The Riemann problem admitting δ'-shock wave type solution (the vanishing viscosity approach)

V. M. Shelkovich

Abstract. In this paper, using the vanishing viscosity method, a solution of the Riemann problem for the system of conservation laws

$$u_t + \big(u^2\big)_x = 0, \quad v_t + 2\big(uv\big)_x = 0, \quad w_t + 2\big(v^2 + uw\big)_x = 0$$

with the initial data

$$(u(x,0), v(x,0), w(x,0)) = \begin{cases} (u_-, v_-, w_-), & x < 0, \\ (u_+, v_+, w_+), & x > 0, \end{cases}$$

is constructed. This problem admits a δ'-shock wave type solution, which is a new type of singular solutions to systems of conservation laws first introduced in [**25**]. Roughly speaking, it is a solution of the above system such that for $t > 0$ its second component v may contain Dirac measures, and the third component w may contain a linear combination of Dirac measures and their derivatives, while the first component u of the solution has bounded variation. Using the above mentioned results, we solve the δ-shock Cauchy problem for the first two equations of the above system.

Since δ'-shocks can be constructed by the vanishing viscosity method, these solutions are "natural" *distributional* solutions to systems of conservation laws. The results of this paper as well as those of the paper [**25**] show that solutions of systems of conservation laws can develop not only Dirac measures (as in the case of δ-shocks) but their derivatives as well.

1. Introduction

1.1. L^∞-type solutions. Let us consider the Cauchy problem for the hyperbolic system of conservation laws

$$\begin{cases} U_t + \big(F(U)\big)_x = 0, & \text{in } \mathbb{R} \times (0,\ \infty), \\ U = U^0, & \text{in } \mathbb{R} \times \{t = 0\}, \end{cases} \tag{1.1}$$

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where $F : \mathbb{R}^m \to \mathbb{R}^m$ and $U^0 : \mathbb{R} \to \mathbb{R}^m$ are given smooth vector-functions, and $U = U(x,t) = (u_1(x,t), \dots, u_m(x,t))$ is the unknown function, $x \in \mathbb{R}$, $t \geq 0$.

As is well known, even in the case of smooth (and, certainly, in the case of discontinuous) initial data $U^0(x)$, we cannot in general find a smooth solution of (1.1). As said in the book [**14**, 3.4.1.a.], in this case "... we must devise some way to interpret a less regular function U as somehow "solving" this initial-value problem. But as it stands, the PDE does not even make sense unless U is differentiable. However, observe that if we *temporary* assume U is smooth, we can as follows rewrite, so that the resulting expression does not directly involve the derivatives of U. The idea is to multiply the PDE in (1.1) by a smooth function φ and then to integrate by parts, thereby transferring the derivatives onto φ". In this way we derive the integral identities which define a L^∞-*generalized solution* of the Cauchy problem (1.1). It is said that $U \in L^\infty\big(\mathbb{R} \times (0,\infty); \mathbb{R}^m\big)$ is a *generalized solution* of the Cauchy problem (1.1) if the integral identities

$$\int_0^\infty \int \Big(U \cdot \widetilde{\varphi}_t + F(U) \cdot \widetilde{\varphi}_x \Big)\, dx\, dt + \int U^0(x) \cdot \widetilde{\varphi}(x,0)\, dx = 0 \tag{1.2}$$

hold for all compactly supported test vector-functions $\widetilde{\varphi} : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$, where $\cdot$ is the scalar product of vectors, $\int f(x)\, dx$ denotes the improper integral $\int_{-\infty}^{\infty} f(x)\, dx$.

1.2. δ-Shock wave type solutions. Consider two particular cases of the above system of conservation laws:

$$u_t + \big(F(u,v)\big)_x = 0, \qquad v_t + \big(G(u,v)\big)_x = 0, \tag{1.3}$$

and

$$v_t + \big(G(u,v)\big)_x = 0, \qquad (uv)_t + \big(H(u,v)\big)_x = 0, \tag{1.4}$$

where $F(u,v)$, $G(u,v)$, $H(u,v)$ are smooth functions, *linear* with respect to v; $u = u(x,t)$, $v = v(x,t) \in \mathbb{R}$; $x \in \mathbb{R}$.

In [**2**]– [**4**], [**8**]– [**17**], [**20**], [**21**], [**29**]– [**33**], [**35**] it is shown that for some cases of hyperbolic systems (1.3), (1.4) "nonclassical" situations may occur, when the Riemann problem does not possess a weak L^∞-solution except for some particular initial data. Here the *linear* component v of the solution may contain Dirac measures and must be sought in the space of measures, while the nonlinear component u of the solution has bounded variation. In order to solve the Cauchy problems in these nonclassical situations, it is necessary to introduce new singularities called δ-shocks, which are solutions of hyperbolic systems (1.3) or (1.4), whose *linear* components have the form $v(x,t) = V(x,t) + e(x,t)\delta(\Gamma)$, Γ is a graph in the upper half-plane $\{(x,t) : x \in \mathbb{R}, t \geq 0\}$, $V \in L^\infty$, $e \in C(\Gamma)$, and the *nonlinear* component $u \in L^\infty\big(\mathbb{R} \times (0,\infty); \mathbb{R}\big)$. We need to define *in which sense* a distributional solution satisfies a nonlinear system. Unfortunately, using the above instruction from the L. C. Evans's book [**14**, 3.4.1.a.], δ-shock wave type solutions *cannot be defined*. Indeed, as can be seen from (1.3), (1.4) (see also (1.5), (1.9)), if integrating by parts we transfer the derivatives onto a test function φ, under the integral sign there

still remain terms *undefined in the distributional sense*, since the component v may contain Dirac measures. In order to introduce *δ-shock type solutions*, we must devise some way to define a *singular superposition* of distributions (for example, a *product of the Heaviside function and the delta function*) (see, for example, [**31**]).

Recently, the theory of δ-shock type solutions for systems of conservation laws has attracted intensive attention. In particular, there are large number of papers where the system of zero-pressure gas dynamics is studied (see, for example, [**1**]–[**4**], [**10**], [**12**], [**28**], [**30**], [**35**]). For one dimensional case this system is a particular case of system (1.4), where $G(u,v)=uv$, $H(u,v)=u^2v$, and $v(x,t)\geq 0$ is density, and $u(x,t)$ is velocity.

Several approaches to solving δ-shock problems are known (see the above cited papers and the references therein). One of them is the *vanishing viscosity method* [**13**], [**16**], [**17**], [**20**], [**27**], [**33**], [**35**], which relates with introducing a *viscosity term* in the right-hand side into a system of conservation laws. Next, we study a zero dissipation limit of the viscous conservation laws obtained in this way. "Although the solution of the viscous conservation laws are expected to approach those of hyperbolic conservation laws as the viscosity tends to zero, this zero dissipation limit is quite complicated" [**23**]. Note that the vanishing viscosity regularization is often physically appropriate.

Note that in [**17**], by using the *vanishing viscosity method*, a δ-shock wave type solution of the system

$$u_t+\Big(\frac{u^2}{2}\Big)_x=0,\qquad v_t+\big(uv\big)_x=0 \tag{1.5}$$

(here $F(u,v)=u^2/2$, $G(u,v)=vu$) with the initial data

$$\big(u^0(x),v^0(x)\big)=\begin{cases}(u_-,v_-), & x<0,\\ (u_+,v_+), & x>0,\end{cases} \tag{1.6}$$

is obtained, where $u_\pm$, $v_\pm$ are given constants. In this paper the following definition is used: the Riemann problem for (1.5) is to find distributions $(u,v)\in\mathcal{D}'(D)\times\mathcal{D}'(D)$ such that

$$\langle u,\varphi_t\rangle+\langle u^2/2,\varphi_x\rangle=0,\quad \langle v,\varphi_t\rangle+\langle uv,\varphi_x\rangle=0, \tag{1.7}$$

for all $\varphi\in\mathcal{D}(D)$, which satisfy the initial data (1.6). In [**17**], to solve the problem (1.5), (1.6), the *weak limit* $(u,v)=(\lim_{\varepsilon\to+0}u_\varepsilon,\lim_{\varepsilon\to+0}v_\varepsilon)$ is constructed, where $(u_\varepsilon(x,t),v_\varepsilon(x,t))$ is a solution of the parabolic problem

$$u_{\varepsilon\,t}+\Big(\frac{u_\varepsilon^2}{2}\Big)_x=\frac{1}{2}\varepsilon u_{xx},\qquad v_{\varepsilon\,t}+\big(u_\varepsilon v_\varepsilon\big)_x=\frac{1}{2}\varepsilon v_{xx} \tag{1.8}$$

with the initial data (1.6). Since the pair of distributions (u,v) is such that u contains the Heaviside function, and v contains both the Heaviside function and the delta function (see [**17**, (2.27),(2.30)] and Corollary 7.2), the product uv is not defined in the sense of distributions, and, consequently, a δ-shock wave type solution of this problem *cannot be defined* by definition (1.7). Moreover, it is clear that in the

sense of distributions $\lim_{\varepsilon\to+0} u_\varepsilon(x,t)v_\varepsilon(x,t) \neq \lim_{\varepsilon\to+0} u_\varepsilon(x,t) \lim_{\varepsilon\to+0} v_\varepsilon(x,t)$. It is also easily to see that the weak limit of the solution to the problem (1.8), (1.6) *can be interpreted* as a δ-shock wave type solution of the Cauchy problem (1.5), (1.6), for example, in the sense of the *measure-valued solutions* considered in [**2**], [**33**], [**35**], or in the sense of the approach [**3**], [**4**]. In the framework of our approach the correct solution of this problem is given below by Corollary 7.2.

Recall that in [**21**], to construct a δ-shock wave type solution of the system

$$u_t + \big(f(u)\big)_x = 0, \qquad v_t + \big(f'(u)v\big)_x = 0, \tag{1.9}$$

(here $F(u,v) = f(u)$, $G(u,v) = f'(u)v$) the problem of *multiplication of distributions* is solved by using the definition of Volpert's averaged superposition [**34**]. In [**24**], a general framework for *nonconservative product*

$$g(u)\frac{du}{dx} \tag{1.10}$$

was introduced, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded Borel function, and $u : (a,b) \to \mathbb{R}^n$ is a discontinuous function of bounded variation. In the framework of approach [**24**] the Cauchy problems for nonlinear hyperbolic systems in nonconservative form can be considered [**21**], [**22**].

In [**5**], [**6**]– [**11**], [**29**]– [**32**], a new asymptotics method (namely, the *weak asymptotics method*) for studying the *dynamics of propagation and interaction* of different singularities of quasilinear differential equations and hyperbolic systems of conservation laws was developed. In [**9**]– [**11**], in the framework of the *weak asymptotics method definitions of a δ-shock wave type solution* by integral identities were introduced for two classes of hyperbolic systems of conservation laws (1.3), (1.4) (for system (1.3) see Definition 2.1 below). These definitions give *natural* generalizations of the classical definition of the weak L^∞-solutions (1.1) relevant to the structure of δ-shocks.

1.3. δ'-Shock wave type solutions. In [**25**] (a short review of some results from [**25**] can be found in [**26**]), a concept of a new type of singular solutions to systems of conservation laws, namely, $\delta^{(n)}$-shock wave, was introduced, where $\delta^{(n)}$ is n-th derivative of the Dirac delta function ($n = 1, 2, \dots$). In this paper the case $n = 1$ was studied in details.

In [**25**], in the framework of the mentioned above *weak asymptotics method*, a *definition of a δ'-shock wave type solution* (Definition 3.1) for the system of conservation laws

$$\begin{aligned} u_t + \big(f(u)\big)_x &= 0, \\ v_t + \big(f'(u)v\big)_x &= 0, \\ w_t + \big(f''(u)v^2 + f'(u)w\big)_x &= 0, \end{aligned} \tag{1.11}$$

was introduced, where $f(u)$ is a smooth function, $f''(u) > 0$, $u = u(x,t), v = v(x,t), w = w(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$. Definition 3.1 is a *natural* generalization of the

δ-shock Definition 2.1. If in Definitions 2.1 and 3.1 there are no δ and δ'-terms (see (2.1) and (3.1), (1.18), respectively), i.e., $e(x,t) = g(x,t) = h(x,t) = 0$, then these definitions coincide with the classical definition (1.2).

Since by differentiating the scalar conservation law $u_t + \big(f(u)\big)_x = 0$ twice with respect to x and denoting $v = u_x$, $w = v_x$, we obtain system (1.11), this system is a 3×3 "prolonged system". System (1.11) is extremely degenerate with repeated eigenvalues $\lambda = f'(u)$ and repeated eigenvectors $(0, 0, 1)$.

In [**25**], within the framework of Definition 3.1, the Rankine–Hugoniot conditions for δ'-shock were derived. In [**25**], a δ'-shock wave type solution to the Cauchy problem of the system of conservation laws

$$u_t + \big(u^2\big)_x = 0, \quad v_t + 2\big(uv\big)_x = 0, \quad w_t + 2\big(v^2 + uw\big)_x = 0 \tag{1.12}$$

with the singular initial data

$$\begin{aligned} u^0(x) &= u_0^0(x) + u_1^0(x)H(-x), \\ v^0(x) &= v_0^0(x) + v_1^0(x)H(-x) + e^0\delta(-x), \\ w^0(x) &= w_0^0(x) + w_1^0(x)H(-x) + g^0\delta(-x) + h^0\delta'(-x), \end{aligned} \tag{1.13}$$

was constructed, where $u_k^0(x)$, $v_k^0(x)$, $w_k^0(x)$, $k = 0, 1$ are given smooth functions; e^0, g^0, h^0 are given constants, $H(x)$ is the Heaviside function, $\delta(x)$ is the delta function, and $\delta'(x)$ is its derivative.

Roughly speaking, a δ'-shock wave type solution is such a solution of system (1.11) that for $t > 0$ its second component v may contain Dirac measures, and the third component w may contain a linear combination of Dirac measures and their derivatives, while the first component u of the solution has bounded variation (the exact structure of a δ'-shock wave type solution see below in (1.18)).

In [**25**], we used the following admissibility condition for δ'-shocks:

$$f'(u_+) \le \dot{\phi}(t) \le f'(u_-), \tag{1.14}$$

where $\dot{\phi}(t)$ is the velocity of the δ'-shock wave, and u_-, u_+ are the respective left- and right-hand values of u on the discontinuity curve. Condition (1.14) means that all characteristics on both sides of the discontinuity are in-coming. For system (1.12) condition (1.14) has the form

$$2u_+ \le \dot{\phi}(t) \le 2u_-. \tag{1.15}$$

Note that the notion of a *weak asymptotic solution* is one of the most important in the *weak asymptotics method* [**9**]– [**11**], [**25**], [**26**]. In these papers *δ-shock and δ'-shock wave type solutions* of the Cauchy problems are constructed as the weak limits of a *weak asymptotic solutions* of the corresponding Cauchy problems. Definition 2.1 and Definition 3.1 are derived only after analyzing *asymptotic solutions* of the Cauchy problems. These definitions are based on the possibility to represent weak limits of nonlinear terms (for example, $f'(u(x,t,\varepsilon))v(x,t,\varepsilon)$, $f''(u(x,t,\varepsilon))v^2(x,t,\varepsilon) + f'(u(x,t,\varepsilon))w(x,t,\varepsilon))$ as $\varepsilon \to 0$, in the form of linear combinations of the Heaviside function, the delta function and its derivative, where

$(u(x,t,\varepsilon),v(x,t,\varepsilon),w(x,t,\varepsilon))$ is a *weak asymptotic solution* to the Cauchy problem (see [**9**]– [**11**], [**25**], [**26**] and Sec. 8).

1.4. Main results and contents of the paper. In this paper we continue studying $\delta^{(n)}$*-shock waves* started in [**25**]. Namely, by using the *vanishing viscosity method*, we construct a δ'-shock wave type solution to the Cauchy problem for system (1.12) with the initial data

$$\bigl(u^0(x),v^0(x),w^0(x)\bigr)=\begin{cases}(u_-,v_-,w_-), & x<0,\\ (u_+,v_+,w_+), & x>0,\end{cases} \tag{1.16}$$

where $u_+=u_0^0$, $v_+=v_0^0$, $w_+=w_0^0$, $u_-=u_0^0+u_1^0$, $v_-=v_0^0+w_1^0$, $w_-=w_0^0+w_1^0$ are given constants. The initial data (1.16) are a particular case of the initial data (1.13).

In Sec. 2, in order to compare our results on δ*-shocks* [**9**]– [**11**], [**30**]– [**32**] with those on δ'-shocks, we give Definition 2.1 for δ-shock type solutions and the Rankine–Hugoniot conditions for δ-shocks. In Sec. 3, we introduce Definition 3.1 of a δ'*-shock wave type solution* for system (1.11) as well as the Rankine–Hugoniot conditions for δ'-shocks from [**25**].

As mentioned above, we construct solutions of the Cauchy problem (1.12), (1.16), in particular, a δ'-shock wave type solution, using the *vanishing viscosity method*. Thereto, in Sec. 4, as the first step, we construct solutions of parabolic approximation of system (1.12)

$$\begin{aligned} u_{\varepsilon\,t}+\bigl(u_\varepsilon^2\bigr)_x &= \varepsilon u_{\varepsilon\,xx},\\ v_{\varepsilon\,t}+2\bigl(u_\varepsilon v_\varepsilon\bigr)_x &= \varepsilon v_{\varepsilon\,xx},\\ w_{\varepsilon\,t}+2\bigl(v_\varepsilon^2+u_\varepsilon w_\varepsilon\bigr)_x &= \varepsilon w_{\varepsilon\,xx}\end{aligned} \tag{1.17}$$

with the initial data (1.16).

By the Hopf-Cole transformations (4.12), system (1.17) is reduced to the triple of linear heat equations (4.10). Solving this system of the heat equations (4.10), by Lemma 4.1 we find a solution of problem (1.17), (1.16).

Next, in Sec. 5, for the case $u_-\geq u_+$, the weak limit (5.15) of a solution to the parabolic problem (1.17), (1.16) is constructed by Theorem 5.1. In Sec. 6, for the case $u_-<u_+$, the weak limit (6.7) of a solution to the parabolic problem (1.17), (1.16) is constructed by Theorem 6.1. The proofs of Theorems 5.1, 6.1 are based on the limiting properties of the functions $\mathrm{Th}\bigl(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\bigr)$ and $\mathrm{Ch}\bigl(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\bigr)$ introduced by (5.7). According to Lemma 5.1, for the case $u_+\leq u_-$ these properties coincide with limiting properties of the corresponding *hyperbolic functions* of the argument $\frac{[u](x-ct)}{2\varepsilon}$, as $\varepsilon\to+0$. Note that Theorems 5.1, 6.1 are the most important results of this paper.

In Sec. 7, using the results of Sec. 5, 6, the Riemann problem (1.12), (1.16) is solved.

In Subsec. 7.1, by Theorems 7.1, 5.1, and Corollary 7.1 we prove the following statements.

(a) If $u_+ \le u_-$, the weak limit of the solution to the parabolic problem (1.17), (1.16) (i.e., the triple of distributions (5.15)) satisfies the integral identities (3.1), and, consequently, it is a δ'-shock wave type solution to the Cauchy problem (1.12), (1.16). This solution has the form

$$\begin{aligned} u(x,t) &= u_+ + [u]H(-x+\phi(t)), \\ v(x,t) &= v_+ + [v]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \\ w(x,t) &= w_+ + [w]H(-x+\phi(t)) + g(t)\delta(-x+\phi(t)) \\ &\qquad + h(t)\delta'(-x+\phi(t)) \end{aligned} \tag{1.18}$$

and satisfies the entropy condition (1.15), where functions $\phi(t)$, $e(t)$, $g(t)$, $h(t)$ are given by (5.16). Thus the problem of propagation of a δ'-shock in system (1.12) is studied.

(a.1) If $u_+ < u_-$ then (1.18), (5.16) imply that the Cauchy problem (1.12), (1.16) has a *classical shock-solution* (1.18), i.e., piecewise constant solution (7.7) if and only if $v_- + v_+ = 0$ and $w_- + w_+ = 0$.

(a.2) If $u_+ < u_-$ and $v_- + v_+ = 0$ and $w_- + w_+ \neq 0$ or $u_+ = u_- = u_0$ then the Cauchy problem (1.12), (1.16) has a *δ-shock solution* (1.18), i.e., (7.8) or (7.9), respectively: w component contains a δ measure, while u and v components are piecewise constant.

(a.3) The Cauchy problem (1.12), (1.16) has a δ'-shock wave type solution (1.18) only if $v_- + v_+ \neq 0$, $w_- + w_+ \neq 0$.

This situation reflects the fact that systems (1.11) and (1.12) are *overdetermined*, so we *cannot solve* the Cauchy problem with *arbitrary* jumps *without introducing δ'-shock*.

Remark 1.1. In [**18**], the system of conservation laws

$$u_t + \Big(\frac{u^2}{2}\Big)_x = 0, \quad v_t + (uv)_x = 0, \quad w_t + \Big(\frac{v^2}{2} + uw\Big)_x = 0 \tag{1.19}$$

was studied. This system has repeated eigenvalues. As it is said in [**18**], system (1.19) *cannot be solved in the classical distributional sense*, therefore it is necessary to define a generalized solution in the Colombeau sense. In [**18**] this is motivated by the following arguments: if $v_- + v_+ \neq 0$ then the v component contains a δ measure along $x = 0$. Though the product uv does not make sense in the classical theory of distributions, it can be defined in the sense of the approach [**24**], but v^2 contains a square of δ measure and cannot be defined in this sense.

It is clear that by the change of variables $u \to 2u$, $v \to 2v$, $w \to w$ system (1.19) can be transformed into system (1.12). Thus, contrary to the assertion from the paper [**18**], according to Theorem 7.1, system (1.19) *admits a δ'-shock wave type solution*. This solution considered in the sense of Definition 3.1 is a *distributional solution*.

In addition, by Corollary 7.2, we prove that the first and second distributions in (1.18) constitute a δ-shock wave type solution (in the sense of Definition 2.1) of the

Cauchy problem

$$u_t + \big(u^2\big)_x = 0, \quad v_t + 2\big(uv\big)_x = 0, \tag{1.20}$$

with the initial data (1.6). Since by the change of variables $u \to \frac{u}{2}$, $v \to v$, system (1.20) can be transformed into system (1.5), Corollary 7.2 gives a correct solution of the above mentioned problem (1.5), (1.6) from [**17**]. More precisely, the weak limit of the solution to the parabolic problem (1.8), (1.6) is a δ-shock wave type solution of the Cauchy problem (1.5), (1.6) in the sense of Definition 2.1. Thus, by using the vanishing viscosity method, the Cauchy problems (1.20), (1.6) and (1.5), (1.6) are solved.

In Subsec. 7.2, by Theorems 7.2, 6.1, we prove the following statement.

(b) If $u_+ > u_-$ then the weak limit of the solution to the parabolic problem (1.17), (1.16) (i.e., the triple of distributions (6.7))

$$\big(u(x,t), v(x,t), w(x,t)\big) = \begin{cases} (u_-, v_-, w_-), & x \le 2u_-t, \\ \big(\frac{x}{2t}, 0, 0\big), & 2u_-t < x < 2u_+t, \\ (u_+, v_+, w_+), & x \ge 2u_+t, \end{cases} \tag{1.21}$$

satisfies the integral identities (3.1), and, consequently, it is a solution to the Cauchy problem (1.12), (1.16). The first component u of solution (1.21) is the rarefaction wave, the second component v and the third component w contain the intermediate *vacuum states* $v = 0$ and $w = 0$.

According to the mentioned above Theorems 7.1, 7.2, Theorem 5.1 and Theorem 6.1 describe the formation of the δ'-shocks and the vacuum states from a smooth solutions $(u_\varepsilon(x,t), v_\varepsilon(x,t), w_\varepsilon(x,t))$ of the problem (1.17), (1.16), respectively.

It is clear that the exact solution $(u_\varepsilon(x,t), v_\varepsilon(x,t), w_\varepsilon(x,t))$ of the problem (1.17), (1.16) is a *weak asymptotic solution* of the Cauchy problem (1.12), (1.16) (see definition of a *weak asymptotic solution* in [**25**]).

In Sec. 8, the *algebraic aspect* of singular solutions (in particular, δ'-shock type solutions) is studied. Namely, we show that according to (8.7), singular solutions (in particular, δ'-shock type solutions) $(u, v, w) = (\lim_{\varepsilon\to+0} u_\varepsilon, \lim_{\varepsilon\to+0} v_\varepsilon, \lim_{\varepsilon\to+0} w_\varepsilon)$ of the Cauchy problem (1.12), (1.16) *generate algebraic relations* between distributional components u, v, w. We construct these *algebraic relations*, i.e., the *"right" singular superpositions of distributions* $u^2(x,t)$, $2u(x,t)v(x,t)$, $2\big(v^2(x,t) + u(x,t)w(x,t)\big)$ given by formulas (8.2)–(8.4) or (8.11)–(8.13) (see Lemmas 8.1, 8.2), as the weak limit of flux-functions

$$\lim_{\varepsilon\to+0} u_\varepsilon^2(x,t), \quad \lim_{\varepsilon\to+0} 2u_\varepsilon(x,t)v_\varepsilon(x,t), \quad \lim_{\varepsilon\to+0} 2\big(v_\varepsilon^2(x,t) + u_\varepsilon(x,t)w_\varepsilon(x,t)\big).$$

Formulas (8.2)–(8.4) show that the above mentioned flux-function limit can be *very singular* and may contain *δ-functions and their derivatives*.

Note that Theorem 7.1 and Theorem 7.2 could be proved by direct substituting the *"right" singular superpositions* (8.2)–(8.4) and (8.11)–(8.13), respectively, into system (1.12).

Since systems (1.11) and (1.12) have singular terms which differ from the terms of the type (1.10), it is *impossible* to construct a *δ'-shock wave type solution* for them by using the well-known *nonconservative product* [**21**]– [**24**].

The *geometric aspects* of δ-shock and δ'-shock type solutions were studied in the papers [**30**], [**32**], and [**25**], respectively. Namely, in these papers δ-shock and δ'-shock *balance relations* connected with *area, mass, and momentum transportation* were proved.

The construction of a δ'-shock type solution gives a new perspective in the theory of singular solutions to systems of conservation laws. This result shows that systems of conservation laws can develop not only Dirac measures (as in the case of δ-shocks) but their derivatives as well.

2. δ-Shocks: generalized solution and the Rankine–Hugoniot conditions

Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a graph in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\} \in \mathbb{R}^2$ containing smooth arcs γ_i, $i \in I$, and I is a finite set. By I_0 we denote a subset of I such that an arc γ_k for $k \in I_0$ starts from the points of the x-axis. Denote by $\Gamma_0 = \{x_k^0 : k \in I_0\}$ the set of initial points of arcs γ_k, $k \in I_0$.

Consider *δ-shock wave type initial data* $(u^0(x), v^0(x))$, where

$$v^0(x) = \widehat{v}^0(x) + e^0\delta(\Gamma_0),$$

$u^0, \widehat{v}^0 \in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, $e^0\delta(\Gamma_0) = \sum_{k\in I_0} e_k^0\delta(x - x_k^0)$, e_k^0 are constants, $k \in I_0$.

Definition 2.1. ([**9**]– [**11**]) A pair of distributions $(u(x,t), v(x,t))$ and a graph Γ, where $v(x,t)$ has the form of the sum

$$v(x,t) = \widehat{v}(x,t) + e(x,t)\delta(\Gamma),$$

$u, \widehat{v} \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}\big)$, $e(x,t)\delta(\Gamma) = \sum_{i\in I} e_i(x,t)\delta(\gamma_i)$, $e_i(x,t) \in C(\Gamma)$, $i \in I$, is called a *generalized δ-shock wave type solution* of system (1.3) with the *δ-shock wave type initial data* $(u^0(x), v^0(x))$ if the integral identities

$$\begin{aligned}
\int_0^\infty \int \Big(u\varphi_t + F(u,\widehat{v})\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty \int \Big(\widehat{v}\varphi_t + G(u,\widehat{v})\varphi_x\Big)\,dx\,dt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl & \\
+ \int \widehat{v}^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) &= 0,
\end{aligned} \tag{2.1}$$

hold for all test functions $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\infty))$, where $\dfrac{\partial\varphi(x,t)}{\partial \mathbf{l}}$ is the tangential derivative on the graph Γ, $\int_{\gamma_i} \cdot\ dl$ is the line integral over the arc γ_i. Here the delta function $\delta(\gamma_i)$ on the curve γ_i is defined in [**15**, ch.III,§1.3.], [**19**, 5.3.].

Suppose that arcs of the graph $\Gamma = \{\gamma_i : i \in I\}$ have the form $\gamma_i = \{(x,t) : x = \phi_i(t)\}$, $i \in I$, and $\mathbf{n} = (\nu_1, \nu_2)$ is the unit oriented normal to the curve γ_i. In this case

$$\mathbf{n} = (\nu_1, \nu_2) = \frac{1}{\sqrt{1 + (\dot\phi_i(t))^2}}\big(1, -\dot\phi_i(t)\big), \qquad \mathbf{l} = (-\nu_2, \nu_1), \tag{2.2}$$

and

$$\begin{aligned} \frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\bigg|_{\gamma_i} &= \frac{\varphi_t(\phi_i(t),t) + \dot\phi_i(t)\varphi_x(\phi_i(t),t)}{\sqrt{1 + (\dot\phi_i(t))^2}} \\ &= \frac{1}{\sqrt{1 + (\dot\phi_i(t))^2}} \frac{d\varphi(\phi_i(t),t)}{dt}. \end{aligned} \tag{2.3}$$

THEOREM 2.1. ([**30**]– [**32**], see also [**25**]) *Let us assume that $\Omega \subset \mathbb{R} \times (0,\infty)$ is some region cut by a smooth curve Γ into a left- and right-hand parts $\Omega_{\mp}$, $(u(x,t), v(x,t))$, Γ is a generalized δ-shock wave type solution of system* (1.3)*, and $(u(x,t), v(x,t))$ is smooth in $\Omega_{\pm}$. Then the Rankine–Hugoniot conditions for δ-shocks*

$$\begin{aligned} \big[F(u,v)\big]_\Gamma \nu_1 + \big[u\big]_\Gamma \nu_2 &= 0, \\ \big[G(u,v)\big]_\Gamma \nu_1 + \big[v\big]_\Gamma \nu_2 &= \tfrac{\partial e(x,t)|_\Gamma}{\partial \mathbf{l}}, \end{aligned} \tag{2.4}$$

hold along Γ, where $\mathbf{n} = (\nu_1, \nu_2)$ is the unit normal to the curve Γ pointing from Ω_- into Ω_+, $\mathbf{l} = (-\nu_2, \nu_1)$ is the tangential vector to Γ,

$$\big[a(u,v)\big] = a(u_-, v_-) - a(u_+, v_+)$$

is, as usual, a jump in function $a(u(x,t), v(x,t))$ across the discontinuity curve Γ, $(u_\mp, v_\mp)$ are respective left- and right-hand values of (u,v) on the discontinuity curve.

If $\Gamma = \{(x,t) : x = \phi(t)\}$, $\Omega_\pm = \{(x,t) : \pm(x - \phi(t)) > 0\}$ then relations (2.4) *can be rewritten as*

$$\begin{aligned} \dot\phi(t) &= \tfrac{[F(u,v)]}{[u]}\Big|_{x=\phi(t)}, \\ \dot e(t) &= \Big([G(u,v)] - [v]\tfrac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)}, \end{aligned} \tag{2.5}$$

where $e(t) \overset{def}{=} e(\phi(t), t)$ and $(\dot{\cdot}) = \frac{d}{dt}(\cdot)$.

The first equation (2.4) (or (2.5)) is the *standard* Rankine–Hugoniot condition. The left-hand side of the second equation in (2.4) (or the right-hand side of the second equation in (2.5)) is called the *Rankine–Hugoniot deficit*.

The system of *δ-shocks* integral identities (2.1) is a *natural generalization* of the *usual* system of integral identities (1.2) (for $m = 2$). The integral identities (2.1)

differ from the integral identities (1.2) (for $m = 2$) by the additional term

$$\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl = \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl$$

in the second identity. This term appears due to the so-called *Rankine–Hugoniot deficit*.

3. δ'-Shocks: generalized solution and the Rankine–Hugoniot conditions

Denote by $\widetilde{\mathcal{C}}\big(\mathbb{R}\times(0,\infty);\mathbb{R}\big)$ the class of piecewise-smooth functions. Let $\Gamma = \{\gamma_i : i\in I\}$ be a graph introduced in Sec. 2. Initial data $(u^0(x), v^0(x), w^0(x))$, where

$$v^0(x) = \widehat{v}^0(x) + e^0\delta(\Gamma_0), \quad w^0(x) = \widehat{w}^0(x) + g^0\delta(\Gamma_0) + h^0\delta'(\Gamma_0),$$

and $u^0, \widehat{v}^0, \widehat{w}^0 \in \widetilde{\mathcal{C}}\big(\mathbb{R};\mathbb{R}\big)$, we call *$\delta'$-shock wave type initial data*. Here, by definition, $e^0\delta(\Gamma_0) \overset{def}{=} \sum_{k\in I_0} e_k^0\delta(x-x_k^0)$, $g^0\delta(\Gamma_0) \overset{def}{=} \sum_{k\in I_0} g_k^0\delta(x-x_k^0)$, $h^0\delta(\Gamma_0) \overset{def}{=} \sum_{k\in I_0} h_k^0\delta'(x-x_k^0)$, where e_k^0, g_k^0, h_k^0 are constants, $k\in I_0$.

Definition 3.1. ([**25**]) A triple of distributions $\big(u(x,t), v(x,t), w(x,t)\big)$ and graph Γ, where $v(x,t)$ and $w(x,t)$ have the form of the sums

$$v(x,t) = \widehat{v}(x,t) + e(x,t)\delta(\Gamma), \quad w(x,t) = \widehat{w}(x,t) + g(x,t)\delta(\Gamma) + h(x,t)\delta'(\Gamma),$$

where $u, \widehat{v}, \widehat{w} \in \widetilde{\mathcal{C}}\big(\mathbb{R}\times(0,\infty);\mathbb{R}\big)$,

$$\begin{aligned} e(x,t)\delta(\Gamma) &\overset{def}{=} \textstyle\sum_{i\in I} e_i(x,t)\delta(\gamma_i),\\ g(x,t)\delta(\Gamma) &\overset{def}{=} \textstyle\sum_{i\in I} g_i(x,t)\delta(\gamma_i),\\ h(x,t)\delta'(\Gamma) &\overset{def}{=} \textstyle\sum_{i\in I} h_i(x,t)\delta'(\gamma_i), \end{aligned}$$

and $e_i(x,t), g_i(x,t), h_i(x,t) \in C^1(\Gamma)$, $i\in I$, is called a *generalized δ'-shock wave type solution* of system (1.11) with *δ'-shock wave type* initial data $(u^0(x), v^0(x), w^0(x))$ if

the integral identities

$$\begin{aligned}
\int_0^\infty \int \Big(u\varphi_t + f(u)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty \int \widehat{v}\Big(\varphi_t + f'(u)\varphi_x\Big)\,dx\,dt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl &\\
+ \int \widehat{v}^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) &= 0,\\
\int_0^\infty \int \Big(\widehat{w}\varphi_t + \big(f''(u)\widehat{v}^2 + f'(u)\widehat{w}\big)\varphi_x\Big)\,dx\,dt &\\
+ \sum_{i\in I}\Big(\int_{\gamma_i} g_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl &\\
+ \int_{\gamma_i} h_i(x,t)\frac{\partial\varphi_x(x,t)}{\partial \mathbf{l}}\,dl + \int_{\gamma_i}\frac{\frac{\partial e_i^2(x,t)}{\partial \mathbf{l}} - h_i(x,t)\frac{\partial[u(x,t)]}{\partial \mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\,dl\Big) &\\
+ \int \widehat{w}^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} g_k^0\varphi(x_k^0,0) + \sum_{k\in I_0} h_k^0\varphi_x(x_k^0,0) &= 0,
\end{aligned} \tag{3.1}$$

hold for all test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\infty))$. Here the derivative of the delta function $\delta'(\gamma)$ on the curve γ is defined in [**15**, ch.III,§1.5.], [**19**, 5.3.;5.5.].

THEOREM 3.1. ([**25**]) *Let us assume that $\Omega\subset\mathbb{R}\times[0,\infty)$ is some region cut by a smooth curve $\Gamma=\{(x,t): x=\phi(t)\}$, $\phi(t)\in C^1(0,+\infty)$ into a left- and right-hand parts $\Omega_\pm=\{(x,t)\in\Omega : \pm(x-\phi(t))>0\}$, $(u(x,t),v(x,t),w(x,t))$, Γ is a generalized δ'-shock wave type solution of system (1.11), functions $(u(x,t),\widehat{v}(x,t),\widehat{w}(x,t))$ are smooth in the domains $\Omega_\pm$ and have one-sided limits $u_\pm$, $V_\pm$, $W_\pm$ on the curve Γ, which are supposed to be continuous functions on Γ. Then the Rankine–Hugoniot conditions for δ'-shock*

$$\dot{\phi}(t) = \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \tag{3.2}$$

$$\dot{e}(t) = \Big([f'(u)v] - [v]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}, \tag{3.3}$$

$$\dot{g}(t) = \Big([f''(u)v^2 + f'(u)w] - [w]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}, \tag{3.4}$$

$$\frac{d}{dt}\Big(h(t)[u(\phi(t),t)]\Big) = \frac{de^2(t)}{dt} \tag{3.5}$$

hold along Γ. Here the functions e,g,h can be treated as functions of the single variable t, so that $e(t)\overset{def}{=}e(\phi(t),t)$, $g(t)\overset{def}{=}g(\phi(t),t)$, $h(t)\overset{def}{=}h(\phi(t),t)$.

The system of the Rankine–Hugoniot conditions (3.2)–(3.5) determines the trajectory $x = \phi(t)$ of a δ'-shock wave and the coefficients $e(t)$, $g(t)$, $h(t)$ of the singularities. The first equation in this system is the "standard" Rankine–Hugoniot condition for the *shock*, while the first and second equations are the "standard" *Rankine–Hugoniot conditions for δ-shock* (2.5). The right-hand sides of the equalities (3.3), (3.4) are the *first Rankine-Hugoniot deficits*, while the right-hand side of (3.5) is the *second Rankine-Hugoniot deficit*.

The integral identities (3.1) differ from classical integral identities (1.2) (for $m = 3$) by additional terms in the second and third identities. Here the terms

$$\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl = \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl,$$

$$\int_\Gamma g(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl = \sum_{i\in I}\int_{\gamma_i} g_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl$$

appear due to the *first Rankine–Hugoniot deficit*, and the term

$$\int_\Gamma h(x,t)\frac{\partial\varphi_x(x,t)}{\partial \mathbf{l}}\,dl + \int_\Gamma \frac{\frac{\partial e^2(x,t)}{\partial \mathbf{l}} - h(x,t)\frac{\partial[u(x,t)]}{\partial \mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\,dl$$

$$= \sum_{i\in I}\left(\int_{\gamma_i} h_i(x,t)\frac{\partial\varphi_x(x,t)}{\partial \mathbf{l}}\,dl + \int_{\gamma_i} \frac{\frac{\partial e_i^2(x,t)}{\partial \mathbf{l}} - h_i(x,t)\frac{\partial[u(x,t)]}{\partial \mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\,dl\right)$$

appears due to the *second Rankine–Hugoniot deficit*. Moreover, the first integral identity in (3.1) is a "standard" type integral identity (see (1.2)), while the first and second integral identities in (3.1) constitute δ-shock type integral identities (see Definition 2.1), and the third integral identity in (3.1) is a special type of δ'-shock type integral identity.

4. Solution of the parabolic problem (1.17), (1.16)

Using the vanishing viscosity method, we study the Cauchy problem (1.12), (1.16). The first step is to find a solution of system (1.17) with the initial data (1.16), where system (1.17) is the parabolic approximation of system (1.12).

Integrating system (1.17) and the initial data (1.16) with respect to x, we obtain the system

$$\begin{aligned} U_{\varepsilon t} + U_{\varepsilon x}^2 &= \varepsilon U_{\varepsilon xx},\\ V_{\varepsilon t} + 2U_{\varepsilon x}V_{\varepsilon x} &= \varepsilon V_{\varepsilon xx},\\ W_{\varepsilon t} + 2\big(V_{\varepsilon x}^2 + U_{\varepsilon x}W_{\varepsilon x}\big) &= \varepsilon W_{\varepsilon xx} \end{aligned} \tag{4.1}$$

with the initial data

$$(U^0(x), V^0(x), W^0(x) = \begin{cases} (u_-x, v_-x, w_-x), & x < 0,\\ (u_+x, v_+x, w_+x), & x > 0, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} U_\varepsilon(x,t) &= \textstyle\int_0^x u_\varepsilon(y,t)\,dy, \\ V_\varepsilon(x,t) &= \textstyle\int_0^x v_\varepsilon(y,t)\,dy, \\ W_\varepsilon(x,t) &= \textstyle\int_0^x w_\varepsilon(y,t)\,dy, \end{aligned} \tag{4.3}$$

$$U^0(x) = \int_0^x u^0(y)\,dy, \quad V^0(x) = \int_0^x v^0(y)\,dy, \quad W^0(x) = \int_0^x w^0(y)\,dy.$$

It is clear that if the triple of functions $(U_\varepsilon(x,t), V_\varepsilon(x,t), W_\varepsilon(x,t))$ solves the problem (4.1), (4.2) then the triple of functions $(u_\varepsilon(x,t), v_\varepsilon(x,t), w_\varepsilon(x,t))$, where

$$u_\varepsilon(x,t) = U_{\varepsilon\,x}(x,t), \quad v_\varepsilon(x,t) = V_{\varepsilon\,x}(x,t), \quad w_\varepsilon(x,t) = W_{\varepsilon\,x}(x,t), \tag{4.4}$$

solves the problem (1.17), (1.16).

It is well known that the first equation of system (4.1) can be linearized by the Hopf-Cole transformation $A_\varepsilon(x,t) = e^{-\frac{U_\varepsilon(x,t)}{\varepsilon}}$. Note that differentiating the first equation in (4.1) twice with respect to x, denoting $V_\varepsilon(x,t) = U_{\varepsilon\,x}(x,t)$ and $W_\varepsilon(x,t) = V_{\varepsilon\,x}(x,t)$, we obtain whole system (4.1). Thus system (4.1) constitutes a 3×3 "prolonged system". Due to this fact, system (4.1) can be linearized by the generalized Hopf-Cole transformations

$$\begin{aligned} A_\varepsilon(x,t) &= e^{-\frac{U_\varepsilon(x,t)}{\varepsilon}}, \\ B_\varepsilon(x,t) &= -\frac{V_\varepsilon(x,t)}{\varepsilon} e^{-\frac{U_\varepsilon(x,t)}{\varepsilon}}, \\ C_\varepsilon(x,t) &= \Big(\frac{V_\varepsilon^2(x,t)}{\varepsilon^2} - \frac{W_\varepsilon(x,t)}{\varepsilon}\Big) e^{-\frac{U_\varepsilon(x,t)}{\varepsilon}}, \end{aligned} \tag{4.5}$$

where the second and third transformations were obtained by successive differentiating the first Hopf-Cole transformation with respect to x and denoting $V_\varepsilon(x,t) = U_{\varepsilon\,x}(x,t)$ and $W_\varepsilon(x,t) = V_{\varepsilon\,x}(x,t)$. Thus

$$U_\varepsilon = -\varepsilon \log A_\varepsilon, \quad V_\varepsilon = -\varepsilon \frac{B_\varepsilon}{A_\varepsilon}, \quad W_\varepsilon = -\varepsilon \frac{A_\varepsilon C_\varepsilon - B_\varepsilon^2}{A_\varepsilon^2}. \tag{4.6}$$

It is easy to calculate that

$$\begin{aligned} A_{\varepsilon\,t} &= -\tfrac{U_{\varepsilon\,t}}{\varepsilon} e^{-\frac{U_\varepsilon}{\varepsilon}}, \\ A_{\varepsilon\,xx} &= \Big(-\tfrac{U_{\varepsilon\,xx}}{\varepsilon} + \tfrac{U_{\varepsilon\,x}^2}{\varepsilon^2}\Big) e^{-\frac{U_\varepsilon}{\varepsilon}}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} B_{\varepsilon\,t} &= \Big(-\tfrac{V_{\varepsilon\,t}}{\varepsilon} + \tfrac{U_{\varepsilon\,t} V_\varepsilon}{\varepsilon^2}\Big) e^{-\frac{U_\varepsilon}{\varepsilon}}, \\ B_{\varepsilon\,xx} &= \Big(-\tfrac{V_{\varepsilon\,xx}}{\varepsilon} + \tfrac{2U_{\varepsilon\,x}V_{\varepsilon\,x} + U_{\varepsilon\,xx}V_\varepsilon}{\varepsilon^2} - \tfrac{U_{\varepsilon\,x}^2 V_\varepsilon}{\varepsilon^3}\Big) e^{-\frac{U_\varepsilon}{\varepsilon}}, \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
C_{\varepsilon t} &= \Big(-\tfrac{W_{\varepsilon t}}{\varepsilon} + \tfrac{2V_\varepsilon V_{\varepsilon t}+U_{\varepsilon t}W_\varepsilon}{\varepsilon^2} - \tfrac{U_{\varepsilon t}V_\varepsilon^2}{\varepsilon^3}\Big) e^{-\frac{U_\varepsilon}{\varepsilon}}, \\
C_{\varepsilon xx} &= \Big(-\tfrac{W_{\varepsilon xx}}{\varepsilon} + \tfrac{2V_{\varepsilon x}^2+2V_\varepsilon V_{\varepsilon xx}+U_{\varepsilon xx}W_\varepsilon+2U_{\varepsilon x}W_{\varepsilon x}}{\varepsilon^2} \\
&\qquad - \tfrac{U_{\varepsilon xx}V_\varepsilon^2+4U_{\varepsilon x}V_\varepsilon V_{\varepsilon x}+U_{\varepsilon x}^2W_\varepsilon}{\varepsilon^3} + \tfrac{U_{\varepsilon x}^2V_\varepsilon^2}{\varepsilon^4}\Big) e^{-\frac{U_\varepsilon}{\varepsilon}},
\end{aligned} \tag{4.9}
$$

From (4.1) and (4.7)–(4.9) it follows that the functions A_ε, B_ε, C_ε satisfy the system of the heat equations

$$
A_{\varepsilon t} = \varepsilon A_{\varepsilon xx}, \quad B_{\varepsilon t} = \varepsilon B_{\varepsilon xx}, \quad C_{\varepsilon t} = \varepsilon C_{\varepsilon xx}. \tag{4.10}
$$

The initial data for the last system read off from the initial data (4.2) and Hopf-Cole transformations (4.5):

$$
\begin{aligned}
&\big(A_\varepsilon^0(x),\, B_\varepsilon^0(x),\, C_\varepsilon^0(x)\big) \\
&= \begin{cases}
\Big(e^{-\frac{u_-x}{\varepsilon}},\, -\dfrac{v_-x}{\varepsilon}e^{-\frac{u_-x}{\varepsilon}},\, \big(\dfrac{v_-^2x^2}{\varepsilon^2} - \dfrac{w_-x}{\varepsilon}\big)e^{-\frac{u_-x}{\varepsilon}}\Big), & x < 0, \\
\Big(e^{-\frac{u_+x}{\varepsilon}},\, -\dfrac{v_+x}{\varepsilon}e^{-\frac{u_+x}{\varepsilon}},\, \big(\dfrac{v_+^2x^2}{\varepsilon^2} - \dfrac{w_+x}{\varepsilon}\big)e^{-\frac{u_-x}{\varepsilon}}\Big), & x > 0.
\end{cases}
\end{aligned} \tag{4.11}
$$

Thus, in view of (4.4), (4.6), by the Hopf-Cole transformations

$$
u_\varepsilon(x,t) = -\varepsilon\frac{A_{\varepsilon x}}{A_\varepsilon}, \quad v_\varepsilon(x,t) = -\varepsilon\left(\frac{B_\varepsilon}{A_\varepsilon}\right)_x, \quad w_\varepsilon(x,t) = -\varepsilon\left(\frac{A_\varepsilon C_\varepsilon - B_\varepsilon^2}{A_\varepsilon^2}\right)_x \tag{4.12}
$$

system (1.17) is reduced to the linear system of the heat equations (4.10).

It is well known that a solution of the heat equation with the initial data

$$
\Phi_{\varepsilon t} = \varepsilon\Phi_{\varepsilon xx}, \qquad \Phi_\varepsilon(x,0) = \Phi_\varepsilon^0(x)
$$

has the following form

$$
\Phi_\varepsilon(x,t) = \frac{1}{\sqrt{4\pi t\varepsilon}}\int_{-\infty}^{\infty}\Phi_\varepsilon^0(y)\exp\Big(-\frac{(x-y)^2}{4t\varepsilon}\Big)\,dy. \tag{4.13}
$$

By substituting the initial data (4.11) into formula (4.13), we obtain a solution of the problem (4.10), (4.11):

$$
\begin{aligned}
A_\varepsilon(x,t) &= a_-^\varepsilon(x,t) + a_+^\varepsilon(x,t), \\
B_\varepsilon(x,t) &= b_-^\varepsilon(x,t) + b_+^\varepsilon(x,t), \\
C_\varepsilon(x,t) &= c_-^\varepsilon(x,t) + c_+^\varepsilon(x,t),
\end{aligned} \tag{4.14}
$$

where

$$
\begin{aligned}
a_-^\varepsilon(x,t) &= \frac{1}{\sqrt{4\pi t\varepsilon}}\int_{-\infty}^{0}\exp\Big(-\frac{(x-y)^2}{4t\varepsilon} - \frac{u_-}{\varepsilon}y\Big)\,dy, \\
a_+^\varepsilon(x,t) &= \frac{1}{\sqrt{4\pi t\varepsilon}}\int_{0}^{\infty}\exp\Big(-\frac{(x-y)^2}{4t\varepsilon} - \frac{u_+}{\varepsilon}y\Big)\,dy,
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
b_-^\varepsilon(x,t) &= \frac{1}{\sqrt{4\pi t\varepsilon}}\int_{-\infty}^{0}\Big(-\frac{v_-}{\varepsilon}y\Big)\exp\Big(-\frac{(x-y)^2}{4t\varepsilon}-\frac{u_-}{\varepsilon}y\Big)\,dy,\\
b_+^\varepsilon(x,t) &= \frac{1}{\sqrt{4\pi t\varepsilon}}\int_{0}^{\infty}\Big(-\frac{v_+}{\varepsilon}y\Big)\exp\Big(-\frac{(x-y)^2}{4t\varepsilon}-\frac{u_+}{\varepsilon}y\Big)\,dy,
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
c_-^\varepsilon(x,t) &= \frac{1}{\sqrt{4\pi t\varepsilon}}\int_{-\infty}^{0}\Big(\frac{v_-^2}{\varepsilon^2}y^2-\frac{w_-}{\varepsilon}y\Big)\exp\Big(-\frac{(x-y)^2}{4t\varepsilon}-\frac{u_-}{\varepsilon}y\Big)\,dy,\\
c_+^\varepsilon(x,t) &= \frac{1}{\sqrt{4\pi t\varepsilon}}\int_{0}^{\infty}\Big(\frac{v_+^2}{\varepsilon^2}y^2-\frac{w_+}{\varepsilon}y\Big)\exp\Big(-\frac{(x-y)^2}{4t\varepsilon}-\frac{u_+}{\varepsilon}y\Big)\,dy.
\end{aligned}
\tag{4.17}
$$

LEMMA 4.1. *A solution* $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ *of the problem* (1.17), (1.16) *is represented in the form*

$$u_\varepsilon(x,t)=\frac{u_-a_-^\varepsilon+u_+a_+^\varepsilon}{a_-^\varepsilon+a_+^\varepsilon},\tag{4.18}$$

$$v_\varepsilon(x,t)=V_{\varepsilon\,x}(x,t),\tag{4.19}$$

$$w_\varepsilon(x,t)=W_{\varepsilon\,x}(x,t),\tag{4.20}$$

where

$$
\begin{aligned}
V_\varepsilon(x,t) &= -\varepsilon\frac{B_\varepsilon}{A_\varepsilon}\\
&= \frac{v_-(x-2u_-t)a_-^\varepsilon+v_+(x-2u_+t)a_+^\varepsilon-(v_--v_+)\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}}{a_-^\varepsilon+a_+^\varepsilon},
\end{aligned}
\tag{4.21}
$$

$$W_\varepsilon(x,t)=-\varepsilon\bigg(\frac{A_\varepsilon C_\varepsilon-B_\varepsilon^2}{A_\varepsilon^2}\bigg)=-\varepsilon\frac{C_\varepsilon}{A_\varepsilon}+\frac{1}{\varepsilon}\Big(V_\varepsilon\Big)^2,\tag{4.22}$$

where A_ε, B_ε, C_ε *are given by formulas* (4.14)–(4.17), *and*

$$B_\varepsilon(x,t)=-\frac{1}{\varepsilon}\Big(v_-(x-2u_-t)a_-^\varepsilon+v_+(x-2u_+t)a_+^\varepsilon-(v_--v_+)\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\Big),\tag{4.23}$$

$$
\begin{aligned}
C_\varepsilon(x,t) &= \frac{2t}{\varepsilon}\big(v_-^2a_-^\varepsilon+v_+^2a_+^\varepsilon\big)+\frac{1}{\varepsilon^2}\big(v_-^2(x-2u_-t)^2a_-^\varepsilon+v_+^2(x-2u_+t)^2a_+^\varepsilon\big)\\
&\quad -\frac{1}{\varepsilon^2}\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\big(v_-^2(x-2u_-t)-v_+^2(x-2u_+t)\big)
\end{aligned}
$$

$$-\frac{1}{\varepsilon}\Big(w_-(x-2u_-t)a_-^\varepsilon+w_+(x-2u_+t)a_+^\varepsilon-(w_--w_+)\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\Big).\tag{4.24}$$

PROOF. According to the above calculations, a solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ of the problem (1.17), (1.16) is represented by formulas (4.12), where A_ε, B_ε, C_ε are given by (4.14)–(4.17).

Integrating by parts, it is easy to calculate that

$$J^1_\pm = \pm\frac{1}{\sqrt{4\pi t\varepsilon}}\int_0^{\pm\infty} y\,\exp\Big(-\frac{(x-y)^2}{4t\varepsilon}-\frac{u_\pm}{\varepsilon}y\Big)\,dy$$

$$= (x-2u_\pm t)a^\varepsilon_\pm(x,t) \pm \sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}. \tag{4.25}$$

Then easy calculations show that (4.14)–(4.16), (4.25) imply

$$A_{\varepsilon\,x}(x,t) = -\frac{u_-}{\varepsilon}a^\varepsilon_-(x,t) - \frac{u_+}{\varepsilon}a^\varepsilon_+(x,t), \tag{4.26}$$

and

$$B_\varepsilon(x,t) = -\frac{v_-}{\varepsilon}\Big((x-2u_-t)a^\varepsilon_-(x,t) - \sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\Big)$$

$$-\frac{v_+}{\varepsilon}\Big((x-2u_+t)a^\varepsilon_+(x,t) + \sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\Big)$$

$$= -\frac{1}{\varepsilon}\Big(v_-(x-2u_-t)a^\varepsilon_- + v_+(x-2u_+t)a^\varepsilon_+ - (v_- - v_+)\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\Big),$$

i.e., (4.23) (see also calculations in Lemma 2.1. from [**17**]). Thus (4.12), (4.14), (4.26), (4.23) imply (4.18), and (4.19), (4.21).

Next, integrating by parts, performing simple calculations, and using (4.25), we obtain

$$J^2_\pm = \pm\frac{1}{\sqrt{4\pi t\varepsilon}}\int_0^{\pm\infty} y^2\,\exp\Big(-\frac{(x-y)^2}{4t\varepsilon}-\frac{u_\pm}{\varepsilon}y\Big)\,dy$$

$$= (x-2u_\pm t)J^1_\pm + 2t\varepsilon a^\varepsilon_\pm(x,t)$$

$$= (x-2u_\pm t)\Big((x-2u_\pm t)a^\varepsilon_\pm(x,t) \pm \sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\Big) + 2t\varepsilon a^\varepsilon_\pm(x,t). \tag{4.27}$$

Using (4.14)–(4.17), (4.25), (4.27), we obtain

$$C_\varepsilon(x,t) = \frac{v_-^2}{\varepsilon^2}J^2_- - \frac{w_-}{\varepsilon}J^1_- + \frac{v_+^2}{\varepsilon^2}J^2_+ - \frac{w_+}{\varepsilon}J^1_+.$$

The last relation can be easily transformed into (4.24). Thus (4.12), (4.14), (4.26), (4.23), (4.24) imply (4.20), (4.22). □

5. Weak limit of the solution to the problem (1.17), (1.16) for $u_+ \le u_-$

Let us construct the weak limit of a solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ to the Cauchy problem (1.17), (1.16), as $\varepsilon \to +0$ for the case $u_+ \le u_-$.

For our calculations we need the following identities

$$-\frac{u_\pm(x-u_\pm t)}{\varepsilon} - \frac{(x-2u_\pm t)^2}{4t\varepsilon} = -\frac{x^2}{4t\varepsilon} \tag{5.1}$$

$$-\frac{[u](x-ct)}{2\varepsilon} - \frac{(x-2u_- t)^2}{4t\varepsilon} = \frac{[u](x-ct)}{2\varepsilon} - \frac{(x-2u_+ t)^2}{4t\varepsilon}$$

$$= -\frac{(x-ct)^2 + [u]^2 t^2}{4t\varepsilon}. \tag{5.2}$$

Denote $c = u_- + u_+$. Since $u_+ \le u_-$, we have $2u_+ \le c \le 2u_-$, i.e.,

$$x - 2u_- t \le x - ct \le x - 2u_+ t. \tag{5.3}$$

In view of the identity

$$-\frac{(x-y)^2}{4t\varepsilon} - \frac{u_\pm}{\varepsilon} y = -\frac{(x-2tu_\pm - y)^2}{4t\varepsilon} - \frac{u_\pm(x-u_\pm t)}{\varepsilon},$$

elementary calculations transform relations (4.15) to the form

$$\begin{aligned} a_-^\varepsilon(x,t) &= e^{-\frac{u_-(x-u_- t)}{\varepsilon}} J\Big(\frac{x-2u_- t}{\sqrt{4t\varepsilon}}\Big), \\ a_+^\varepsilon(x,t) &= e^{-\frac{u_+(x-u_+ t)}{\varepsilon}} J\Big(-\frac{x-2u_+ t}{\sqrt{4t\varepsilon}}\Big), \end{aligned} \tag{5.4}$$

where

$$J(z) = \frac{1}{\sqrt{\pi}} \int_z^\infty e^{-y^2}\, dy \sim \begin{cases} 1 + \dfrac{e^{-z^2}}{2\sqrt{\pi}\, z}, & z \to -\infty, \\ \dfrac{e^{-z^2}}{2\sqrt{\pi}\, z}\Big(1 - \dfrac{1}{2z^2}\Big), & z \to +\infty. \end{cases} \tag{5.5}$$

By using (5.4) and taking into account that

$$\frac{(x-2u_+ t)^2}{4t\varepsilon} - \frac{(x-2u_- t)^2}{4t\varepsilon}$$

$$= \frac{u_-(x-u_- t)}{\varepsilon} - \frac{u_+(x-u_+ t)}{\varepsilon} = \frac{[u](x-ct)}{\varepsilon}, \tag{5.6}$$

we introduce the following functions

$$
\begin{aligned}
\mathrm{Sh}\Big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\Big) &\overset{def}{=} \frac{e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)-e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)}{2},\\
\mathrm{Ch}\Big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\Big) &\overset{def}{=} \frac{e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)+e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)}{2},\\
\mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\Big) &\overset{def}{=} \frac{\mathrm{Sh}\big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\big)}{\mathrm{Ch}\big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\big)}=\frac{a_+^\varepsilon-a_-^\varepsilon}{a_+^\varepsilon+a_-^\varepsilon}\\
&= \frac{e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)-e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)}{e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)+e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)}.
\end{aligned}
\tag{5.7}
$$

Here

$$
J\big(\mp\frac{x-2u_\pm t}{\sqrt{4t\varepsilon}}\big)=J\big(\mp\frac{x-ct}{\sqrt{4t\varepsilon}}-\frac{[u]}{2}\sqrt{\frac{t}{\varepsilon}}\big).
\tag{5.8}
$$

To solve our problem, we need to study the limiting properties of functions in (5.7), as $\varepsilon\to+0$.

Lemma 5.1. *We have*

$$
\begin{aligned}
\lim_{\varepsilon\to+0}\frac{1}{2}\Big(1-\mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\Big)\Big) &= \lim_{\varepsilon\to+0}\frac{a_-^\varepsilon}{a_+^\varepsilon+a_-^\varepsilon}=H(-x+ct),\\
\lim_{\varepsilon\to+0}\mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\Big) &= \lim_{\varepsilon\to+0}\frac{a_+^\varepsilon-a_-^\varepsilon}{a_+^\varepsilon+a_-^\varepsilon}\\
&= 1-2H(-x+ct),
\end{aligned}
\tag{5.9}
$$

$$
\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\frac{1}{\mathrm{Ch}^2\big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\big)}=\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\frac{4a_-^\varepsilon a_+^\varepsilon}{(a_-^\varepsilon+a_+^\varepsilon)^2}=\frac{4}{[u]}\delta(-x+ct),
\tag{5.10}
$$

where the limits are understood in the weak sense.

Proof. 1. Taking into account (5.4) and (5.6), one can see that

$$
\begin{aligned}
\frac{a_-^\varepsilon}{a_+^\varepsilon+a_-^\varepsilon} &= \frac{1}{2}\Big(1-\mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\Big)\Big)\\
&= \frac{e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)}{e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)+e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}.
\end{aligned}
\tag{5.11}
$$

Let $x<ct$. Taking into account the inequalities $x-ct<0$, (5.3), $[u]>0$, and relation (5.5), we obtain that $\lim_{\varepsilon\to+0}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)=1$, and $\lim_{\varepsilon\to+0}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)\le 1$. Consequently, $\lim_{\varepsilon\to+0}e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)=0$, $\lim_{\varepsilon\to+0}e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)=\infty$.

Thus for the first function in (5.7) we have that $\lim_{\varepsilon\to+0}\operatorname{Th}\big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\big)=-1$ for $x<ct$.

Let $x>ct$. Taking into account the inequalities $x-ct>0$, (5.3), $[u]>0$, and relation (5.5), we have that $\lim_{\varepsilon\to+0}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)=1$, $\lim_{\varepsilon\to+0}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)\le 1$, and $\lim_{\varepsilon\to+0}e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)=\infty$, $\lim_{\varepsilon\to+0}e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)=0$. Thus for the first function in (5.7) we have that $\lim_{\varepsilon\to+0}\operatorname{Th}\big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\big)=1$ for $x>ct$.

Thus according to (5.7) and (5.11), we have *in the weak sense* that

$$\lim_{\varepsilon\to+0}\operatorname{Th}\Big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\Big)=\lim_{\varepsilon\to+0}\frac{a_+^\varepsilon-a_-^\varepsilon}{a_+^\varepsilon+a_-^\varepsilon}=\begin{cases}1, & x>ct,\\ -1, & x<ct,\end{cases}$$

i.e., (5.9) holds.

2. In view of (5.7), (5.8), by changing $\xi=\frac{[u](x-ct)}{2\varepsilon}$, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{1}{\varepsilon}\frac{\varphi(x,t)}{\operatorname{Ch}^2\big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\big)}\,dx\\ &\quad=\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{4}{\varepsilon}\frac{\varphi(x,t)}{\Big(e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)+e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)\Big)^2}\,dx\\ &\quad=\frac{8}{[u]}\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{\varphi\big(ct+\frac{2\xi}{[u]}\varepsilon,t\big)}{\Big(e^{\xi}J\big(-\frac{\xi\sqrt{\varepsilon}}{[u]\sqrt{t}}-\frac{[u]}{2}\sqrt{\frac{t}{\varepsilon}}\big)+e^{-\xi}J\big(\frac{\xi\sqrt{\varepsilon}}{[u]\sqrt{t}}-\frac{[u]}{2}\sqrt{\frac{t}{\varepsilon}}\big)\Big)^2}\,d\xi.\end{aligned}$$

Since according to (5.8),

$$\lim_{\varepsilon\to+0}J\Big(\mp\frac{x-2u_\pm t}{\sqrt{4t\varepsilon}}\Big)\Big|_{x=ct+\frac{2\xi}{[u]}\varepsilon}=\lim_{\varepsilon\to+0}J\Big(\mp\frac{\xi\sqrt{\varepsilon}}{[u]\sqrt{t}}-\frac{[u]}{2}\sqrt{\frac{t}{\varepsilon}}\Big)=1,\tag{5.12}$$

and, according to the above calculations, a denominator of the integrand tends to ∞ sufficiently rapidly, as $|\xi|\to\infty$, one can see that

$$\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{1}{\varepsilon}\frac{\varphi(x,t)}{\operatorname{Ch}^2\big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\big)}\,dx=\frac{8\varphi\big(ct,t\big)}{[u]}\int_{-\infty}^{\infty}\frac{d\xi}{\big(e^{\xi}+e^{-\xi}\big)^2}=\frac{4\varphi\big(ct,t\big)}{[u]},\tag{5.13}$$

for all $\varphi\in\mathcal{D}(\mathbb{R}\times[0,\infty))$. Thus in the weak sense we have the first relation in (5.10).

Taking into account (5.6), according to (5.4), (5.7), we have

$$\begin{aligned}\frac{1}{\varepsilon}\frac{4a_-^\varepsilon a_+^\varepsilon}{(a_-^\varepsilon+a_+^\varepsilon)^2}&=\frac{1}{\varepsilon}\frac{4J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}{\Big(e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)+e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)\Big)^2}\\ &=J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)\frac{1}{\varepsilon}\frac{1}{\operatorname{Ch}^2\big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\big)}.\end{aligned}\tag{5.14}$$

Thus, in view of (5.13), (5.12), relation (5.14) implies the second relation in (5.10). □

It is clear that for $u_+ \le u_-$ limiting properties of functions (5.7) coincide with limiting properties of the corresponding hyperbolic functions of the argument $\frac{[u](x-ct)}{2\varepsilon}$, as $\varepsilon \to +0$.

THEOREM 5.1. *Let $u_+ \le u_-$. If $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a solution of the parabolic problem (1.17), (1.16) then for $t \in [0, \infty)$ we have in the weak sense*

$$\begin{array}{rcl} u(x,t) &=& \lim_{\varepsilon\to+0} u_\varepsilon(x,t) = u_+ + [u]H(-x+\phi(t)), \\ v(x,t) &=& \lim_{\varepsilon\to+0} v_\varepsilon(x,t) = v_+ + [v]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \\ w(x,t) &=& \lim_{\varepsilon\to+0} w_\varepsilon(x,t) = w_+ + [w]H(-x+\phi(t)) + g(t)\delta(-x+\phi(t)) \\ && \qquad\qquad +h(t)\delta'(-x+\phi(t)), \end{array} \tag{5.15}$$

where

$$\begin{array}{rcl} \phi(t) &=& ct = \frac{[u^2]}{[u]}t = \big(u_- + u_+\big)t, \\ e(t) &=& \Big(2[uv] - [v]\dot\phi(t)\Big)t = [u](v_- + v_+)t, \\ g(t) &=& \Big(2[v^2+uw] - [w]\dot\phi(t)\Big)t \\ &=& \big(2[v](v_- + v_+) + [u](w_- + w_+)\big)t, \\ h(t) &=& [u](v_- + v_+)^2t^2. \end{array} \tag{5.16}$$

Moreover,

$$h(t) = \frac{e^2(t)}{[u]}. \tag{5.17}$$

PROOF. 1. Taking into account (5.6), by substituting (5.4) into (4.18), we obtain

$$\begin{aligned} u_\varepsilon(x,t) &= \frac{u_- e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + u_+ e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}{e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)} \\ &= u_+ + [u]\frac{e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)}{e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)} \\ &= u_+ + [u]\frac{1}{2}\Big(1 - \mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon}; t, \varepsilon\Big)\Big). \end{aligned} \tag{5.18}$$

Using (5.18) and (5.9), we have in the weak sense

$$\begin{aligned} \langle u(x,t), \varphi(x,t)\rangle &= \lim_{\varepsilon\to+0}\langle u_\varepsilon(x,t), \varphi(x,t)\rangle = \lim_{\varepsilon\to+0}\Big\langle \frac{u_- a_-^\varepsilon + u_+ a_+^\varepsilon}{a_-^\varepsilon + a_+^\varepsilon}, \varphi(x,t)\Big\rangle \\ &= \lim_{\varepsilon\to+0}\Big\langle u_+ + [u]\frac{1}{2}\Big(1 - \mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon}; t, \varepsilon\Big)\Big), \varphi(x,t)\Big\rangle \\ &= \big\langle u_+ + [u]H(-x+\phi(t)), \varphi(x,t)\big\rangle, \end{aligned} \tag{5.19}$$

for all $\varphi \in \mathcal{D}(\mathbb{R} \times [0,\infty))$, where $\phi(t) = ct = (u_- + u_+)t$. Here the passage to the limit under the integral sign is justified by the Lebesgue dominated theorem. The first equality in (5.15) is thus proved.

2. Similarly to the above calculations, taking into account relations (5.6), (5.1), (5.2), we transform $V_\varepsilon(x,t)$ given by (4.21), to the form

$$
\begin{aligned}
V_\varepsilon(x,t) = {} & x\frac{v_- e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + v_+ e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}{e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)} \\
& - \frac{[v]\sqrt{\frac{t\varepsilon}{\pi}} e^{-\frac{(x-2u_-t)^2}{4t\varepsilon}}}{J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)} \\
& - 2t\frac{u_- v_- e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + u_+ v_+ e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}{e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}. \qquad (5.20)
\end{aligned}
$$

Since

$$
\frac{1}{\sqrt{\pi\varepsilon}} e^{-\frac{x^2}{\varepsilon}} \xrightarrow{\mathcal{D}'} \delta(x), \qquad \varepsilon \to +0, \qquad (5.21)
$$

and, according to the proof of Lemma 5.1, $\lim_{\varepsilon\to+0} e^{\mp\frac{[u](x-ct)}{2\varepsilon}} J\big(\pm\frac{x-2u_\mp t}{\sqrt{4t\varepsilon}}\big) = \infty$ for $\mp(x-ct) > 0$, we have

$$
\begin{aligned}
\frac{[v]\sqrt{\frac{t\varepsilon}{\pi}} e^{-\frac{x^2}{4t\varepsilon}}}{a_-^\varepsilon + a_+^\varepsilon} &= \frac{[v]\sqrt{\frac{t\varepsilon}{\pi}} e^{-\frac{(x-2u_-t)^2}{4t\varepsilon}}}{J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)} \\
&= \varepsilon\frac{[v]\sqrt{\frac{t}{\pi\varepsilon}} e^{-\frac{(x-ct)^2}{4t\varepsilon}} e^{-\frac{[u]^2 t}{4\varepsilon}}}{e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)} \xrightarrow{\mathcal{D}'} 0, \quad \varepsilon \to +0. \qquad (5.22)
\end{aligned}
$$

Passing to the weak limit in $V_\varepsilon(x,t)$ as $\varepsilon \to +0$, and taking into account relations (5.9), (5.18), (5.19), we have

$$
\begin{aligned}
&\frac{v_- e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + v_+ e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}{e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)} \\
&\quad = v_+ + [v]\frac{1}{2}\Big(1 - \mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon}; t, \varepsilon\Big)\Big) \xrightarrow{\mathcal{D}'} v_+ + [v]H(-x+\phi(t)), \\
&\frac{u_- v_- e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + u_+ v_+ e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}{e^{-\frac{[u](x-ct)}{2\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{2\varepsilon}} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)} \\
&\quad = u_+v_+ + [uv]\frac{1}{2}\Big(1 - \mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon}; t, \varepsilon\Big)\Big) \xrightarrow{\mathcal{D}'} u_+v_+ + [uv]H(-x+\phi(t)).
\end{aligned}
$$

Thus taking into account (5.22), similarly to (5.19), we obtain that in the weak sense

$$V(x,t)=\lim_{\varepsilon\to+0}V_\varepsilon(x,t)$$

$$=x\Big(v_+ +[v]H(-x+\phi(t))\Big)-2t\Big(u_+v_+ +[uv]H(-x+\phi(t))\Big). \tag{5.23}$$

According to (4.21), (5.4), $v_\varepsilon(x,t)=(V_\varepsilon(x,t))_x$. Hence, taking into account the well-known relation $(-x+\phi(t))\delta(-x+\phi(t))=0$, and differentiating relation (5.23) with respect to x, we obtain

$$\langle v(x,t),\varphi(x,t)\rangle=\lim_{\varepsilon\to+0}\langle v_\varepsilon(x,t),\varphi(x,t)\rangle=\lim_{\varepsilon\to+0}\langle (V_\varepsilon(x,t))_x,\varphi(x,t)\rangle$$

$$=-\lim_{\varepsilon\to+0}\langle V_\varepsilon(x,t),\varphi_x(x,t)\rangle=\langle V_x(x,t),\varphi(x,t)\rangle$$

$$=\big\langle v_+ +[v]H(-x+\phi(t))-x[v]\delta(-x+\phi(t))+2t[uv]\delta(-x+\phi(t)),\varphi(x,t)\big\rangle$$

$$=\big\langle v_+ +[v]H(-x+\phi(t))+e(t)\delta(-x+\phi(t)),\varphi(x,t)\big\rangle, \tag{5.24}$$

for all $\varphi\in\mathcal{D}(\mathbb{R}\times[0,\infty))$, where $e(t)$ is given by the second relation in (5.16). The second equality in (5.15) is thus proved.

3. According to (4.20), (4.22), $w_\varepsilon(x,t)=W_{\varepsilon\,x}$, where

$$W_\varepsilon(x,t)=-\varepsilon\frac{C_\varepsilon}{A_\varepsilon}+\frac{1}{\varepsilon}\Big(V_\varepsilon\Big)^2.$$

To achieve our goal, we shall use the same transforms of $W_\varepsilon(x,t)$ as above.

In view of (4.24), (4.14), (4.15), we have

$$-\varepsilon\frac{C_\varepsilon}{A_\varepsilon}=-2t\frac{v_-^2a_-^\varepsilon+v_+^2a_+^\varepsilon}{a_-^\varepsilon+a_+^\varepsilon}+\frac{w_-(x-2u_-t)a_-^\varepsilon+w_+(x-2u_+t)a_+^\varepsilon}{a_-^\varepsilon+a_+^\varepsilon}$$

$$-\frac{[w]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}}{a_-^\varepsilon+a_+^\varepsilon}+\frac{1}{\varepsilon}\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\frac{v_-^2(x-2u_-t)-v_+^2(x-2u_+t)}{a_-^\varepsilon+a_+^\varepsilon}$$

$$-\frac{1}{\varepsilon}\frac{v_-^2(x-2u_-t)^2a_-^\varepsilon+v_+^2(x-2u_+t)^2a_+^\varepsilon}{a_-^\varepsilon+a_+^\varepsilon}. \tag{5.25}$$

Next, using (4.21), one can easily see that

$$\frac{1}{\varepsilon}\Big(V_\varepsilon\Big)^2=\frac{1}{\varepsilon}\frac{v_-^2(x-2u_-t)^2(a_-^\varepsilon)^2+v_+^2(x-2u_+t)^2(a_+^\varepsilon)^2}{(a_-^\varepsilon+a_+^\varepsilon)^2}$$

$$+\frac{1}{\varepsilon}\frac{2v_-v_+(x-2u_-t)(x-2u_+t)a_-^\varepsilon a_+^\varepsilon}{(a_-^\varepsilon+a_+^\varepsilon)^2}$$

$$+\frac{[v]^2\frac{t}{\pi}e^{-\frac{x^2}{2t\varepsilon}}}{(a_-^\varepsilon+a_+^\varepsilon)^2}-\frac{1}{\varepsilon}2[v]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\frac{v_-(x-2u_-t)a_-^\varepsilon+v_+(x-2u_+t)a_+^\varepsilon}{(a_-^\varepsilon+a_+^\varepsilon)^2}. \tag{5.26}$$

Summarizing (5.25) and (5.26), we obtain

$$W_\varepsilon(x,t) = -\varepsilon\frac{C_\varepsilon}{A_\varepsilon} + \frac{1}{\varepsilon}\Big(V_\varepsilon\Big)^2 = Z_{1\varepsilon} + Z_{2\varepsilon} + Z_{3\varepsilon} + Z_{4\varepsilon}, \tag{5.27}$$

where

$$Z_{1\varepsilon} = -2t\frac{v_-^2 a_-^\varepsilon + v_+^2 a_+^\varepsilon}{a_-^\varepsilon + a_+^\varepsilon} + \frac{w_-(x-2u_-t)a_-^\varepsilon + w_+(x-2u_+t)a_+^\varepsilon}{a_-^\varepsilon + a_+^\varepsilon}, \tag{5.28}$$

$$\begin{aligned} Z_{2\varepsilon} &= \frac{1}{\varepsilon}\frac{v_-^2(x-2u_-t)^2(a_-^\varepsilon)^2 + v_+^2(x-2u_+t)^2(a_+^\varepsilon)^2}{(a_-^\varepsilon + a_+^\varepsilon)^2} \\ &\quad + \frac{1}{\varepsilon}\frac{2v_-v_+(x-2u_-t)(x-2u_+t)a_-^\varepsilon a_+^\varepsilon}{(a_-^\varepsilon + a_+^\varepsilon)^2} \\ &\quad - \frac{1}{\varepsilon}\frac{v_-^2(x-2u_-t)^2 a_-^\varepsilon + v_+^2(x-2u_+t)^2 a_+^\varepsilon}{a_-^\varepsilon + a_+^\varepsilon} \end{aligned}$$

$$= -\frac{1}{\varepsilon}\frac{a_-^\varepsilon a_+^\varepsilon}{(a_-^\varepsilon + a_+^\varepsilon)^2}\big(v_-(x-2u_-t) - v_+(x-2u_+t)\big)^2, \tag{5.29}$$

$$Z_{3\varepsilon} = -\frac{[w]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}}{a_-^\varepsilon + a_+^\varepsilon} + \frac{[v]^2\frac{t}{\pi}e^{-\frac{x^2}{2t\varepsilon}}}{(a_-^\varepsilon + a_+^\varepsilon)^2}, \tag{5.30}$$

$$\begin{aligned} Z_{4\varepsilon} &= \frac{1}{\varepsilon}\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\frac{v_-^2(x-2u_-t) - v_+^2(x-2u_+t)}{a_-^\varepsilon + a_+^\varepsilon} \\ &\quad - \frac{1}{\varepsilon}2[v]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\frac{v_-(x-2u_-t)a_-^\varepsilon + v_+(x-2u_+t)a_+^\varepsilon}{(a_-^\varepsilon + a_+^\varepsilon)^2}. \end{aligned} \tag{5.31}$$

Passing to the weak limit in $Z_{1\varepsilon}$ as $\varepsilon \to +0$, and taking into account relations (5.9), (5.18), (5.19), one can easily see that

$$\begin{aligned} \lim_{\varepsilon\to+0} Z_{1\varepsilon} &\overset{\mathcal{D}'}{=} x\Big(w_+ + [w]H(-x+\phi(t))\Big) \\ &\quad -2t\Big(v_+^2 + u_+w_+ + [v^2+uw]H(-x+\phi(t))\Big). \end{aligned} \tag{5.32}$$

In view of (5.10), relation (5.29) implies that in the weak sense

$$\begin{aligned} \lim_{\varepsilon\to+0} Z_{2\varepsilon} &\overset{\mathcal{D}'}{=} -\frac{1}{[u]}\big(v_-(ct-2u_-t) - v_+(ct-2u_+t)\big)^2\delta(-x+\phi(t)) \\ &= -\frac{1}{[u]}[u]^2(v_- + v_+)^2t^2\delta(-x+\phi(t)) = -[u](v_- + v_+)^2t^2\delta(-x+\phi(t)). \end{aligned} \tag{5.33}$$

In view of (5.21), by using (5.30), (5.6), (5.1), (5.2), and repeating the proof of relation (5.22) almost word for word, we obtain

$$Z_{3\varepsilon}=-\varepsilon\frac{[w]\sqrt{\frac{t}{\pi\varepsilon}}e^{-\frac{(x-ct)^2}{4t\varepsilon}}e^{-\frac{[u]^2t}{4\varepsilon}}}{e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)+e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}$$

$$+\sqrt{\varepsilon}\frac{[v]^2\frac{t}{\pi\sqrt{\varepsilon}}e^{-\frac{(x-ct)^2}{2t\varepsilon}}e^{-\frac{[u]^2t}{2\varepsilon}}}{\Big(e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)+e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)\Big)^2}\overset{\mathcal{D}'}{\to}0,\quad \varepsilon\to+0. \tag{5.34}$$

Taking into account (5.6), (5.1), (5.2), by the above elementary calculations, we transform (5.31) to the form

$$Z_{4\varepsilon}=\sqrt{\frac{t}{\pi\varepsilon}}e^{-\frac{(x-ct)^2}{4t\varepsilon}}e^{-\frac{[u]^2t}{4\varepsilon}}\Bigg\{\frac{v_-^2(x-2u_-t)-v_+^2(x-2u_+t)}{e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)+e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}$$

$$-2[v]\frac{v_-(x-2u_-t)e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)+v_+(x-2u_+t)e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}{\Big(e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)+e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)\Big)^2}\Bigg\}.$$

Applying the last relation to a test function $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\infty))$ and making the change of variables $\xi=\frac{[u](x-ct)}{2\varepsilon}$, $\eta=\frac{t}{\varepsilon}$, we obtain for the first term in $Z_{4\varepsilon}$:

$$\int_0^\infty\int Z_{4\varepsilon}(x,t)\varphi(x,t)\,dx\,dt$$

$$=2\varepsilon^3\int_0^\infty\int\sqrt{\frac{\eta}{\pi}}\frac{e^{-\frac{\xi^2}{[u]^2\eta}-\frac{[u]^2\eta}{4}}\big(2\xi[v^2]-[u]^2(v_-^2+v_+)\eta\big)}{e^{-\xi}J\big(\frac{2\xi-[u]^2\eta}{2[u]\sqrt{\eta}}\big)+e^{\xi}J\big(-\frac{2\xi+[u]^2\eta}{2[u]\sqrt{\eta}}\big)}\varphi\big((c\eta+\frac{2\xi}{[u]})\varepsilon,\eta\varepsilon\big)\,d\xi\,d\eta\to0,$$

as $\varepsilon\to+0$. Similar simple calculations show that the second term in $Z_{4\varepsilon}$ tends to zero in the weak sense, as $\varepsilon\to+0$. Thus

$$\lim_{\varepsilon\to+0}Z_{4\varepsilon}\overset{\mathcal{D}'}{=}0. \tag{5.35}$$

Here we use the fact that $\sqrt{\frac{t}{\pi\varepsilon}}e^{-\frac{(x-ct)^2}{4t\varepsilon}}e^{-\frac{[u]^2t}{4\varepsilon}}\overset{\mathcal{D}'}{\to}0$, $\varepsilon\to+0$.

Thus, according to (5.32), (5.33), (5.34), (5.35), we have

$$\lim_{\varepsilon\to+0}W_\varepsilon(x,t)=\lim_{\varepsilon\to+0}(Z_{1\varepsilon}+Z_{2\varepsilon}+Z_{3\varepsilon}+Z_{4\varepsilon})\overset{\mathcal{D}'}{=}x\Big(w_++[w]H(-x+\phi(t))\Big)$$

$$-2t\Big(v_+^2+u_+w_++[v^2+uw]H(-x+\phi(t))\Big)$$

$$-[u](v_-+v_+)^2t^2\delta(-x+\phi(t)). \tag{5.36}$$

Taking into account that $(-x+\phi(t))\delta(-x+\phi(t))=0$, and differentiating relation (5.36) with respect to x, we obtain

$$\begin{aligned}\big\langle w(x,t),\varphi(x,t)\big\rangle &= \lim_{\varepsilon\to+0}\big\langle (W_\varepsilon(x,t))_x,\varphi(x,t)\big\rangle = -\lim_{\varepsilon\to+0}\big\langle W_\varepsilon(x,t),\varphi_x(x,t)\big\rangle\\ &= \big\langle W_x(x,t),\varphi(x,t)\big\rangle = \big\langle w_+ + [w]H(-x+\phi(t)-x[w]\delta(-x+\phi(t))\\ &\quad +2t[v^2+uw]\delta(-x+\phi(t))+[u](v_-+v_+)^2t^2\delta'(-x+\phi(t)),\varphi(x,t)\big\rangle\\ &= \big\langle w_+ + [w]H(-x+\phi(t))+g(t)\delta(-x+\phi(t))+h(t)\delta'(-x+\phi(t)),\varphi(x,t)\big\rangle, \end{aligned}\tag{5.37}$$

for all $\varphi\in\mathcal{D}(\mathbb{R}\times[0,\infty))$, where $g(t)=(2[v^2+uw]-[w]c)t$ and $h(t)=[u](v_-+v_+)^2t^2$ are given by the third and fourth relations in (5.16). Thus, the third equality in (5.15) is proved. □

6. Weak limit of the solution to the problem (1.17), (1.16) for $u_+ > u_-$

In this case we have $[u]=u_--u_+<0$, $2u_-<c=u_-+u_+<2u_+$, and

$$x-2u_+t<x-ct<x-2u_-t.\tag{6.1}$$

According to (5.4), (5.5), (5.1), we have

$$\begin{aligned} a_-^\varepsilon(x,t) &\sim e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_-t)}\Big(1-\frac{4t\varepsilon}{2(x-2u_-t)^2}\Big), \quad x>2u_-t,\\ a_+^\varepsilon(x,t) &\sim e^{-\frac{x^2}{4t\varepsilon}}\frac{-\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_+t)}\Big(1-\frac{4t\varepsilon}{2(x-2u_+t)^2}\Big), \quad x<2u_+t, \end{aligned}\tag{6.2}$$

as $\varepsilon\to+0$.

Lemma 6.1. *We have*

$$\begin{aligned}\lim_{\varepsilon\to+0}\frac12\Big(1-\mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon};t,\varepsilon\Big)\Big) &= \lim_{\varepsilon\to+0}\frac{a_-^\varepsilon}{a_+^\varepsilon+a_-^\varepsilon}\\ &= \begin{cases} 1, & x\le 2u_-t,\\ \frac{1}{[u]}\big(\frac{x}{2t}-u_+\big), & 2u_-t<x<2u_+t,\\ 0, & x\ge 2u_+t, \end{cases}\\ &= \frac{1}{2[u]t}\Big((-x+2u_-t)H(-x+2u_-t)-(-x+2u_+t)H(-x+2u_+t)\Big), \end{aligned}\tag{6.3}$$

where the limit is understood in the weak sense.

Proof. 1. Let $x\le 2u_-t$. In view of (6.1), we have $x-ct$, $x-2u_+t<0$, i.e., according to (5.5), $\lim_{\varepsilon\to+0}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)=0$, and $\lim_{\varepsilon\to+0}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)=1$, for $x<2u_-t$; $J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)\big|_{x=2u_-t}=\frac12$. Applying estimates (5.5) and formulas (5.1), (5.2) to (5.11), we see that

$$\lim_{\varepsilon\to+0}\frac{a_-^\varepsilon}{a_+^\varepsilon+a_-^\varepsilon}=\lim_{\varepsilon\to+0}\frac{J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)}{J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)+e^{\frac{[u](x-ct)}{\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}$$

$$= \lim_{\varepsilon\to+0} \frac{J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)}{J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) - e^{\frac{[u](x-ct)}{\varepsilon}} \frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_+t)} e^{-\frac{(x-2u_+t)^2}{4t\varepsilon}}}$$

$$= \lim_{\varepsilon\to+0} \frac{J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)}{J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) - \frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_+t)} e^{-\frac{(x-2u_-t)^2}{4t\varepsilon}}} = 1. \tag{6.4}$$

Let $x \geq 2u_+t$. In view of (6.1), we have $x - ct$, $x - 2u_-t > 0$, and, consequently, $\lim_{\varepsilon\to+0} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) = 0$, and $\lim_{\varepsilon\to+0} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big) = 1$, for $x > 2u_+t$; $J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)\big|_{x=2u_+t} = \frac{1}{2}$. Applying estimates (5.5) to (5.11), and taking into account (5.1), (5.2), just as above, we have

$$\lim_{\varepsilon\to+0} \frac{a_-^\varepsilon}{a_+^\varepsilon + a_-^\varepsilon} = \lim_{\varepsilon\to+0} \frac{e^{-\frac{[u](x-ct)}{\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)}{e^{-\frac{[u](x-ct)}{\varepsilon}} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}$$

$$= \lim_{\varepsilon\to+0} \frac{e^{-\frac{[u](x-ct)}{\varepsilon}} \frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_-t)} e^{-\frac{(x-2u_-t)^2}{4t\varepsilon}}}{e^{-\frac{[u](x-ct)}{\varepsilon}} \frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_-t)} e^{-\frac{(x-2u_-t)^2}{4t\varepsilon}} + J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}$$

$$= \lim_{\varepsilon\to+0} \frac{\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_-t)} e^{-\frac{(x-2u_+t)^2}{4t\varepsilon}}}{\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_-t)} e^{-\frac{(x-2u_+t)^2}{4t\varepsilon}} + J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)} = 0. \tag{6.5}$$

Let $2u_-t < x < 2u_+t$. In this case we have $x - 2u_-t > 0$ and $x - 2u_+t < 0$, and, consequently, $\lim_{\varepsilon\to+0} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big) = 0$, and $\lim_{\varepsilon\to+0} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) = 0$. Thus applying (5.5), (5.2), (5.1), (6.2), to (5.11), we calculate

$$\lim_{\varepsilon\to+0} \frac{a_-^\varepsilon}{a_+^\varepsilon + a_-^\varepsilon} = \lim_{\varepsilon\to+0} \frac{e^{-\frac{x^2}{4t\varepsilon}} \frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_-t)}}{e^{-\frac{x^2}{4t\varepsilon}} \frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_-t)} + e^{-\frac{x^2}{4t\varepsilon}} \frac{-\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_+t)}}$$

$$= \frac{\frac{1}{x-2u_-t}}{\frac{1}{x-2u_-t} - \frac{1}{x-2u_+t}} = \frac{x-2u_+t}{2[u]t}. \tag{6.6}$$

Summarizing the above relations (6.4)–(6.6), we conclude that relation (6.3) holds. □

THEOREM 6.1. *Let $u_+ > u_-$. If $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a solution of the parabolic problem (1.17), (1.16) then for $t \in [0, \infty)$ we have in the weak sense*

$$\big(u(x,t), v(x,t), w(x,t)\big) = \lim_{\varepsilon\to+0} \big(u_\varepsilon(x,t), v_\varepsilon(x,t), w_\varepsilon(x,t)\big)$$

$$= \begin{cases} (u_-, v_-, w_-), & x \leq 2u_-t, \\ \big(\frac{x}{2t}, 0, 0\big), & 2u_-t < x < 2u_+t, \\ (u_+, v_+, w_+), & x \geq 2u_+t, \end{cases}$$

$$= (u_+, v_+, w_+)\big(1 - H(-x + 2u_+t)\big) + (u_-, v_-, w_-)H(-x + 2u_-t)$$

$$+\big(\frac{x}{2t}, 0, 0\big)\big(H(-x + 2u_+t) - H(-x + 2u_-t)\big). \tag{6.7}$$

Proof. 1. According to (5.18) and (6.3), we have in the weak sense

$$u(x,t) = \lim_{\varepsilon\to+0} u_\varepsilon(x,t) = \lim_{\varepsilon\to+0} \frac{u_-a_-^\varepsilon + u_+a_+^\varepsilon}{a_-^\varepsilon + a_+^\varepsilon}$$

$$= \lim_{\varepsilon\to+0}\left(u_+ + [u]\frac{1}{2}\Big(1 - \mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon}; t, \varepsilon\Big)\Big)\right)$$

$$= u_+ + \frac{1}{2t}\Big((-x + 2u_-t)H(-x + 2u_-t) - (-x + 2u_+t)H(-x + 2u_+t)\Big). \tag{6.8}$$

The first equality in (5.15) thus holds.

2. Similarly to the above calculations, we can calculate the weak limit of the following terms of $V_\varepsilon(x,t)$ given by (5.20), as $\varepsilon \to +0$:

$$\frac{v_-e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + v_+e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}{e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}$$

$$= v_+ + [v]\frac{1}{2}\Big(1 - \mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon}; t, \varepsilon\Big)\Big) \xrightarrow{\mathcal{D}'}$$

$$v_+ + \frac{[v]}{2[u]t}\Big((-x + 2u_-t)H(-x + 2u_-t) - (-x + 2u_+t)H(-x + 2u_+t)\Big), \tag{6.9}$$

$$\frac{u_-v_-e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + u_+v_+e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}{e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) + e^{\frac{[u](x-ct)}{2\varepsilon}}J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)}$$

$$= u_+v_+ + [uv]\frac{1}{2}\Big(1 - \mathrm{Th}\Big(\frac{[u](x-ct)}{2\varepsilon}; t, \varepsilon\Big)\Big) \xrightarrow{\mathcal{D}'}$$

$$u_+v_+ + \frac{[uv]}{2[u]t}\Big((-x + 2u_-t)H(-x + 2u_-t) - (-x + 2u_+t)H(-x + 2u_+t)\Big). \tag{6.10}$$

If $x \le 2u_-t$, according to the proof of Lemma 6.1, $\lim_{\varepsilon\to+0} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) = 1$, for $x < 2u_-t$; $J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)\big|_{x=2u_-t} = \frac{1}{2}$; $\lim_{\varepsilon\to+0} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big) = 0$. Next, using estimates (5.5), equalities (5.1), (5.2), and repeating the proof of Lemma 6.1, we see that

$$\lim_{\varepsilon\to+0}\frac{[v]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}}{a_-^\varepsilon + a_+^\varepsilon} = \lim_{\varepsilon\to+0}\frac{[v]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{(x-ct)^2}{4t\varepsilon}}e^{-\frac{[u]^2t}{4\varepsilon}}}{e^{-\frac{[u](x-ct)}{2\varepsilon}}J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) - e^{\frac{[u](x-ct)}{2\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_+t)}e^{-\frac{(x-2u_+t)^2}{4t\varepsilon}}}$$

$$= \lim_{\varepsilon\to+0} \frac{[v]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{(x-2u_-t)^2}{4t\varepsilon}}}{J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) - \frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_+t)}e^{-\frac{(x-2u_-t)^2}{4t\varepsilon}}} = 0. \tag{6.11}$$

If $x \geq 2u_+t$, taking into account that $\lim_{\varepsilon\to+0} J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big) = 1$; $J\big(-\frac{x-2u_+t}{\sqrt{4t\varepsilon}}\big)\big|_{x=2u_+t} = \frac{1}{2}$; $\lim_{\varepsilon\to+0} J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big) = 0$, and repeating the proof of Lemma 6.1, we obtain

$$\lim_{\varepsilon\to+0} \frac{[v]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}}{a_-^\varepsilon + a_+^\varepsilon} = 0. \tag{6.12}$$

If $2u_-t < x < 2u_+t$, according to the proof of Lemma 6.1, $\lim_{\varepsilon\to+0} J\big(\pm\frac{x-2u_\mp t}{\sqrt{4t\varepsilon}}\big) = 0$. Using (5.5), (5.1), (5.2), (6.2), and repeating the proof of Lemma 6.1, we obtain

$$\lim_{\varepsilon\to+0} \frac{[v]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}}{a_-^\varepsilon + a_+^\varepsilon} = \lim_{\varepsilon\to+0} \frac{[v]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}}{e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_-t)} + e^{-\frac{x^2}{4t\varepsilon}}\frac{-\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_+t)}}$$

$$= \lim_{\varepsilon\to+0} \frac{[v]\sqrt{\frac{t\varepsilon}{\pi}}}{\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_-t)} - \frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_+t)}} = \frac{[v](-x+2u_-t)(-x+2u_+t)}{2[u]t}. \tag{6.13}$$

Thus (6.11), (6.12), (6.13) imply

$$\lim_{\varepsilon\to+0} \frac{[v]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}}{a_-^\varepsilon + a_+^\varepsilon}$$

$$= \frac{[v](-x+2u_-t)(-x+2u_+t)}{2[u]t}\Big(H(-x+2u_+t) - H(-x+2u_-t)\Big). \tag{6.14}$$

Taking into account (6.9), (6.10), (6.14), by easy calculations we derive

$$V(x,t) = \lim_{\varepsilon\to+0} V_\varepsilon(x,t)$$

$$= x\Big\{v_+ + \frac{[v]}{2[u]t}\Big((-x+2u_-t)H(-x+2u_-t) - (-x+2u_+t)H(-x+2u_+t)\Big)\Big\}$$

$$-2t\Big\{u_+v_+ + \frac{[uv]}{2[u]t}\Big((-x+2u_-t)H(-x+2u_-t) - (-x+2u_+t)H(-x+2u_+t)\Big)\Big\}$$

$$-\frac{[v](-x+2u_-t)(-x+2u_+t)}{2[u]t}\Big(H(-x+2u_+t) - H(-x+2u_-t)\Big)$$

$$= -v_+\big(-x+2u_+t\big)\big(1 - H(-x+2u_+t)\big) - v_-\big(-x+2u_-t\big)H(-x+2u_-t)$$

$$= \begin{cases} -v_-\big(-x+2u_-t\big), & x \leq 2u_-t, \\ 0, & 2u_-t < x < 2u_+t. \\ -v_+\big(-x+2u_+t\big), & x \geq 2u_+t, \end{cases} \tag{6.15}$$

Consequently, taking into account the relations $(-x+2u_\pm t)\delta(-x+2u_\pm t)=0$, we have

$$\langle v(x,t),\varphi(x,t)\rangle=\lim_{\varepsilon\to+0}\langle (V_\varepsilon(x,t))_x,\varphi(x,t)\rangle=\langle V_x(x,t),\varphi(x,t)\rangle$$

$$=\big\langle v_+\big(1-H(-x+2u_+t)\big)+v_-H(-x+2u_-t),\varphi(x,t)\big\rangle, \tag{6.16}$$

for all $\varphi\in\mathcal{D}(\mathbb{R}\times(0,\infty))$, i.e., the second equality in (6.7) holds.

3. According to (5.27),

$$W_\varepsilon(x,t)=Z_{1\varepsilon}+Z_{2\varepsilon}+Z_{3\varepsilon}+Z_{4\varepsilon},$$

where $Z_{1\varepsilon}$, $Z_{2\varepsilon}$, $Z_{3\varepsilon}$, $Z_{4\varepsilon}$ are given by (5.28)–(5.31). We set $Z_{3\varepsilon}=Z^1_{3\varepsilon}+Z^2_{3\varepsilon}$, where

$$Z^1_{3\varepsilon}=-\frac{[w]\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}}{a^\varepsilon_-+a^\varepsilon_+},\qquad Z^2_{3\varepsilon}=\frac{[v]^2\frac{t}{\pi}e^{-\frac{x^2}{2t\varepsilon}}}{(a^\varepsilon_-+a^\varepsilon_+)^2}. \tag{6.17}$$

Repeating the calculations for deriving formulas (6.8), (6.9), (6.10), (6.14), we see that

$$\lim_{\varepsilon\to+0}\big(Z_{1\varepsilon}+Z^1_{3\varepsilon}\big)_x=\begin{cases} w_-, & x\le 2u_-t,\\ -\frac{[v^2]}{[u]}, & 2u_-t<x<2u_+t,\\ w_+, & x\ge 2u_+t.\end{cases} \tag{6.18}$$

Using (5.29) and taking into account (5.21), by repeating the above calculations, we obtain for $x<2u_-t$

$$\lim_{\varepsilon\to+0}Z_{2\varepsilon}$$

$$=\lim_{\varepsilon\to+0}\frac{J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)\frac{1}{x-2u_+t}\sqrt{\frac{t}{\pi\varepsilon}}e^{-\frac{(x-2u_-t)^2}{4t\varepsilon}}}{\big(J\big(\frac{x-2u_-t}{\sqrt{4t\varepsilon}}\big)-\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}(x-2u_+t)}e^{-\frac{(x-2u_-t)^2}{4t\varepsilon}}\big)^2}\big(v_-(x-2u_-t)-v_+(x-2u_+t)\big)^2=0.$$

Similarly to proving the above equality and (6.11), (6.12), it is easy to prove that if $x<2u_-t$ or $x>2u_+t$ then

$$\lim_{\varepsilon\to+0}Z_{2\varepsilon}=\lim_{\varepsilon\to+0}Z^2_{3\varepsilon}=\lim_{\varepsilon\to+0}Z_{4\varepsilon}=0. \tag{6.19}$$

Let $2u_-t<x<2u_+t$. Denote $X_\pm=x-2u_\pm t$. Applying formulas (5.1), (5.2), (6.2) to (5.29), (6.17), (5.31), and taking into account that $X_+-X_-=2[u]t$, we calculate that

$$\lim_{\varepsilon\to+0}\big(Z_{2\varepsilon}+Z^2_{3\varepsilon}+Z_{4\varepsilon}\big)$$

$$=\lim_{\varepsilon\to+0}\Bigg\{-\frac{1}{\varepsilon}\frac{e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_-}\big(1-\frac{2t\varepsilon}{X_-^2}\big)e^{-\frac{x^2}{4t\varepsilon}}\frac{-\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_+}\big(1-\frac{2t\varepsilon}{X_+^2}\big)}{\big(e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_-}\big(1-\frac{2t\varepsilon}{X_-^2}\big)-e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_+}\big(1-\frac{2t\varepsilon}{X_+^2}\big)\big)^2}\big(v_-X_--v_+X_+\big)^2$$

$$
\begin{aligned}
&+\frac{[v]^2\frac{t}{\pi}e^{-\frac{x^2}{2t\varepsilon}}}{\big(e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_-}\big(1-\frac{2t\varepsilon}{X_-^2}\big)-e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_+}\big(1-\frac{2t\varepsilon}{X_+^2}\big)\big)^2}\\
&+\frac{1}{\varepsilon}\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\frac{v_-^2X_--v_+^2X_+}{e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_-}\big(1-\frac{2t\varepsilon}{X_-^2}\big)-e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_+}\big(1-\frac{2t\varepsilon}{X_+^2}\big)}\\
&-\frac{2[v]}{\varepsilon}\sqrt{\frac{t\varepsilon}{\pi}}e^{-\frac{x^2}{4t\varepsilon}}\frac{v_-X_-\frac{e^{-\frac{x^2}{4t\varepsilon}}\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_-}\big(1-\frac{2t\varepsilon}{X_-^2}\big)-v_+X_+\frac{e^{-\frac{x^2}{4t\varepsilon}}\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_+}\big(1-\frac{2t\varepsilon}{X_+^2}\big)}{\big(e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_-}-e^{-\frac{x^2}{4t\varepsilon}}\frac{\sqrt{4t\varepsilon}}{2\sqrt{\pi}X_+}\big)^2}\Bigg\}\\
&=\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\Bigg\{\frac{X_-X_+}{4[u]^2t^2}\big(v_-X_--v_+X_+\big)^2+\frac{[v]^2X_-^2X_+^2}{4[u]^2t^2}\\
&\qquad\qquad+\frac{X_-X_+\big(v_-^2X_--v_+^2X_+\big)}{2[u]t}-2[v]^2\frac{X_-^2X_+^2}{4[u]^2t^2}\Bigg\}\\
&+\Bigg\{\frac{1}{2[u]^2t}\frac{(X_-+X_+)^2}{X_-X_+}\big(v_-X_--v_+X_+\big)^2+\frac{[v]^2}{[u]^2t}\big(X_-^2+X_-X_++X_+^2\big)^2\\
&\quad-\frac{1}{2[u]^2t}\frac{X_-^3-X_+^3}{X_-X_+}\big(v_-^2X_--v_+^2X_+\big)\\
&\quad-\frac{[v]}{[u]^2t}\Big(v_-(X_+^2+2X_-X_++2X_-^2)-v_+(X_-^2+2X_-X_++2X_+^2)\Big)\Bigg\}.
\end{aligned}
\tag{6.20}
$$

It is easy to examine that the expression in the first braces in (6.20) is equal to zero, i.e.,

$$
\begin{aligned}
&\frac{X_-X_+}{4[u]^2t^2}\big(v_-X_--v_+X_+\big)^2+\frac{[v]^2X_-^2X_+^2}{4[u]^2t^2}\\
&\qquad+\frac{X_-X_+\big(v_-^2X_--v_+^2X_+\big)}{2[u]t}-2[v]^2\frac{X_-^2X_+^2}{4[u]^2t^2}=0.
\end{aligned}
\tag{6.21}
$$

Let us consider the expression in the second braces in (6.20). Easy calculations show that

$$
\begin{aligned}
&\frac{1}{2[u]^2t}\frac{(X_-+X_+)^2}{X_-X_+}\big(v_-X_--v_+X_+\big)^2+\frac{[v]^2}{[u]^2t}\big(X_-^2+X_-X_++X_+^2\big)^2\\
&\quad-\frac{1}{2[u]^2t}\frac{X_-^3-X_+^3}{X_-X_+}\big(v_-^2X_--v_+^2X_+\big)\\
&\quad-\frac{[v]}{[u]^2t}\Big(v_-(X_+^2+2X_-X_++2X_-^2)-v_+(X_-^2+2X_-X_++2X_+^2)\Big)\\
&=\frac{1}{2[u]^2t}\Bigg(\frac{1}{X_-X_+}\Big(X_-^2X_+^2(v_-^2-4v_-v_++v_+^2)\\
&\qquad+X_-^3X_+(2v_-^2-2v_-v_++v_+^2)+X_-X_+^3(2v_+^2-2v_-v_++v_-^2)\Big)
\end{aligned}
$$

$$+\Big(X_-X_+(-2v_-^2+4v_-v_+-2v_+^2)+X_-^2(-2v_-^2+2v_-v_+)+X_+^2(-2v_+^2+2v_-v_+)\Big)\Big)$$

$$= \frac{1}{2[u]^2t}\Big(X_-^2v_+^2 - X_-X_+(v_-^2+v_+^2)+X_+^2v_-^2\Big). \tag{6.22}$$

Thus (6.20), (6.21), (6.22) imply that

$$\lim_{\varepsilon\to+0}(Z_{2\varepsilon}+Z_{3\varepsilon}^2+Z_{4\varepsilon}) = \frac{1}{2[u]^2t}\Big(X_-^2v_+^2 - X_-X_+(v_-^2+v_+^2)+X_+^2v_-^2\Big)$$

for $2u_-t < x < 2u_+t$. Consequently,

$$\lim_{\varepsilon\to+0}(Z_{2\varepsilon}+Z_{3\varepsilon}^2+Z_{4\varepsilon})_x = \frac{[v^2]}{2[u]^2t}\big(X_+-X_-\big) = \frac{[v^2]}{[u]}, \quad 2u_-t < x < 2u_+t. \tag{6.23}$$

Summarizing (6.18), (6.19), (6.23), we conclude that

$$w(x,t) = \lim_{\varepsilon\to+0}(W_\varepsilon(x,t))_x = \begin{cases} w_-, & x \le 2u_-t, \\ 0, & 2u_-t < x < 2u_+t, \\ w_+, & x \ge 2u_+t, \end{cases} \tag{6.24}$$

i.e., the third equality in (6.7) holds. □

7. Solutions of the Riemann problem (1.12), (1.16)

7.1. Propagation of δ'-shock wave in system (1.12). Now we prove that the triple of distributions (5.15) constructed by Theorem 5.1 is a δ'-shock wave type solution of the Cauchy problem (1.12), (1.16) for $u_+ \le u_-$.

THEOREM 7.1. *Let $u_+ \le u_-$. Then for $t \in [0, \infty)$, the Cauchy problem* (1.12), (1.16) *has a unique generalized δ'-shock wave type solution* (1.18) (*see* (5.15))

$$\begin{aligned} u(x,t) &= u_+ + [u]H(-x+\phi(t)), \\ v(x,t) &= v_+ + [v]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \\ w(x,t) &= w_+ + [w]H(-x+\phi(t)) + g(t)\delta(-x+\phi(t)) \\ &\qquad + h(t)\delta'(-x+\phi(t)), \end{aligned}$$

which satisfies the integral identities (3.1)*:*

$$
\begin{aligned}
\int_0^\infty \int \Big(u(x,t)\varphi_t + u^2(x,t)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty \int \Big(\widehat{v}(x,t)\varphi_t + 2u(x,t)\widehat{v}(x,t)\varphi_x\Big)\,dx\,dt&\\
+ \int_\Gamma e(t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl + \int \widehat{v}^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty \int \Big(\widehat{w}(x,t)\varphi_t + 2\big(\widehat{v}^2(x,t) + u(x,t)\widehat{w}(x,t)\big)\varphi_x\Big)\,dx\,dt&\\
+ \int_\Gamma g(t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl + \int_\Gamma h(x,t)\frac{\partial\varphi_x(x,t)}{\partial \mathbf{l}}\,dl + \int_\Gamma \frac{\frac{\partial e^2(x,t)}{\partial \mathbf{l}}}{[u]}\varphi_x(x,t)\,dl&\\
+ \int \widehat{w}^0(x)\varphi(x,0)\,dx &= 0,
\end{aligned}
\tag{7.1}
$$

for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\ \infty))$*, where functions* $e(t)$*,* $g(t)$*,* $h(t)$ *are given by* (5.16)*. Here* $\Gamma = \{(x,t) : x = \phi(t) = ct,\ t \geq 0\}$*,* $\widehat{v}(x,t) = v_+ + [v]H(-x+\phi(t))$*,* $\widehat{w}(x,t) = w_+ + [w]H(-x+\phi(t))$*, and (see* (2.3)*)*

$$
\begin{aligned}
\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl &= \int_0^\infty e(t)\frac{d\varphi(\phi(t),t)}{dt}\,dt,\\
\int_\Gamma g(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl &= \int_0^\infty g(t)\frac{d\varphi(\phi(t),t)}{dt}\,dt,\\
\int_\Gamma h(x,t)\frac{\partial\varphi_x(x,t)}{\partial \mathbf{l}}\,dl &= \int_0^\infty h(t)\frac{d\varphi_x(\phi(t),t)}{dt}\,dt,\\
\int_\Gamma \frac{\frac{\partial e^2(x,t)}{\partial \mathbf{l}} - h(x,t)\frac{\partial[u(x,t)]}{\partial \mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\,dl &= \int_0^\infty \frac{\frac{de^2(t)}{dt}}{[u]}\varphi_x(\phi(t),t)\,dt.
\end{aligned}
$$

Moreover, for this solution the admissibility condition (1.15) *holds.*

Proof. Let $\Omega \subset \mathbb{R}\times[0,\infty)$ be some region and suppose that the curve $\Gamma = \{(x,t) : x = \phi(t) = ct,\ t \geq 0\}$ cuts it into a left- and right-hand parts $\Omega_\pm = \{(x,t) : \pm(x-ct) > 0\}$. Let $\mathbf{n} = (\nu_1,\nu_2) = \frac{(1,-\dot\phi(t))}{\sqrt{1+(\dot\phi(t))^2}} = \frac{(1,-c)}{\sqrt{1+c^2}}$ be the unit normal to the curve Γ pointing from Ω_- into Ω_+, and $\mathbf{l} = (-\nu_2,\nu_1) = \frac{(c,1)}{\sqrt{1+c^2}}$ be the tangential vector to Γ (see (2.2)).

Choosing a test function $\varphi(x,t)$ with support in Ω, we deduce that the left-hand side of the first relation in (7.1) transforms to the form

$$
\begin{aligned}
&\int_0^\infty \int \Big(u(x,t)\varphi_t + u^2(x,t)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx\\
&= \int\int_{\Omega_-} \big(u_-\varphi_t + u_-^2\varphi_x\big)\,dx\,dt + \int\int_{\Omega_+} \big(u_+\varphi_t + u_+^2\varphi_x\big)\,dx\,dt
\end{aligned}
$$

$$+\int_{-\infty}^{0} u^0(x)\varphi(x,0)\,dx+\int_{0}^{\infty} u^0(x)\varphi(x,0)\,dx. \tag{7.2}$$

Next, integrating by parts, taking into account that $\frac{dx}{dt}=-\frac{\nu_2}{\nu_1}=-c$ and $\nu_1 dl=dt$, we obtain

$$\begin{aligned}
&\int\int_{\Omega_\pm}\big(u_\pm\varphi_t+u_\pm^2\varphi_x\big)\,dx\,dt\\
&\qquad=\mp\int_\Gamma\big(\nu_2u_\pm+\nu_1u_\pm^2\big)\varphi\,dl\mp\int_0^{\pm\infty}u^0(x)\varphi(x,0)\,dx\\
&\qquad=\mp\int_0^{\infty}\big(-cu_\pm+u_\pm^2\big)\varphi(ct,t)\,dt\mp\int_0^{\pm\infty}u^0(x)\varphi(x,0)\,dx. \qquad (7.3)
\end{aligned}$$

Since according to the first equation in (5.16) $\dot\phi(t)=c=\frac{[u^2]}{[u]}$, relations (7.2), (7.3) imply

$$\begin{aligned}
&\int_0^\infty\int\Big(u(x,t)\varphi_t+u^2(x,t)\varphi_x\Big)\,dx\,dt+\int u^0(x)\varphi(x,0)\,dx\\
&\qquad=\int_0^\infty\big(-c[u]+[u^2]\big)\varphi(ct,t)\,dt=0. \qquad (7.4)
\end{aligned}$$

Thus the first identity in (7.1) holds.

Applying the above calculations to the left-hand side of the second relation in (7.1), we obtain

$$\begin{aligned}
&\int_0^\infty\int\Big(\widehat v(x,t)\varphi_t+2u(x,t)\widehat v(x,t)\varphi_x\Big)\,dx\,dt+\int\widehat v^0(x)\varphi(x,0)\,dx\\
&\qquad=\int_\Gamma\big(\nu_2v_-+\nu_12u_-v_-\big)\varphi\,dl-\int_\Gamma\big(\nu_2v_++\nu_12u_+v_+\big)\varphi\,dl\\
&\qquad=\int_\Gamma\big(\nu_2[v]+\nu_12[uv]\big)\varphi\,dl=\int_0^\infty\big(-c[v]+2[uv]\big)\varphi(ct,t)\,dt. \qquad (7.5)
\end{aligned}$$

Since by integration by parts we have

$$\int_0^\infty t\frac{d\varphi(ct,t)}{dt}\,dt=-\int_0^\infty\varphi(ct,t)\,dt,$$

in view of the second equation in (5.16), and (2.3), we deduce that

$$\begin{aligned}
&\int_0^\infty\int\Big(\widehat v(x,t)\varphi_t+2u(x,t)\widehat v(x,t)\varphi_x\Big)\,dx\,dt+\int\widehat v^0(x)\varphi(x,0)\,dx\\
&=\int_0^\infty\big(-c[v]+2[uv]\big)\varphi(ct,t)\,dt=-\int_0^\infty\big(-c[v]+2[uv]\big)t\frac{d\varphi(ct,t)}{dt}\,dt\\
&\qquad=-\int_0^\infty e(t)\frac{d\varphi(ct,t)}{dt}\,dt=-\int_\Gamma e(t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl.
\end{aligned}$$

By substituting the last relation into the left-hand side of the second relation in (7.1) we see that the second identity in (7.1) holds.

Now, applying the above calculations to the left-hand side of the third relation in (7.1), we obtain

$$\int_0^\infty \int \Big(\widehat{w}(x,t)\varphi_t + 2\big(\widehat{v}^2(x,t) + u(x,t)\widehat{w}(x,t)\big)\varphi_x\Big)\,dx\,dt$$

$$+ \int \widehat{w}^0(x)\varphi(x,0)\,dx = -\int_0^\infty g(t)\frac{d\varphi(ct,t)}{dt}\,dt = -\int_\Gamma g(t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl,$$

where according to (5.16), $g(t) = \big(2[v^2+uw] - [w]\frac{[u^2]}{[u]}\big)t$. Thus,

$$\int_0^\infty \int \Big(\widehat{w}(x,t)\varphi_t + 2\big(\widehat{v}^2(x,t) + u(x,t)\widehat{w}(x,t)\big)\varphi_x\Big)\,dx\,dt$$

$$+ \int \widehat{w}^0(x)\varphi(x,0)\,dx + \int_\Gamma g(t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl = 0. \tag{7.6}$$

According to (5.16), (5.17), $e(t) = [u](v_- + v_+)t$, $h(t) = \frac{e^2(t)}{[u]} = [u](v_- + v_+)^2t^2$. Consequently, taking into account that $[u]$ is a constant, and integrating by parts, we have

$$\int_0^\infty h(t)\frac{d\varphi_x(ct,t)}{dt}\,dt = \int_0^\infty [u](v_- + v_+)^2t^2\frac{d\varphi_x(ct,t)}{dt}\,dt$$

$$= -\int_0^\infty 2[u](v_- + v_+)^2t\varphi_x(ct,t)\,dt = -\int_0^\infty \frac{\frac{de^2(t)}{dt}}{[u]}\varphi_x(ct,t)\,dt,$$

i.e., in view of (2.3),

$$\int_\Gamma h(x,t)\frac{\partial\varphi_x(x,t)}{\partial \mathbf{l}}\,dl + \int_\Gamma \frac{\frac{\partial e^2(x,t)}{\partial \mathbf{l}}}{[u]}\varphi_x(x,t)\,dl = 0.$$

By summing (7.6) and the last relation, we deduce that the third identity in (7.1) holds.

The proof is complete. □

In view of Remark 1.1, a solution of the Cauchy problem (1.19), (1.16) formulated in [**18**], can be obtained from solution (5.15) of the Cauchy problem (1.12), (1.16) by the change of variables $u \to \frac{1}{2}u$, $v \to \frac{1}{2}v$, $w \to w$.

Note that system (5.16) which determines the trajectory $x = \phi(t)$ of a δ'-shock wave and the coefficients $e(t)$, $g(t)$, $h(t)$ of the singularities is a solution of the system the Rankine–Hugoniot conditions for δ'-shock (3.2)–(3.5).

If $u_+ \le u_-$, it follows from Theorems 5.1, 7.1 that $c = u_+ + u_- = \dot{\phi}(t)$ and $x = \phi(t) = ct$ are the velocity of motion and the trajectory of a δ'-shock wave, respectively. Moreover, Theorems 5.1, 7.1 imply the following statements.

COROLLARY 7.1. *Let $u_+ < u_-$. The Cauchy problem* (1.12), (1.16) *has*
(a.1) *a classical shock-solution* (1.18) *of the form*

$$\begin{array}{rcl} u(x,t) &=& u_+ + [u]H(-x+\phi(t)), \\ v(x,t) &=& v_+ + [v]H(-x+\phi(t)), \\ w(x,t) &=& w_+ + [w]H(-x+\phi(t)), \end{array} \tag{7.7}$$

if and only if $v_- + v_+ = 0$ and $w_- + w_+ = 0$;
(a.2) *a δ-shock solution* (1.18) *of the form*

$$\begin{array}{rcl} u(x,t) &=& u_+ + [u]H(-x+\phi(t)), \\ v(x,t) &=& v_+ + [v]H(-x+\phi(t)), \\ w(x,t) &=& w_+ + [w]H(-x+\phi(t)) + [u](w_- + w_+)t\delta(-x+\phi(t)). \end{array} \tag{7.8}$$

if $v_- + v_+ = 0$ and $w_- + w_+ \neq 0$, or

$$\begin{array}{rcl} u(x,t) &=& u_0, \\ v(x,t) &=& v_+ + [v]H(-x+\phi_0(t)), \\ w(x,t) &=& w_+ + [w]H(-x+\phi_0(t)) + 2[v^2]t\delta(-x+\phi_0(t)), \end{array} \tag{7.9}$$

if $u_+ = u_- = u_0$, where $\phi_0(t) = 2u_0 t$;
(a.3) *a δ'-shock wave type solution* (1.18) *only if $v_- + v_+ \neq 0$, $w_- + w_+ \neq 0$.*

PROOF. Let $u_+ < u_-$. In this case, according to (1.18) (5.15), and (5.16), the Cauchy problem (1.12), (1.16) has a classical shock-solution (7.7) if and only if $v_- + v_+ = 0$, $w_- + w_+ = 0$.

If $v_- + v_+ = 0$, $w_- + w_+ \neq 0$, in view of (5.16), the Cauchy problem has a δ-shock wave type solution (1.18) of the form (7.8).

According to (5.16), the Cauchy problem (1.12), (1.16) has a δ'-shock wave type solution (1.18) (see (5.15)) only if $v_- + v_+ \neq 0$, $w_- + w_+ \neq 0$.

Let $u_+ = u_- = u_0$. In this case the Cauchy problem (1.12), (1.16) has a δ-shock wave type solution (1.18) of the form (7.9)), where $\phi_0(t) = 2u_0 t$. Here $x = \phi_0(t) = 2u_0 t$ is a characteristic line of the first equation $u_t + \left(u^2\right)_x = 0$ in system (1.12) issued from $(0, 0)$. □

COROLLARY 7.2. *Let $u_+ \leq u_-$. Then for $t \in [0, \infty)$, the Cauchy problem* (1.20), (1.6) *has a unique generalized δ-shock wave type solution*

$$\begin{array}{rcl} u(x,t) &=& u_+ + [u]H(-x+\phi(t)), \\ v(x,t) &=& v_+ + [v]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{array} \tag{7.10}$$

which satisfies the integral identities (2.1), *i.e., the first two integral identities in* (7.1), *where $\phi(t)$, $e(t)$ are given by* (5.16).

The correct δ-shock wave type solution of the Cauchy problem (1.5), (1.6) (*see* [**17**] *and Subsec.* 1.2) *is reduced to solution* (7.10) *by the transform $u_\pm \to \frac{1}{2}u_\pm$, $v_\pm \to v_\pm$.*

7.2. Vacuum states in solution of the Riemann problem (1.12), (1.16). Now we consider the case $u_+ > u_-$. Substituting the triple of distributions (6.7) constructed by Theorem 6.1 into the left-hand side of (7.1), it is easy to prove the following assertion.

THEOREM 7.2. *Let $u_+ > u_-$. Then for $t \in [0, \infty)$ the triple of distributions* (1.21)

$$\big(u(x,t), v(x,t), w(x,t)\big) = \begin{cases} (u_-, v_-, w_-), & x \le 2u_-t, \\ \big(\frac{x}{2t}, 0, 0\big), & 2u_-t < x < 2u_+t, \\ (u_+, v_+, w_+), & x \ge 2u_+t, \end{cases}$$

is a unique generalized solution of the Cauchy problem (1.12), (1.16), *which satisfies the integral identities* (7.1), *where $\widehat{v}(x,t) = v(x,t)$, $\widehat{w}(x,t) = w(x,t)$, and $e(t) \equiv 0$, $g(t) \equiv 0$, $h(t) \equiv 0$.*

Here the first component u of the solution (1.21) is a rarefaction wave, and the second component v and the third component w contain the intermediate *vacuum states* $v = 0$ and $w = 0$.

8. Algebraic aspect of singular solutions

As mentioned in Introduction, the problem of defining a δ'-shock wave type solution of the Cauchy problem is connected with the construction of *singular superpositions* (*products*) of distributions.

It seems natural to define a product of the Heaviside function and delta function as the weak limit of the product of their regularizations. For example, choosing regularizations of the delta function and the Heaviside function in the form

$$\delta(x,\varepsilon) = \frac{1}{\varepsilon}\omega_\delta\Big(\frac{x}{\varepsilon}\Big), \qquad H(x,\varepsilon) = \int_{-\infty}^{\frac{x}{\varepsilon}} \omega(\eta)\, d\eta,$$

respectively, where ω, ω_δ are the mollifiers, it is easy to calculate that in the weak sense

$$\overbrace{H(x)\delta(x)} \overset{def}{=} \lim_{\varepsilon\to+0} H(x,\varepsilon)\delta(x,\varepsilon) = B\delta(x), \tag{8.1}$$

where $B = \int \omega_0(\eta)\omega_\delta(\eta)\, d\eta$. Product (8.1) defined in this way *depends* on the mollifiers ω, ω_δ, i.e., on the regularizations of distributions $H(x)$, $\delta(x)$.

In [**25**], in a similar way, using regularizations $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$, $w(x,t,\varepsilon)$ of distributions $u(x,t)$, $v(x,t)$, $w(x,t)$ given by (5.15), *singular superpositions*

$$\overbrace{u^2(x,t)}, \qquad \overbrace{2u(x,t)v(x,t)}, \qquad \overbrace{2\big(v^2(x,t) + u(x,t)w(x,t)\big)}$$

were constructed. As shown in [**25**], these *singular superpositions depend* on the regularizations of the Heaviside function, delta function, and its derivative. Moreover, the last superposition is *unbounded*. Nevertheless, according to [**25**], using instead of *arbitrary* regularizations of distributions $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$, $w(x,t,\varepsilon)$ the special

regularizations of distributions, namely, the *weak asymptotic solution* of the Cauchy problem, we shall construct *unique "right" singular superpositions*

$$u^2(x,t), \qquad 2u(x,t)v(x,t), \qquad 2\big(v^2(x,t)+u(x,t)w(x,t)\big),$$

which are the Schwartz distributions.

Now we prove that these *unique "right" singular superpositions* can be constructed by using a solution $(u_\varepsilon(x,t), v_\varepsilon(x,t), w_\varepsilon(x,t))$ of the parabolic problem (1.17), (1.16).

LEMMA 8.1. *Let $u_+ \le u_-$. Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be a solution of the parabolic problem* (1.17), (1.16) *and (u,v,w) be a triple of limiting distributions* (5.15)*, which is a δ'-shock type solution of the Cauchy problem* (1.12), (1.16)*. Then for $t \in [0,\infty)$ we can define explicit formulas for the "right" singular superpositions:*

$$u^2(x,t) \overset{def}{=} \lim_{\varepsilon\to+0} u_\varepsilon^2(x,t) = u_+^2 + [u^2]H(-x+\phi(t)), \tag{8.2}$$

$$2u(x,t)v(x,t) \overset{def}{=} \lim_{\varepsilon\to+0} 2u_\varepsilon(x,t)v_\varepsilon(x,t)$$

$$= 2u_+v_+ + 2[uv]H(-x+\phi(t)) + e(t)\dot\phi(t)\delta(-x+\phi(t)), \tag{8.3}$$

$$2\big(v^2(x,t)+u(x,t)w(x,t)\big) \overset{def}{=} \lim_{\varepsilon\to+0} 2\big(v_\varepsilon^2(x,t)+u_\varepsilon(x,t)w_\varepsilon(x,t)\big)$$

$$= 2(v_+^2+u_+w_+) + 2[v^2+uw]H(-x+\phi(t))$$

$$+\left(\frac{1}{[u]}\frac{de^2(t)}{dt} + g(t)\dot\phi(t)\right)\delta(-x+\phi(t)) + h(t)\dot\phi(t)\delta'(-x+\phi(t)), \tag{8.4}$$

where functions $e(t)$, $g(t)$, $h(t)$ are given by (5.16)*, and the limits are understood in the weak sense. Here $\frac{1}{[u]}\frac{de^2(t)}{dt} + g(t)\dot\phi(t) = 4[uv](v_- + v_+) + [u^2](w_- + w_+)$.*

PROOF. According to (4.18), we have

$$u_\varepsilon^2(x,t) = u_+^2 + [u^2]\frac{a_-^\varepsilon}{a_-^\varepsilon + a_+^\varepsilon} - (u_- + u_+)^2\frac{a_-^\varepsilon a_+^\varepsilon}{(a_-^\varepsilon + a_+^\varepsilon)^2}. \tag{8.5}$$

In view of (5.9) (5.10), formula (8.5) implies (8.2).

Next, using the direct representation (4.19)–(4.22) of $v_\varepsilon(x,t)$, $w_\varepsilon(x,t)$, we can prove that relations (8.3)–(8.4) hold. However, we shall use another approach. According to Theorem 5.1, if $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ is a solution of the parabolic problem (1.17), (1.16) then

$$\begin{aligned}
\lim_{\varepsilon\to+0}\langle u_{\varepsilon\,t},\varphi\rangle + \lim_{\varepsilon\to+0}\big\langle \big(u_\varepsilon^2\big)_x,\varphi\big\rangle &= \lim_{\varepsilon\to+0}\varepsilon\big\langle u_{\varepsilon\,xx},\varphi\big\rangle &= 0,\\
\lim_{\varepsilon\to+0}\langle v_{\varepsilon\,t},\varphi\rangle + \lim_{\varepsilon\to+0}\big\langle 2\big(u_\varepsilon v_\varepsilon\big)_x,\varphi\big\rangle &= \lim_{\varepsilon\to+0}\varepsilon\langle v_{\varepsilon\,xx},\varphi\rangle &= 0,\\
\lim_{\varepsilon\to+0}\big\langle w_{\varepsilon\,t},\varphi\big\rangle + \lim_{\varepsilon\to+0}\big\langle 2\big(v_\varepsilon^2+u_\varepsilon w_\varepsilon\big)_x,\varphi\big\rangle &= \lim_{\varepsilon\to+0}\varepsilon\langle w_{\varepsilon\,xx},\varphi\rangle &= 0,
\end{aligned} \tag{8.6}$$

for all $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\infty))$. By definition, the "right" singular superpositions are given as the weak limits

$$\begin{aligned} u^2(x,t) &\overset{def}{=} \lim_{\varepsilon\to+0} u_\varepsilon^2(x,t),\\ 2u(x,t)v(x,t) &\overset{def}{=} \lim_{\varepsilon\to+0} 2u_\varepsilon(x,t)v_\varepsilon(x,t),\\ 2\big(v^2(x,t)+u(x,t)w(x,t)\big) &\overset{def}{=} \lim_{\varepsilon\to+0} 2\big(v_\varepsilon^2(x,t)+u_\varepsilon(x,t)w_\varepsilon(x,t)\big), \end{aligned}$$

and, consequently, (8.6) implies that

$$\begin{aligned} \big\langle \big(u^2(x,t)\big)_x,\varphi\big\rangle &= \lim_{\varepsilon\to+0}\big\langle \big(u_\varepsilon^2\big)_x,\varphi\big\rangle = -\langle u_t,\varphi\rangle,\\ \big\langle \big(2u(x,t)v(x,t)\big)_x,\varphi\big\rangle &= \lim_{\varepsilon\to+0}\big\langle 2\big(u_\varepsilon v_\varepsilon\big)_x,\varphi\big\rangle = -\langle v_t,\varphi\rangle,\\ \big\langle 2\big(v^2(x,t)+u(x,t)w(x,t)\big)_x,\varphi\big\rangle &= \lim_{\varepsilon\to+0}\big\langle 2\big(v_\varepsilon^2+u_\varepsilon w_\varepsilon\big)_x,\varphi\big\rangle\\ &\qquad = -\langle w_t,\varphi\rangle, \end{aligned} \tag{8.7}$$

for all $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\ \infty))$.

Using the first formula in (5.15) and the first relation in (8.7), we have in the weak sense

$$\begin{aligned} \big(2u(x,t)v(x,t)\big)_x &= -v_t = -\big(v_+ + [v]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t))\big)_t\\ &= -([v]\dot\phi(t)+\dot e(t))\delta(-x+\phi(t)) - e(t)\dot\phi(t)\delta'(-x+\phi(t)). \end{aligned}$$

According to (5.16), $\dot\phi(t)=(u_-+u_+)$, $\dot e(t)=[u](v_-+v_+)$, and, consequently, $[v]\dot\phi(t)+\dot e(t)=2[uv]$. Thus

$$\big(2u(x,t)v(x,t)\big)_x = -2[uv]\delta(-x+\phi(t)) - e(t)\dot\phi(t)\delta'(-x+\phi(t)).$$

By integrating the last relation with respect to x, we obtain the relation

$$2u(x,t)v(x,t) = 2[uv]H(-x+\phi(t)) + e(t)\dot\phi(t)\delta(-x+\phi(t)) + C, \tag{8.8}$$

where C is a constant. Since $\lim_{\varepsilon\to+0} 2u_\varepsilon(x,t)v_\varepsilon(x,t) = 2u_+v_+$ for $x>\phi(t)$, we conclude that (8.8) implies $C=2u_+v_+$. Relation (8.3) is thus proved.

Using the second formula in (5.15) and the second relation in (8.7), we have in the weak sense

$$\begin{aligned} &2\big(v^2(x,t)+u(x,t)w(x,t)\big)_x = -w_t\\ &\quad = -\big(w_+ + [w]H(-x+\phi(t)) + g(t)\delta(-x+\phi(t)) + h(t)\delta'(-x+\phi(t))\big)_t\\ &\quad = -\big([w]\dot\phi(t)+\dot g(t)\big)\delta(-x+\phi(t))\\ &\qquad -\big(g(t)\dot\phi(t)+\dot h(t)\big)\delta'(-x+\phi(t)) - h(t)\dot\phi(t)\delta''(-x+\phi(t)). \end{aligned} \tag{8.9}$$

According to (5.16), $\dot\phi(t)=(u_-+u_+)$, $g(t)=\big(2[v](v_-+v_+)+[u](w_-+w_+)\big)t$, $h(t)=[u](v_-+v_+)^2t^2$, we have $\dot g(t)+[w]\dot\phi(t)=2[v^2+uw]$ and $g(t)\dot\phi(t)+\dot h(t)=\frac{1}{[u]}\frac{de^2(t)}{dt}+g(t)\dot\phi(t)=4[uv](v_-+v_+)+[u^2](w_-+w_+)$. Integrating relation (8.9) with respect to x, we obtain

$$2\big(v^2(x,t)+u(x,t)w(x,t)\big) = 2[v^2+uw]H(-x+\phi(t))$$

$$+\big(4[uv](v_-+v_+)+[u^2](w_-+w_+)\big)\delta(-x+\phi(t))+h(t)\dot\phi(t)\delta'(-x+\phi(t))+C, \tag{8.10}$$

where C is a constant. If $x > \phi(t)$ then $\lim_{\varepsilon\to+0} 2\big(v_\varepsilon^2(x,t)+u_\varepsilon(x,t)w_\varepsilon(x,t)\big) = 2(v_+^2+u_+w_+)$, and, consequently, $C = 2(v_+^2+u_+w_+)$. Thus relation (8.4) holds. □

Now using Theorem 6.1 and formulas (8.7), it is easy to prove the following assertion.

LEMMA 8.2. *Let $u_+ < u_-$. Let $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ be a solution of the parabolic problem* (1.17), (1.16) *and (u, v, w) be a triple of limiting distributions* (6.7), *which is a solution of the Cauchy problem* (1.12), (1.16). *Then for $t \in [0, \infty)$ we can define explicit formulas for the "right" singular superpositions:*

$$u^2(x,t) \overset{def}{=} \lim_{\varepsilon\to+0} u_\varepsilon^2(x,t) = \begin{cases} u_-^2, & x \le 2u_-t, \\ \left(\frac{x}{2t}\right)^2, & 2u_-t < x < 2u_+t, \\ u_+^2, & x \ge 2u_+t, \end{cases} \tag{8.11}$$

$$2u(x,t)v(x,t) \overset{def}{=} \lim_{\varepsilon\to+0} 2u_\varepsilon(x,t)v_\varepsilon(x,t)$$

$$= \begin{cases} 2u_-v_-, & x \le 2u_-t, \\ 0, & 2u_-t < x < 2u_+t, \\ 2u_+u_+, & x \ge 2u_+t, \end{cases} \tag{8.12}$$

$$2\big(v^2(x,t)+u(x,t)w(x,t)\big) \overset{def}{=} \lim_{\varepsilon\to+0} 2\big(v_\varepsilon^2(x,t)+u_\varepsilon(x,t)w_\varepsilon(x,t)\big)$$

$$= \begin{cases} 2(v_-^2+u_-w_-), & x \le 2u_-t, \\ 0, & 2u_-t < x < 2u_+t, \\ 2(v_+^2+u_+w_+), & x \ge 2u_+t. \end{cases} \tag{8.13}$$

Thus one can see that the generalized solution (u, v, w) of the Cauchy problem (1.12), (1.16), constructed by Theorem 5.1 and Theorem 6.1, *generates algebraic relations* (8.2)–(8.4) and (8.11)–(8.13), respectively, between distributions u, v, w.

Note that Theorem 7.1 and Theorem 7.2 could be proved by direct substituting the *"right" singular superpositions of distributions* (8.2)–(8.4) and (8.11)–(8.13), respectively, into system (1.12).

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Department of Mathematics, St.-Petersburg State Architecture and Civil Engineering University, 2 Krasnoarmeiskaya 4, 190005, St. Petersburg, Russia.
E-mail address: shelkv@vs1567.spb.edu