GLOBAL CONSERVATIVE SOLUTIONS OF THE CAMASSA–HOLM EQUATION — A LAGRANGIAN POINT OF VIEW

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Abstract. We show that the Camassa–Holm equation $u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$ possesses a global continuous semigroup of weak conservative solutions for initial data $u|_{t=0}$ in H^1. The result is obtained by introducing a coordinate transformation into Lagrangian coordinates. To characterize conservative solutions it is necessary to include the energy density given by the positive Radon measure μ with $\mu_{\rm ac} = (u^2 + u_x^2)\,dx$.

1. Introduction

The Cauchy problem for the Camassa–Holm equation [7, 8]

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad u|_{t=0} = u_0, \tag{1.1}$$

has received considerable attention the last decade. With κ positive it models, see [25], propagation of unidirectional gravitational waves in a shallow water approximation, with u representing the fluid velocity. The Camassa–Holm equation has a bi-Hamiltonian structure and is completely integrable. It has infinitely many conserved quantities. In particular, for smooth solutions the quantities

$$\int u\,dx, \quad \int (u^2 + u_x^2)\,dx, \quad \int (u^3 + uu_x^2)\,dx \tag{1.2}$$

are all time independent.

In this article we consider the case $\kappa = 0$ on the real line, that is,

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.3}$$

and henceforth we refer to (1.3) as the Camassa–Holm equation. The equation can be rewritten as the following system

$$u_t + uu_x + P_x = 0, \tag{1.4a}$$

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2. \tag{1.4b}$$

A highly interesting property of the equation is that for a wide class of initial data the solution experiences wave breaking in finite time in the sense that the solution u remains bounded pointwise while the spatial derivative u_x becomes unbounded pointwise. However, the H^1 norm of u remains finite. More precisely, Constantin, Escher, and Molinet [12, 14] showed the following result: If the initial data $u|_{t=0} = u_0 \in H^1(\mathbb{R})$ and $m_0 := u_0 - u_0''$ is a positive Radon measure, then equation (1.3) has a unique global weak solution $u \in C([0,T], H^1(\mathbb{R}))$, for any T positive, with initial data u_0. However, any solution with odd initial data u_0 in $H^3(\mathbb{R})$ such that $u_{0,x}(0) < 0$ blows up in a finite time.

The problem how to extend the solution beyond wave breaking can nicely be illustrated by studying an explicit class of solutions. The Camassa–Holm equation possesses solutions, denoted (multi)peakons, of the form

$$u(t,x) = \sum_{i=1}^{n} p_i(t) e^{-|x - q_i(t)|}, \tag{1.5}$$

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where the $(p_i(t), q_i(t))$ satisfy the explicit system of ordinary differential equations

$$\dot{q}_i = \sum_{j=1}^{n} p_j e^{-|q_i - q_j|}, \quad \dot{p}_i = \sum_{j=1}^{n} p_i p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i - q_j|}.$$

Observe that the solution (1.5) is not smooth even with continuous functions $(p_i(t), q_i(t))$; one possible way to interpret (1.5) as a weak solution of (1.3) is to rewrite the equation (1.3) as

$$u_t + \big(\frac{1}{2}u^2 + (1 - \partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2)\big)_x = 0.$$

Peakons interact in a way similar to that of solitons of the Korteweg–de Vries equation, and wave breaking may appear when at least two of the q_i's coincide. If all the $p_i(0)$ have the same sign, the peakons move in the same direction. Furthermore, in that case the solution experiences no wave breaking, and one has a global solution. Higher peakons move faster than the smaller ones, and when a higher peakon overtakes a smaller, there is an exchange of mass, but no wave breaking takes place. Furthermore, the $q_i(t)$ remain distinct. However, if some of $p_i(0)$ have opposite sign, wave breaking may incur, see, e.g., [3, 26]. For simplicity, consider the case with $n = 2$ and one peakon $p_1(0) > 0$ (moving to the right) and one antipeakon $p_2(0) < 0$ (moving to the left). In the symmetric case ($p_1(0) = -p_2(0)$ and $q_1(0) = -q_2(0) < 0$) the solution will vanish pointwise at the collision time t^* when $q_1(t^*) = q_2(t^*)$, that is, $u(t^*, x) = 0$ for all $x \in \mathbb{R}$. Clearly, at least two scenarios are possible; one is to let $u(t, x)$ vanish identically for $t > t^*$, and the other possibility is to let the peakon and antipeakon "pass through" each other in a way that is consistent with the Camassa–Holm equation. In the first case the energy $\int (u^2 + u_x^2)\, dx$ decreases to zero at t^*, while in the second case, the energy remains constant except at t^*. Clearly, the well-posedness of the equation is a delicate matter in this case. The first solution could be denoted a dissipative solution, while the second one could be called conservative. Other solutions are also possible. Global dissipative solutions of a more general class of equations were recently derived by Coclite, Holden, and Karlsen [9, 10]. In their approach the solution was obtained by first regularizing the equation by adding a small diffusion term ϵu_{xx} to the equation, and subsequently analyzing the vanishing viscosity limit $\epsilon \to 0$. Multipeakons are fundamental building blocks for general solutions. Indeed, if the initial data u_0 is in H^1 and $m_0 := u_0 - u_0''$ is a positive Radon measure, then it can proved, see [23], that one can construct a sequence of multipeakons that converges in $L^\infty_{\text{loc}}(\mathbb{R}; H^1_{\text{loc}}(\mathbb{R}))$ to the unique global solution of the Camassa–Holm equation.

The problem of continuation beyond wave breaking was recently considered by Bressan and Constantin [4]. They reformulated the Camassa–Holm equation as a semilinear system of ordinary differential equations taking values in a Banach space. This formulation allowed them to continue the solution beyond collision time, giving a global conservative solution where the energy is conserved for almost all times. Thus in the context of peakon-antipeakon collisions they considered the solution where the peakons and antipeakons "passed through" each other. Local existence of the semilinear system is obtained by a contraction argument. Furthermore, the clever reformulation allows for a global solution where all singularities disappear. Going back to the original function u, one obtains a global solution of the Camassa–Holm equation. The well-posedness, i.e., the uniqueness and stability of the solution, is resolved as follows. In addition to the solution u, one includes a family of non-negative Radon measures μ_t with density $u_x^2\, dx$ with respect to the Lebesgue measure. The pair (u, μ_t) constitutes a continuous semigroup, in particular, one has uniqueness and stability.

Very recently, Bressan and Fonte [5, 20] presented another approach to the Camassa–Holm equation. The flow map $u_0 \mapsto u(t)$ is, as we have seen, neither a continuous map on H^1 nor on L^2. However, they introduced a new distance $J(u, v)$ with the property

$$c_1 \|u - v\|_{L^1} \le J(u, v) \le c_2 \|u - v\|_{H^1}.$$

Furthermore, it satisfies

$$J(u(t), u_0) \le c_3 |t|, \quad J(u(t), v(t)) \le J(u_0, v_0) e^{c_4 |t|},$$

where $u(t), v(t)$ are solutions with initial data u_0, v_0, respectively. The distance is introduced by first defining it for multipeakons, using the global, conservative solution described above. Subsequently it is shown that multipeakons are dense in the space H^1. This enables them to construct a semi-group of conservative solutions for the Camassa–Holm equation which is continuous with respect to the distance J.

In this paper, as Bressan and Constantin [4], we reformulate the equation using a different set of variables and obtain a semilinear system of ordinary differential equations. However, the change of variables we use is distinct from that of Bressan and Constantin and simply corresponds to the transformation between Eulerian and Lagrangian coordinates. Let $u = u(t,x)$ denote the solution, and $y(t,\xi)$ the corresponding characteristics, thus $y_t(t,\xi) = u(t, y(t,\xi))$. Our new variables are $y(t,\xi)$,

$$U(t,\xi) = u(t, y(t,\xi)), \quad H(t,\xi) = \int_{-\infty}^{y(t,\xi)} (u^2 + u_x^2)\, dx \tag{1.6}$$

where U corresponds to the Lagrangian velocity while H could be interpreted as the Lagrangian cumulative energy distribution. Furthermore, let

$$\begin{aligned} Q(t,\xi) &= -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2 y_\xi + H_\xi\big)(\eta)\, d\eta, \\ P(t,\xi) &= \frac{1}{4}\int_{\mathbb{R}} \exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2 y_\xi + H_\xi\big)\, d\eta. \end{aligned}$$

Then one can show that

$$\begin{cases} y_t = U, \\ U_t = -Q, \\ H_t = U^3 - 2PU, \end{cases} \tag{1.7}$$

is equivalent to the Camassa–Holm equation. Global existence of solutions of (1.7) is obtained starting from a contraction argument, see Theorem 2.8. The uniqueness issue is resolved by considering the set $\mathcal{D}$ (see Definition 3.1) which consists of pairs (u,μ) such that $(u,\mu) \in \mathcal{D}$ if $u \in H^1(\mathbb{R})$ and μ is a positive Radon measure whose absolutely continuous part satisfies $\mu_{\mathrm{ac}} = (u^2 + u_x^2)\, dx$. With three Lagrangian variables (y, U, H) versus two Eulerian variables (u,μ), it is clear that there can be no bijection between the two coordinates systems. However, we define a group of relabeling transformations which acts on the Lagrangian variables and let the system of equations (1.7) invariant. Using this group, we are able to establish a bijection between the space of Eulerian variables and the space of Lagrangian variables when we identify variables that are invariant under relabeling. This bijection allows us to transform the results obtained in the Lagrangian framework (in which the equation is well-posed) into the Eulerian framework (in which the situation is much more subtle). In particular, and this constitutes the main result of this paper, we obtain a metric $d_{\mathcal{D}}$ on $\mathcal{D}$ and a continuous semi-group of solutions on $(\mathcal{D}, d_{\mathcal{D}})$. The distance $d_{\mathcal{D}}$ gives $\mathcal{D}$ the structure of a complete metric space. This metric is compared with some more standard topologies, and we obtain that convergence in $H^1(\mathbb{R})$ implies convergence in $(\mathcal{D}, d_{\mathcal{D}})$ which itself implies convergence in $L^\infty(\mathbb{R})$, see Propositions 5.1 and 5.2. The properties of the spaces as well as the various mappings between them are described in great detail, see Section 3. Our main result, Theorem 4.2, states that for given initial data in $\mathcal{D}$ there exists a unique weak solution of the Camassa–Holm equation. The associated measure $\mu(t)$ has constant total mass, i.e., $\mu(t)(\mathbb{R}) = \mu(0)(\mathbb{R})$ for all t, which corresponds to the total energy of the system. This is the reason why our solutions are called conservative.

The method described here can be studied in detail for multipeakons, see [22] for details. By suitably modifying the techniques described in this paper, the results can be extended to show global existence of conservative solutions for the generalized hyperelastic-rod equation

$$\begin{cases} u_t + f(u)_x + P_x = 0 \\ P - P_{xx} = g(u) + \frac{1}{2} f''(u) u_x^2. \end{cases} \tag{1.8}$$

where $f, g \in C^\infty(\mathbb{R})$ and f is strictly convex. Observe that if $g(u) = \kappa u + u^2$ and $f(u) = \frac{u^2}{2}$, then (1.8) is the classical Camassa–Holm equation (1.1). With $g(u) = \frac{3-\gamma}{2}u^2$ and $f(u) = \frac{\gamma}{2}u^2$, Dai [15, 16, 17] derived (1.8) as an equation describing finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods, and the equation is often referred to as the hyperelastic-rod wave equation. See [9, 10] for a recent proof of existence of dissipative solutions of (1.8). The details will be described in a forthcoming paper.

Furthermore, the methods presented in this paper can be used to derive numerical methods that converge to conservative solutions rather than dissipative solutions. This contrasts finite difference methods that normally converge to dissipative solutions, see [24] and [21] for the related Hunter–Saxton equation. See also [23]. Results will be presented separately.

2. Global solutions in Lagrangian coordinates

2.1. **Equivalent system.**

Assuming that u is smooth, it is not hard to check that

$$(u^2 + u_x^2)_t + (u(u^2 + u_x^2))_x = (u^3 - 2Pu)_x. \tag{2.1}$$

Let us introduce the characteristics $y(t,\xi)$ defined as the solutions of

$$y_t(t,\xi) = u(t, y(t,\xi)) \tag{2.2}$$

for a given $y(0,\xi)$. Equation (2.1) gives us information about the evolution of the amount of energy contained between two characteristics. Indeed, given ξ_1, ξ_2 in $\mathbb{R}$, let $H(t) = \int_{y(t,\xi_1)}^{y(t,\xi_2)} \left(u^2 + u_x^2\right) dx$ be the energy contained between the two characteristic curves $y(t,\xi_1)$ and $y(t,\xi_2)$. Then, using (2.1) and (2.2), we obtain

$$\frac{dH}{dt} = \left[(u^3 - 2Pu) \circ y\right]_{\xi_1}^{\xi_2}. \tag{2.3}$$

Solutions of the Camassa–Holm blow up when characteristics arising from different points collide. It is important to notice that we do not get shocks as the Camassa–Holm preserves the H^1 norm and therefore solutions remain continuous. However, it is not obvious how to continue the solution after collision time. It turns out that, when two characteristics collide, the energy contained between these two characteristics has a limit which can be computed from (2.3). As we will see, knowing this energy enables us to prolong the characteristics and thereby the solution, after collisions.

We now derive a system equivalent to (1.4). All the derivations in this section are formal and will be justified later. Let y still denote the characteristics. We introduce two other variables, the Lagrangian velocity and cumulative energy distribution, U and H, defined as $U(t,\xi) = u(t, y(t,\xi))$ and

$$H(t,\xi) = \int_{-\infty}^{y(t,\xi)} \left(u^2 + u_x^2\right) dx. \tag{2.4}$$

From the definition of the characteristics, it follows that

$$U_t(t,\xi) = u_t(t,y) + y_t(t,\xi) u_x(t,y) = -P_x \circ y\,(t,\xi). \tag{2.5}$$

This last term can be expressed uniquely in term of U, y, and H. From (1.4b), we obtain the following explicit expression for P,

$$P(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}(u^2(t,z) + \frac{1}{2}u_x^2(t,z))\,dz. \tag{2.6}$$

Thus we have

$$P_x \circ y\,(t,\xi) = -\frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(y(t,\xi) - z) e^{-|y(t,\xi)-z|}(u^2(t,z) + \frac{1}{2}u_x^2(t,z))\,dz$$

and, after the change of variables $z = y(t,\eta)$,

$$P_x \circ y(t,\xi) = -\frac{1}{2}\int_{\mathbb{R}} \Big[\operatorname{sgn}(y(t,\xi) - y(t,\eta)) e^{-|y(t,\xi)-y(t,\eta)|} \\ \times \left(u^2(t, y(t,\eta)) + \frac{1}{2}u_x^2(t, y(t,\eta))\right) y_\xi(t,\eta)\Big]\,d\eta.$$

Finally, since $H_\xi = (u^2+u_x^2)\circ y\, y_\xi$,

$$P_x\circ y\,(\xi) = -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(y(\xi)-y(\eta))\exp(-|y(\xi)-y(\eta)|)\big(U^2y_\xi+H_\xi\big)(\eta)\,d\eta \tag{2.7}$$

where the t variable has been dropped to simplify the notation. Later we will prove that y is an increasing function for any fixed time t. If, for the moment, we take this for granted, then $P_x\circ y$ is equivalent to Q where

$$Q(t,\xi) = -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+H_\xi\big)(\eta)\,d\eta, \tag{2.8}$$

and, slightly abusing the notation, we write

$$P(t,\xi) = \frac{1}{4}\int_{\mathbb{R}} \exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+H_\xi\big)(\eta)\,d\eta. \tag{2.9}$$

Thus $P_x\circ y$ and $P\circ y$ can be replaced by equivalent expressions given by (2.8) and (2.9) which only depend on our new variables U, H, and y. We introduce yet another variable, $\zeta(t,\xi)$, simply defined as $\zeta(t,\xi)=y(t,\xi)-\xi$. It will turn out that $\zeta\in L^\infty(\mathbb{R})$. We now derive a new system of equations, formally equivalent to the Camassa–Holm equation. Equations (2.5), (2.3) and (2.2) give us

$$\begin{cases} \zeta_t = U,\\ U_t = -Q,\\ H_t = U^3-2PU. \end{cases} \tag{2.10}$$

As we will see, the system (2.10) of ordinary differential equations for $(\zeta,U,H)\colon[0,T]\to E$ is well-posed, where E is Banach space to be defined in the next section. We have

$$Q_\xi = -\frac{1}{2}H_\xi - \left(\frac{1}{2}U^2-P\right)y_\xi \quad\text{and}\quad P_\xi = Qy_\xi. \tag{2.11}$$

Hence, differentiating (2.10) yields

$$\begin{cases} \zeta_{\xi t} = U_\xi \ (\text{or } y_{\xi t}=U_\xi),\\ U_{\xi t} = \dfrac{1}{2}H_\xi + \left(\dfrac{1}{2}U^2-P\right)y_\xi,\\ H_{\xi t} = -2Q\,Uy_\xi + \big(3U^2-2P\big)\,U_\xi. \end{cases} \tag{2.12}$$

The system (2.12) is semilinear with respect to the variables y_ξ, U_ξ and H_ξ.

2.2. **Existence and uniqueness of solutions of the equivalent system.** In this section, we focus our attention on the system of equations (2.10) and prove, by a contraction argument, that it admits a unique solution. Let V be the Banach space defined by

$$V = \{f\in C_b(\mathbb{R}) \mid f_\xi\in L^2(\mathbb{R})\}$$

where $C_b(\mathbb{R}) = C(\mathbb{R})\cap L^\infty(\mathbb{R})$ and the norm of V is given by $\|f\|_V = \|f\|_{L^\infty(\mathbb{R})}+\|f_\xi\|_{L^2(\mathbb{R})}$. Of course $H^1(\mathbb{R})\subset V$ but the converse is not true as V contains functions that do not vanish at infinity. We will employ the Banach space E defined by

$$E = V\times H^1(\mathbb{R})\times V$$

to carry out the contraction mapping argument. For any $X=(\zeta,U,H)\in E$, the norm on E is given by $\|X\|_E = \|\zeta\|_V+\|U\|_{H^1(\mathbb{R})}+\|H\|_V$. The following lemma gives the Lipschitz bounds we need on Q and P.

Lemma 2.1. *For any $X=(\zeta,U,H)$ in E, we define the maps $\mathcal{Q}$ and $\mathcal{P}$ as $\mathcal{Q}(X)=Q$ and $\mathcal{P}(X)=P$ where Q and P are given by* (2.8) *and* (2.9)*, respectively. Then, $\mathcal{P}$ and $\mathcal{Q}$ are Lipschitz maps on bounded sets from E to $H^1(\mathbb{R})$. Moreover, we have*

$$Q_\xi = -\frac{1}{2}H_\xi - \left(\frac{1}{2}U^2-P\right)(1+\zeta_\xi), \tag{2.13}$$

$$P_\xi = Q(1+\zeta_\xi). \tag{2.14}$$

Proof. We rewrite $\mathcal{Q}$ as

$$\mathcal{Q}(X)(\xi) = -\frac{e^{-\zeta(\xi)}}{4}\int_{\mathbb{R}} \chi_{\{\eta<\xi\}} e^{-|\xi-\eta|} e^{\zeta(\eta)} \left[U(\eta)^2(1+\zeta_\xi(\eta)) + H_\xi(\eta)\right] d\eta$$
$$+ \frac{e^{\zeta(\xi)}}{4}\int_{\mathbb{R}} \chi_{\{\eta>\xi\}} e^{-|\xi-\eta|} e^{-\zeta(\eta)} \left[U(\eta)^2(1+\zeta_\xi(\eta)) + H_\xi(\eta)\right] d\eta, \quad (2.15)$$

where χ_B denotes the indicator function of a given set B. We decompose $\mathcal{Q}$ into the sum $\mathcal{Q}_1 + \mathcal{Q}_2$ where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the operators corresponding to the two terms on the right-hand side of (2.15). We know that the operator $(1-\partial_{xx})^{-1}$ that we denote by A, is a continuous linear operator from $H^{-1}(\mathbb{R})$ to $H^1(\mathbb{R})$, see for example, [6]. It is explicitly given by $A(f)(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-y|} f(y)\,dy$, and we can rewrite $\mathcal{Q}_1$ as

$$\mathcal{Q}_1(X)(\xi) = -\frac{e^{-\zeta(\xi)}}{2} A \circ R(\zeta, U, H)(\xi) \quad (2.16)$$

where R is the operator from E to $L^2(\mathbb{R})$ given by $R(\zeta, U, H)(t,\xi) = \chi_{\{\eta<\xi\}} e^{\zeta}(U^2 + U^2\zeta_\xi + H_\xi)$. Since $L^2(\mathbb{R})$ is continuously embedded in $H^{-1}(\mathbb{R})$, we have $A\circ R(\zeta, U, H) \in H^1$. We say that an operator is B-Lipschitz when it is Lipschitz on bounded sets. Let us prove that $\mathcal{Q}_1 \colon E \to H^1(\mathbb{R})$ is B-Lipschitz. It is not hard to prove that R is B-Lipschitz from E into $L^2(\mathbb{R})$ and therefore from E into $H^{-1}(\mathbb{R})$. Since $A\colon H^{-1}(\mathbb{R}) \to H^1(\mathbb{R})$ is linear and continuous, $A\circ R$ is B-Lipschitz from E to $H^1(\mathbb{R})$. Then, we use the following lemma whose proof is left to the reader.

Lemma 2.2. *Let $\mathcal{R}_1\colon E \to V$ and $\mathcal{R}_2\colon E \to H^1(\mathbb{R})$, or $\mathcal{R}_2\colon E\to V$, be two B-Lipschitz maps. Then, the product $X \mapsto R_1(X)R_2(X)$ is also a B-Lipschitz map from E to $H^1(\mathbb{R})$, or from E to V.*

Since the mapping $X \mapsto e^{-\zeta}$ is B-Lipschitz from E to V, Q_1 is the product of two B-Lipschitz maps, one from E to $H^1(\mathbb{R})$ and the other from E to V, it is B-Lipschitz map from E to $H^1(\mathbb{R})$. Similarly, one proves that $\mathcal{Q}_2$ is B-Lipschitz and therefore $\mathcal{Q}$ is B-Lipschitz. Furthermore, $\mathcal{P}$ is B-Lipschitz. The formulas (2.13) and (2.14) are obtained by direct computation using the product rule, see [18, p. 129]. □

In the next theorem, by using a contraction argument, we prove the short-time existence of solutions to (2.10).

Theorem 2.3. *Given $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{H})$ in E, there exists a time T depending only on $\left\|\bar{X}\right\|_E$ such that the system* (2.10) *admits a unique solution in $C^1([0,T], E)$ with initial data $\bar{X}$.*

Proof. Solutions of (2.10) can be rewritten as

$$X(t) = \bar{X} + \int_0^t F(X(\tau))\,d\tau \quad (2.17)$$

where $F\colon E \to E$ is given by $F(X) = (U, -\mathcal{Q}(X), U^3 - 2\mathcal{P}(X)U)$ where $X = (\zeta, U, H)$. The integrals are defined as Riemann integrals of continuous functions on the Banach space E. Using Lemma 2.1, we can check that each component of $F(X)$ is a product of functions that satisfy one of the assumptions of Lemma 2.2 and using this same lemma, we obtain that $F(X)$ is a Lipchitz function on any bounded set of E. Since E is a Banach space, we use the standard contraction argument to prove the theorem. □

We now turn to the proof of existence of global solutions of (2.10). We are interested in a particular class of initial data that we are going to make precise later, see Definition 2.6. In particular, we will only consider initial data that belong to $E \cap \left[W^{1,\infty}(\mathbb{R})\right]^3$ where $W^{1,\infty}(\mathbb{R}) = \{f \in C_b(\mathbb{R}) \mid f_\xi \in L^\infty(\mathbb{R})\}$. Given $(\bar{\zeta}, \bar{U}, \bar{H}) \in E \cap [W^{1,\infty}(\mathbb{R})]^3$, we consider the short-time solution $(\zeta, U, H) \in C([0,T], E)$ of (2.10) given by Theorem 2.3. Using the fact that $\mathcal{Q}$ and $\mathcal{P}$ are Lipschitz on bounded sets (Lemma 2.1) and, since $X \in C([0,T], E)$, we can prove that P and Q belongs to

$C([0,T],H^1(\mathbb{R}))$. We now consider U, P and Q as given function in $C([0,T],H^1(\mathbb{R}))$. Then, for any *fixed* $\xi\in\mathbb{R}$, we can solve the system of ordinary differential equations

$$\begin{cases}\dfrac{d}{dt}\alpha(t,\xi)=\beta(t,\xi),\\ \dfrac{d}{dt}\beta(t,\xi)=\dfrac{1}{2}\gamma(t,\xi)+\left[(\dfrac{1}{2}U^2-P)(t,\xi)\right](1+\alpha(t,\xi)),\\ \dfrac{d}{dt}\gamma(t,\xi)=-\left[2(Q\,U)(t,\xi)\right](1+\alpha(t,\xi))+\left[(3U^2-2P)(t,\xi)\right]\beta(t,\xi),\end{cases}\tag{2.18}$$

which is obtained by substituting ζ_ξ, U_ξ and H_ξ in (2.12) by the unknowns α, β, and γ, respectively. We have to specify the initial conditions for (2.18). Let $\mathcal{A}$ be the following set

$$\mathcal{A}=\{\xi\in\mathbb{R}\mid\ \left|\bar U_\xi(\xi)\right|\leq\left\|\bar U_\xi\right\|_{L^\infty(\mathbb{R})},\ \left|\bar H_\xi(\xi)\right|\leq\left\|\bar H_\xi\right\|_{L^\infty(\mathbb{R})},\ \left|\bar\zeta_\xi(\xi)\right|\leq\left\|\bar\zeta_\xi\right\|_{L^\infty(\mathbb{R})}\},$$

We have that $\mathcal{A}$ has full measure, that is, $\operatorname{meas}(\mathcal{A}^c)=0$. For $\xi\in\mathcal{A}$ we define $(\alpha(0,\xi),\beta(0,\xi),\gamma(0,\xi))=(\bar U_\xi(\xi),\bar H_\xi(\xi),\bar\zeta_\xi(\xi))$. However, for $\xi\in\mathcal{A}^c$ we take $(\alpha(0,\xi),\beta(0,\xi),\gamma(0,\xi))=(0,0,0)$.

Lemma 2.4. *Given initial condition $\bar X=(\bar U,\bar H,\bar\zeta)\in E\cap[W^{1,\infty}(\mathbb{R})]^3$, we consider the solution $X=(\zeta,U,H)\in C^1([0,T],E)$ of (2.18) given by Theorem 2.3. Then, $X\in C^1([0,T],E\cap[W^{1,\infty}(\mathbb{R})]^3)$. The functions $\alpha(t,\xi)$, $\beta(t,\xi)$ and $\gamma(t,\xi)$ which are obtained by solving (2.18) for any fixed given ξ with the initial condition specified above, coincide for almost every ξ and for all time t with ζ_ξ, U_ξ and H_ξ, respectively, that is, for all $t\in[0,T]$, we have*

$$(\alpha(t,\xi),\beta(t,\xi),\gamma(t,\xi))=(\zeta_\xi(t,\xi),U_\xi(t,\xi),H_\xi(t,\xi))\tag{2.19}$$

for almost every $\xi\in\mathbb{R}$.

Thus, this lemma allows us to pick up a special representative for (ζ_ξ,U_ξ,H_ξ) given by (α,β,γ), which is defined for all $\xi\in\mathbb{R}$ and which, for any given ξ, satisfies the ordinary differential equation (2.18). In the remaining we will of course identify the two and set (ζ_ξ,U_ξ,H_ξ) equal to (α,β,γ). To prove this lemma, we will need the following proposition which is adapted from [27, p. 134, Corollary 2].

Proposition 2.5. *Let R be a bounded linear operator on a Banach space X into a Banach space Y. Let f be in $C([0,T],X)$. Then, Rf belongs to $C([0,T],Y)$ and therefore is Riemann integrable, and $\int_{[0,T]}Rf(t)\,dt=R\int_{[0,T]}f(t)\,dt$.*

Proof of Lemma 2.4. We introduce the Banach space of *everywhere* bounded function $B^\infty(\mathbb{R})$ whose norm is naturally given by $\|f\|_{B^\infty(\mathbb{R})}=\sup_{\xi\in\mathbb{R}}|f(\xi)|$. Obviously, $C_b(\mathbb{R})$ is included in $B^\infty(\mathbb{R})$. We define (α,β,γ) as the solution of (2.18) in $[B^\infty(\mathbb{R})]^3\cap\left[L^2(\mathbb{R})\right]^3$ with initial data as given above. Thus, strictly speaking, this is a different definition than the one given in the lemma but we will see that they are in fact equivalent. We note that the system (2.18) is affine (it consists of a sum of a linear transformation and a constant) and therefore it is not hard to prove, by using a contraction argument in $[B^\infty(\mathbb{R})]^3\cap\left[L^2(\mathbb{R})\right]^3$, the short-time existence of solutions. Moreover, the solution exists on $[0,T]$, the interval on which (ζ,U,H) is defined. Let us assume the opposite. Then, $Z_1(t)=\|\alpha(t,\cdot)\|_{B^\infty(\mathbb{R})\cap L^2(\mathbb{R})}+\|\beta(t,\cdot)\|_{B^\infty(\mathbb{R})\cap L^2(\mathbb{R})}+\|\gamma(t,\cdot)\|_{B^\infty(\mathbb{R})\cap L^2(\mathbb{R})}$ has to blow up when t approaches some time strictly smaller than T. We rewrite (2.18) in integral form:

$$\begin{cases}\alpha(t)=\alpha(0)+\displaystyle\int_0^t\beta(\tau)\,d\tau,\\ \beta(t)=\beta(0)+\displaystyle\int_0^t\left(\frac{1}{2}\gamma+(\frac{1}{2}U^2-P)(1+\alpha)\right)(\tau)\,d\tau,\\ \gamma(t)=\gamma(0)+\displaystyle\int_0^t\left(-2Q\,U(1+\alpha)+(3U^2-2P)\beta\right)(\tau)\,d\tau.\end{cases}\tag{2.20}$$

Note that in (2.20) all the terms belong to $B^\infty(\mathbb{R})\cap L^2(\mathbb{R})$ and the equalities hold in this space. After taking the norms on both sides of the three equations in (2.20) and adding them term by term, we obtain the following inequality

$$Z_1(t) \leq Z_1(0) + CT + C \int_0^t Z_1(\tau)\, d\tau$$

where C is a constant which depends on the $C([0,T], H^1(\mathbb{R}))$-norms of U, P and Q, which, by assumption, are bounded. From Gronwall's lemma, we get $Z_1(t) \leq (Z_1(0)+CT)e^{CT}$ and therefore $Z_1(t)$ cannot blow up and α, β and γ belong to $C^1([0,T], B^\infty(\mathbb{R}) \cap L^2(\mathbb{R}))$. For any given ξ, the map $f \mapsto f(\xi)$ from $B^\infty(\mathbb{R})$ to $\mathbb{R}$ is linear and continuous. Hence, after applying this map to each term in (2.20) and using Proposition 2.5, we recover the original definition of α, β and γ as solutions, for any given $\xi \in \mathbb{R}$, of the system (2.18) of ordinary differential equations in $\mathbb{R}^3$. The derivation map ∂_ξ is continuous from V and $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$. We can apply it to each term in (2.10) written in integral from and, by Proposition 2.5, this map commutes with the integral. We end up with, after using (2.13) and (2.14),

$$\begin{cases} \zeta_\xi(t) = \bar{\zeta}_\xi + \displaystyle\int_0^t U_\xi(\tau)\, d\tau, \\ U_\xi(t) = \bar{U}_\xi + \displaystyle\int_0^t \left(\frac{1}{2} H_\xi + (\frac{1}{2}U^2 - P)(1+\zeta_\xi)\right)(\tau)\, d\tau, \\ H_\xi(t) = \bar{H}_\xi + \displaystyle\int_0^t \left(-2Q\, U(1+\zeta_\xi) + (3U^2 - 2P)U_\xi\right)(\tau)\, d\tau. \end{cases} \tag{2.21}$$

The injection map from $B^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ is of course continuous, we can apply it to (2.20) and again use Proposition 2.5. Then, we can subtract each equation in (2.21) from the corresponding one in (2.20), take the norm and add them. After introducing $Z_2(t) = \|\alpha(t,\,\cdot\,) - \zeta_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|\beta(t,\,\cdot\,) - U_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|\gamma(t,\,\cdot\,) - H_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})}$, we end up with the following equation

$$Z_2(t) \leq Z_2(0) + C \int_0^t Z_2(\tau)\, d\tau$$

where C is a constant which, again, only depends on the $C([0,T], H^1(\mathbb{R}))$-norms, of U, P and Q. By assumption on the initial conditions, we have $Z_2(0) = 0$ because $\alpha(0) = \bar{\zeta}_\xi$, $\beta(0) = \bar{U}_\xi$, $\gamma(0) = \bar{H}_\xi$ almost everywhere and therefore, by Gronwall's lemma, we get $Z_2(t) = 0$ for all $t \in [0,T]$. This is just a reformulation of (2.19), and this concludes the proof of the lemma. □

It is possible to carry out the contraction argument of Theorem 2.3 in the Banach space $[W^{1,\infty}(\mathbb{R})]^3$ but the topology on $[W^{1,\infty}(\mathbb{R})]^3$ turns out to be too strong for our purpose and that is why we prefer E whose topology is in some sense weaker. Our goal is to find solutions of (1.4) with initial data $\bar{u}$ in H^1 because H^1 is the natural space for the equation. Theorem 2.3 gives us the existence of solutions to (2.10) for initial data in E. Therefore we have to find initial conditions that match the initial data $\bar{u}$ and belong to E. A natural choice would be to use $\bar{y}(\xi) = y(0,\xi) = \xi$ and $\bar{U}(\xi) = u(\xi)$. Then $y(t,\xi)$ gives the position of the particle which is at ξ at time $t = 0$. But, if we make this choice, then $\bar{H}_\xi = \bar{u}^2 + \bar{u}_x^2$ and H_ξ does not belong to $L^2(\mathbb{R})$ in general. We consider instead $(\bar{y}, \bar{U}, \bar{H})$ given by the relations

$$\xi = \int_{-\infty}^{\bar{y}(\xi)} (\bar{u}^2 + \bar{u}_x^2)\, dx + \bar{y}(\xi),\ \bar{U} = \bar{u}\circ\bar{y}\,,\ \text{and } \bar{H} = \int_{-\infty}^{\bar{y}} \left(\bar{u}^2 + \bar{u}_x^2\right)\, dx. \tag{2.22}$$

Later (see Remark 3.10), we will prove that $(\bar{y} - \mathrm{Id}, \bar{U}, \bar{H})$ belongs to $\mathcal{G}$ where $\mathcal{G}$ is defined as follows.

Definition 2.6. The set $\mathcal{G}$ is composed of all $(\zeta, U, H) \in E$ such that

$$(\zeta, U, H) \in \left[W^{1,\infty}(\mathbb{R})\right]^3, \tag{2.23a}$$

$$y_\xi \geq 0, H_\xi \geq 0, y_\xi + H_\xi > 0 \text{ almost everywhere, and } \lim_{\xi\to-\infty} H(\xi) = 0, \tag{2.23b}$$

$$y_\xi H_\xi = y_\xi^2 U^2 + U_\xi^2 \text{ almost everywhere,} \tag{2.23c}$$

where we denote $y(\xi) = \zeta(\xi) + \xi$.

Note that if all functions are smooth and $y_\xi > 0$, we have $u_x \circ y = \frac{U_\xi}{y_\xi}$ and condition (2.23c) is equivalent to (2.4). For initial data in $\mathcal{G}$, the solution of (2.10) exists globally.

Lemma 2.7. *Given initial data $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{H})$ in $\mathcal{G}$, let $X(t) = (\zeta(t), U(t), H(t))$ be the short-time solution of (2.10) in $C([0,T], E)$ for some $T > 0$ with initial data $(\bar{\zeta}, \bar{U}, \bar{H})$. Then,*
(*i*) *$X(t)$ belongs to $\mathcal{G}$ for all $t \in [0,T]$,*
(*ii*) *for almost every $t \in [0,T]$, $y_\xi(t,\xi) > 0$ for almost every $\xi \in \mathbb{R}$,*
(*iii*) *For all $t \in [0,T]$, $\lim_{\xi\to\pm\infty} H(t,\xi)$ exists and is independent of time.*

We denote by $\mathcal{A}$ the set where the absolute values of $\bar{\zeta}_\xi(\xi)$, $\bar{H}_\xi(\xi)$, and $\bar{U}_\xi(\xi)$ all are smaller than $\|\bar{X}\|_{[W^{1,\infty}(\mathbb{R})]^3}$ and where the inequalities in (2.23b) and (2.23c) are satisfied for y_ξ, U_ξ and H_ξ. By assumption, we have $\operatorname{meas}(\mathcal{A}^c) = 0$ and we set $(\bar{U}_\xi, \bar{H}_\xi, \bar{\zeta}_\xi)$ equal to zero on $\mathcal{A}^c$. Thus, as allowed by Lemma 2.4, we choose a special representative for $(\zeta(t,\xi), U(t,\xi), H(t,\xi))$ which satisfies (2.12) as an ordinary differential equation, for every $\xi \in \mathbb{R}$.

Proof. (*i*) We already proved in Lemma 2.4 that the space $[W^{1,\infty}(\mathbb{R})]^3$ is preserved and therefore $X(t)$ satisfies (2.23a) for all $t \in [0,T]$. Let us prove that (2.23c) and the inequalities in (2.23b) hold for any $\xi \in \mathcal{A}$ and therefore almost everywhere. We consider a fixed ξ in $\mathcal{A}$ and drop it in the notations when there is no ambiguity. From (2.12), we have, on the one hand,

$$(y_\xi H_\xi)_t = y_{\xi t} H_\xi + H_{\xi t} y_\xi = U_\xi H_\xi + (3U^2 U_\xi - 2y_\xi QU - 2PU_\xi) y_\xi,$$

and, on the other hand,

$$\begin{aligned}(y_\xi^2 U^2 + U_\xi^2)_t &= 2y_{\xi t} y_\xi U^2 + 2y_\xi^2 U_t U + 2U_{\xi t} U_\xi \\ &= 3U_\xi U^2 y_\xi - 2PU_\xi y_\xi + H_\xi U_\xi - 2y_\xi^2 QU.\end{aligned}$$

Thus, $(y_\xi H_\xi - y_\xi^2 U^2 - U_\xi^2)_t = 0$, and since $y_\xi H_\xi(0) = (y_\xi^2 U^2 + U_\xi^2)(0)$, we have $y_\xi H_\xi(t) = (y_\xi^2 U^2 + U_\xi^2)(t)$ for all $t \in [0,T]$. We have proved (2.23c). Let us introduce t^* given by

$$t^* = \sup\{t \in [0,T] \mid y_\xi(t') \geq 0 \text{ for all } t' \in [0,t]\}.$$

Here we recall that we consider a fixed $\xi \in \mathcal{A}$ and dropped it in the notation. Assume that $t^* < T$. Since $y_\xi(t)$ is continuous with respect to time, we have

$$y_\xi(t^*) = 0. \tag{2.24}$$

Hence, from (2.23c) that we just proved, $U_\xi(t^*) = 0$ and, by (2.12),

$$y_{\xi t}(t^*) = U_\xi(t^*) = 0. \tag{2.25}$$

From (2.12), since $y_\xi(t^*) = U_\xi(t^*) = 0$, we get

$$y_{\xi tt}(t^*) = U_{\xi t}(t^*) = \frac{1}{2} H_\xi(t^*). \tag{2.26}$$

If $H_\xi(t^*) = 0$, then $(y_\xi, U_\xi, H_\xi)(t^*) = (0,0,0)$ and, by the uniqueness of the solution of (2.12), seen as a system of ordinary differential equations, we must have $(y_\xi, U_\xi, H_\xi)(t) = 0$ for all $t \in [0,T]$. This contradicts the fact that $y_\xi(0)$ and $H_\xi(0)$ cannot vanish at the same time ($\bar{y}_\xi + \bar{H}_\xi > 0$ for all $\xi \in \mathcal{A}$). If $H_\xi(t^*) < 0$, then $y_{\xi tt}(t^*) < 0$ and, because of (2.24) and (2.25), there exists a neighborhood $\mathcal{U}$ of t^* such that $y(t) < 0$ for all $t \in \mathcal{U} \setminus \{t^*\}$. This contradicts the definition of t^*. Hence, $H_\xi(t^*) > 0$ and, since we now have $y_\xi(t^*) = y_{\xi t}(t^*) = 0$ and $y_{\xi tt}(t^*) > 0$, there exists a neighborhood of t^* that we again denote $\mathcal{U}$ such that $y_\xi(t) > 0$ for all $t \in \mathcal{U} \setminus \{t^*\}$. This contradicts the fact that $t^* < T$, and we have proved the first inequality in (2.23b), namely that $y_\xi(t) \geq 0$ for all $t \in [0,T]$. Let us prove that $H_\xi(t) \geq 0$ for all $t \in [0,T]$. This follows from (2.23c) when $y_\xi(t) > 0$. Now, if $y_\xi(t) = 0$, then $U_\xi(t) = 0$ from (2.23c) and we have seen that $H_\xi(t) < 0$ would imply that $y_\xi(t') < 0$ for some t' in a punctured neighborhood of t, which is impossible. Hence, $H_\xi(t) \geq 0$ and we have proved the second inequality in (2.23b). Assume that the third inequality in (2.23c) does not hold. Then, by continuity, there exists a time $t \in [0,T]$ such that $(y_\xi + H_\xi)(t) = 0$. Since y_ξ and H_ξ are positive, we must have $y_\xi(t) = H_\xi(t) = 0$ and, by (2.23c), $U_\xi(t) = 0$. Since zero is a solution of (2.12), this implies that $y_\xi(0) = U_\xi(0) = H_\xi(0)$, which contradicts $(y_\xi + H_\xi)(0) > 0$. The fact that $\lim_{\xi\to-\infty} H(t,\xi) = 0$ will be proved below in (*iii*).

(*ii*) We define the set

$$\mathcal{N} = \{(t,\xi) \in [0,T] \times \mathbb{R} \mid y_\xi(t,\xi) = 0\}.$$

Fubini's theorem gives us

$$\operatorname{meas}(\mathcal{N}) = \int_{\mathbb{R}} \operatorname{meas}(\mathcal{N}_\xi)\, d\xi = \int_{[0,T]} \operatorname{meas}(\mathcal{N}_t)\, dt \tag{2.27}$$

where $\mathcal{N}_\xi$ and $\mathcal{N}_t$ are the ξ-section and t-section of $\mathcal{N}$, respectively, that is,

$$\mathcal{N}_\xi = \{t \in [0,T] \mid y_\xi(t,\xi) = 0\} \text{ and } \mathcal{N}_t = \{\xi \in \mathbb{R} \mid y_\xi(t,\xi) = 0\}.$$

Let us prove that, for all $\xi \in \mathcal{A}$, $\operatorname{meas}(\mathcal{N}_\xi) = 0$. If we consider the sets $\mathcal{N}_\xi^n$ defined as

$$\mathcal{N}_\xi^n = \{t \in [0,T] \mid y_\xi(t,\xi) = 0 \text{ and } y_\xi(t',\xi) > 0 \text{ for all } t' \in [t - 1/n, t + 1/n] \setminus \{t\}\},$$

then

$$\mathcal{N}_\xi = \bigcup_{n\in\mathbb{N}} \mathcal{N}_\xi^n. \tag{2.28}$$

Indeed, for all $t \in \mathcal{N}_\xi$, we have $y_\xi(t,\xi) = 0$, $y_{\xi t}(t,\xi) = 0$ from (2.23c) and (2.12) and $y_{\xi tt}(t,\xi) = \frac{1}{2} H_\xi(t,\xi) > 0$ from (2.12) and (2.23b) (y_ξ and H_ξ cannot vanish at the same time for $\xi \in \mathcal{A}$). This implies that, on a small punctured neighborhood of t, y_ξ is strictly positive. Hence, t belongs to some $\mathcal{N}_\xi^n$ for n large enough. This proves (2.28). The set $\mathcal{N}_\xi^n$ consists of isolated points that are countable since, by definition, they are separated by a distance larger than $1/n$ from one another. This means that $\operatorname{meas}(\mathcal{N}_\xi^n) = 0$ and, by the subadditivity of the measure, $\operatorname{meas}(\mathcal{N}_\xi) = 0$. It follows from (2.27) and since $\operatorname{meas}(\mathcal{A}^c) = 0$ that

$$\operatorname{meas}(\mathcal{N}_t) = 0 \text{ for almost every } t \in [0,T]. \tag{2.29}$$

We denote by $\mathcal{K}$ the set of times such that $\operatorname{meas}(\mathcal{N}_t) > 0$, i.e.,

$$\mathcal{K} = \{t \in \mathbb{R}_+ \mid \operatorname{meas}(\mathcal{N}_t) > 0\}. \tag{2.30}$$

By (2.29), $\operatorname{meas}(\mathcal{K}) = 0$. For all $t \in \mathcal{K}^c$, $y_\xi > 0$ almost everywhere and, therefore, $y(t,\xi)$ is strictly increasing and invertible (with respect to ξ).

(*iii*) For any given $t \in [0,T]$, since $H_\xi(t,\xi) \geq 0$, $H(t,\xi)$ is an increasing function with respect to ξ and therefore, as $H(t,\,\cdot\,) \in L^\infty(\mathbb{R})$, $H(t,\xi)$ has a limit when $\xi \to \pm\infty$. We denote those limits $H(t,\pm\infty)$. Since $U(t,\,\cdot\,) \in H^1(\mathbb{R})$, we have $\lim_{\xi\to\pm\infty} U(t,\xi) = 0$ for all $t \in [0,T]$. We have

$$H(t,\xi) = H(0,\xi) + \int_0^t \left[U^3 - 2PU\right](\tau,\xi)\, d\tau. \tag{2.31}$$

We let ξ tend to $\pm\infty$. Since U and P are bounded in $L^\infty([0,T] \times \mathbb{R})$, we can apply the Lebesgue dominated convergence theorem and it follows from (2.31), as $\lim_{\xi\to\pm\infty} U(t,\xi) = 0$, that $H(t,\pm\infty) = H(0,\pm\infty)$ for all $t \in [0,T]$. Since $\bar{X} \in \mathcal{G}$, $H(0,-\infty) = 0$ and therefore $H(t,-\infty) = 0$ for all $t \in [0,T]$. ☐

We are now ready to prove global existence of solutions to (2.10).

Theorem 2.8. *For any $\bar{X} = (\bar{y},\bar{U},\bar{H}) \in \mathcal{G}$, the system (2.10) admits a unique global solution $X(t) = (y(t),U(t),H(t))$ in $C^1(\mathbb{R}_+,E)$ with initial data $\bar{X} = (\bar{y},\bar{U},\bar{H})$. We have $X(t) \in \mathcal{G}$ for all times. If we equip $\mathcal{G}$ with the topology inducted by the E-norm, then the mapping $S\colon \mathcal{G} \times \mathbb{R}_+ \to \mathcal{G}$ defined as*

$$S_t(\bar{X}) = X(t)$$

is a continuous semigroup.

Proof. The solution has a finite time of existence T only if $\|(\zeta,U,H)(t,\,\cdot\,)\|_E$ blows up when t tends to T because, otherwise, by Theorem 2.3, the solution can be prolongated by a small time interval beyond T. Let (ζ,U,H) be a solution of (2.10) in $C([0,T),E)$ with initial data $(\bar{\zeta},\bar{U},\bar{H})$. We want to prove that

$$\sup_{t\in[0,T)} \|(\zeta(t,\,\cdot\,),U(t,\,\cdot\,),H(t,\,\cdot\,))\|_E < \infty. \tag{2.32}$$

We have already seen that $H(t,\xi)$ is an increasing function in ξ for all t and, from Lemma 2.7, we have $\lim_{\xi\to\infty} H(t,\xi) = \lim_{\xi\to\infty} H(0,\xi)$. This shows that $\sup_{t\in[0,T)} \|H(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}$ is bounded by $\|\bar H\|_{L^\infty(\mathbb{R})}$ and therefore is finite. To simplify the notation we suppress the dependence in t for the moment. We have

$$U^2(\xi) = 2\int_{-\infty}^{\xi} U(\eta)U_\xi(\eta)\,d\eta = 2\int_{\{\eta\le\xi \mid y_\xi(\eta)>0\}} U(\eta)U_\xi(\eta)\,d\eta \tag{2.33}$$

since, from (2.23c), $U_\xi(\xi)=0$ when $y_\xi(\xi)=0$. For almost every ξ such that $y_\xi(\xi)>0$, we have

$$|U(\xi)U_\xi(\xi)| = \left|\sqrt{y_\xi}\,U(\xi)\frac{U_\xi(\xi)}{\sqrt{y_\xi(\xi)}}\right| \le \frac12\left(U(\xi)^2 y_\xi(\xi) + \frac{U_\xi^2(\xi)}{y_\xi(\xi)}\right) = \frac12 H_\xi(\xi),$$

from (2.23c). Inserting this inequality in (2.33), we obtain $U^2(\xi)\le H(\xi)$ and $\sup_{t\in[0,T)}\|U(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}$ is therefore finite. Then, from the governing equation (2.10), it follows that

$$|\zeta(t,\xi)| \le |\zeta(0,\xi)| + \sup_{t\in[0,T)} \|U(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}\,T$$

and $\sup_{t\in[0,T)}\|\zeta(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}<\infty$. Next we prove that $\sup_{t\in[0,T)}\|Q(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}<\infty$. After one integration by parts, Q can be rewritten as

$$Q(t,\xi) = -\frac14\int_{\mathbb{R}} e^{-|y(\xi)-y(\eta)|}y_\xi(\eta)\left[\operatorname{sgn}(\xi-\eta)U(\eta)^2 - H(\eta)\right]\,d\eta - \frac12 H(t,\xi).$$

Hence, we get, after a change of variable,

$$|Q(t,\xi)| \le C\int_{\mathbb{R}} e^{-|y(\xi)-y(\eta)|}y_\xi(\eta)\,d\eta + C \le 3C,$$

where the constant C depends only on $\sup_{t\in[0,T)}\|H(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}$ and $\sup_{t\in[0,T)}\|U(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}$. Similarly, one proves that that $\|P(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}<\infty$. We denote

$$\begin{aligned} C_1 = \sup_{t\in[0,T)}\{&\|U(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} + \|H(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} \\ &+ \|\zeta(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} + \|P(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} + \|Q(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}\}. \end{aligned}$$

We have just proved that $C_1<\infty$. Let $t\in[0,T)$. Looking back at (2.16) and the definition of R, we obtain that

$$\|R(t,\,\cdot\,)\|_{L^2(\mathbb{R})} \le C(\|U(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})})$$

for some constant C depending only on C_1. Since A is a continuous linear mapping from $L^2(\mathbb{R})$ to $H^1(\mathbb{R})$, we get

$$\|A\circ R(t,\,\cdot\,)\|_{L^2(\mathbb{R})} \le C(\|U(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})})$$

for another constant C which again only depends on C_1. From now on, we denote generically by C such constants that only depends on C_1. From (2.16), as $\|e^{-\zeta(t,\,\cdot\,)}\|_{L^\infty(\mathbb{R})}\le C$, we obtain that

$$\|Q_1(t,\,\cdot\,)\|_{L^2(\mathbb{R})} \le C(\|U(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})})$$

The same bound holds for Q_2 and therefore

$$\|Q(t,\,\cdot\,)\|_{L^2(\mathbb{R})} \le C(\|U(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})}). \tag{2.34}$$

Similarly, one proves

$$\|P(t,\,\cdot\,)\|_{L^2(\mathbb{R})} \le C(\|U(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})}). \tag{2.35}$$

Let $Z(t) = \|U(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|U_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\,\cdot\,)\|_{L^2(\mathbb{R})}$, then the theorem will be proved once we have established that $\sup_{t\in[0,T)} Z(t)<\infty$. From the integrated version of (2.10) and (2.21), after taking the $L^2(\mathbb{R})$-norms on both sides, adding the relevant terms and using (2.35), we obtain

$$Z(t) \le Z(0) + C\int_0^t Z(\tau)\,d\tau.$$

Hence, Gronwall's lemma gives us that $\sup_{t\in[0,T)} Z(t) < \infty$. From standard ordinary differential equation theory, we infer that S_t is a continuous semi-group. $\square$

3. From Eulerian to Lagrangian coordinates and vice versa

As noted in [4], even if $H^1(\mathbb{R})$ is a natural space for the equation, there is no hope to obtain a semigroup of solutions by only considering $H^1(\mathbb{R})$. Thus, we introduce the following space $\mathcal{D}$, which characterizes the solutions in *Eulerian coordinates*:

Definition 3.1. The set $\mathcal{D}$ is composed of all pairs (u, μ) such that u belongs to $H^1(\mathbb{R})$ and μ is a positive finite Radon measure whose absolute continuous part, $\mu_{\rm ac}$, satisfies

$$\mu_{\rm ac} = (u^2 + u_x^2)\,dx. \tag{3.1}$$

We derived the equivalent system (2.10) by using characteristics. Since y satisfies (2.2), y, for a given ξ, can also be seen as the position of a particle evolving in the velocity field u, where u is the solution of the Camassa–Holm equation. We are then working in *Lagrangian coordinates.* In [13], the Camassa–Holm equation is derived as a geodesic equation on the group of diffeomorphism equipped with a right-invariant metric. In the present paper, the geodesic curves correspond to $y(t, \cdot)$. Note that y does not remain a diffeomorphism since it can become non invertible, which agrees with the fact that the solutions of the geodesic equation may break down, see [11]. The right-invariance of the metric can be interpreted as an invariance with respect to relabeling as noted in [2]. This is a property that we also observe in our setting. We denote by G the subgroup of the group of homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f - \mathrm{Id} \ \text{ and } \ f^{-1} - \mathrm{Id} \ \text{ both belong to } W^{1,\infty}(\mathbb{R}) \tag{3.2}$$

where Id denotes the identity function. The set G can be interpreted as the set of relabeling functions. For any $\alpha > 1$, we introduce the subsets G_α of G defined by

$$G_\alpha = \{f \in G \mid \|f - \mathrm{Id}\|_{W^{1,\infty}(\mathbb{R})} + \left\|f^{-1} - \mathrm{Id}\right\|_{W^{1,\infty}(\mathbb{R})} \leq \alpha\}.$$

The subsets G_α do not possess the group structure of G. The next lemma provides a useful characterization of G_α.

Lemma 3.2. *Let $\alpha \geq 0$. If f belongs to G_α, then $1/(1+\alpha) \leq f_\xi \leq 1+\alpha$ almost everywhere. Conversely, if f is absolutely continuous, $f - \mathrm{Id} \in L^\infty(\mathbb{R})$ and there exists $c \geq 1$ such that $1/c \leq f_\xi \leq c$ almost everywhere, then $f \in G_\alpha$ for some α depending only on c and $\|f - \mathrm{Id}\|_{L^\infty(\mathbb{R})}$.*

Proof. Given $f \in G_\alpha$, let B be the set of points where f^{-1} is differentiable. Rademacher's theorem says that $\mathrm{meas}(B^c) = 0$. For any $\xi \in f^{-1}(B)$, we have

$$\lim_{\xi'\to\xi} \frac{f^{-1}(f(\xi')) - f^{-1}(f(\xi))}{f(\xi') - f(\xi)} = (f^{-1})_\xi(f(\xi))$$

because f is continuous and f^{-1} is differentiable at $f(\xi)$. On the other hand, we have

$$\frac{f^{-1}(f(\xi')) - f^{-1}(f(\xi))}{f(\xi') - f(\xi)} = \frac{\xi' - \xi}{f(\xi') - f(\xi)}.$$

Hence, f is differentiable for any $\xi \in f^{-1}(B)$ and

$$f_\xi(\xi) \geq \frac{1}{\|(f^{-1})_\xi\|_{L^\infty(\mathbb{R})}} \geq \frac{1}{1+\alpha}. \tag{3.3}$$

The estimate (3.3) holds only on $f^{-1}(B)$ but, since $\mathrm{meas}(B^c) = 0$ and f^{-1} is Lipschitz and one-to-one, $\mathrm{meas}(f^{-1}(B^c)) = 0$ (see, e.g., [1, Remark 2.72]), and therefore (3.3) holds almost everywhere. We have $f_\xi \leq 1 + \|f_\xi - 1\|_{L^\infty(\mathbb{R})} \leq 1 + \alpha$.

Let us now consider a function f which is absolutely continuous and such that $f - \mathrm{Id} \in L^\infty(\mathbb{R})$ and $1/c \leq f_\xi \leq c$ almost everywhere for some $c \geq 1$. Since f_ξ is bounded, f and therefore $f - \mathrm{Id}$ are Lipschitz and $f - \mathrm{Id} \in W^{1,\infty}(\mathbb{R})$. Since $f_\xi \geq 1/c$ almost everywhere, f is strictly increasing and, since it is also continuous, it is invertible. As f is Lipschitz, we can make the following change of variables (see, for example, [1]) and get that, for all ξ_1, ξ_2 in $\mathbb{R}$ such that $\xi_1 < \xi_2$,

$$f^{-1}(\xi_2) - f^{-1}(\xi_1) = \int_{[f^{-1}(\xi_1), f^{-1}(\xi_2)]} \frac{f_\xi}{f_\xi}\, d\xi \leq c(\xi_2 - \xi_1).$$

Hence, f^{-1} is Lipschitz and $(f^{-1})_\xi \leq c$. We have $f^{-1}(\xi') - \xi' = \xi - f(\xi)$ for $\xi' = f(\xi)$ and therefore $\|f - \mathrm{Id}\|_{L^\infty(\mathbb{R})} = \left\|f^{-1} - \mathrm{Id}\right\|_{L^\infty(\mathbb{R})}$. Finally, we get

$$\begin{aligned}\|f - \mathrm{Id}\|_{W^{1,\infty}(\mathbb{R})} + \left\|f^{-1} - \mathrm{Id}\right\|_{W^{1,\infty}(\mathbb{R})} &\leq 2\,\|f - \mathrm{Id}\|_{W^{1,\infty}(\mathbb{R})} + 2 \\ &\quad + \|f_\xi\|_{L^\infty(\mathbb{R})} + \left\|(f^{-1})_\xi\right\|_{L^\infty(\mathbb{R})} \\ &\leq 2\,\|f - \mathrm{Id}\|_{L^\infty(\mathbb{R})} + 2 + 2c.\end{aligned}$$

□

We define the subsets $\mathcal{F}_\alpha$ and $\mathcal{F}$ of $\mathcal{G}$ as follows

$$\mathcal{F}_\alpha = \{X = (y, U, H) \in \mathcal{G} \mid y + H \in G_\alpha\},$$

and

$$\mathcal{F} = \{X = (y, U, H) \in \mathcal{G} \mid y + H \in G\}.$$

For $\alpha = 0$, $G_0 = \{\mathrm{Id}\}$. As we will see, the space $\mathcal{F}_0$ will play a special role. These sets are relevant only because they are in some sense preserved by the governing equation (2.10) as the next lemma shows.

Lemma 3.3. *The space $\mathcal{F}$ is preserved by the governing equation* (2.10). *More precisely, given $\alpha, T \geq 0$ and $\bar{X} \in \mathcal{F}_\alpha$, we have*

$$S_t(\bar{X}) \in \mathcal{F}_{\alpha'}$$

for all $t \in [0, T]$ where α' only depends on T, α and $\left\|\bar{X}\right\|_E$.

Proof. Let $\bar{X} = (\bar{y}, \bar{U}, \bar{H}) \in \mathcal{F}_\alpha$, we denote $X(t) = (y(t), U(t), H(t))$ the solution of (2.10) with initial data $\bar{X}$ and set $h(t,\xi) = y(t,\xi) + H(t,\xi)$, $\bar{h}(\xi) = \bar{y}(\xi) + \bar{H}(\xi)$. By definition, we have $\bar{h} \in G_\alpha$ and, from Lemma 3.2, $1/c \leq \bar{h}_\xi \leq c$ almost everywhere, for some constant $c > 1$ depending only α. We consider a fixed ξ and drop it in the notation. Applying Gronwall's inequality backward in time to (2.12), we obtain

$$|y_\xi(0)| + |H_\xi(0)| + |U_\xi(0)| \leq e^{CT}\left(|y_\xi(t)| + |H_\xi(t)| + |U_\xi(t)|\right) \tag{3.4}$$

for some constant C which depends on $\|X(t)\|_{C([0,T],E)}$, which itself depends only on $\left\|\bar{X}\right\|_E$ and T. From (2.23c), we have

$$|U_\xi(t)| \leq \sqrt{y_\xi(t) H_\xi(t)} \leq \frac{1}{2}(y_\xi(t) + H_\xi(t)).$$

Hence, since y_ξ and H_ξ are positive, (3.4) gives us

$$\frac{1}{c} \leq \bar{y}_\xi + \bar{H}_\xi \leq \frac{3}{2} e^{CT}(y_\xi(t) + H_\xi(t)),$$

and $h_\xi(t) = y_\xi(t) + H_\xi(t) \geq \frac{2}{3c} e^{-CT}$. Similarly, by applying Gronwall's lemma forward in time, we obtain $y_\xi(t) + H_\xi(t) \leq \frac{3}{2} c e^{CT}$. We have $\|(y + H)(t) - \xi\|_{L^\infty(\mathbb{R})} \leq \|X(t)\|_{C([0,T],E)} \leq C$ for another constant C which also only depends on $\left\|\bar{X}\right\|_E$ and T. Hence, applying Lemma 3.2, we obtain that $y(t, \cdot\,) + H(t, \cdot\,) \in G_{\alpha'}$ and therefore $X(t) \in \mathcal{F}_{\alpha'}$ for some α' depending only on α, T and $\left\|\bar{X}\right\|_E$. □

For the sake of simplicity, for any $X = (y, U, H) \in \mathcal{F}$ and any function $f \in G$, we denote $(y \circ f, U \circ f, H \circ f)$ by $X \circ f$.

Proposition 3.4. *The map from $G \times \mathcal{F}$ to $\mathcal{F}$ given by $(f, X) \mapsto X \circ f$ defines an action of the group G on $\mathcal{F}$.*

Proof. We have to prove that $X \circ f$ belongs to $\mathcal{F}$ for any $X = (y, U, H) \in \mathcal{F}$ and $f \in G$. We denote $\bar{X} = (\bar{y}, \bar{U}, \bar{H}) = X \circ f$. As compositions of two Lipschitz maps, $\bar{y}$, $\bar{U}$ and $\bar{H}$ are Lipschitz. We have

$$\begin{aligned}\|\bar{y} - \mathrm{Id}\|_{L^\infty(\mathbb{R})} &\leq \|\bar{y} \circ f - f\|_{L^\infty(\mathbb{R})} + \|f - \mathrm{Id}\|_{L^\infty(\mathbb{R})} \\ &\leq \|\bar{y} - \mathrm{Id}\|_{L^\infty(\mathbb{R})} + \|f - \mathrm{Id}\|_{L^\infty(\mathbb{R})} < +\infty.\end{aligned}$$

Hence, $(\bar{y} - \mathrm{Id}, \bar{U}, \bar{H}) \in [W^{1,\infty}(\mathbb{R})]^3$. Let us prove that

$$\bar{y}_\xi = y_\xi \circ f\, f_\xi, \;\; \bar{U}_\xi = U_\xi \circ f\, f_\xi \text{ and } \bar{H}_\xi = H_\xi \circ f\, f_\xi \tag{3.5}$$

almost everywhere. Let B_1 be the set where y is differentiable and B_2 the set where $\bar{y}$ and f are differentiable. Using Radamacher's theorem, we get that $\mathrm{meas}(B_1^c) = \mathrm{meas}(B_2^c) = 0$. For $\xi \in B_3 = B_2 \cap f^{-1}(B_1)$, we consider a sequence ξ_i converging to ξ ($\xi_i \neq \xi$). We have

$$\frac{y(f(\xi_i)) - y(f(\xi))}{f(\xi_i) - f(\xi)} \frac{f(\xi_i) - f(\xi)}{\xi_i - \xi} = \frac{\bar{y}(\xi_i) - \bar{y}(\xi)}{\xi_i - \xi}. \tag{3.6}$$

Since f is continuous, $f(\xi_i)$ converges to $f(\xi)$ and, as y is differentiable at $f(\xi)$, the left-hand side of (3.6) tends to $y_\xi \circ f(\xi)\, f_\xi(\xi)$. The right-hand side of (3.6) tends to $\bar{y}_\xi(\xi)$, and we get that

$$y_\xi(f(\xi)) f_\xi(\xi) = \bar{y}_\xi(\xi) \tag{3.7}$$

for all $\xi \in B_3$. Since f^{-1} is Lipschitz, one-to-one and $\mathrm{meas}(B_1^c) = 0$, we have $\mathrm{meas}(f^{-1}(B_1)^c) = 0$ and therefore (3.7) holds everywhere. One proves the two other identities in (3.5) similarly. From Lemma 3.2, we have that $f_\xi > 0$ almost everywhere. Then, using (3.5) we easily check that (2.23b) and (2.23c) are fulfilled. Thus, we have proved that $(\bar{y} - \mathrm{Id}, \bar{U}, \bar{H})$ fulfills (2.23). It remains to prove that $(\bar{y} - \mathrm{Id}, \bar{U}, \bar{H}) \in E$. Since $f \in G$, $f \in G_\alpha$ for some large enough α and, by Lemma 3.2, there exists a constant $c > 0$ such that $1/c \leq f_\xi \leq c$ almost everywhere. We have, after a change of variables,

$$\|\bar{U}\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} (U \circ f)^2 \, d\xi \leq c \int_{\mathbb{R}} (U \circ f)^2 f_\xi \, d\xi = c \|U\|_{L^2(\mathbb{R})}^2 .$$

Hence, $\bar{U} \in L^2(\mathbb{R})$. Similarly, one proves that $y_\xi - 1$, U_ξ and H_ξ belong to $L^2(\mathbb{R})$ and therefore $(y, U, H) \in \mathcal{G}$. We have $\bar{y} + \bar{H} = (y + H) \circ f$ which implies, since $y + H$ and f belongs to G and G is a group, that $\bar{y} + \bar{H} \in G$. Therefore $\bar{X} \in \mathcal{F}$ and the proposition is proved. □

Since G is acting on $\mathcal{F}$, we can consider the quotient space $\mathcal{F}/G$ of $\mathcal{F}$ with respect to the action of the group G. The equivalence relation on $\mathcal{F}$ is defined as follows: For any $X, X' \in \mathcal{F}$, X and X' are equivalent if there exists $f \in G$ such that $X' = X \circ f$. We denote by $\Pi(X) = [X]$ the projection of $\mathcal{F}$ into the quotient space $\mathcal{F}/G$. We introduce the mapping $\Gamma\colon \mathcal{F} \to \mathcal{F}_0$ given by

$$\Gamma(X) = X \circ (y + H)^{-1}$$

for any $X = (y, U, H) \in \mathcal{F}$. We have $\Gamma(X) = X$ when $X \in \mathcal{F}_0$. It is not hard to prove Γ is invariant under the G action, that is, $\Gamma(X \circ f) = \Gamma(X)$ for any $X \in \mathcal{F}$ and $f \in G$. Hence, there corresponds to Γ a mapping $\tilde{\Gamma}$ from the quotient space $\mathcal{F}/G$ to $\mathcal{F}_0$ given by $\tilde{\Gamma}([X]) = \Gamma(X)$ where $[X] \in \mathcal{F}/G$ denotes the equivalence class of $X \in \mathcal{F}$. For any $X \in \mathcal{F}_0$, we have $\tilde{\Gamma} \circ \Pi(X) = \Gamma(X) = X$. Hence, $\tilde{\Gamma} \circ \Pi_{|\mathcal{F}_0} = \mathrm{Id}_{|\mathcal{F}_0}$. Any topology defined on $\mathcal{F}_0$ is naturally transported into $\mathcal{F}/G$ by this isomorphism. We equip $\mathcal{F}_0$ with the metric induced by the E-norm, i.e., $d_{\mathcal{F}_0}(X, X') = \|X - X'\|_E$ for all $X, X' \in \mathcal{F}_0$. Since $\mathcal{F}_0$ is closed in E, this metric is complete. We define the metric on $\mathcal{F}/G$ as

$$d_{\mathcal{F}/G}([X], [X']) = \|\Gamma(X) - \Gamma(X')\|_E ,$$

for any $[X], [X'] \in \mathcal{F}/G$. Then, $\mathcal{F}/G$ is isometrically isomorphic with $\mathcal{F}_0$ and the metric $d_{\mathcal{F}/G}$ is complete.

Lemma 3.5. *Given $\alpha \geq 0$. The restriction of Γ to $\mathcal{F}_\alpha$ is a continuous mapping from $\mathcal{F}_\alpha$ to $\mathcal{F}_0$.*

Remark 3.6. The mapping Γ is not continuous from $\mathcal{F}$ to $\mathcal{F}_0$. The spaces $\mathcal{F}_\alpha$ were precisely introduced in order to make the mapping Γ continuous.

Proof. As for $\mathcal{F}_0$, we equip $\mathcal{F}_\alpha$ with the topology induced by the E-norm. Let $X_n=(y_n,U_n,H_n)\in\mathcal{F}_\alpha$ be a sequence that converges to $X=(y,U,H)$ in $\mathcal{F}_\alpha$. We denote $\bar{X}_n=(\bar{y}_n,\bar{U}_n,\bar{H}_n)=\Gamma(X_n)$ and $\bar{X}=(\bar{y},\bar{U},\bar{H})=\Gamma(X)$. By definition of $\mathcal{F}_0$, we have $\bar{H}_n=-\bar{\zeta}_n$ (recall that $\zeta_n=y_n-\mathrm{Id}$). Let us prove first that $\bar{H}_n$ tends to $\bar{H}$ in $L^\infty(\mathbb{R})$. We denote $f_n=y_n+H_n$, $f=y+H$, and we have $f_n,f\in G_\alpha$. Thus $\bar{H}_n-\bar{H}=(H_n-H)\circ {f_n}^{-1}+\bar{H}\circ f\circ {f_n}^{-1}-\bar{H}$ and we have

$$\left\|\bar{H}_n-\bar{H}\right\|_{L^\infty(\mathbb{R})}\leq\left\|H_n-H\right\|_{L^\infty(\mathbb{R})}+\left\|\bar{H}\circ f-\bar{H}\circ f_n\right\|_{L^\infty(\mathbb{R})}. \tag{3.8}$$

From the definition of $\mathcal{F}_0$, we know that $\bar{H}$ is Lipschitz with Lipschitz constant smaller than one. Hence,

$$\left\|\bar{H}\circ f-\bar{H}\circ f_n\right\|_{L^\infty(\mathbb{R})}\leq\left\|f_n-f\right\|_{L^\infty(\mathbb{R})}. \tag{3.9}$$

Since H_n and f_n converges to H and f, respectively, in $L^\infty(\mathbb{R})$, from (3.8) and (3.9), we get that $\bar{H}_n$ converges to $\bar{H}$ in $L^\infty(\mathbb{R})$. Let us prove now that $\bar{H}_{n,\xi}$ tend to $\bar{H}_\xi$ in $L^2(\mathbb{R})$. We have $\bar{H}_{n,\xi}-\bar{H}_\xi=\frac{H_{n,\xi}}{f_{n,\xi}}\circ {f_n}^{-1}-\frac{H_\xi}{f_\xi}\circ f^{-1}$ which can be decomposed into

$$\bar{H}_{n,\xi}-\bar{H}_\xi=\left(\frac{H_{n,\xi}-H_\xi}{f_{n,\xi}}\right)\circ {f_n}^{-1}+\frac{H_\xi}{f_{n,\xi}}\circ {f_n}^{-1}-\frac{H_\xi}{f_\xi}\circ f^{-1}. \tag{3.10}$$

Since $f_n\in G_\alpha$, there exists a constant $c>0$ independent of n such that $1/c\geq f_{n,\xi}\geq c$ almost everywhere, see Lemma 3.2. We have

$$\left\|\left(\frac{H_{n,\xi}-H_\xi}{f_{n,\xi}}\right)\circ {f_n}^{-1}\right\|_{L^2(\mathbb{R})}^2=\int_{\mathbb{R}}(H_{n,\xi}-H_\xi)^2\frac{1}{f_{n,\xi}}\,d\xi\leq c\left\|H_{n,\xi}-H_\xi\right\|_{L^2(\mathbb{R})}^2, \tag{3.11}$$

where we have made the change of variables $\xi'={f_n}^{-1}(\xi)$. Hence, the left-hand side of (3.11) converges to zero. If we can prove that $\frac{H_\xi}{f_{n,\xi}}\circ {f_n}^{-1}\to\frac{H_\xi}{f_\xi}\circ f^{-1}$ in $L^2(\mathbb{R})$, then, using (3.10), we get that $\bar{H}_{n,\xi}\to\bar{H}_\xi$ in $L^2(\mathbb{R})$, which is the desired result. We have

$$\frac{H_\xi}{f_{n,\xi}}\circ {f_n}^{-1}=\frac{(\bar{H}_\xi\circ f)f_\xi}{f_{n,\xi}}\circ {f_n}^{-1}=(\bar{H}_\xi\circ g_n)g_{n,\xi}$$

where $g_n=f\circ {f_n}^{-1}$. Let us prove that $\lim_{n\to\infty}\left\|g_{n,\xi}-1\right\|_{L^2(\mathbb{R})}=0$. We have, after using a change of variables,

$$\left\|g_{n,\xi}-1\right\|_{L^2(\mathbb{R})}^2=\int_{\mathbb{R}}\left(\frac{f_\xi}{f_{n,\xi}}\circ {f_n}^{-1}-1\right)^2d\xi=c\left\|f_\xi-f_{n,\xi}\right\|_{L^2(\mathbb{R})}^2. \tag{3.12}$$

Hence, since $f_{n,\xi}\to f_\xi$ in $L^2(\mathbb{R})$, $\lim_{n\to\infty}\left\|g_{n,\xi}-1\right\|_{L^2(\mathbb{R})}=0$. We have

$$\left\|\bar{H}_\xi\circ g_n g_{n,\xi}-\bar{H}_\xi\right\|_{L^2(\mathbb{R})}\leq\left\|\bar{H}_\xi\circ g_n\right\|_{L^\infty(\mathbb{R})}\left\|g_{n,\xi}-1\right\|_{L^2(\mathbb{R})}+\left\|\bar{H}_\xi\circ g_n-\bar{H}_\xi\right\|_{L^2(\mathbb{R})}. \tag{3.13}$$

We have $\left\|\bar{H}_\xi\circ g_n\right\|_{L^\infty(\mathbb{R})}\leq 1$ since, as we already noted, $\bar{H}$ is Lipschitz with Lipschitz constant smaller than one. Hence, the first term in the sum in (3.13) converges to zero. As far as the second term is concerned, one can always approximate $\bar{H}_\xi$ in $L^2(\mathbb{R})$ by a continuous function h with compact support. After observing that $1/c^2\leq g_{n,\xi}\leq c^2$ almost everywhere, we can prove, as we have done several times now, that $\left\|H_\xi\circ g_n-h\circ g_n\right\|_{L^2(\mathbb{R})}^2\leq c^2\left\|H_\xi-h\right\|_{L^2(\mathbb{R})}^2$ and $h\circ g_n$ can be chosen arbitrarily close to $H_\xi\circ g_n$ in $L^2(\mathbb{R})$ independently of n, that is, for all $\varepsilon>0$, there exists h such that

$$\left\|H_\xi\circ g_n-h\circ g_n\right\|_{L^2(\mathbb{R})}\leq\frac{\varepsilon}{3}\ \text{ and }\ \left\|H_\xi-h\right\|_{L^2(\mathbb{R})}\leq\frac{\varepsilon}{3} \tag{3.14}$$

for all n. Since $f_n\to f$ in $L^\infty(\mathbb{R})$, $g_n\to\mathrm{Id}$ in $L^\infty(\mathbb{R})$ and there exists a compact K independent of n such that $\mathrm{supp}(h\circ g_n)\subset K$. Then, by the Lebesgue dominated convergence theorem, we obtain that $h\circ g_n\to h$ in $L^2(\mathbb{R})$. Hence, for n large enough, we have $\left\|h\circ g_n-h\right\|_{L^2(\mathbb{R})}\leq\frac{\varepsilon}{3}$ which, together with (3.14), implies $\left\|\bar{H}_\xi\circ g_n-\bar{H}_\xi\right\|_{L^2(\mathbb{R})}\leq\varepsilon$, and $\bar{H}_\xi\circ g_n\to\bar{H}_\xi$ in $L^2(\mathbb{R})$. From (3.10), (3.11), (3.12) and (3.13), we obtain that $\bar{H}_{n,\xi}\to\bar{H}_\xi$ in $L^2(\mathbb{R})$. It follows that $\bar{\zeta}_{n,\xi}\to\bar{\zeta}_\xi$ in $L^2(\mathbb{R})$ and, similarly, one proves that $\bar{U}_{n,\xi}\to\bar{U}_\xi$ in $L^2(\mathbb{R})$. It remains to prove that $U_n\to U$ in $L^2(\mathbb{R})$. We write

$$\bar{U}_n-\bar{U}=(U_n-U)\circ {f_n}^{-1}+U\circ {f_n}^{-1}-U\circ f^{-1}. \tag{3.15}$$

We have, after a change of variable,

$$\left\|(U_n - U)\circ {f_n}^{-1}\right\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} (U_n - U)^2 f_{n,\xi}\,d\xi \leq c\left\|U_n - U\right\|_{L^2(\mathbb{R})}^2. \tag{3.16}$$

We also have, after the same change of variable, that

$$\left\|U\circ {f_n}^{-1} - U\circ f^{-1}\right\|_{L^2(\mathbb{R})}^2 \leq c\int_{\mathbb{R}} (U - U\circ f^{-1}\circ f_n)^2\,d\xi. \tag{3.17}$$

By approximating U by continuous functions with compact support as we did before, we prove that $\int_{\mathbb{R}}(U - U\circ f^{-1}\circ f_n)^2$ tends to zero. Hence, by (3.15), (3.16) and (3.17), we get that $\bar{U}_n \to U$ in $L^2(\mathbb{R})$, which concludes the proof of the lemma. □

3.1. **Continuous semigroup of solutions in $\mathcal{F}/G$.** We denote by $S\colon \mathcal{F}\times\mathbb{R}_+ \to \mathcal{F}$ the continuous semigroup which to any initial data $\bar{X}\in\mathcal{F}$ associates the solution $X(t)$ of the system of differential equation (2.10) at time t. As we indicated earlier, the Camassa–Holm equation is invariant with respect to relabeling, more precisely, using our terminology, we have the following result.

Theorem 3.7. *For any $t > 0$, the mapping $S_t\colon \mathcal{F}\to\mathcal{F}$ is G-equivariant, that is,*

$$S_t(X\circ f) = S_t(X)\circ f \tag{3.18}$$

for any $X\in\mathcal{F}$ and $f\in G$. Hence, the mapping $\tilde{S}_t$ from $\mathcal{F}/G$ to $\mathcal{F}/G$ given by

$$\tilde{S}_t([X]) = [S_t X]$$

is well-defined. It generates a continuous semigroup.

Proof. For any $X_0 = (y_0, U_0, H_0)\in\mathcal{F}$ and $f\in G$, we denote $\bar{X}_0 = (\bar{y}_0, \bar{U}_0, \bar{H}_0) = X_0\circ f$, $X(t) = S_t(X_0)$ and $\bar{X}(t) = S_t(\bar{X}_0)$. We claim that $X(t)\circ f$ satisfies (2.10) and therefore, since $X(t)\circ f$ and $\bar{X}(t)$ satisfy the same system of differential equation with the same initial data, they are equal. We denote $\hat{X}(t) = (\hat{y}(t), \hat{U}(t), \hat{H}(t)) = X(t)\circ f$. We have

$$\hat{U}_t = \frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(\hat{y}(\xi) - y(\eta))\big)\left[U(\eta)^2 y_\xi(\eta) + H_\xi(\eta)\right]d\eta. \tag{3.19}$$

We have $\hat{y}_\xi(\xi) = y_\xi(f(\xi))f_\xi(\xi)$ and $\hat{H}_\xi(\xi) = H_\xi(f(\xi))f_\xi(\xi)$ for almost every $\xi\in\mathbb{R}$. Hence, after the change of variable $\eta = f(\eta')$, we get from (3.19) that

$$\hat{U}_t = \frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(\hat{y}(\xi) - \hat{y}(\eta))\big)\left[\hat{U}(\eta)^2 \hat{y}_\xi(\eta) + \hat{H}_\xi(\eta)\right]d\eta.$$

We treat similarly the other terms in (2.10), and it follows that $(\hat{y}, \hat{U}, \hat{H})$ is a solution of (2.10). Since $(\hat{y}, \hat{U}, \hat{H})$ and $(\bar{y}, \bar{U}, \bar{H})$ satisfy the same system of ordinary differential equations with the same initial data, they are equal, i.e., $\bar{X}(t) = X(t)\circ f$ and (3.18) is proved. We have the following diagram:

$$\begin{tikzcd}
\mathcal{F}_0 \arrow[r, "\Pi"] & \mathcal{F}/G \\
\mathcal{F}_\alpha \arrow[u, "\Gamma"] & \\
\mathcal{F}_0 \arrow[u, "S_t"] \arrow[r, "\Pi"] & \mathcal{F}/G \arrow[uu, "\tilde{S}_t"']
\end{tikzcd} \tag{3.20}$$

on a bounded domain of $\mathcal{F}_0$ whose diameter together with t determines the constant α, see Lemma 3.3. By the definition of the metric on $\mathcal{F}/G$, the mapping Π is an isometry from $\mathcal{F}_0$ to $\mathcal{F}/G$. Hence, from the diagram (3.20), we see that $\tilde{S}_t\colon \mathcal{F}/G\to\mathcal{F}/G$ is continuous if and only if $\Gamma\circ S_t\colon \mathcal{F}_0\to\mathcal{F}_0$ is continuous. Let us prove that $\Gamma\circ S_t\colon \mathcal{F}_0\to\mathcal{F}_0$ is sequentially continuous. We consider a sequence $X_n\in\mathcal{F}_0$ that converges to $X\in\mathcal{F}_0$ in $\mathcal{F}_0$, that is, $\lim_{n\to\infty}\left\|X_n - X\right\|_E = 0$. From Theorem 2.8, we get that $\lim_{n\to\infty}\left\|S_t(X_n) - S_t(X)\right\|_E = 0$. Since $X_n\to X$ in E, there exists a constant $C\geq 0$ such that $\|X_n\|\leq C$ for all n. Lemma 3.3 gives us that $S_t(X_n)\in\mathcal{F}_\alpha$ for some α

which depends on C and t. Hence, $S_t(X_n) \to S_t(X)$ in $\mathcal{F}_\alpha$. Then, by Lemma 3.5, we obtain that $\Gamma \circ S_t(X_n) \to \Gamma \circ S_t(X)$ in $\mathcal{F}_0$. □

3.2. **Mappings between the two coordinate systems.** Our next task is to derive the correspondence between Eulerian coordinates (functions in $\mathcal{D}$) and Lagrangian coordinates (functions in $\mathcal{F}/G$). Earlier we considered initial data in $\mathcal{D}$ with a special structure: The energy density μ was given by $(u^2+u_x^2)\,dx$ and therefore μ did not have any singular part. The set $\mathcal{D}$ however allows the energy density to have a singular part and a positive amount of energy can concentrate on a set of Lebesgue measure zero. We constructed corresponding initial data in $\mathcal{F}_0$ by the means of (2.22). This construction can be generalized in the following way. Let us denote by $L\colon \mathcal{D} \to \mathcal{F}/G$ the mapping transforming Eulerian coordinates into Lagrangian coordinates whose definition is contained in the following theorem.

Theorem 3.8. *For any (u, μ) in $\mathcal{D}$, let*

$$y(\xi) = \sup\{y \mid \mu((-\infty, y)) + y < \xi\}\,, \tag{3.21a}$$

$$H(\xi) = \xi - y(\xi), \tag{3.21b}$$

$$U(\xi) = u\circ y(\xi)\,. \tag{3.21c}$$

Then $(y, U, H) \in \mathcal{F}_0$. We define $L(u, \mu) \in \mathcal{F}/G$ to be the equivalence class of (y, U, H).

Proof. Clearly the definition of y yields an increasing function and $\lim_{\xi\to\pm\infty} y(\xi) = \pm\infty$. For any $z > y(\xi)$, we have $\xi \le z + \mu((-\infty, z))$. Hence, $\xi - z \le \mu(\mathbb{R})$ and, since we can choose z arbitrarily close to $y(\xi)$, we get $\xi - y(\xi) \le \mu(\mathbb{R})$. It is not hard to check that $y(\xi) \le \xi$. Hence,

$$|y(\xi) - \xi| \le \mu(\mathbb{R}) \tag{3.22}$$

and $\|y - \mathrm{Id}\|_{L^\infty(\mathbb{R})} \le \mu(\mathbb{R})$ and $y - \mathrm{Id} \in L^\infty(\mathbb{R})$. Let us prove that y is Lipschitz with Lipschitz constant at most one. We consider ξ, ξ' in $\mathbb{R}$ such that $\xi < \xi'$ and $y(\xi) < y(\xi')$ (the case $y(\xi) = y(\xi')$ is straightforward). It follows from the definition that there exists an increasing sequence, x_i', and a decreasing one, x_i such that $\lim_{i\to\infty} x_i = y(\xi)$, $\lim_{i\to\infty} x_i' = y(\xi')$ with $\mu((-\infty, x_i')) + x_i' < \xi'$ and $\mu((-\infty, x_i)) + x_i \ge \xi$. Subtracting the these two inequalities one to the other, we obtain

$$\mu((-\infty, x_i')) - \mu((-\infty, x_i)) + x_i' - x_i < \xi' - \xi. \tag{3.23}$$

For i large enough, since by assumption $y(\xi) < y(\xi')$, we have $x_i < x_i'$ and therefore $\mu((-\infty, x_i')) - \mu((-\infty, x_i)) = \mu([x_i, x_i')) \ge 0$. Hence, $x_i' - x_i < \xi' - \xi$. Letting i tend to infinity, we get $y(\xi') - y(\xi) \le \xi' - \xi$. Hence, y is Lipschitz with Lipschitz constant bounded by one and, by Rademacher's theorem, differentiable almost everywhere. Following [19], we decompose μ into its absolute continuous, singular continuous and singular part, denoted μ_{ac}, μ_{sc} and μ_{s}, respectively. Here, since $(u, \mu) \in \mathcal{D}$, we have $\mu_{\mathrm{ac}} = (u^2 + u_x^2)\,dx$. The support of μ_{s} consists of a countable set of points. Let $F(x) = \mu((-\infty, x))$, then F is lower semi-continuous and its points of continuity exactly coincide with the support of μ_{s} (see [19]). Let A denote the complement of $y^{-1}(\mathrm{supp}(\mu_{\mathrm{s}}))$. We claim that for any $\xi \in A$, we have

$$\mu((-\infty, y(\xi))) + y(\xi) = \xi. \tag{3.24}$$

From the definition of $y(\xi)$ follows the existence of an increasing sequence x_i which converges to $y(\xi)$ and such that $F(x_i) + x_i < \xi$. Since F is lower semi-continuous, $\lim_{i\to\infty} F(x_i) = F(y(\xi))$ and therefore

$$F(y(\xi)) + y(\xi) \le \xi. \tag{3.25}$$

Let us assume that $F(y(\xi)) + y(\xi) < \xi$. Since $y(\xi)$ is a point of continuity of F, we can then find an x such that $x > y(\xi)$ and $F(x) + x < \xi$. This contradicts the definition of $y(\xi)$ and proves our claim (3.24). In order to check that (2.23c) is satisfied, we have to compute y_ξ and U_ξ. We define the set B_1 as

$$B_1 = \left\{x \in \mathbb{R} \mid \lim_{\rho\downarrow 0} \frac{1}{2\rho}\mu((x-\rho, x+\rho)) = (u^2 + u_x^2)(x)\right\}.$$

Since $(u^2 + u_x^2)\,dx$ is the absolutely continuous part of μ, we have, from Besicovitch's derivation theorem (see [1]), that $\mathrm{meas}(B_1^c) = 0$. Given $\xi \in y^{-1}(B_1)$, we denote $x = y(\xi)$. We claim that for all $i \in \mathbb{N}$, there exists $0 < \rho < \frac{1}{i}$ such that $x-\rho$ and $x+\rho$ both belong to $\operatorname{supp}(\mu_{\mathrm{s}})^c$. Assume namely the opposite. Then for any $z \in (x-\frac{1}{i}, x+\frac{1}{i})\setminus\operatorname{supp}(\mu_{\mathrm{s}})$, we have that $z' = 2x-z$ belongs to $\operatorname{supp}(\mu_{\mathrm{s}})$. Thus we can construct an injection between the uncountable set $(x-\frac{1}{i}, x+\frac{1}{i})\setminus\operatorname{supp}(\mu_{\mathrm{s}})$ and the countable set $\operatorname{supp}(\mu_{\mathrm{s}})$. This is impossible, and our claim is proved. Hence, since y is surjective, we can find two sequences ξ_i and ξ_i' in A such that $\frac{1}{2}(y(\xi_i)+y(\xi_i')) = y(\xi)$ and $y(\xi_i') - y(\xi_i) < \frac{1}{i}$. We have, by (3.24), since $y(\xi_i)$ and $y(\xi_i')$ belong to A,

$$\mu([y(\xi_i), y(\xi_i'))) + y(\xi_i') - y(\xi_i) = \xi_i' - \xi_i. \tag{3.26}$$

Since $y(\xi_i) \notin \operatorname{supp}(\mu_{\mathrm{s}})$, $\mu(\{y(\xi_i)\}) = 0$ and $\mu([y(\xi_i), y(\xi_i'))) = \mu((y(\xi_i), y(\xi_i')))$. Dividing (3.26) by $\xi_i' - \xi_i$ and letting i tend to ∞, we obtain

$$y_\xi(\xi)(u^2 + u_x^2)(y(\xi)) + y_\xi(\xi) = 1 \tag{3.27}$$

where y is differentiable in $y^{-1}(B_1)$, that is, almost everywhere in $y^{-1}(B_1)$. We now derive a short lemma which will be useful several times in this proof.

Lemma 3.9. *Given a Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$, for any set B of measure zero, we have $f_\xi = 0$ almost everywhere in $f^{-1}(B)$.*

Proof of Lemma 3.9. The Lemma follows directly from the area formula:

$$\int_{f^{-1}(B)} f_\xi(\xi)\, d\xi = \int_{\mathbb{R}} \mathcal{H}^0\left(f^{-1}(B) \cap f^{-1}(\{x\})\right)\, dx \tag{3.28}$$

where $\mathcal{H}^0$ is the multiplicity function, see [1] for the formula and the precise definition of $\mathcal{H}^0$. The function $\mathcal{H}^0\left(f^{-1}(B)\cap f^{-1}(\{x\})\right)$ is Lebesgue measurable (see [1]) and it vanishes on B^c. Hence, $\int_{f^{-1}(B)} f_\xi\, d\xi = 0$ and therefore, since $f_\xi \geq 0$, $f_\xi = 0$ almost everywhere in $f^{-1}(B)$. $\square$

We apply Lemma 3.9 to B_1^c and get, since $\operatorname{meas}(B_1^c) = 0$, that $y_\xi = 0$ almost everywhere on $y^{-1}(B_1^c)$. On $y^{-1}(B_1)$, we proved that y_ξ satisfies (3.27). It follows that $0 \leq y_\xi \leq 1$ almost everywhere, which implies, since $H_\xi = 1 - y_\xi$, that $H_\xi \geq 0$. In the same way as we proved that y was Lipschitz with Lipschitz constant at most one, we can prove that the function $\xi \mapsto \int_{-\infty}^{y(\xi)} u_x^2\, dx$ is also Lipschitz with Lipschitz constant at most one. Indeed, from (3.23), for i large enough, we have

$$\int_{x_i}^{x_i'} u_x^2\, dx \leq \mu_{\mathrm{ac}}([x_i, x_i')) \leq \mu([x_i, x_i')) < \xi' - \xi.$$

Since $\lim_{i\to\infty} x_i' = y(\xi')$ and $\lim_{i\to\infty} x_i = y(\xi)$, letting i tend to infinity, we obtain $\int_{y(\xi)}^{y(\xi')} u_x^2\, dx < \xi' - \xi$ and the function $\xi \mapsto \int_{-\infty}^{y(\xi)} u_x^2\, dx$ is Lipchitz with Lipschitz coefficient at most one. For all $(\xi, \xi') \in \mathbb{R}^2$, we have, after using the Cauchy–Schwarz inequality,

$$\begin{aligned} |U(\xi') - U(\xi)| &= \int_{y(\xi)}^{y(\xi')} u_x\, dx \\ &\leq \sqrt{y(\xi') - y(\xi)}\sqrt{\int_{y(\xi)}^{y(\xi')} u_x^2\, dx} \\ &\leq |\xi' - \xi| \end{aligned} \tag{3.29}$$

because y and $\int_{-\infty}^{y(\xi)} u_x^2\, dx$ are Lipschitz with Lipschitz constant at most one. Hence, U is also Lipschitz and therefore differentiable almost everywhere. We denote by B_2 the set of Lebesgue points of u_x in B_1, i.e.,

$$B_2 = \{x \in B_1 \mid \lim_{\rho\to 0} \frac{1}{\rho}\int_{x-\rho}^{x+\rho} u_x(t)\, dt = u_x(x)\}.$$

We have $\text{meas}(B_2^c) = 0$. We choose a sequence ξ_i and ξ_i' such that $\frac{1}{2}(y(\xi_i) + y(\xi_i')) = x$ and $y(\xi_i') - y(\xi_i) \leq \frac{1}{i}$. Thus

$$\frac{U(\xi_i') - U(\xi_i)}{\xi_i' - \xi_i} = \frac{\int_{y(\xi_i)}^{y(\xi_i')} u_x(t)\,dt}{y(\xi_i') - y(\xi_i)} \frac{y(\xi_i') - y(\xi_i)}{\xi_i' - \xi_i}.$$

Hence, letting i tend to infinity, we get that for every ξ in $y^{-1}(B_2)$ where U and y are differentiable, that is, almost everywhere on $y^{-1}(B_2)$,

$$U_\xi(\xi) = y_\xi(\xi)u_x(y(\xi)). \tag{3.30}$$

From (3.29) and using the fact that $\int_{-\infty}^{y(\xi)} u_x^2\,dx$ is Lipschitz with Lipschitz constant at most one, we get

$$\left|\frac{U(\xi') - U(\xi)}{\xi' - \xi}\right| \leq \sqrt{\frac{y(\xi') - y(\xi)}{\xi' - \xi}}.$$

Hence, for almost every ξ in $y^{-1}(B_2^c)$, we have

$$|U_\xi(\xi)| \leq \sqrt{y_\xi(\xi)}. \tag{3.31}$$

Since $\text{meas}(B_2^c) = 0$, we have by Lemma 3.9, that $y_\xi = 0$ almost everywhere on $y^{-1}(B_2^c)$. Hence, $U_\xi = 0$ almost everywhere on $y^{-1}(B_2^c)$. Thus, we have computed U_ξ almost everywhere. It remains to verify (2.23c). We have, after using (3.27) and (3.30), that $y_\xi H_\xi = y_\xi(1 - y_\xi) = y_\xi^2(u^2 + u_x^2) \circ y$ and, finally, $y_\xi H_\xi = y_\xi^2 U^2 + U_\xi^2$ almost everywhere on $y^{-1}(B_2)$. On $y^{-1}(B_2^c)$, we have $y_\xi = U_\xi = 0$ almost everywhere. Therefore (2.23c) is satisfied almost everywhere. Up to now we have proved that $X = (y, U, H)$ satisfies (2.23a), (2.23c), the three inequalities in (2.23b) and, by definition, $y + H = \text{Id}$. It remains to prove that $X \in E$ and $\lim_{\xi\to-\infty} H(\xi) = 0$. From (3.24), we have $H(\xi) = \mu((-\infty, y(\xi)))$ for any $\xi \in A$. We can find a sequence $\xi_i \in A$ such that $\lim_{i\to\infty} \xi_i = -\infty$ and we have $\lim_{i\to\infty} H(\xi_i) = 0$. Since H is monotone, it implies that $\lim_{\xi\to-\infty} H(\xi) = 0$. From (3.22) and (3.21b), we obtain $\|H\|_{L^\infty(\mathbb{R})} \leq \mu(\mathbb{R})$. We have, since $H_\xi \geq 0$,

$$\|H_\xi\|_{L^2(\mathbb{R})}^2 \leq \|H_\xi\|_{L^\infty(\mathbb{R})} \|H_\xi\|_{L^1(\mathbb{R})} \leq \|H\|_{L^\infty(\mathbb{R})}^2 \leq \mu(\mathbb{R})$$

and $H \in V$. Since $\zeta = -H$, we have $\zeta \in V$. From (2.23c) we obtain

$$\|U_\xi\|_{L^2(\mathbb{R})}^2 \leq \|y_\xi H_\xi\|_{L^1(\mathbb{R})} \leq (1 + \|\zeta_\xi\|_{L^\infty(\mathbb{R})}) \|H\|_{L^\infty(\mathbb{R})}.$$

Hence, $U_\xi \in L^2(\mathbb{R})$. Let $B_3 = \{\xi \in \mathbb{R} \mid y_\xi < \frac{1}{2}\}$. Since $\zeta_\xi = y_\xi - 1$ and $y_\xi \geq 0$, $B_3 = \{\xi \in \mathbb{R} \mid |\zeta_\xi| > \frac{1}{2}\}$ and, after using the Chebychev inequality, as $\zeta_\xi \in L^2(\mathbb{R})$, we obtain $\text{meas}(B_3) < \infty$. Hence,

$$\begin{aligned}
\int_{\mathbb{R}} U^2(\xi)\,d\xi &= \int_{B_3} U^2(\xi)\,d\xi + \int_{B_3^c} U^2(\xi)\,d\xi \\
&\leq \text{meas}(B_3) \|u\|_{L^\infty(\mathbb{R})}^2 + 2\int_{B_3^c} (u \circ y)^2 y_\xi\,d\xi \\
&\leq \text{meas}(B_3) \|U\|_{L^\infty(\mathbb{R})}^2 + 2\|u\|_{L^2(\mathbb{R})}^2,
\end{aligned}$$

after a change of variables. Hence, $U \in L^2(\mathbb{R})$ and, finally, we have $(y - \text{Id}, U, H) \in E$. □

Remark 3.10. If μ is absolutely continuous, then $\mu = (u^2 + u_x^2)dx$ and, from (3.24), we get

$$\int_{-\infty}^{y(\xi)} (u^2 + u_x^2)\,dx + y(\xi) = \xi$$

for all $\xi \in \mathbb{R}$.

At the very beginning, $H(t, \xi)$ was introduced as the energy contained in a strip between $-\infty$ and $y(t, \xi)$, see (2.4). This interpretation still holds. We obtain μ, the energy density in Eulerian coordinates, by pushing forward by y the energy density in Lagrangian coordinates, $H_\xi\,d\xi$. We recall that the push-forward of a measure ν by a measurable function f is the measure $f_{\#}\nu$ defined as

$$f_{\#}\nu(B) = \nu(f^{-1}(B))$$

for all Borel set B. We are led to the mapping M which transforms Lagrangian coordinates into Eulerian coordinates and whose definition is contained in the following theorem.

Theorem 3.11. *Given any element $[X]$ in $\mathcal{F}/G$. Then, (u,μ) defined as follows*

$$u(x) = U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi), \tag{3.32a}$$

$$\mu = y_{\#}(H_\xi \, d\xi) \tag{3.32b}$$

belongs to $\mathcal{D}$ and is independent of the representative $X = (y,U,H) \in \mathcal{F}$ we choose for $[X]$. We denote by $M\colon \mathcal{F}/G \to \mathcal{D}$ the mapping which to any $[X]$ in $\mathcal{F}/G$ associates (u,μ) as given by (3.32).

Proof. First we have to prove that the definition of u makes sense. Since y is surjective, there exists ξ, which may not be unique, such that $x = y(\xi)$. It remains to prove that, given ξ_1 and ξ_2 such that $x = y(\xi_1) = y(\xi_2)$, we have

$$U(\xi_1) = U(\xi_2). \tag{3.33}$$

Since $y(\xi)$ is an increasing function in ξ, we must have $y(\xi) = x$ for all $\xi \in [\xi_1, \xi_2]$ and therefore $y_\xi(\xi) = 0$ in $[\xi_1,\xi_2]$. From (2.23c), we get that $U_\xi(\xi) = 0$ for all $\xi \in [\xi_1,\xi_2]$ and (3.33) follows.

Since y is proper and $H_\xi \, d\xi$ is a Radon measure, we have, see [1, Remark 1.71], that μ is also a Radon measure. For any $\bar X = (\bar y, \bar U, \bar H) \in \mathcal{F}$ which is equivalent to X, we denote $(\bar u, \bar\mu)$ the pair given by (3.32) when we replace X by $\bar X$. There exists $f \in G$ such that $X = \bar X \circ f$. For any x, there exists ξ' such that $x = \bar y(\xi')$ and $\bar u(x) = \bar U(\xi')$. Let $\xi = f^{-1}(\xi')$. As $x = \bar y(\xi') = y(\xi)$, by (3.32a), we get $u(x) = U(\xi)$ and, since $U(\xi) = \bar U(\xi')$, we finally obtain $\bar u(x) = u(x)$. For any function $\phi \in C_b(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \phi \, d\bar\mu = \int_{\mathbb{R}} \phi \circ \bar y(\xi') \bar H_\xi(\xi') \, d\xi',$$

see [1]. Hence, after making the change of variables $\xi' = f(\xi)$, we obtain

$$\int_{\mathbb{R}} \phi \, d\bar\mu = \int_{\mathbb{R}} \phi \circ \bar y \circ f(\xi) \, \bar H_\xi \circ f(\xi) \, f_\xi(\xi) \, d\xi$$

and, since $H_\xi = \bar H_\xi \circ f f_\xi$ almost everywhere,

$$\int_{\mathbb{R}} \phi \, d\bar\mu = \int_{\mathbb{R}} \phi \circ y(\xi) H_\xi(\xi) \, d\xi = \int_{\mathbb{R}} \phi \, d\mu.$$

Since ϕ was arbitrary in $C_b(\mathbb{R})$, we get $\bar\mu = \mu$. This proves that X and $\bar X$ give raise to the same pair (u,μ), which therefore does not depend on the representative of $[X]$ we choose.

Let us prove that $u \in H^1(\mathbb{R})$. We start by proving that $u_x \in L^2(\mathbb{R})$. For any smooth function ϕ, we have, using the change of variable $x = y(\xi)$,

$$\int_{\mathbb{R}} u(x)\phi_x(x) \, dx = \int_{\mathbb{R}} U(\xi)\phi_x(y(\xi)) y_\xi(\xi) \, d\xi = -\int_{\mathbb{R}} U_\xi(\xi)(\phi \circ y)(\xi) \, d\xi, \tag{3.34}$$

after integrating by parts. Let $B_1 = \{\xi \in \mathbb{R} \mid y_\xi(\xi) > 0\}$. Because of (2.23c), and since $y_\xi \geq 0$ almost everywhere, we have $U_\xi = 0$ almost everywhere on B_1^c. Hence, we can restrict the integration domain in (3.34) to B_1. We divide and multiply by $\sqrt{y_\xi}$ the integrand in (3.34) and obtain, after using the Cauchy–Schwarz inequality,

$$\left| \int_{\mathbb{R}} u\phi_x \, dx \right| = \left| \int_{B_1} \frac{U_\xi}{\sqrt{y_\xi}} (\phi \circ y) \sqrt{y_\xi} \, d\xi \right| \leq \sqrt{\int_{B_1} \frac{U_\xi^2}{y_\xi} \, d\xi} \sqrt{\int_{B_1} (\phi \circ y)^2 y_\xi \, d\xi}.$$

By (2.23c), we have $\frac{U_\xi^2}{y_\xi} \leq H_\xi$. Hence, after another change of variables, we get

$$\left| \int_{\mathbb{R}} u\phi_x \, dx \right| \leq \sqrt{H(\infty)} \, \|\phi\|_{L^2(\mathbb{R})},$$

which implies that $u_x \in L^2(\mathbb{R})$. Similarly, taking again a smooth function ϕ, we have

$$\left|\int_{\mathbb{R}} u\phi\,dx\right| = \left|\int_{\mathbb{R}} U(\phi\circ y)y_\xi\,d\xi\right| \leq \|\phi\|_{L^2(\mathbb{R})}\sqrt{\int_{\mathbb{R}} U^2 y_\xi\,d\xi} \leq \sqrt{H(\infty)}\,\|\phi\|_{L^2(\mathbb{R})}$$

because $U^2 y_\xi \leq H_\xi$ from (2.23c). Hence, $u \in L^2(\mathbb{R})$.

Let us prove that the absolute continuous part of μ is equal to $(u^2+u_x^2)\,dx$. We introduce the sets Z and B defined as follows

$$Z = \Big\{\xi\in\mathbb{R} \mid \ y \text{ is differentiable at } \xi \text{ and } y_\xi(\xi)=0 \\ \text{or } y \text{ or } U \text{ are not differentiable at } \xi\Big\}$$

and

$$B = \{x \in y(Z)^c \mid u \text{ is differentiable at } x\}\,.$$

Since u belongs to $H^1(\mathbb{R})$, it is differentiable almost everywhere. We have, since y is Lipschitz and by the definition of Z, that $\operatorname{meas}(y(Z)) = \int_Z y_\xi(\xi)\,d\xi = 0$. Hence, $\operatorname{meas}(B^c)=0$. For any $\xi\in y^{-1}(B)$, we denote $x=y(\xi)$. By necessity, we have $\xi\in Z^c$. Let ξ_i be a sequence converging to ξ such that $\xi_i\neq\xi$ for all i. We write $x_i = y(\xi_i)$. Since $y_\xi(\xi)>0$, for i large enough, $x_i\neq x$. The following quantity is well-defined

$$\frac{U(\xi_i)-U(\xi)}{\xi_i-\xi} = \frac{u(x_i)-u(x)}{x_i-x}\,\frac{x_i-x}{\xi_i-\xi}.$$

Since u is differentiable at x and ξ belongs to Z^c, we obtain, after letting i tend to infinity, that

$$U_\xi(\xi) = u_x(y(\xi))y_\xi(\xi). \tag{3.35}$$

For all subsets B' of B, we have

$$\mu(B') = \int_{y^{-1}(B')} H_\xi\,d\xi = \int_{y^{-1}(B')}\left(U^2+\frac{U_\xi^2}{y_\xi^2}\right)y_\xi\,d\xi.$$

We can divide by y_ξ in the integrand above because y_ξ does not vanish on $y^{-1}(B)$. After a change of variables and using (3.35), we obtain

$$\mu(B') = \int_{B'}(u^2+u_x^2)\,dx. \tag{3.36}$$

Since (3.36) holds for any set $B'\subset B$ and $\operatorname{meas}(B^c)=0$, we have $\mu_{\mathrm{ac}} = (u^2+u_x^2)\,dx$. □

The next theorem shows that the transformation from Eulerian to Lagrangian coordinates is a bijection.

Theorem 3.12. *The mapping M and L are invertible. We have*

$$L\circ M = \mathrm{Id}_{\mathcal{F}/G} \ \text{ and } M\circ L = \mathrm{Id}_{\mathcal{D}}\,.$$

Proof. Given $[X]$ in $\mathcal{F}/G$, we choose $X=(y,U,H)=\tilde\Gamma([X])$ as a representative of $[X]$ and consider (u,μ) given by (3.32) for this particular X. Note that, from the definition of $\tilde\Gamma$, we have $X\in\mathcal{F}_0$. Let $\bar X = (\bar y,\bar U,\bar H)$ be the representative of $L(u,\mu)$ in $\mathcal{F}_0$ given by the formulas (3.21). We claim that $(\bar y,\bar U,\bar H)=(y,U,H)$ and therefore $L\circ M=\mathrm{Id}_{\mathcal{F}/G}$. Let

$$g(x) = \sup\{\xi\in\mathbb{R} \mid y(\xi)<x\}. \tag{3.37}$$

It is not hard to prove, using the fact that y is increasing and continuous, that

$$y(g(x)) = x \tag{3.38}$$

and $y^{-1}((-\infty,x)) = (-\infty,g(x))$. For any $x\in\mathbb{R}$, we have, by (3.32b), that

$$\mu((-\infty,x)) = \int_{y^{-1}((-\infty,x))} H_\xi\,d\xi = \int_{-\infty}^{g(x)} H_\xi\,d\xi = H(g(x))$$

because $H(-\infty)=0$. Since $X\in\mathcal{F}_0$, $y+H=\mathrm{Id}$ and we get

$$\mu((-\infty,x))+x=g(x). \tag{3.39}$$

From the definition of $\bar{y}$, we then obtain that

$$\bar{y}(\xi)=\sup\{x\in\mathbb{R}\mid g(x)<\xi\}. \tag{3.40}$$

For any given $\xi\in\mathbb{R}$, let us consider an increasing sequence x_i tending to $\bar{y}(\xi)$ such that $g(x_i)<\xi$; such sequence exists by (3.40). Since y is increasing and using (3.38), it follows that $x_i\leq y(\xi)$. Letting i tend to ∞, we obtain $\bar{y}(\xi)\leq y(\xi)$. Assume that $\bar{y}(\xi)<y(\xi)$. Then, there exists x such that $\bar{y}(\xi)<x<y(\xi)$ and equation (3.40) then implies that $g(x)\geq\xi$. On the other hand, $x=y(g(x))<y(\xi)$ implies $g(x)<\xi$ because y is increasing, which gives us a contradiction. Hence, we have $\bar{y}=y$. It follows directly from the definitions, since $y+H=\mathrm{Id}$, that $\bar{H}=H$ and $\bar{U}=U$ and we have proved that $L\circ M=\mathrm{Id}_{\mathcal{F}/G}$.

Given (u,μ) in $\mathcal{D}$, we denote by (y,U,H) the representative of $L(u,\mu)$ in $\mathcal{F}_0$ given by (3.21). Then, let $(\bar{u},\bar{\mu})=M\circ L(u,\mu)$. We claim that $(\bar{u},\bar{\mu})=(u,\mu)$. Let g be the function defined as before by (3.37). The same computation that leads to (3.39) now gives

$$\bar{\mu}((-\infty,x))+x=g(x). \tag{3.41}$$

Given $\xi\in\mathbb{R}$, we consider an increasing sequence x_i which converges to $y(\xi)$ and such that $\mu((-\infty,x_i))+x_i<\xi$. The existence of such sequence is guaranteed by (3.21a). Passing to the limit and since $F(x)=\mu((-\infty,x))$ is lower semi-continuous, we obtain $\mu((-\infty,y(\xi)))+y(\xi)\leq\xi$. We take $\xi=g(x)$ and get

$$\mu((-\infty,x))+x\leq g(x). \tag{3.42}$$

From the definition of g, there exists an increasing sequence ξ_i which converges to $g(x)$ such that $y(\xi_i)<x$. The definition (3.21a) of y tells us that $\mu((-\infty,x))+x\geq\xi_i$. Letting i tend to infinity, we obtain $\mu((-\infty,x))+x\geq g(x)$ which, together with (3.42), yields

$$\mu((-\infty,x))+x=g(x). \tag{3.43}$$

Comparing (3.43) and (3.41) we get that $\mu=\bar{\mu}$. It is clear from the definitions that $\bar{u}=u$. Hence, $(\bar{u},\bar{\mu})=(u,\mu)$ and $M\circ L=\mathrm{Id}_{\mathcal{D}}$. □

4. Continuous semigroup of solutions on $\mathcal{D}$

Now comes the justification of all the analysis done in the previous section. The fact that we have been able to establish a bijection between the two coordinate systems, $\mathcal{F}/G$ and $\mathcal{D}$, enables us now to transport the topology defined in $\mathcal{F}/G$ into $\mathcal{D}$. On $\mathcal{D}$ we define the distance $d_{\mathcal{D}}$ which makes the bijection L between $\mathcal{D}$ and $\mathcal{F}/G$ into an isometry:

$$d_{\mathcal{D}}((u,\mu),(\bar{u},\bar{\mu}))=d_{\mathcal{F}/G}(L(u,\mu),L(\bar{u},\bar{\mu})).$$

Since $\mathcal{F}/G$ equipped with $d_{\mathcal{F}/G}$ is a complete metric space, we have the following theorem.

Theorem 4.1. *$\mathcal{D}$ equipped with the metric d_D is a complete metric space.*

For each $t\in\mathbb{R}$, we define the mapping T_t from $\mathcal{D}$ to $\mathcal{D}$ as

$$T_t=M\tilde{S}_tL.$$

We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{D} & \mathcal{F}/G \arrow[l, "M"'] \\ \mathcal{D} \arrow[u, "T_t"] \arrow[r, "L"] & \mathcal{F}/G \arrow[u, "\tilde{S}_t"'] \end{tikzcd} \tag{4.1}$$

Our main theorem reads as follows.

Theorem 4.2. *$T\colon \mathcal{D}\times\mathbb{R}_+\to\mathcal{D}$ (where $\mathcal{D}$ is defined by Definition* 3.1*) defines a continuous semigroup of solutions of the Camassa–Holm equation, that is, given $(\bar u,\bar\mu)\in\mathcal{D}$, if we denote $t\mapsto(u(t),\mu(t))=T_t(\bar u,\bar\mu)$ the corresponding trajectory, then u is a weak solution of the Camassa–Holm equation* (1.4a)*. Moreover μ is a weak solution of the following transport equation for the energy density*

$$\mu_t+(u\mu)_x=(u^3-2Pu)_x. \tag{4.2}$$

Furthermore, we have that

$$\mu(t)(\mathbb{R})=\mu(0)(\mathbb{R})\text{ for all }t \tag{4.3}$$

and

$$\mu(t)(\mathbb{R})=\mu_{ac}(t)(\mathbb{R})=\|u(t)\|_{H^1}^2=\mu(0)(\mathbb{R})\text{ for almost all }t. \tag{4.4}$$

Remark 4.3. We denote the unique solution described in the theorem as a *conservative* weak solution of the Camassa–Holm equation.

Proof. We want to prove that, for all $\phi\in C^\infty(\mathbb{R}_+\times\mathbb{R})$ with compact support,

$$\int_{\mathbb{R}_+\times\mathbb{R}}[-u(t,x)\phi_t(t,x)+u(t,x)u_x(t,x)\phi(t,x)]\,dxdt=\int_{\mathbb{R}_+\times\mathbb{R}}-P_x(t,x)\phi(t,x)\,dxdt \tag{4.5}$$

where P is given by (1.4b) or equivalently (2.6). Let $(y(t),U(t),H(t))$ be a representative of $L(u(t),\mu(t))$ which is solution of (2.10). Since y is Lipschitz in ξ and invertible for $t\in\mathcal{K}^c$ (see (2.30) for the definition of $\mathcal{K}$, in particular, we have $\operatorname{meas}(\mathcal{K})=0$), we can use the change of variables $x=y(t,\xi)$ and, using (3.30), we get

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}}&[-u(t,x)\phi_t(t,x)+u(t,x)u_x(t,x)\phi(t,x)]\,dxdt\\&=\int_{\mathbb{R}_+\times\mathbb{R}}[-U(t,\xi)y_\xi(t,\xi)\phi_t(t,y(t,\xi))+U(t,\xi)U_\xi(t,\xi)\phi(t,y(t,\xi))]\,d\xi dt.\end{aligned} \tag{4.6}$$

Using the fact that $y_t=U$ and $y_{\xi t}=U_\xi$, one easily check that

$$(Uy_\xi\phi\circ y)_t-(U^2\phi)_\xi=Uy_\xi\phi_t\circ y-UU_\xi\phi\circ y+U_ty_\xi\phi\circ y. \tag{4.7}$$

After integrating (4.7) over $\mathbb{R}_+\times\mathbb{R}$, the left-hand side of (4.7) vanishes and we obtain

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}}&[-Uy_\xi\,\phi_t{\circ}y+UU_\xi\,\phi{\circ}y]\,d\xi dt\\&=\frac14\int_{\mathbb{R}_+\times\mathbb{R}^2}\Big[\operatorname{sgn}(\xi-\eta)e^{-\{\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta)\}}\times\big(U^2y_\xi+H_\xi\big)(\eta)y_\xi(\xi)\phi{\circ}y(\xi)\Big]\,d\eta d\xi dt\end{aligned} \tag{4.8}$$

by (2.10). Again, to simplify the notation, we deliberately omitted the t variable. On the other hand, by using the change of variables $x=y(t,\xi)$ and $z=y(t,\eta)$ when $t\in\mathcal{K}^c$, we have

$$\begin{aligned}-\int_{\mathbb{R}_+\times\mathbb{R}}P_x(t,x)\phi(t,x)\,dxdt=\frac12\int_{\mathbb{R}_+\times\mathbb{R}^2}&\Big[\operatorname{sgn}(y(\xi)-y(\eta))e^{-|y(\xi)-y(\eta)|}\\&\times\big(u^2(t,y(\eta))+\frac12u_x^2(t,y(\eta))\big)\phi(t,y(\xi))y_\xi(\eta)y_\xi(\xi)\Big]\,d\eta d\xi dt.\end{aligned}$$

Since, from Lemma 2.7, y_ξ is strictly positive for $t\in\mathcal{K}^c$ and almost every ξ, we can replace $u_x(t,y(t,\eta))$ by $U_\xi(t,\eta)/y_\xi(t,\eta)$, see (3.30), in the equation above and, using the fact that y is an increasing function and the identity (2.23c), we obtain

$$\begin{aligned}-\int_{\mathbb{R}_+\times\mathbb{R}}P_x(t,x)\phi(t,x)\,dxdt=\frac14\int_{\mathbb{R}_+\times\mathbb{R}^2}&\Big[\operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\\&\times\big(U^2y_\xi+H_\xi\big)(\eta)y_\xi(\xi)\phi(t,y(\xi))\Big]\,d\eta d\xi dt.\end{aligned} \tag{4.9}$$

Thus, comparing (4.8) and (4.9), we get

$$\int_{\mathbb{R}_+\times\mathbb{R}}[-Uy_\xi\,\phi_t(t,y)+UU_\xi\,\phi]\,d\xi dt=-\int_{\mathbb{R}_+\times\mathbb{R}}P_x(t,x)\phi(t,x)\,dxdt$$

and (4.5) follows from (4.6). Similarly, one proves that $\mu(t)$ is solution of (4.2). From (3.32a), we obtain

$$\mu(t)(\mathbb{R}) = \int_{\mathbb{R}} H_\xi \, d\xi = H(t,\infty)$$

which is constant in time, see Lemma 2.7 (*iii*). Hence, (4.3) is proved. We know from Lemma 2.7 (*ii*) that, for $t \in \mathcal{K}^c$, $y_\xi(t,\xi) > 0$ for almost every $\xi \in \mathbb{R}$. Given $t \in \mathcal{K}^c$ (the time variable is suppressed in the notation when there is no ambiguity), we have, for any Borel set B,

$$\mu(t)(B) = \int_{y^{-1}(B)} H_\xi \, d\xi = \int_{y^{-1}(B)} \left(U^2 + \frac{U_\xi^2}{y_\xi^2} \right) y_\xi \, d\xi \tag{4.10}$$

from (2.23c) and because $y_\xi(t,\xi) > 0$ almost everywhere for $t \in \mathcal{K}^c$. Since y is one-to-one when $t \in \mathcal{K}^c$ and $u_x \circ y y_\xi = U_\xi$ almost everywhere, we obtain from (4.10) that

$$\mu(t)(B) = \int_B (u^2 + u_x^2)(t,x) \, dx.$$

Hence, as $\operatorname{meas}(\mathcal{K}) = 0$, (4.4) is proved. ◻

5. The topology on $\mathcal{D}$

The metric $d_{\mathcal{D}}$ gives to $\mathcal{D}$ the structure of a complete metric space while it makes continuous the semigroup T_t of conservative solutions for the Camassa–Holm equation as defined in Theorem 4.2. In that respect, it is a suitable metric for the Camassa–Holm equation. However, as the definition of $d_{\mathcal{D}}$ is not straightforward, this metric is not so easy to manipulate and in this section we compare it with more standard topologies. More precisely, we establish that convergence in $H^1(\mathbb{R})$ implies convergence in $(\mathcal{D}, d_{\mathcal{D}})$, which itself implies convergence in $L^\infty(\mathbb{R})$.

Proposition 5.1. *The mapping*

$$u \mapsto (u, (u^2 + u_x^2)dx)$$

is continuous from $H^1(\mathbb{R})$ into $\mathcal{D}$. In other words, given a sequence $u_n \in H^1(\mathbb{R})$ converging to u in $H^1(\mathbb{R})$, then $(u_n, (u_n^2 + u_{nx}^2)dx)$ converges to $(u, (u^2 + u_x^2)dx)$ in $\mathcal{D}$.

Proof. We write $g_n = u_n^2 + u_{n,x}^2$ and $g = u^2 + u_x^2$. Let $X_n = (y_n, U_n, H_n)$ and $X = (y, U, H)$ be the representatives in $\mathcal{F}_0$ given by (3.21) of $L(u_n, (u_n^2 + u_{nx}^2)dx)$ and $L(u, (u^2 + u_x^2)dx)$, respectively. Following Remark 3.10, we have

$$\int_{-\infty}^{y(\xi)} g(x) \, dx + y(\xi) = \xi \ , \ \int_{-\infty}^{y_n(\xi)} g_n(x) \, dx + y_n(\xi) = \xi \tag{5.1}$$

and, after taking the difference between the two equations, we obtain

$$\int_{-\infty}^{y(\xi)} (g - g_n)(x) \, dx + \int_{y_n(\xi)}^{y(\xi)} g_n(x) \, dx + y(\xi) - y_n(\xi) = 0. \tag{5.2}$$

Since g_n is positive, $\left| y - y_n + \int_{y_n}^{y} g_n(x) \, d\xi \right| = |y - y_n| + \left| \int_{y_n}^{y} g_n(x) \, d\xi \right|$ and (5.2) implies

$$|y(\xi) - y_n(\xi)| \le \int_{-\infty}^{y(\xi)} |g - g_n| \ dx \le \|g - g_n\|_{L^1(\mathbb{R})} .$$

Since $u_n \to u$ in $H^1(\mathbb{R})$, $g_n \to g$ in $L^1(\mathbb{R})$ and it follows that $\zeta_n \to \zeta$ and $H_n \to H$ in $L^\infty(\mathbb{R})$. We recall that $\zeta(\xi) = y(\xi) - \xi$ and $H = -\zeta$ (as $X, X_n \in \mathcal{F}_0$). The measures $(u^2 + u_x^2)dx$ and $(u_n^2 + u_{n,x}^2)dx$ have, by definition, no singular part and in that case (3.27) holds almost everywhere, that is,

$$y_\xi = \frac{1}{g \circ y + 1} \quad \text{and} \quad y_{n,\xi} = \frac{1}{g_n \circ y + 1} \tag{5.3}$$

almost everywhere. Hence,

$$\begin{aligned} \zeta_{n,\xi} - \zeta_\xi &= (g \circ y - g_n \circ y_n) y_{n,\xi} y_\xi \\ &= (g \circ y - g \circ y_n) y_{n,\xi} y_\xi + (g \circ y_n - g_n \circ y_n) y_{n,\xi} y_\xi. \end{aligned} \tag{5.4}$$

Since $0 \leq y_\xi \leq 1$, we have

$$\int_{\mathbb{R}} |g \circ y_n - g_n \circ y_n| \, y_{n,\xi} y_\xi \, d\xi \leq \int_{\mathbb{R}} |g \circ y_n - g_n \circ y_n| \, y_{n,\xi} \, d\xi = \|g - g_n\|_{L^1(\mathbb{R})} . \tag{5.5}$$

For any $\varepsilon > 0$, there exists a continuous function h with compact support such that $\|g - h\|_{L^1(\mathbb{R})} \leq \varepsilon/3$. We can decompose the first term in the right-hand side of (5.4) into

$$\begin{aligned}(g \circ y - g \circ y_n) y_{n,\xi} y_\xi = (g \circ y - h \circ y) y_{n,\xi} y_\xi \\ + (h \circ y - h \circ y_n) y_{n,\xi} y_\xi + (h \circ y_n - g \circ y_n) y_{n,\xi} y_\xi .\end{aligned} \tag{5.6}$$

Then, we have

$$\int_{\mathbb{R}} |g \circ y - h \circ y| \, y_{n,\xi} y_\xi \, d\xi \leq \int |g \circ y - h \circ y| \, y_\xi \, d\xi = \|g - h\|_{L^1(\mathbb{R})} \leq \varepsilon/3$$

and, similarly, we obtain $\int_{\mathbb{R}} |g \circ y_n - h \circ y_n| \, y_{n,\xi} y_\xi \, d\xi \leq \varepsilon/3$. Since $y_n \to y$ in $L^\infty(\mathbb{R})$ and h is continuous with compact support, by applying Lebesgue dominated convergence theorem, we obtain $h \circ y_n \to h \circ y$ in $L^1(\mathbb{R})$ and we can choose n big enough so that

$$\int_{\mathbb{R}} |h \circ y - h \circ y_n| \, y_{n,\xi} y_\xi \, d\xi \leq \|h \circ y - h \circ y_n\|_{L^1(\mathbb{R})} \leq \varepsilon/3.$$

Hence, from (5.6), we get that $\int_{\mathbb{R}} |g \circ y - g \circ y_n| \, y_{n,\xi} y_\xi \, d\xi \leq \varepsilon$ so that

$$\lim_{n \to \infty} \int_{\mathbb{R}} |g \circ y - g \circ y_n| \, y_{n,\xi} y_\xi \, d\xi = 0,$$

and, from (5.4) and (5.5), it follows that $\zeta_{n,\xi} \to \zeta_\xi$ in $L^1(\mathbb{R})$. Since $X_n \in \mathcal{F}_0$, $\zeta_{n,\xi}$ is bounded in $L^\infty(\mathbb{R})$ and we finally get that $\zeta_{n,\xi} \to \zeta_\xi$ in $L^2(\mathbb{R})$ and, by (3.21b), $H_{n,\xi} \to H_\xi$ in $L^2(\mathbb{R})$. It remains to prove that $U_n \to U$ in $H^1(\mathbb{R})$. Let $C_n = \{x \in \mathbb{R} \mid g_n(x) > 1\}$. Chebychev's inequality yields $\operatorname{meas}(C_n) \leq \|g_n\|_{L^1(\mathbb{R})}$. Let $B_n = \{\xi \in \mathbb{R} \mid y_{n,\xi}(\xi) < \frac{1}{2}\}$. Since $y_{n,\xi}(g_n \circ y_n + 1) = 1$ almost everywhere, $g_n \circ y_n > 1$ on B_n and therefore $y_n(B_n) \subset C_n$. From (5.1), we get that

$$\operatorname{meas}(y_n(B)) + \int_{y_n(B)} g_n(\xi) \, d\xi = \operatorname{meas}(B) \tag{5.7}$$

for any set B equal to a countable union of disjoint open intervals. Any Borel set B can be "approximated" by such countable union of disjoint open intervals and therefore, using the fact that y_n is Lipschitz and one-to-one, we infer that (5.7) holds for any Borel set B. After taking $B = B_n$, (5.7) yields

$$\begin{aligned}\operatorname{meas}(B_n) &\leq \operatorname{meas}(y_n(B_n)) + \|g_n\|_{L^1(\mathbb{R})} \\ &\leq \operatorname{meas}(C_n) + \|g_n\|_{L^1(\mathbb{R})}\end{aligned}$$

and therefore $\operatorname{meas}(B_n) \leq 2 \|g_n\|_{L^1(\mathbb{R})}$. For any function $f_1, f_2 \in H^1(\mathbb{R})$, we have

$$\|f_1 \circ y_n - f_2 \circ y_n\|^2_{L^2(\mathbb{R})} = \int_{B_n} (f_1 \circ y_n - f_2 \circ y_n)^2 \, d\xi + \int_{B_n^c} (f_1 \circ y_n - f_2 \circ y_n)^2 \, d\xi \tag{5.8}$$

and, as $y_{n,\xi} \geq 0$ on B_n^c,

$$\int_{B_n^c} (f_1 \circ y_n - f_2 \circ y_n)^2 \, d\xi \leq 2 \int_{B_n^c} (f_1 \circ y_n - f_2 \circ y_n)^2 y_{n,\xi} \, d\xi \leq 2 \|f_1 - f_2\|^2_{L^2(\mathbb{R})} .$$

Hence,

$$\|f_1 \circ y_n - f_2 \circ y_n\|^2_{L^2(\mathbb{R})} \leq \operatorname{meas}(B_n) \|f_1 - f_2\|^2_{L^\infty(\mathbb{R})} + 2 \|f_1 - f_2\|^2_{L^2(\mathbb{R})}$$

and, since $\operatorname{meas}(B_n) \leq 2 \|g_n\|_{L^1(\mathbb{R})}$,

$$\begin{aligned}\|f_1 \circ y_n - f_2 \circ y_n\|^2_{L^2(\mathbb{R})} &\leq 2 \|g_n\| \|f_1 - f_2\|^2_{L^\infty(\mathbb{R})} + 2 \|f_1 - f_2\|^2_{L^2(\mathbb{R})} \\ &\leq C \|f_1 - f_2\|^2_{H^1(\mathbb{R})}\end{aligned} \tag{5.9}$$

for some constant C which is independent of n. We have

$$\|U_n - U\|_{L^2(\mathbb{R})} \leq \|u_n \circ y_n - u \circ y_n\|_{L^2(\mathbb{R})} + \|u \circ y_n - u \circ y\|_{L^2(\mathbb{R})} . \tag{5.10}$$

After using (5.9) for $f_1 = u_n$ and $f_2 = u$ and since, by assumption, $u_n \to u$ in $H^1(\mathbb{R})$, we obtain that $\lim_{n\to\infty} \|u_n \circ y_n - u \circ y_n\|_{L^2(\mathbb{R})} = 0$. We can find continuous functions with compact support h which are arbitrarily close to u in $H^1(\mathbb{R})$. Then, from (5.9), $h \circ y_n$ and $h \circ y$ are arbitrarily closed in $L^2(\mathbb{R})$ to $u \circ y_n$ and $u \circ y$, respectively, and independently of n. By the Lebesgue dominated convergence theorem, as $y_n \to y$ in $L^\infty(\mathbb{R})$, we get that $h \circ y_n \to h \circ y$ in $L^2(\mathbb{R})$. Hence,

$$\|u \circ y_n - u \circ y\|_{L^2(\mathbb{R})} \leq \|u \circ y_n - h \circ y_n\|_{L^2(\mathbb{R})} + \|h \circ y_n - h \circ y\|_{L^2(\mathbb{R})} + \|h \circ y - u \circ y\|_{L^2(\mathbb{R})}$$

implies that $\lim_{n\to\infty} \|u \circ y_n - u \circ y\|_{L^2(\mathbb{R})} = 0$ and, finally, from (5.10), we conclude that $U_n \to U$ in $L^2(\mathbb{R})$. It remains to prove that $U_{n,\xi} \to U_\xi$ in $L^2(\mathbb{R})$. Since $H_{n,\xi} = 1 - y_{n,\xi}$, (2.23c) can be rewritten as

$$U_{n,\xi}^2 = H_{n,\xi} - H_{n,\xi}^2 - U_n^2 + H_{n,\xi}^2 U_n^2 \tag{5.11}$$

and there holds the corresponding identity holds for U_ξ. We have $\|U_n\|_{L^\infty(\mathbb{R})} = \|u_n\|_{L^\infty(\mathbb{R})}$ and therefore $\|U_n\|_{L^\infty(\mathbb{R})}$ is uniformly bounded in n. Hence, since $U_n \to U$ in $L^2(\mathbb{R})$, $H_n \to H$ in V and $\|U_n\|_{L^\infty(\mathbb{R})}$, $\|H_{n,\xi}\|_{L^\infty(\mathbb{R})}$ are uniformly bounded in n, we get from (5.11) that

$$\lim_{n\to\infty} \|U_{n,\xi}\|_{L^2(\mathbb{R})} = \|U_\xi\|_{L^2(\mathbb{R})}\,, \tag{5.12}$$

Once we have proved that $U_{n,\xi}$ converges weakly to U_ξ, then (5.11) will imply that $U_{n,\xi} \to U_\xi$ strongly in $L^2(\mathbb{R})$, see, for example, [27, section V.1]. For any continuous function ϕ with compact support, we have

$$\int_{\mathbb{R}} U_{n,\xi}\phi\, d\xi = \int_{\mathbb{R}} u_{n,x} \circ y_n y_{n,\xi}\phi\, d\xi = \int_{\mathbb{R}} u_{n,x}\, \phi \circ {y_n}^{-1}\, d\xi. \tag{5.13}$$

By assumption, we have $u_{n,x} \to u_x$ in $L^2(\mathbb{R})$. Since $y_n \to y$ in $L^\infty(\mathbb{R})$, the support of $\phi \circ {y_n}^{-1}$ is contained in some compact that can be chosen to be independent of n. Thus, after using Lebesgue's dominated convergence theorem, we obtain that $\phi \circ {y_n}^{-1} \to \phi \circ y^{-1}$ in $L^2(\mathbb{R})$ and therefore

$$\lim_{n\to\infty} \int_{\mathbb{R}} U_{n,\xi}\phi\, d\xi = \int_{\mathbb{R}} u_x\, \phi \circ y^{-1}\, d\xi = \int_{\mathbb{R}} U_\xi \phi\, d\xi. \tag{5.14}$$

From (5.12), we have that $U_{n,\xi}$ is bounded and therefore, by a density argument, (5.14) holds for any function ϕ in $L^2(\mathbb{R})$ and $U_{n,\xi} \rightharpoonup U_\xi$ weakly in $L^2(\mathbb{R})$. □

Proposition 5.2. *Let (u_n, μ_n) be a sequence in $\mathcal{D}$ that converges to (u, μ) in $\mathcal{D}$. Then*

$$u_n \to u \text{ in } L^\infty(\mathbb{R}) \quad \text{and} \quad \mu_n \overset{*}{\rightharpoonup} \mu.$$

Proof. We denote by $X_n = (y_n, U_n, H_n)$ and $X = (y, U, H)$ the representative of $L(u_n, \mu_n)$ and $L(u, \mu)$ given by (3.21). For any $x \in \mathbb{R}$, there exists ξ_n and ξ, which may not be unique, such that $x = y_n(\xi_n)$ and $x = y(\xi)$. We set $x_n = y_n(\xi)$. We have

$$u_n(x) - u(x) = u_n(x) - u_n(x_n) + U_n(\xi) - U(\xi) \tag{5.15}$$

and

$$\begin{aligned}
|u_n(x)-u_n(x_n)| &= \left|\int_\xi^{\xi_n} U_{n,\xi}(\eta)\,d\eta\right| \\
&\leq \sqrt{\xi_n-\xi}\left(\int_\xi^{\xi_n} U_{n,\xi}^2\,d\eta\right)^{1/2} && \text{(Cauchy–Schwarz)} \\
&\leq \sqrt{\xi_n-\xi}\left(\int_\xi^{\xi_n} y_{n,\xi}H_{n,\xi}\,d\eta\right)^{1/2} && \text{(from (2.23c))} \\
&\leq \sqrt{\xi_n-\xi}\sqrt{|y_n(\xi_n)-y_n(\xi)|} && \text{(since } H_{n,\xi}\leq 1) \\
&= \sqrt{\xi_n-\xi}\sqrt{y(\xi)-y_n(\xi)} \\
&\leq \sqrt{\xi_n-\xi}\,\|y-y_n\|_{L^\infty(\mathbb{R})}^{1/2}. \qquad (5.16)
\end{aligned}$$

From (3.22), we get

$$|\xi_n-\xi| \leq 2\mu_n(\mathbb{R}) + |y_n(\xi_n)-y_n(\xi)| = 2\lim_{\xi\to\infty} H_n(\xi) + |y(\xi)-y_n(\xi)|$$

and, therefore, since $H_n \to H$ and $y_n \to y$ in $L^\infty(\mathbb{R})$, $|\xi_n-\xi|$ is bounded by a constant C independent of n. Then, (5.16) implies

$$|u_n(x)-u_n(x_n)| \leq C\,\|y-y_n\|_{L^\infty(\mathbb{R})}^{1/2}. \tag{5.17}$$

Since $y_n \to y$ and $U_n \to U$ in $L^\infty(\mathbb{R})$, it follows from (5.15) and (5.17) that $u_n \to u$ in $L^\infty(\mathbb{R})$. By weak-star convergence, we mean that

$$\lim_{n\to\infty}\int_{\mathbb{R}} \phi\,d\mu_n = \int_{\mathbb{R}} \phi\,d\mu \tag{5.18}$$

for all continuous functions with compact support. It follows from (3.32b) that

$$\int_{\mathbb{R}} \phi\,d\mu_n = \int_{\mathbb{R}} \phi\circ y_n H_{n,\xi}\,d\xi \quad\text{and}\quad \int_{\mathbb{R}} \phi\,d\mu = \int_{\mathbb{R}} \phi\circ y H_\xi\,d\xi \tag{5.19}$$

see [1, definition 1.70]. Since $y_n \to y$ in $L^\infty(\mathbb{R})$, the support of $\phi\circ y_n$ is contained in some compact which can be chosen independently of n and, from Lebesgue's dominated convergence theorem, we have that $\phi\circ y_n \to \phi\circ y$ in $L^2(\mathbb{R})$. Hence, since $H_{n,\xi} \to H_\xi$ in $L^2(\mathbb{R})$,

$$\lim_{n\to\infty}\int_{\mathbb{R}} \phi\circ y_n H_{n,\xi}\,d\xi = \int_{\mathbb{R}} \phi\circ y H_\xi\,d\xi,$$

and (5.18) follows from (5.19).

□

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