On the algebraic aspect of singular solutions to conservation laws systems [1]

A. Yu. Khrennikov* and V. M. Shelkovich†

*International Center for Mathematical Modeling in Physics, Engineering and Cognitive science MSI, Växjö University, S-35195, Sweden
†Department of Mathematics, St.-Petersburg State Architecture and Civil Engineering University, 2 Krasnoarmeiskaya 4, 190005, St. Petersburg, Russia

Abstract. The algebraic aspect of singular solutions (δ-shocks) to systems of conservation laws is studied. Namely, we show that singular solution of the Cauchy problem generates algebraic relations between distributional components of a singular solution ("right" singular superpositions of distributions). These "right" singular superpositions of distributions are constructed.

Keywords: Systems of conservation laws, δ-shocks, δ'-shocks, singular superposition of distributions, product of distributions
PACS: 02.30.Jr

1. SINGULAR SOLUTIONS TO SYSTEMS OF CONSERVATION LAWS

Let us consider the Cauchy problem for the hyperbolic system of conservation laws

$$\begin{cases} U_t + \big(F(U)\big)_x = 0, & \text{in } \mathbb{R}\times(0,\,\infty), \\ U = U^0, & \text{in } \mathbb{R}\times\{t=0\}, \end{cases} \tag{1}$$

where $F:\mathbb{R}^m\to\mathbb{R}^m$ and $U^0:\mathbb{R}\to\mathbb{R}^m$ are given smooth vector-functions, and $U = U(x,t) = (u_1(x,t),\dots,u_m(x,t))$ is the unknown function, $x\in\mathbb{R}$, $t\ge 0$.

As is well known, even in the case of smooth (and, certainly, in the case of discontinuous) initial data $U^0(x)$, we cannot in general find a smooth solution of (1). In this case, it is said that $U\in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}^m\big)$ is a *generalized solution* of the Cauchy problem (1) if the integral identities

$$\int_0^\infty\!\int \Big(U\cdot\widetilde{\varphi}_t + F(U)\cdot\widetilde{\varphi}_x\Big)\,dx\,dt + \int U^0(x)\cdot\widetilde{\varphi}(x,0)\,dx = 0 \tag{2}$$

hold for all compactly supported test vector-functions $\widetilde{\varphi}:\mathbb{R}\times[0,\infty)\to\mathbb{R}^m$, where $\cdot$ is the scalar product of vectors, $\int f(x)\,dx$ denotes the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$.

[1] This paper was supported in part by the grant of The Swedish Royal Academy of Sciences on collaboration with scientists of former Soviet Union and the EU-Network "Quantum Probability and Applications". The second author (V. S.) was also supported in part by DFG Project 436 RUS 113/823/0-1 and Grant 05-01-00912 of Russian Foundation for Basic Research.

Consider two particular cases of the above system of conservation laws:

$$L_1[u,v] = u_t + \big(F(u,v)\big)_x = 0, \quad L_2[u,v] = v_t + \big(G(u,v)\big)_x = 0, \tag{3}$$

and

$$v_t + \big(G(u,v)\big)_x = 0, \qquad (uv)_t + \big(H(u,v)\big)_x = 0, \tag{4}$$

where $F(u,v)$, $G(u,v)$, $H(u,v)$ are smooth functions, *linear* with respect to v; $u = u(x,t)$, $v = v(x,t) \in \mathbb{R}$; $x \in \mathbb{R}$. The well-known zero-pressure gas dynamics system is a particular case of system (4), where $G(u,v) = uv$, $H(u,v) = u^2v$, and $v(x,t) \geq 0$ is density, and $u(x,t)$ is velocity.

In numerous papers (e.g., see [1], [4]– [7], [13]– [18] and the reference therein) it is shown that for some cases of hyperbolic systems (3) and (4) "nonclassical" situations may occur, when the Riemann problem does not possess a weak L^∞-solution except for some particular initial data. Here the *linear* component v of the solution may contain Dirac measures and must be sought in the space of measures, while the nonlinear component u of the solution has bounded variation. In order to solve the Cauchy problems in these nonclassical situations, it is necessary to introduce new singularities called δ-shocks, which are solutions of hyperbolic systems (3) or (4), whose *linear* components have the form $v(x,t) = V(x,t) + e(x,t)\delta(\Gamma)$, Γ is a graph in the upper half-plane $\{(x,t) : x \in \mathbb{R}, t \geq 0\}$, $V \in L^\infty$, $e \in C(\Gamma)$, and the *nonlinear* component $u \in L^\infty\big(\mathbb{R} \times (0,\infty); \mathbb{R}\big)$.

Unfortunately, by using Definition (2), δ-shocks *cannot be defined*. Indeed, as can be seen from (3), (4), if integrating by parts we transfer the derivatives onto a test function φ, under the integral sign there still remain terms *undefined in the distributional sense*, since the component v may contain Dirac measures.

Recently, the theory of δ-shock type solutions for systems of conservation laws has attracted intensive attention and several approaches to solving δ-shock problems are known (see the above cited papers and the references therein). One of them is the *weak asymptotics method* which was developed in [3]– [5], [14]– [16]. In [4], [5], in the framework of the *weak asymptotics method definitions of a δ-shock wave type solution* by integral identities were introduced for two classes of hyperbolic systems of conservation laws (3), (4) (for system (3) see Definition 1 below). These definitions give *natural* generalizations of the classical definition of the weak L^∞-solutions (2) relevant to the structure of δ-shocks.

Moreover, in [13], in the framework of the *weak asymptotics method*, for the system of conservation laws

$$u_t + \big(f(u)\big)_x = 0, \quad v_t + \big(f'(u)v\big)_x = 0, \quad w_t + \big(f''(u)v^2 + f'(u)w\big)_x = 0, \tag{5}$$

a definition of a new type of singular solutions to systems of conservation laws, namely, δ'-shock wave, was introduced. Roughly speaking, a δ'-shock wave type solution is a such a solution of system (12) such that its second component v may contain Dirac measures, and the third component w may contain a linear combination of Dirac measures and their derivatives, while the first component u of the solution has bounded variation. It is clear that by using Definition (2), δ'-shocks *cannot be also defined*.

In fact, to introduce *δ-shocks* and *δ'-shocks*, we must devise some way to define a *singular superposition* of distributions (for example, a *product of the Heaviside function* *and the delta function*). This problem in connection with the constructing of *singular solutions* of systems of conservation laws in the framework of the *weak asymptotics method* was discussed in [3], [4], [8]– [10], [13], [15]– [17]. For some cases this problem can be solved by using *nonconservative product* [11]: $g(u)\frac{du}{dx}$, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded Borel function, and $u : (a,b) \to \mathbb{R}^n$ is a discontinuous function of bounded variation. There is another approach associated with the name of J. Colombeau (see [2], [12] and the reference therein).

As is well known, in the general case, the product of distributions is either not a Schwartz distribution or it is a Schwartz distribution which is not uniquely defined. Nevertheless, in this paper, we show that a *singular solution* of the Cauchy problem *generates algebraic relations* between distributional components of a singular solution ("right" singular superpositions of distributions). Our *"right" singular superpositions* (contextual singular superpositions) are *well defined and unique* Schwartz distributions. To illustrate our results we consider δ-shock type solutions for hyperbolic system of conservation laws (3) with the simplest initial data

$$\big(u^0(x), v^0(x)\big) = \begin{cases} (u_-, v_-), & x < 0, \\ (u_+, v_+), & x > 0, \end{cases} \tag{6}$$

where $u_\pm$, $v_\pm$ are given constants. We also consider two important particular cases of system (3). Namely, the system

$$u_t + \big(f(u)\big)_x = 0, \quad v_t + \big(g(u)v\big)_x = 0, \tag{7}$$

(here $F(u,v) = f(u)$, $G(u,v) = vg(u)$) and the well known Keyfitz–Kranzer system

$$u_t + (u^2 - v)_x = 0, \qquad v_t + \Big(\frac{1}{3}u^3 - u\Big)_x = 0 \tag{8}$$

(here $F(u,v) = u^2 - v$, $G(u,v) = \frac{1}{3}u^3 - u$).

In this paper explicit formulas (14), (15), (22), (23), (25), (26) for the *contextual singular superpositions*, generated by solutions of the corresponding Cauchy problems are constructed. According to Theorem 2, in fact, the contextual singular superpositions are determined by the structure of linear terms u_t, v_t and the Rankine–Hugoniot conditions.

It remains to note that, since in the systems (8) there is no terms of the type of $g(u)\frac{du}{dx}$, it is *impossible* to construct a *δ-shock wave type solution* for it by using the *nonconservative product* [11].

In this way explicit formulas for the "right" singular superpositions of distributions generated by δ'-shock type solution can be constructed (see [13], [17]).

2. δ-SHOCK GENERALIZED SOLUTION AND THE RANKINE–HUGONIOT CONDITIONS

2.1. Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a graph in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\} \in \mathbb{R}^2$ containing smooth arcs γ_i, $i \in I$, and I is a finite set. By I_0 we denote a subset of I such that an arc γ_k for $k \in I_0$ starts from the points of the x-axis. Denote by $\Gamma_0 = \{x_k^0 : k \in I_0\}$ the set of initial points of arcs γ_k, $k \in I_0$.

Consider δ*-shock wave type initial data* $(u^0(x), v^0(x))$, where

$$v^0(x) = V^0(x) + e^0\delta(\Gamma_0),$$

$u^0, V^0 \in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, $e^0\delta(\Gamma_0) = \sum_{k\in I_0} e_k^0\delta(x - x_k^0)$, e_k^0 are constants, $k \in I_0$.

Definition 1. ([4], [5]) A pair of distributions $(u(x,t), v(x,t))$ and a graph Γ, where $v(x,t)$ is represented in the form of the sum

$$v(x,t) = V(x,t) + e(x,t)\delta(\Gamma),$$

$u, V \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}\big)$, $e(x,t)\delta(\Gamma) = \sum_{i\in I} e_i(x,t)\delta(\gamma_i)$, $e_i(x,t) \in C(\Gamma)$, $i \in I$, is called a *generalized* δ*-shock wave type solution* of system (3) with the δ*-shock wave type initial data* $(u^0(x), v^0(x))$ if the integral identities

$$\begin{aligned}
\int_0^\infty \int \Big(u\varphi_t + F(u,V)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0, \\
\int_0^\infty \int \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl & \\
+ \int V^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) &= 0,
\end{aligned} \tag{9}$$

hold for all test functions $\varphi(x,t) \in \mathscr{D}(\mathbb{R}\times[0,\infty))$, where $\dfrac{\partial\varphi(x,t)}{\partial\mathbf{l}}$ is the tangential derivative on the graph Γ, $\int_{\gamma_i} \cdot\, dl$ is the line integral over the arc γ_i.

2.2. In [15], [16], within the framework of Definition 1, the Rankine–Hugoniot conditions for δ-shock were derived.

Theorem 1. ([15], [16]) *Let us assume that* $\Omega \subset \mathbb{R}\times(0,\ \infty)$ *is some region cut by a smooth curve* $\Gamma = \{(x,t) : x = \phi(t)\}$ *into a left- and right-hand parts* $\Omega_\pm = \{(x,t) : \pm(x-\phi(t)) > 0\}$, $(u(x,t), v(x,t))$ *and* Γ *is a generalized* δ*-shock wave type solution of system* (3)*, where* $u(x,t)$, $v(x,t)$ *are smooth in* $\Omega_\pm$*. Then the Rankine–Hugoniot conditions for* δ*-shocks*

$$\dot\phi(t) = \frac{[F(u,v)]}{[u]}\Big|_{x=\phi(t)}, \quad \dot e(t) = \Big([G(u,v)] - [v]\frac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)}, \tag{10}$$

hold along Γ*, where* $e(t) \stackrel{def}{=} e(\phi(t),t)$*. Here* $\big[a(u,v)\big] = a(u_-,v_-) - a(u_+,v_+)$ *is a jump in function* $a(u(x,t), v(x,t))$ *across the discontinuity curve* Γ.

2.3. Denote by $O_{\mathscr{D}'}(\varepsilon^\alpha)$ the collection of distributions $f(x,t,\varepsilon) \in \mathscr{D}'(\mathbb{R}_x)$ such that $\langle f(x,t,\varepsilon),\ \psi(x)\rangle = O(\varepsilon^\alpha)$, for any test function $\psi(x) \in \mathscr{D}(\mathbb{R}_x)$. Moreover, $\langle f(x,t,\varepsilon), \psi(x)\rangle$ is a continuous function in t, and the estimate $O(\varepsilon^\alpha)$ is understood in the standard sense and is uniform with respect to t. The relation $o_{\mathscr{D}'}(\varepsilon^\alpha)$ is understood in a corresponding way.

Definition 2. ([4], [5]) A pair of functions $\big(u(x,t,\varepsilon),v(x,t,\varepsilon)\big)$ smooth as $\varepsilon>0$ is called a *weak asymptotic solution* of system (3) with the initial data $(u^0(x),v^0(x))$ if

$$\begin{array}{rclcrcl} L_1[u(x,t,\varepsilon),v(x,t,\varepsilon)] & = & o_{\mathscr{D}'}(1), & & u(x,0,\varepsilon) & = & u^0(x)+o_{\mathscr{D}'}(1), \\ L_2[u(x,t,\varepsilon),v(x,t,\varepsilon)] & = & o_{\mathscr{D}'}(1), & & v(x,0,\varepsilon) & = & v^0(x)+o_{\mathscr{D}'}(1), \end{array} \tag{11}$$

as $\varepsilon\to+0$, where the first two estimates are uniform in t.

Roughly speaking, the *weak asymptotic solution* is a smooth (as $\varepsilon>0$) *approximate solution* of a system of conservation laws which satisfies this system up to $o_{\mathscr{D}'}(1)$, $\varepsilon\to+0$ (for details, see [3]– [5], [14]– [16]).

Let us recall that within the framework of the *weak asymptotics method*, we find a δ-*shock wave type solution* of the Cauchy problem as the weak limit of a *weak asymptotic solution*:

$$u(x,t)=\lim_{\varepsilon\to+0}u_\varepsilon(x,t),\quad v(x,t)=\lim_{\varepsilon\to+0}v_\varepsilon(x,t). \tag{12}$$

Constructing the *weak asymptotic solution* and multiplying the first two relations (11) by a test function $\varphi(x,t)\in\mathscr{D}(\mathbb{R}\times[0,\infty))$, integrating these relations by parts and then passing to the limit as $\varepsilon\to+0$, we will see that the pair of distributions (12) satisfy the integral identities (9).

3. δ-SHOCK SINGULAR SUPERPOSITIONS

Let us consider the Cauchy problem (3), (6). According to [4], [5], [14]– [16], δ-shock wave type solution of this Cauchy problem has the form

$$\begin{array}{rcl} u(x,t) & = & u_+ + [u]H(-x+\phi(t)), \\ v(x,t) & = & v_+ + [v]H(-x+\phi(t))+e(t)\delta(-x+\phi(t)), \end{array} \tag{13}$$

where $e(t)$, $\phi(t)$ are the desired functions, $x=\phi(t)$ is the discontinuity curve, $H(x)$ is the Heaviside function, $\delta(x)$ is the delta-function.

Theorem 2. *Let* (u,v) *be a* δ'*-shock type solution* (13) *and let* $(u_\varepsilon,v_\varepsilon)$ *be a weak asymptotic solution of the Cauchy problem* (3), (6)*. Then for* $t\in[0,\infty)$ *we can define explicit formulas for the "right" singular superpositions:*

$$F(u,v)\stackrel{def}{=}\lim_{\varepsilon\to+0}F(u_\varepsilon,v_\varepsilon)=F(u_+,v_+)+[F(u,v)]\big|_{x=\phi(t)}H(-x+\phi(t)), \tag{14}$$

$$\begin{aligned} G(u,v) &\stackrel{def}{=} \lim_{\varepsilon\to+0}G(u_\varepsilon,v_\varepsilon) \\ &= G(u_+,v_+)+[G(u,v)]\big|_{x=\phi(t)}H(-x+\phi(t))+e(t)\dot{\phi}(t)\delta(-x+\phi(t)), \end{aligned} \tag{15}$$

where functions $\phi(t)$, $e(t)$ *are given by* (10)*, and the limits are understood in the weak sense.*

Proof. Let $(u_\varepsilon(x,t),v_\varepsilon(x,t))$ be a *weak asymptotic solution* of the Cauchy problem (3), (6), i.e., in view of (11), the relation

$$u_{\varepsilon t}+\big(F(u_\varepsilon,v_\varepsilon)\big)_x=o_{\mathscr{D}'}(1), \qquad v_{\varepsilon t}+\big(G(u_\varepsilon,v_\varepsilon)\big)_x=o_{\mathscr{D}'}(1), \quad \varepsilon\to+0. \tag{16}$$

and relations (12) hold, where $(u(x,t),v(x,t))$ is a δ-shock wave type solution (13) of the Cauchy problem (3), (6).

By definition, the *"right" singular superpositions* are defined as the weak limits

$$F(u,v)\stackrel{def}{=}\lim_{\varepsilon\to+0}F(u_\varepsilon,v_\varepsilon), \quad G(u,v)\stackrel{def}{=}\lim_{\varepsilon\to+0}G(u_\varepsilon,v_\varepsilon), \tag{17}$$

where (u,v) is given by (13).

Next, according to (16), (12), we have

$$\begin{array}{rcl}\lim\limits_{\varepsilon\to+0}\langle u_{\varepsilon t},\varphi\rangle+\lim\limits_{\varepsilon\to+0}\Big\langle\big(F(u_\varepsilon,v_\varepsilon)\big)_x,\varphi\Big\rangle=\lim\limits_{\varepsilon\to+0}\big\langle o_{\mathscr{D}'}(1),\varphi\big\rangle & = & 0,\\ \lim\limits_{\varepsilon\to+0}\langle v_{\varepsilon t},\varphi\rangle+\lim\limits_{\varepsilon\to+0}\Big\langle\big(G(u_\varepsilon,v_\varepsilon)\big)_x,\varphi\Big\rangle=\lim\limits_{\varepsilon\to+0}\big\langle o_{\mathscr{D}'}(1),\varphi\big\rangle & = & 0,\end{array} \tag{18}$$

for all $\varphi(x,t)\in\mathscr{D}(\mathbb{R}\times[0,\infty))$. Thus (18), (17) imply that

$$\begin{array}{rcl}\Big\langle\big(F(u,v)\big)_x,\varphi\Big\rangle & = & \lim_{\varepsilon\to+0}\Big\langle\big(F(u_\varepsilon,v_\varepsilon)\big)_x,\varphi\Big\rangle=-\langle u_t,\varphi\rangle,\\ \Big\langle\big(G(u,v)\big)_x,\varphi\Big\rangle & = & \lim_{\varepsilon\to+0}\Big\langle G(u_\varepsilon,v_\varepsilon)\big)_x,\varphi\Big\rangle=-\langle v_t,\varphi\rangle,\end{array} \tag{19}$$

for all $\varphi(x,t)\in\mathscr{D}(\mathbb{R}\times[0,\,\infty))$.

Using the first relation in (19), formulas (13), and the first Rankine–Hugoniot condition for δ-shocks (10), we obtain in the weak sense

$$\begin{aligned}\big(F(u,v)\big)_x&=-u_t=-\big(u_++[u]H(-x+\phi(t))\big)_t\\ &=-[u]\dot\phi(t)\delta(-x+\phi(t))=-[F(u,v)]\delta(-x+\phi(t)).\end{aligned}$$

Integrating the last relation with respect to x, we have

$$F(u,v)=[F(u,v)]H(-x+\phi(t))+C, \tag{20}$$

where C is a constant. Since according to (13), $\lim_{\varepsilon\to+0}F(u_\varepsilon(x,t),v_\varepsilon(x,t))=F(u_+,v_+)$ for $x>\phi(t)$, we conclude that (20) implies $C=F(u_+,v_+)$. Thus relation (14) is proved.

Using the second formula in (19), formulas (13), and the Rankine–Hugoniot conditions for δ-shocks (10), we have in the weak sense

$$\begin{aligned}\big(G(u,v)\big)_x&=-v_t=-\big(v_++[v]H(-x+\phi(t))+e(t)\delta(-x+\phi(t))\big)_t\\ &=-\big([v]\dot\phi(t)+\dot e(t)\big)\delta(-x+\phi(t))-e(t)\dot\phi(t)\delta'(-x+\phi(t))\\ &=-[G(u,v)]\delta(-x+\phi(t))-e(t)\dot\phi(t)\delta'(-x+\phi(t)).\end{aligned}$$

Integrating the last relation with respect to x, we obtain

$$G(u,v)=[G(u,v)]H(-x+\phi(t))+e(t)\dot\phi(t)\delta(-x+\phi(t))+C, \tag{21}$$

where C is a constant. Since $\lim_{\varepsilon\to+0}G(u_\varepsilon(x,t),v_\varepsilon(x,t))=G(u_+,v_+)$ for $x>\phi(t)$, we have $C=F(u_+,v_+)$. Consequently, relation (15) is proved. □

Thus one can see that the generalized solution (13) of the Cauchy problem (3), (6), *generates algebraic relations* (14), (15) between distributional components *u*, *v* of solution (13).

4. TWO PARTICULAR EXAMPLES

4.1. Let us consider the Cauchy problem (7), (6). This Cauchy problem was solved in [4], [5]. In this case for a δ-shock wave type solution (13) of the Cauchy problem (7), (6), according to Theorem 2, the "right" singular superpositions are defined as

$$f(u) = f(u_+) + [f(u)]H(-x+\phi(t)), \tag{22}$$

$$vg(u) = v_+g(u_+) + [vg(u)]H(-x+\phi(t)) + e(t)\dot{\phi}(t)\delta(-x+\phi(t)). \tag{23}$$

Here in view of (10), the Rankine–Hugoniot conditions for δ-shocks are given as $\dot{\phi}(t) = \frac{[f(u)]}{[u]}$, $\dot{e}(t) = \big([vg(u)] - [v]\frac{[f(u)]}{[u]}\big)$.

In fact, by (23), we define the *unique "right" product* of the step function and the delta function:

$$e(t)\delta(-x+\phi(t))u(x,t) = e(t)\delta(-x+\phi(t))\begin{cases} u_-, & x < \phi(t), \\ u_+, & x > \phi(t), \end{cases}$$

$$= e(t)\frac{[f(u)]}{[u]}\delta(-x+\phi(t)). \tag{24}$$

4.2. Let us consider the Cauchy problem (8), (6). This Cauchy problem was solved in [14], [15]. Similarly to the previous example, for a δ-shock wave type solution (13) of the Cauchy problem (8), (6), Theorem 2 implies that the "right" singular superpositions are defined as

$$u^2 - v = u_+^2 - v_+ + \big[u^2 - v\big]H(-x+\phi(t)), \tag{25}$$

$$\frac{1}{3}u^3 - u = \frac{1}{3}u_+^3 - u_+ + \big[\frac{1}{3}u^3 - u\big]H(-x+\phi(t)) + e(t)\frac{[u^2-v]}{[u]}\delta(-x+\phi(t)), \tag{26}$$

where in view of (10), $\dot{\phi}(t) = \frac{[u^2-v]}{[u]}$, $\dot{e}(t) = \big([\frac{1}{3}u^3 - u] - [v]\frac{[u^2-v]}{[u]}\big)$.

The Keyfitz–Kranzer system (8) has a *specific* property. We stress that, in contrast to system (7), in the case of systems (8) we *do not define (!)* the *product of the Heaviside function and the δ-function*. Moreover, although according to (13), $u(x,t)$ *does not depend* on the term $e(t)\delta(-x+\phi(t))$, the *"right" singular superposition* $\frac{1}{3}u^3 - u$ determined by (26), *does depend* on this term. Thus one can say that the term $e(t)\delta(-x+\phi(t))$ "appears from nothing". Analogously, the left-hand side in (25) *depends* on the term $e(t)\delta(-x+\phi(t))$, but the right-hand side in (25) *does not depend* on this term.

Thus a *"right" singular superposition* is determined only in the *context* of solving the Cauchy problem. Moreover, this is the Schwartz distribution.

REFERENCES

1. Albeverio S., Shelkovich V.M. 2005. *On the delta-shock front problem*, in the book: “Analytical Approaches to Multidimensional Balance Laws”, Ch. 2, (Ed. O. Rozanova), Nova Science Publishers, Inc.
2. Colombeau J.F. 1985. *Elementary introduction to new generalized functions*, North Holland.
3. Danilov V.G., Omel′yanov G.A., Shelkovich V.M. 2003. Weak Asymptotics Method and Interaction of Nonlinear Waves. *Asymptotic Methods for Wave and Quantum Problems*, Amer. Math. Society Translations, Ser. 2, **208**, 33–165.
4. Danilov V.G., Shelkovich V.M. 2005. *Delta-shock wave type solution of hyperbolic systems of conservation laws*, Quarterly of Applied Mathematics, **63**, no. 3, 401–427.
5. Danilov V.G., Shelkovich V.M. 2005. *Dynamics of propagation and interaction of delta-shock waves in conservation law systems*, Journal of Differential Equations, **211**, 333–381.
6. E Weinan, Rykov Yu., Sinai Ya.G. 1996. *Generalized variational principles, global weak solutions and behavior with random initial data for dystems of conservation laws arising in adhesion particlae dynamics*, Comm. Math. Phys., **177**, 349–380.
7. Keyfitz B. Lee, Kranzer H.C. 1995. *Spaces of weighted measures for conservation laws with singular shock solutions*, Journal of Differential Equations, **118**, 420–451.
8. Khrennikov A.Yu., Shelkovich V.M., Smolyanov O.G. 2002. *Multiplicative structures in the linear space of vector-valued distributions*, Doklady of Russian Acad. of Sci., **383**, no. 1, 1-4.
9. Khrennikov A.Yu., Shelkovich V.M., Smolyanov O.G. 2002. *Locally convex spaces of vector-valued distributions with multiplicative structures*, Infinite-Dimensional Analysis, Quantum Probability and Related Topics, **5**, no. 4, 1-20.
10. Khrennikov A.Yu., Shelkovich V.M., Smolyanov O.G. 2004. *An associative algebra of vector-valued distributions and singular solutions of nonlinear equations*, in Mathematical modelling in physics, engineering and cognitive sciences. v. 7. Proceedings of the conference “Mathematical Modelling of Wave Phenomena”, November 2002. Edited by B. Nilsson and L. Fishman, Växjö University Press, 2004, 191–205.
11. Maso G.D., Le Floch P.G., Murat F. 1995. *Definition and weak stability of nonconservative products*, J. Math. Pures Appl., **74**, 483–548.
12. Oberguggenberger M. 1992. *Multiplication of Distributions and Applications to Partial Differential Equations*, Pitman Res. Not. Math. 259, Longman Sci. Techn., Essex.
13. Panov E.Yu., Shelkovich V.M. 2005. *$\delta^{(n)}$-Shock wave as a new type solutions of hyperbolic systems of conservation laws*, pp.34, Preprint 2005-012 at the url: http://www.math.ntnu.no/conservation/2005/012.html (To appear in Journal of Differential Equations, 2006)
14. Shelkovich V.M. 2003. *Delta-shock waves of a class of hyperbolic systems of conservation laws*, in: A. Abramian, S. Vakulenko, V. Volpert (Eds.), “Patterns and Waves”, AkademPrint, St. Petersburg, pp. 155–168.
15. Shelkovich V.M. 2003. *A specific hyperbolic system of conservation laws admitting delta-shock wave type solutions*, pp.29, Preprint 2003-059 at the url: http://www.math.ntnu.no/conservation/2003/059.html
16. Shelkovich V.M. 2004. *Delta-shocks, the Rankine–Hugoniot conditions, and singular superposition of distributions*, Proceedings of International Seminar Days on Difraction’2004, June 29– July 2, 2004, St.Petersburg, 2004, 175–196. (See as Preprint 2004-059 at the url: http://www.math.ntnu.no/conservation/2004/059.html)
17. Shelkovich V.M. 2005. *The Riemann problem admitting δ'-shock wave type solution* (*the vanishing viscosity approach*), pp.41, Preprint 2005-053 at the url: http://www.math.ntnu.no/conservation/2005/053.html
18. Shen Wancheg, Zhang Tong 1999. *The Riemann problem for the transportaion equations in gas dynamics*, Memoirs of the Amer. Math. Soc., **137**, N 654, 1–77.