{l} \partial_t \rho + \partial_x \left[\rho \cdot v\right] = 0 \\ \partial_t y + \partial_x \left[y \cdot v\right] = 0 \end{array} \right. & (\rho, y) \in \Omega_c\,, \quad t \geq 0\,, \quad x \geq 0 \\ (\rho, y)(0, x) = (\bar\rho, \bar y) & \qquad\qquad\qquad x \geq 0\,, \\ (\rho v)(t, 0) = \tilde f & \qquad\qquad\qquad t \geq 0\,. \end{cases} \tag{3.7}$$

We denote the maximum possible traffic flow along the considered road by $F = R_f V_f$.

Proposition 3.1 *With reference to* (3.7)*, if*

$$V_{ref} \geq V \left(1 - \frac{r}{eR}\right)\,, \tag{3.8}$$

then for all $(\bar\rho, \bar y) \in \Omega_f \cup \Omega_c$*, there exists a threshold* $f^{\max} = f^{\max}(\bar\rho, \bar y)$ *such that for all* $\tilde f \in [0, f^{\max}]$ *the Riemann problem for* (3.7) *admits a solution in the sense of [17, Definition NC]. More precisely, there exists a unique state* $(\tilde\rho, \tilde y) \in \Omega_f \cup \Omega_c$ *such that the flow at* $(\tilde\rho, \tilde y)$ *is* $\tilde f$ *and the standard solution to the Riemann problem* (2.2) *with data* $(\tilde\rho, \tilde y)$ *and* $(\bar\rho, \bar y)$ *consists only of waves having positive speed.*

1. *If* $(\bar\rho, \bar y) \in \Omega_f$*, then* $f^{\max} = F$ *and* $(\tilde\rho, \tilde y)$ *is in* Ω_f*. The solution consists of a* 2*-wave in the free phase.*
2. *If* $(\bar\rho, \bar y) \in \Omega_c$*, then there exist a* $f^{\min} = f^{\min}(\bar\rho, \bar y)$ *such that:*
   - *(a) If* $f^{\min} \leq \tilde f \leq f^{\max}$*,* $(\tilde\rho, \tilde y)$ *is the unique intersection between the curve* $\rho v(\rho, y) = \tilde f$ *and the* 2*-wave through* $(\bar\rho, \bar y)$*. The solution consists of a simple* 2*-wave.*
   - *(b) If* $\tilde f < f^{\min}$*, then* $(\tilde\rho, \tilde y)$ *is the unique state in* Ω_f *such that* $\tilde\rho v_f(\tilde\rho) = \tilde f$*. The solution consists of a phase boundary and a* 2*-wave.*

Moreover, the Riemann Solver is continuous in $\mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}$.

Note that condition (3.8) ensures that $\sup_{\Omega_f \cup \Omega_c} \lambda_1 < 0$. It means that, if the maximal speed V is not too high, the anticipation factor, which is proportional to V_{ref}, forces informations to move backward.

The proof is the same as in [15]. We recall it for completeness. Note that, as remarked in [15], the incoming flow $\tilde f$ can be slightly greater than the flow $\bar\rho\, v(\bar\rho)$ present on the road.

Proof. Condition (3.8) implies $\sup_{\Omega_f \cup \Omega_c} \lambda_1 < 0$, hence all the waves of the first family are exiting the domain $x \geq 0$, $t \geq 0$.

1. If $(\bar\rho, \bar y)$ is in Ω_f, then for any $\tilde f \in [0, F]$, the line $\rho v = \tilde f$ intersects Ω_f at a unique point $(\tilde\rho, \tilde y)$. The standard Riemann problem with data $(\tilde\rho, \tilde y)$, $(\bar\rho, \bar y)$ admits a solution consisting of a simple wave with positive speed. The restriction of this solution to $x \geq 0$, $t \geq 0$ is a solution to the Riemann problem for (3.7).

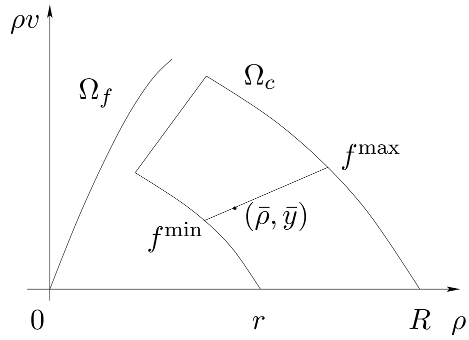


Figure 6: Notation for the proof of Proposition 3.1

2. (a) If $(\bar\rho, \bar y)$ is in Ω_c, then the 2-Lax curve through $(\bar\rho, \bar y)$ has a unique intersection with the line $\rho v = \tilde f$ at a point $(\tilde\rho, \tilde y)$ if and only if $\tilde f \in [f^{\min}, f^{\max}]$, see Figure 6.
   (b) If $(\bar\rho, \bar y)$ is in Ω_c and $\tilde f \in \left[0, f^{\min}\right[$, then the line $\rho v = \tilde f$ intersects Ω_f at a single point, say $(\tilde\rho, \tilde y)$. The standard Riemann problem with data $(\tilde\rho, \tilde y)$, $(\bar\rho, \bar y)$ has a solution consisting of a phase boundary having positive speed and a 2 contact discontinuity. The restriction of this solution to $x \geq 0$, $t \geq 0$ is a solution to the Riemann problem for (3.7).

□

Once the Riemann solvers are available, well posedness for the Cauchy and the Initial-Boundary Value Problems can be proved as in [15], for all initial (and boundary) data with bounded total variation. For sake of completeness, the corresponding results are recalled in the Appendix.

Remark 3.2 In [7] the function $p(\cdot)$ is chosen to be

$$p(\rho) = \rho^\gamma, \quad \gamma > 0.$$

In particular, the function is positive and its behavior near the vacuum is qualitatively different from the one considered here. More precisely, consider the 1-Lax curves in the $(\rho, \rho v)$ coordinates

$$\begin{aligned} m(\rho; \rho_-) := \rho\, v(\rho) &= -\rho\, p(\rho) + \rho\, p(\rho_-), \qquad \rho_- \in [r, R], \\ &= -\rho^{\gamma+1} + \rho\, \rho_-^\gamma. \end{aligned}$$

These curves intersect Ω_f if and only if $m'(0; \rho_-) = \rho_-^\gamma > V$ and $V < R^\gamma < V/\gamma$ (derived from the condition $m'(R; \rho_-) > -V$, and which implies $\gamma < 1$). Under these hypotheses, one can recover all the previous results.

We conjecture that this construction can also be applied to the Modified AR model introduced in [18], in which the pressure takes the form

$$p(\rho) = \left(\frac{1}{\rho} - \frac{1}{R}\right)^{-\gamma}, \qquad \rho \leq R.$$

Remark 3.3 Coupling the AR model with the LWR equation let us correct some drawbacks of the original AR model. First of all, model (2.2) is well-posed and stable near the vacuum, which is not the case for the AR system. Second, as noted in [7, Section 5], when there is a rarefaction wave connecting a state $(\rho_-, \rho_- v_-)$ to the vacuum, the maximal velocity v reached by the cars is $v_{max} = +\infty$ if $p(\rho) = V_{ref} \ln(\rho/R)$, or $v_{max} = v_- + p(\rho_-)$ if $p(\rho) = \rho^\gamma$, i.e. the maximal speed reached by the cars on an empty road is either infinite or it depends on the initial data ρ_-, v_-, which is clearly wrong. On the contrary, the solution given by model (2.2) reaches the maximal velocity V independently from the choice of the pression and the initial data.

4 Traffic flow models with phase transitions

In this section, we compare the model introduced here with the biphasic model introduced by Colombo in [9] and the LWR model (1.1).

The LWR-Colombo coupling introduced in [9] reads:

$$
\begin{array}{ll}
\text{Free flow:} & \text{Congested flow:} \\
(\rho, q) \in \Omega_f & (\rho, q) \in \Omega_c \\
\partial_t \rho + \partial_x[\rho v] = 0 & \left\{ \begin{array}{l} \partial_t \rho + \partial_x[\rho v] = 0 \\ \partial_t q + \partial_x[(q - Q)v] = 0 \end{array} \right. \\
v = v_f(\rho) & v = v_c(\rho, q)
\end{array}
\tag{4.9}
$$

Here, Q is a parameter of the road under consideration and the velocity v_c in the congested phase is given by

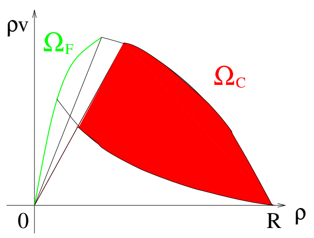
$$
v_c(\rho, q) = \left(1 - \frac{\rho}{R}\right) \frac{q}{\rho}.
$$

The weighted flow q is a variable originally motivated by the linear momentum in gas dynamics. It approximates the real flow ρv for ρ small compared to R. The two phases are defined by

$$
\begin{aligned}
\Omega_f &= \left\{ (\rho, q) \in [0, R_f] \times [0, +\infty[\, : v_f(\rho) \geq V_f, \; q = \rho \cdot V \right\} \\
\Omega_c &= \left\{ (\rho, q) \in [0, R] \times [0, +\infty[\, : v_c(\rho, q) \leq V_c, \; \frac{q - Q}{\rho} \in \left[\frac{Q_- - Q}{R}, \frac{Q_+ - Q}{R} \right] \right\},
\end{aligned}
$$

where the parameters $Q_- \in]0, Q[$ and $Q_+ \in]Q, +\infty[$ depend on the environmental conditions and determines the width of the congested region.

A detailed description of the Riemann solver, and analogies between solutions to (4.9) and real traffic features are given in [9] (see [15] for further analytical results). Note that, even for this model, the invariant domain given in Figure 7, left, is in good agreement with experimental data as reproduced in Figure 7, right.



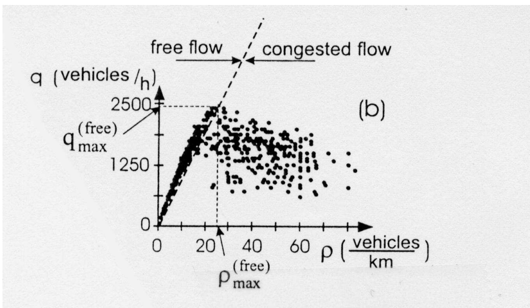


Figure 7: Left: invariant domain for (4.9). Right: experimental data, taken from [3]

The numerical integrations that lead to the figures showed in the following sections rely on the choices: $R = 1$, $r = 0.47$, $V = 2$, $V_f = 1$, $Q = 0.5$, $Q_- = 0.25$, $V_c = 0.85$, $R_f = 0.5$ and $V_{ref} = V \, \frac{1 - R_f/R}{\ln(R/R_f)}$.

4.1 Red traffic light

Assume that a traffic light is placed at $x = 0$ and turns red at $t = 0$. In other words, as in [5], we compare the restrictions to the quadrant $t \geq 0$, $x \leq 0$ of the solutions to suitable Riemann problems for (1.1), (4.9) and (2.2).[1]

- In the case of the LWR equation (1.1), the initial data is of the form

$$\rho(0, x) = \begin{cases} \rho_i & \text{if } x < 0, \\ R & \text{if } x > 0. \end{cases}$$

  The solution is a shock with negative propagation speed, located at the end of the queue of cars. Each driver, as soon as reaches it, brakes and immediately stops the car (see Figures 8–12).

- In the model (4.9), we choose the following initial data

$$(\rho, q)(0, x) = \begin{cases} (\rho_i, q_i) \in \Omega_f & \text{if } x < 0, \\ (R, q) & \text{if } x > 0. \end{cases}$$

  for some q admissible. Figures 8–12 show that there exist two threshold parameters ρ_i^- and ρ_i^+ such that

  - if $\rho_i \notin [\rho_i^-, \rho_i^+]$, we have the same solution as in the LWR model (Figures 8, 11, 12);
  - if $\rho_i \in [\rho_i^-, \rho_i^+]$, the solution consists of a phase transition followed by a rarefaction attached to it (Figures 9 and 10 show two mutual positions with respect to the solution given by (2.2), depending on the initial data).

  According to this model, drivers brake suddenly to zero speed in two cases: when density is high and speed is low, or when density is low and speed is high. In the intermediate situation, drivers brake and enter the congested region, where the car speed smoothly decreases to zero.

- In the case of the LWR-AR coupling (2.2), the initial data are given by

$$(\rho, v)(0, x) = \begin{cases} (\rho_i, v_i) \in \Omega_f & \text{if } x < 0, \\ (R, 0) & \text{if } x > 0. \end{cases}$$

  The solution exhibits the same behavior as in the LWR model, but cars are allowed to form queues at any density. In fact, in the AR model, car density along the queue depends on the density of cars that line up.

While the models (1.1) and (4.9) have a rather similar behavior for all initial data, the model (2.2) provides the same solution only for $(\rho_i, v_i) = (R_f, V_{ref}\, p(R_f) - V_{ref}\, p(R))$, i.e. the extreme point of Ω_f, that lies on the same 1-Lax curve as the right initial data $(R, 0)$ (see Figure 12).

[1] A preliminary study of this example has been presented in [10].

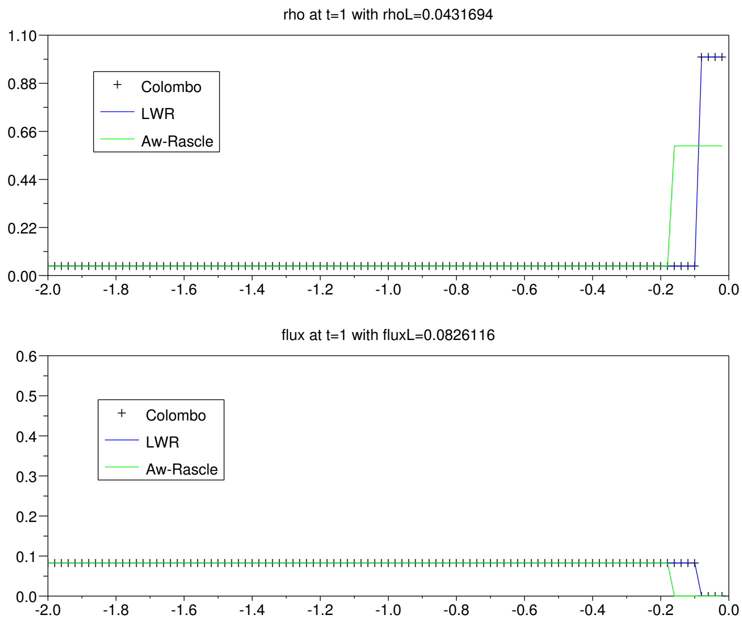


Figure 8: Red traffic light at low density.

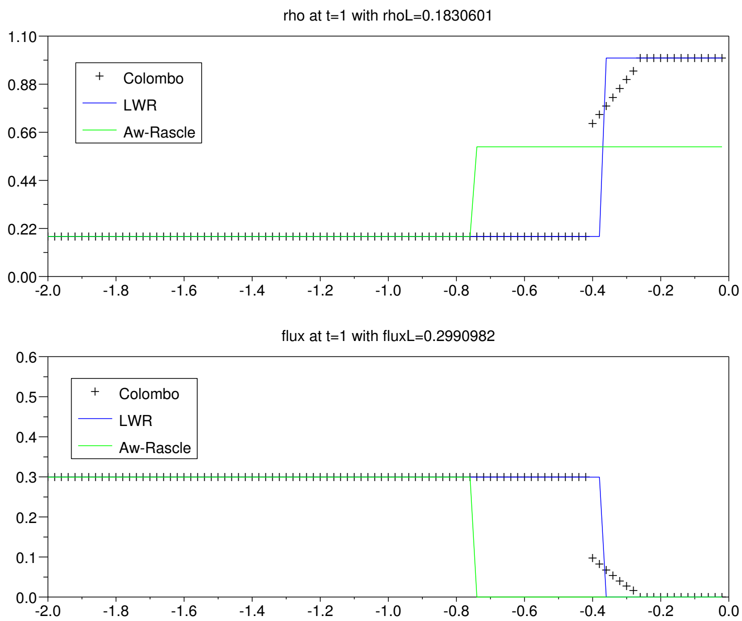


Figure 9: Red traffic light at medium density, Case 1.

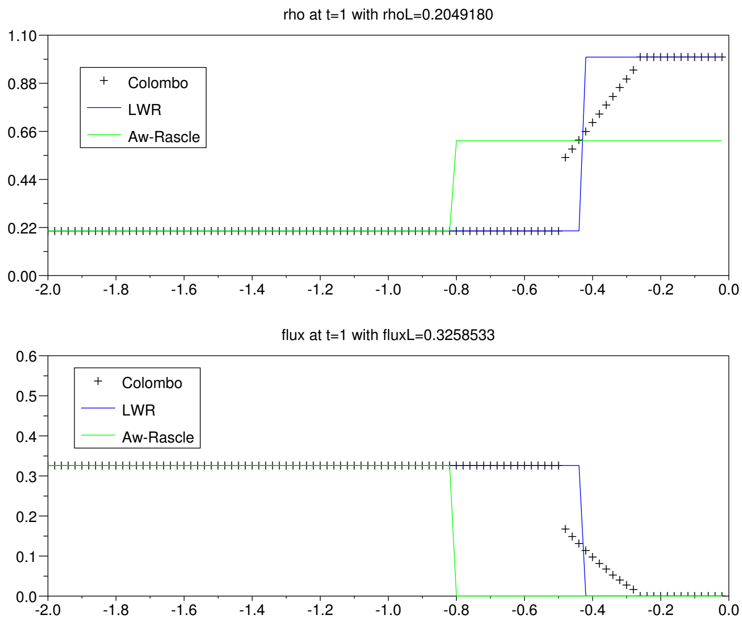


Figure 10: Red traffic light at medium density, Case 2.

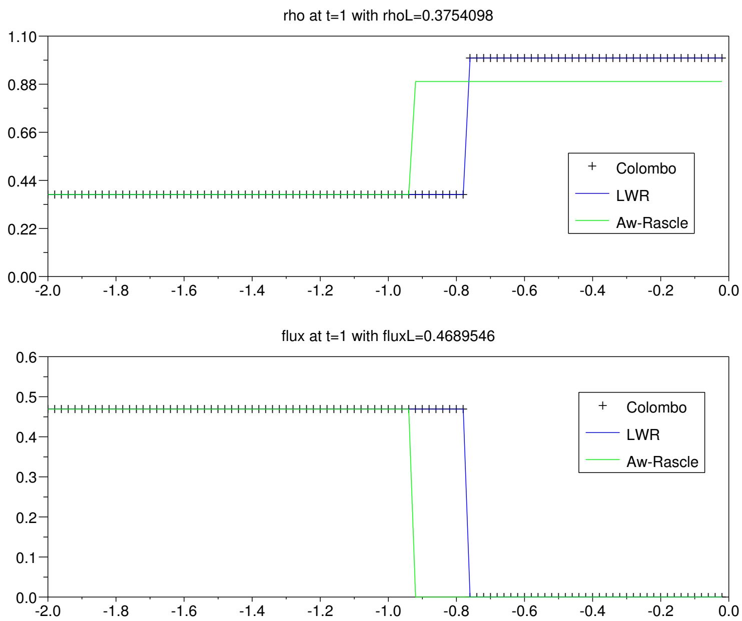


Figure 11: Red traffic light at high density, Case 1.

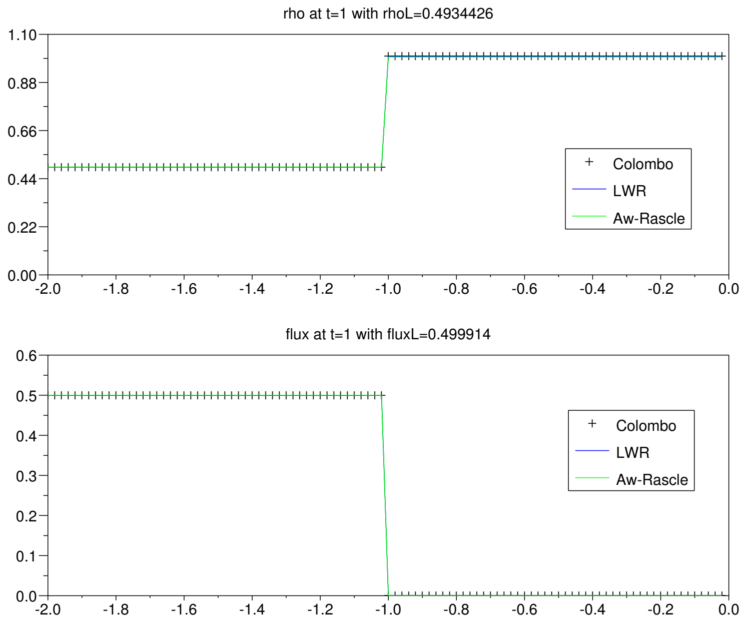


Figure 12: Red traffic light at high density, Case 2.

4.2 Green traffic light

A traffic light placed at $x = 0$ turns green at $t = 0$. The corresponding Riemann problems to be considered are as follows:

- For the LWR equation (1.1),
$$\rho(0, x) = \begin{cases} R & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$
The solution is a rarefaction wave spreading between $-V\,t$ and $V\,t$.

- For the LWR-Colombo coupling (4.9), we take
$$(\rho, q)(0, x) = \begin{cases} (R, q) \in \Omega_c & \text{if } x < 0, \\ (0, 0) & \text{if } x > 0. \end{cases} \tag{4.10}$$
for some $q \in [Q_-, Q_+]$. For $q \in [Q_-, Q]$, the solution consists in a shock-like phase transition with negative speed, followed by a rarefaction wave with positive speed in the free phase (Figures 13 and 14). For $q \in]Q, Q_+]$, the solution exhibits a rarefaction with negative speed, followed by a phase transition and a rarefaction wave with positive speed in the free phase (Figure 15).

- For the LWR-AR coupling (2.2), the initial data to be taken are
$$(\rho, v)(0, x) = \begin{cases} (R, 0) \in \Omega_c & \text{if } x < 0, \\ (0, 0) & \text{if } x > 0. \end{cases}$$
The solution presents a rarefaction wave with negative speed in the congested phase, followed by a phase transition and a rarefaction wave with positive speed in the free phase.

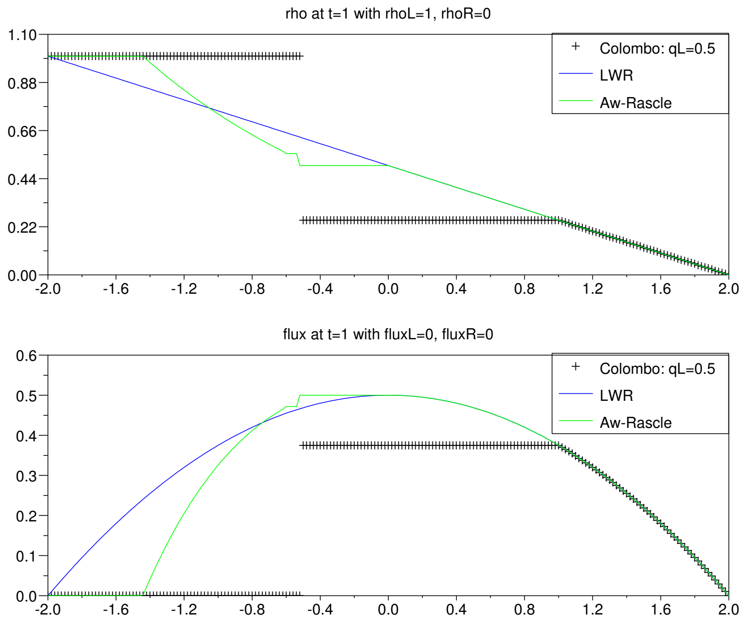


Figure 13: Green traffic light. In (4.10) $q = Q$.

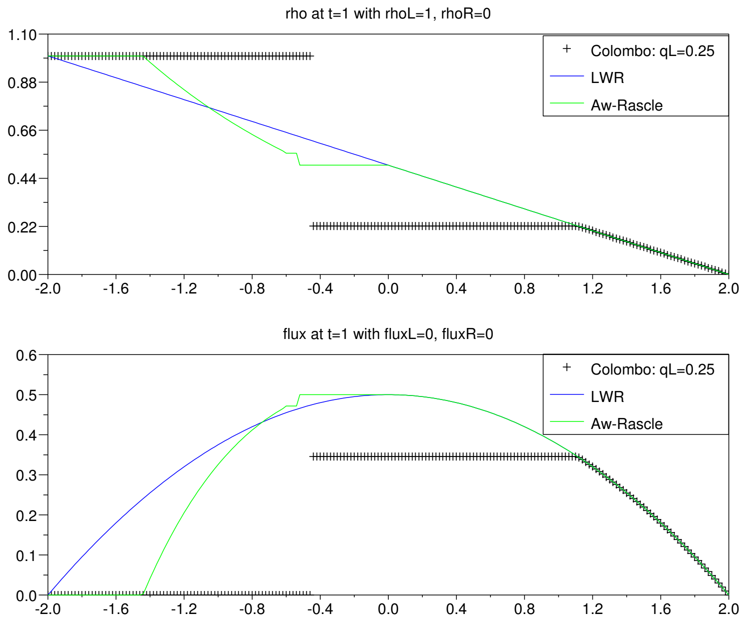


Figure 14: Green traffic light. In (4.10) $q = Q_-$.

Figures 13–15 show the solutions given by the tree models, for different values of $q \in [Q_-, Q_+]$ in (4.10). Models (1.1) and (2.2) are in good agreement, while in general model (4.9) exhibits a quite different solution. Only for q close to Q_+, its solution almost coincide with the one given by the model presented in this paper (see Figure 15, where $q = Q_+$).

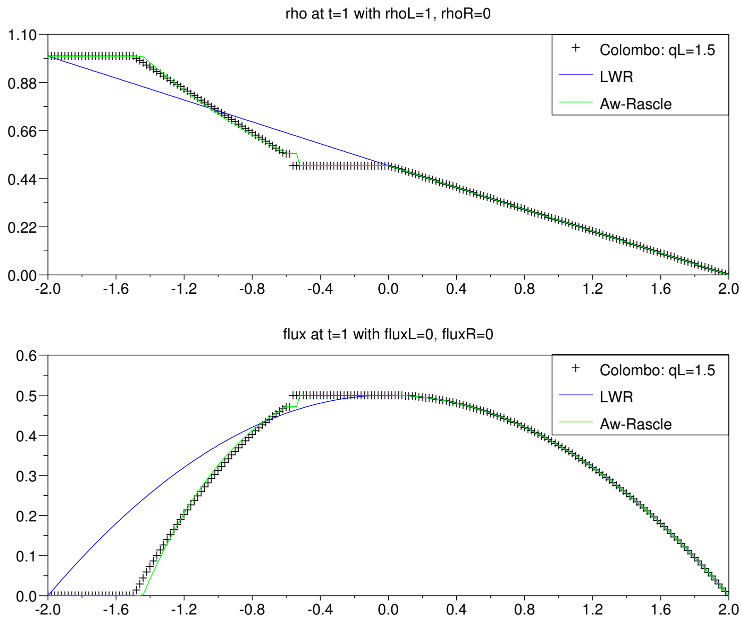


Figure 15: Green traffic light. In (4.10) $q = Q_+$.

4.3 Bottleneck

As a last example, we consider traffic on a highway described by the interval $[-2, 2]$, in which the number of lanes is reduced from three to two at $x = 0$. This is simulated by setting the maximal density $R = 1$ for $x < 0$, and $R = 2/3$ for $x > 0$. All the parameters are changed consequently in the region $x > 0$: $r = 0.41$, $V_f = 1$, $Q = 1/3$, $Q_- = 1/6$, $V_c = 0.85$, $R_f = 1/3$.

Modeling this problem requires the solution of two Riemann problems with boundary, namely (3.7) on the right and

$$\left\{\begin{array}{lll} \partial_t\rho + \partial_x\left[\rho\cdot v_f(\rho)\right] = 0 & (\rho, y)\in\Omega_f\,, & t\geq 0\,,\quad x\leq 0 \\ \left\{\begin{array}{l} \partial_t\rho + \partial_x\left[\rho\cdot v\right] = 0 \\ \partial_t y + \partial_x\left[y\cdot v\right] = 0 \end{array}\right. & (\rho, y)\in\Omega_c\,, & t\geq 0\,,\quad x\leq 0 \\ (\rho, y)(0, x) = (\bar\rho, \bar y) & & x\leq 0\,, \\ (\rho v)(t, 0)\leq\tilde f & & t\geq 0\,, \end{array}\right. \tag{4.11}$$

on the left. Details on the construction of the solution of the above Riemann problem with unilateral constraints are given in [19].

In the example showed here, we have chosen initial data on the left-hand side so that the incoming flux is higher than the maximal possible flux in the two-lane region. This causes the traffic congestion showed by Figures 16–18.

Initial data u_l for $x < 0$ and u_r for $x > 0$ are taken in the free phase Ω_f. The solutions given by the three models present the same behavior:

- a shock (hiding a phase transition for models (2.2) and (4.9)) moving backward in the three lane region, upstream the congested traffic;

- a discontinuity (under-compressive shock) at $x = 0$, corresponding to the bottleneck;

- a rarefaction wave moving forward in the free phase.

In particular, the LWR-Aw-Rascle coupling (2.2) and the LWR-Colombo coupling (4.9) are in good agreement, especially if the flux of the incoming traffic is equal to F (Figure 18).

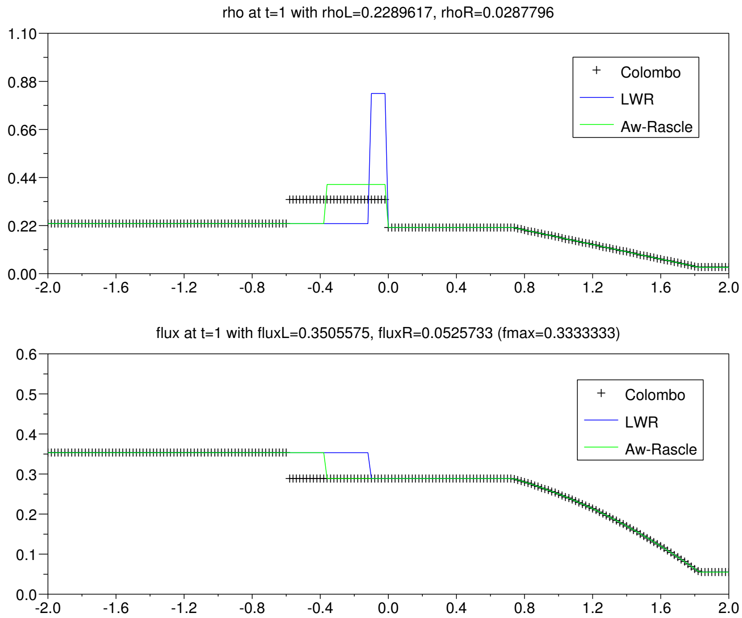


Figure 16: Bottleneck at $x = 0$ with incoming flux f_l slightly higher than the maximal possible flux at $x > 0$.

5 Conclusions

We have showed that a phase transition can be added to the AR model. This allows both to correct some drawbacks of the original 2×2 system, and to obtain results that well fit experimental data. Comparisons with the LWR model and the LWR-Colombo coupling show a satisfying rate of agreement in the resulting solutions. In particular, in the examples that have been considered, the reader can guess which models better describe each situation.

It would be interesting to investigate the continuous dependence of the solutions from the flow function, and apply it to the problem of parameter identification. The model could also be improved to cover various control and optimization problems of interest in the traffic management.

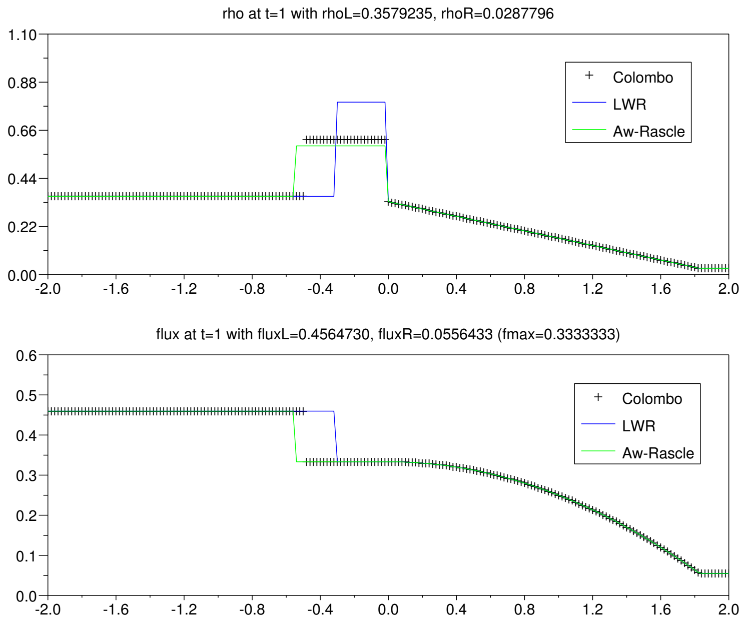


Figure 17: Bottleneck at $x = 0$ with incoming flux f_l higher than the maximal possible flux at $x > 0$.

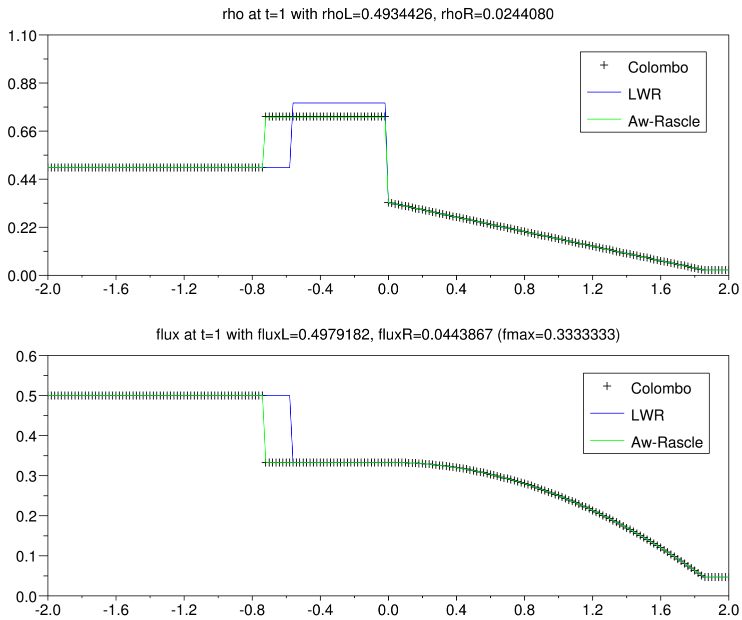


Figure 18: Bottleneck at $x = 0$ with incoming flux $f_l = F$, the maximal possible incoming flux at $x < 0$.

Appendix: Cauchy and Initial-Boundary Value Problems

Let us introduce the following notations:

$$\begin{array}{rcl} X & = & \mathbf{L^1}\left(\mathbb{R};\Omega_f \cup \Omega_c\right) , \\ u & = & (\rho, y) , \\ \|u\|_{\mathbf{L^1}} & = & \|\rho\|_{\mathbf{L^1}(\mathbb{R})} + \|y\|_{\mathbf{L^1}(\mathbb{R})} , \\ \mathrm{TV}(u) & = & \mathrm{TV}(\rho) + \mathrm{TV}(y) . \end{array}$$

Definition 5.1 *Let $M > 0$ and a function space X. A map $S\colon [0,+\infty[\times \mathcal{D} \mapsto \mathcal{D}$ is an M-*Standard Riemann Semigroup *(M-SRS) if the following holds:*

(SRS1) $\mathcal{D} \supseteq \left\{u \in X\colon \mathrm{TV}(u) \leq M\right\}$;

(SRS2) $S_0 = \mathrm{Id}$ *and* $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$;

(SRS3) there exists a constant $L > 0$ such that for all t_1, t_2 in $[0,+\infty[$ and u_1, u_2 in $\mathcal{D}$,

$$\left\|S_{t_1}u_1 - S_{t_2}u_2\right\|_{\mathbf{L^1}} \leq L \cdot \left(\|u_1 - u_2\|_{\mathbf{L^1}} + |t_1 - t_2|\right) ;$$

(SRS4) if $u \in \mathcal{D}$ is piecewise constant, then for t small, $S_t u$ coincides with the gluing of solutions to Riemann problems.

By *"solutions to Riemann problems"* we refer to the solutions to (2.2) defined in Section 3. We are now ready to state the existence of a SRS generated by (2.2).

Theorem 5.2 *For any positive M, the system* (2.2) *generates an M-SRS $S\colon [0,+\infty[\times \mathcal{D} \mapsto \mathcal{D}$. Moreover*

(CP1) for all $(\rho_o, y_o) \in \mathcal{D}$, the orbit $t \mapsto S_t(\rho_o, y_o)$ is a weak entropic solution to (2.2) *with initial data (ρ_o, y_o);*

(CP2) any two M–SRS coincide up to the domain;

(CP3) the solutions yielded by S can be characterized as viscosity solutions, in the sense of [20, Theorem 9.2].

(CP4) $\mathcal{D} \subseteq \left\{u \in X\colon \mathrm{TV}(u) \leq \widehat{M}\right\}$ *for a positive $\widehat{M}$ dependent only on M.*

Observe that the description of several realistic situations requires suitable source terms in the right hand sides of (2.2). Thanks to our choice of the pression, the techniques in [12, 21] can then be applied.

We consider now the IBVP (3.6), relying on the Riemann solver for (3.7) constructed in Section 3. Following [15], in the case of (3.6), we denote:

$$\begin{array}{rcl} X & = & \mathbf{L^1}\left([0,+\infty[\,;(\Omega_f \cup \Omega_c) \times [0,F]\right) \\ u & = & (\rho, y, f) \\ \|u\|_{\mathbf{L^1}} & = & \|\rho\|_{\mathbf{L^1}\left([0,+\infty[\right)} + \|y\|_{\mathbf{L^1}\left([0,+\infty[\right)} + \|f\|_{\mathbf{L^1}\left([0,+\infty[\right)} \\ \mathrm{TV}(u) & = & \mathrm{TV}(\rho) + \mathrm{TV}(y) + \mathrm{TV}(f) + \left|(\rho v)(0) - f(0)\right| . \end{array}$$

With this notation, Definition 5.1 applies also to the case of (3.6) provided *"solutions to Riemann problems"* is now intended as the solutions to (2.2) previously defined where $x > 0$ and as the solution constructed in Proposition 3.1 at $x = 0$.

Theorem 5.3 *For every positive M, the IBVP* (3.6) *generates a M–SRS*

$$\begin{array}{rcccccc} S & : & [0,+\infty[& \times & \mathcal{D} & \mapsto & \mathcal{D} \\ & & t & , & \left(\bar\rho,\bar y,\tilde f\right) & \mapsto & \left(\rho(t),y(t),\mathcal{T}_t\tilde f\right). \end{array}$$

Moreover

(IBVP1) for all $(\bar\rho,\bar y,\tilde f)\in\mathcal{D}$, the map $t\mapsto\big(\rho(t),y(t)\big)$ is a weak entropic solution to (3.6) *with initial data $(\bar\rho,\bar y)$ and boundary data $\tilde f$;*

(IBVP2) any two M–SRS coincide up to the domain;

(IBVP3) the solutions yielded by S can be characterized as viscosity solutions, in the sense of [22, Section 5];

(IBVP4) $\mathcal{D}\subseteq\left\{u\in X\colon \mathrm{TV}(u)\le\widehat{M}\right\}$ for a positive $\widehat{M}$ dependent only on M.

Above, $\mathcal{T}$ is the translation operator, i.e. $(\mathcal{T}_t f)(s)=f(t+s)$. In the case of (3.6), *(SRS3)* implies that

$$\begin{aligned} &\left\|(\rho_1,y_1)(t_1)-(\rho_2,y_2)(t_2)\right\|_{\mathbf{L}^1} \\ \le\ & L\cdot\left(\left\|(\bar\rho_1,\bar y_1)-(\bar\rho_2,\bar y_2)\right\|_{\mathbf{L}^1}+\left\|\tilde f_1-\tilde f_2\right\|_{\mathbf{L}^1}+|t_1-t_2|\right) \end{aligned}$$

Thanks to Proposition 3.1, the IBVP (3.6) fits in the framework of non characteristic problems. Moreover, the techniques used in [23] apply also to the present case.

When source terms need to be added on the right hand sides in (3.6), the techniques in [12, 23] can be applied.

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