CONTRACTIVITY OF WASSERSTEIN METRICS AND ASYMPTOTIC PROFILES FOR SCALAR CONSERVATION LAWS

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Abstract. The aim of this paper is to analyze contractivity properties of Wasserstein-type metrics for one-dimensional scalar conservation laws with nonnegative, L^∞ and compactly supported initial data and its implications on the long time asymptotics. The flux is assumed to be convex and without any growth condition at the zero state. We propose a time–parameterized family of functions as *intermediate asymptotics* and prove the solutions, after a time–depending scaling, converge toward this family in the d_∞–Wasserstein metric. This asymptotic behavior relies on the aforementioned contraction property for conservation laws in the space of probability densities metrized with the d_∞–Wasserstein distance. Finally, we also give asymptotic profiles for initial data whose distributional derivative is a probability measure.

1. Introduction

In this paper we show a contractivity of the flow of one–dimensional scalar conservation laws with respect to a distance related to optimal transportation theory and as a byproduct, we obtain new results concerning the asymptotic behavior of their solutions. Given

$$u_t + f(u)_x = 0, \tag{1.1}$$

with initial condition $u(x,0) = \bar{u}(x)$, we assume the flux function $f(u)$ to be convex and $\bar{u} \in L^\infty(\mathbb{R})$, $\mathrm{supp}(\bar{u})$ compact, $\bar{u} \geq 0$ and, without loss of generality, $\int_{\mathbb{R}} \bar{u}(x)dx = 1$. We restrict ourselves to nonnegative initial data mainly because our aim is to treat the solutions of (1.1) as curves in the space of probability densities metrized with the Wasserstein distance.

Our investigations are inspired by the arguments performed in [2], where the asymptotic behavior of the general nonlinear diffusion equation $u_t = \triangle\phi(u)$ is studied. In particular, as a starting point, we shall prove a contraction in a Wasserstein distance. For scalar conservation laws, we prove such property for the d_∞–Wasserstein metric in Theorem 2.5. The proof relies on the explicit Lax-Hopf formula for the

Hamilton–Jacobi equation

$$v_t + f(v_x) = 0$$

satisfied by the primitive $v(x,t)$ of our solution of (1.1) and on standard approximation procedures.

Next, the asymptotic behavior of the Cauchy problem for (1.1) is studied. At it is well known, the asymptotic structure of a scalar conservation law, with flux f satisfying $f''(u) \geq 0$, $f''(u) > 0$ for $u > 0$, is given by the so-called N–wave, that is

$$N_{P,Q}(x,t) = \begin{cases} g(x/t), & -a_P(t) < x < b_Q(t), \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

In (1.2), $g(x)$ is the inverse of $f'(x)$ with $g(0) = 0$, P and Q stand for the invariants of the Cauchy problem for (1.1), namely

$$P = -\inf_{x\in\mathbb{R}} \int_{-\infty}^{x} \bar{u}(y)dy, \quad Q = \sup_{x\in\mathbb{R}} \int_{x}^{+\infty} \bar{u}(y)dy, \quad Q - P = \int_{\mathbb{R}} \bar{u}(y)dy,$$

and $a_P(t)$, $b_Q(t) \geq 0$ verify

$$P = -\int_{-a_P(t)}^{0} g(y/t)dy, \quad Q = \int_{0}^{b_Q(t)} g(y/t)dy.$$

The asymptotic convergence of solutions of (1.1) toward (1.2) has been proved in various cases, all of them requiring a control of the growth of the flux function $f(u)$ near the zero state $u = 0$. The first result in this direction is contained in [10], where the author proved the asymptotic structure of a uniformly convex conservation law is given by the N–wave of the Burgers' equation

$$u_t + \frac{1}{2}(u^2)_x = 0.$$

The same kind of techniques are employed in [7] to prove optimal rate of convergence toward the N–wave, again in the uniform convex case. The case of the power law

$$f(u) = \frac{1}{\gamma}|u|^\gamma, \ \gamma > 1,$$

is investigated in [11]. In that paper, the authors proved the convergence in L^p (with also an algebraic rate for $p > 1$) for solutions with arbitrary L^1 initial data toward the corresponding N–wave. Finally, the most recent results are contained in [8, 9]. In [9] an optimal rate of convergence toward the N–wave is established for solutions of (1.1) with

L^1, compactly supported initial data without sign restrictions and for convex fluxes satisfying the following generalization of the power law

$$\lim_{u\to 0}\frac{uf'(u)}{f(u)} = \gamma, \quad \gamma > 1.$$

The work [8] is devoted to obtain decay rates in L^1, L^∞ and some weighted norms in the particular case of power laws $f(u) = \frac{1}{\gamma}|u|^\gamma$, $\gamma > 1$, by scaling and entropy dissipation like methods.

In the present paper we propose an *intermediate asymptotics* for nonnegative, L^∞ and compactly supported solutions of (1.1), without imposing any growth condition to the convex flux $f(u)$. More precisely (see Theorem 3.5) we prove the solution, after a time–scaling, asymptotically converges in d_∞ toward a uniquely fixed family of functions, the *asymptotic profile*, parameterized by the time variable t. We remark that such a one parameter family of functions reduces to the usual (properly rescaled) N–wave in the power law case, namely whenever the conservation law meets properties of self–similar invariance. Moreover, for fluxes of the form $f(u) = \frac{1}{\gamma}u^\gamma + h(u)$, $\gamma > 1$, with $h(u)u^{-\gamma} \to 0$ as $u \to 0$, we show that the asymptotic profile tends as $t \to +\infty$ to the N–wave corresponding to leading term $\frac{1}{\gamma}u^\gamma$ of the flux f. The existence of this asymptotic profile is based on strict contractivity properties for suitable scaled solutions of the scalar conservation law with respect to the d_∞.

In this context, let us mention that contractivity properties in Wasserstein distances for the derivatives of the solutions to (1.1) were obtained in [1] for non–decreasing initial data whose distributional derivative is a probability measure. We will also make use of these results for obtaining asymptotic properties of scaled solutions. In fact, we will show that the simplest rarefaction and shock waves give the asymptotic profiles for convex and concave fluxes, respectively, for these initial data.

The rest of this paper is organized as follows. In Section 2 we prove the basic contraction property in the d_∞–Wasserstein metric for (1.1), while Section 3 is devoted to the proof of the asymptotic behavior of its solutions. We devote Section 4 to the asymptotic behavior of initial data whose distributional derivatives are probability measures. Finally, Section 5 is devoted to the proof of several approximation results involving the Wasserstein metric d_∞, needed to perform the proof of the main results of this paper.

2. Contraction in the d_∞–Wasserstein metric

In this section we shall obtain some contraction properties for weak, entropy solutions of (1.1), with respect to the d_∞–Wasserstein metric. We shall work within the set of initial data

$$\mathcal{B} = \left\{ u \in L^\infty(\mathbb{R}),\ u \geq 0,\ \mathrm{supp}(u) \text{ compact},\ \int_{\mathbb{R}} u(x)dx = 1 \right\}. \quad (2.1)$$

We remark here that the assumption of unit mass does not affect the generality of the problem, due to the conservation of the total mass

$$\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u(x,0)dx. \quad (2.2)$$

Relation (2.2) enables us to interpret the solutions to (1.1) as curves in the space of probability densities $\mathcal{B}$ metrized by the Wasserstein metric d_∞ defined as the following limit

$$d_\infty(u_1,u_2) = \lim_{p\to+\infty} d_p(u_1,u_2).$$

The Wasserstein (or Monge–Kantorovich) distance of order p [12] is defined by

$$d_p(u_1,u_2)^p = \inf_{T:u_2=T_\sharp u_1} \int_{-\infty}^{+\infty} |x - T(x)|^p u_1(x)dx,$$

where the constraint $u_2 = T_\sharp u_1$ (which is usually referred to as the density u_2 being the *push forward* by T of the density u_1) is expressed by the condition

$$\int_{\mathbb{R}} \varphi(x)u_2(x)dx = \int_{\mathbb{R}} \varphi(T(x))u_1(x)dx,$$

for any $\varphi \in C_0^0(\mathbb{R})$. We recall here that such a distance is well-defined in our framework when the initial data belong to $\mathcal{B}$. Indeed, the comparison principle guarantees the solutions remain bounded in L^∞ and nonnegative for any $t > 0$ and their supports remain bounded (but growing in time), due to finite speed of propagation.

Remark 2.1. In one space dimension, the Wasserstein metrics d_p, $p \in [1,+\infty]$, have a simple interpretation in terms of the pseudo-inverses of the primitive of the involved densities [4, 12]. Let us denote

$$v_i(x) = \int_{-\infty}^{x} u_i(y)dy, \quad i = 1,2$$

and define their pseudo–inverses $v_i^{-1} : [0,1] \to \mathbb{R}$ as follows

$$v_i^{-1}(\xi) = \inf\{x : v_i(x) > \xi\}.$$

Then, for any $p \in [1, +\infty]$,

$$d_p(u_1, u_2) = \|v_1^{-1} - v_2^{-1}\|_{L^p([0,1])}.$$

We start by proving the desired result in the case of an uniformly convex flux f, and when the initial data belongs to the subset of $\mathcal{B}$

$$\mathcal{B}_c = \{u \in \mathcal{B},\ \mathrm{supp}(u) \text{ connected}\}$$

by using the characterization given in Remark 2.1. The general result will be obtained via approximation procedure. Moreover, we assume, without loss of generality, $f(0) = f'(0) = 0$.

Remark 2.2. For further use, we remark that $\mathcal{B}_c$ is dense in $\mathcal{B}$ with respect to the d_p only if p is finite. Indeed, a probability density u having a support with two connected components cannot be approximated in d_∞ by a sequence in $\mathcal{B}_c$, because this would imply uniform convergence of the corresponding (continuous) pseudo–inverses to a discontinuous function. Therefore, $(\mathcal{B}_c, d_\infty)$ is *not* dense in $(\mathcal{B}, d_\infty)$.

As it is well known, given a solution u to (1.1) with initial datum $\bar{u} \in \mathcal{B}$, the primitive

$$v(x,t) = \int_{-\infty}^{x} u(y)dy$$

solves the Cauchy problem for the Hamilton–Jacobi equation

$$\begin{cases} v_t + f(v_x) = 0 \\ v(x,0) = \bar{v}(x) = \displaystyle\int_{-\infty}^{x} \bar{u}(y)dy. \end{cases} \tag{2.3}$$

Since the total mass of $u(t)$ is finite, we have $v(t) \in L^\infty$ at any $t > 0$. Moreover, since u stays bounded in L^∞ for any $t > 0$, the function $v(t)$ is Lipschitz continuous at any $t > 0$. Therefore, the Lax–Hopf formula gives the following explicit expression for the function v

$$v(x,t) = \min_{y \in \mathbb{R}} \left\{ t f^* \left(\frac{x-y}{t} \right) + \bar{v}(y) \right\}, \tag{2.4}$$

where f^* stands for the Legendre transform of the flux f. We remark that f^* is also uniformly convex and satisfies $f^*(0) = (f^*)'(0) = 0$. In the next lemma, we shall state an explicit formula for the pseudo–inverse v^{-1} under the assumption $\bar{u} \in \mathcal{B}_c$.

Lemma 2.3. *Let f be a uniformly convex function and let $\bar{u} \in \mathcal{B}_c$. Let $\bar{v}$ the primitive of $\bar{u}$ defined in* (2.3). *Then, the function $v(t)$ defined* *in* (2.4) *is strictly increasing from* 0 *to* 1 *on a connected interval of* $\mathbb{R}$. *Moreover, for any* $\xi \in (0,1)$, $v^{-1}(\xi,t)$ *verifies*

$$v^{-1}(\xi,t) = \max_{0\leq w\leq \xi}\left\{tF\left(\frac{\xi-w}{t}\right)+\bar{v}^{-1}(w)\right\}, \tag{2.5}$$

where F *is the inverse of* f^* *restricted to* $[0,+\infty)$.

Proof. As a straightforward consequence of generalized characteristic method [6], we can assert that the support of $u(t)$ remains connected for any $t>0$, and this implies that $v(t)$ is a strictly increasing function from 0 to 1 on an (time–depending) interval. Given $\xi \in (0,1)$, the inverse function $v^{-1}(\xi,t)$ is implicitly defined by the relation

$$\xi = \min_{y\in\mathbb{R}}\left\{tf^*\left(\frac{v^{-1}(\xi,t)-y}{t}\right)+\bar{v}(y)\right\}.$$

Let us start by proving that

$$v^{-1}(\xi,t) = \sup_{\{y\in\mathbb{R}:\ \bar{v}(y)\leq\xi\}}\left\{x:\ tf^*\left(\frac{x-y}{t}\right)+\bar{v}(y)=\xi\right\} =: x_0(\xi,t). \tag{2.6}$$

Indeed, let us fix (ξ,t) and assume there exists y with $\bar{v}(y)\leq\xi$ such that

$$v^{-1}(\xi,t) < x,$$

where x is chosen such that

$$tf^*\left(\frac{x-y}{t}\right)+\bar{v}(y)=\xi. \tag{2.7}$$

Then, we apply $v(\cdot,t)$ to obtain

$$\xi < v(x,t),$$

which gives a contradiction because of (2.7) and (2.4). Thus $v^{-1}(\xi,t)\geq x_0(\xi,t)$. Assume now that $v^{-1}(\xi,t) > x_0(\xi,t)$. Then $v^{-1}(\xi,t) > x$ for any y such that

$$tf^*\left(\frac{x-y}{t}\right)+\bar{v}(y)=\xi.$$

We apply once again the function $v(\cdot,t)$ to that relation to conclude

$$\xi > v(x,t) = \min_{y\in\mathbb{R}}\left\{tf^*\left(\frac{x-y}{t}\right)+\bar{v}(y)\right\} = \xi,$$

which is impossible and therefore (2.6) is proved. Now, let us denote with $x = x(y,\xi,t)$ the biggest value x such that

$$f^*\left(\frac{x-y}{t}\right) = \frac{\xi-\bar{v}(y)}{t},$$

for any fixed ξ, y, t. Due to the convexity of f^*, such a value is given by

$$x = tF\left(\frac{\xi - \bar{v}(y)}{t}\right) + y,$$

where $F : [0, +\infty) \to [0, +\infty)$ denotes the inverse of the 'positive branch' of f^*. Therefore

$$\begin{aligned} v^{-1}(\xi, t) &= \sup_{\{y \in \mathbb{R}:\ \bar{v}(y) \leq \xi\}} \left\{ tF\left(\frac{\xi - \bar{v}(y)}{t}\right) + y \right\} \\ &= \sup_{0 \leq w \leq \xi} \left\{ tF\left(\frac{\xi - w}{t}\right) + \bar{v}^{-1}(w) \right\}, \end{aligned} \tag{2.8}$$

where the last step is justified by the strict monotonicity of $\bar{v}$ on the support of $\bar{u}$. Finally, since for any fixed $\xi \in (0, 1)$ the function

$$tF\left(\frac{\xi - w}{t}\right) + \bar{v}^{-1}(w)$$

is continuous with respect to $w \in [0, \xi]$, then the supremum in (2.8) is indeed a maximum and the proof is complete. $\square$

At this point, we are ready to prove the first result of contraction in Wasserstein metric.

Proposition 2.4. *Let us assume that the flux in equation* (1.1) *verifies* $f''(u) > 0$ *for any* u. *Let us consider two solutions* u_1 *and* u_2 *of such equation with initial data* $\bar{u}_1$, $\bar{u}_2 \in \mathcal{B}_c$. *Then, for any* $t > 0$,

$$d_\infty(u_1(t), u_2(t)) \leq d_\infty(\bar{u}_1, \bar{u}_2). \tag{2.9}$$

Proof. Due to maximum principle, the solutions u_1 and u_2 verify

$$\|u_i\|_\infty \leq M = \max\{\|\bar{u}_1\|_\infty,\ \|\bar{u}_2\|_\infty\},\ i = 1, 2$$

and therefore $f''(u) \geq c(M) > 0$ for any u under consideration. Hence, denoting with v_i the primitive of our solutions u_i, $i = 1, 2$, we are in the hypotheses of Lemma 2.3, namely

$$v_i^{-1}(\xi, t) = \max_{0 \leq w \leq \xi} \left\{ tF\left(\frac{\xi - w}{t}\right) + \bar{v}_i^{-1}(w) \right\},\ i = 1, 2.$$

Let $\bar{w}_1$ be the point where the maximum is attained in the formula for v_1, that is

$$v_1^{-1}(\xi, t) = tF\left(\frac{\xi - \bar{w}_1}{t}\right) + \bar{v}_1^{-1}(\bar{w}_1).$$

Then

$$\begin{aligned} v_1^{-1}(\xi,t) - v_2^{-1}(\xi,t) &\leq tF\left(\frac{\xi-\bar{w}_1}{t}\right) + \bar{v}_1^{-1}(\bar{w}_1) - tF\left(\frac{\xi-\bar{w}_1}{t}\right) + \bar{v}_2^{-1}(\bar{w}_1) \\ &= \bar{v}_1^{-1}(\bar{w}_1) - \bar{v}_2^{-1}(\bar{w}_1) \\ &\leq \sup_{0\leq w\leq \xi} |\bar{v}_1^{-1}(w) - \bar{v}_1^{-1}(w)|. \end{aligned}$$

Finally, interchanging the role of v_1 and v_2 we get

$$|v_1^{-1}(\xi,t) - v_2^{-1}(\xi,t)| \leq \sup_{0\leq w\leq \xi} |\bar{v}_1^{-1}(\bar{w}) - \bar{v}_1^{-1}(\bar{w})|,$$

which reduces to (2.9) taking the supremum over all $\xi \in [0,1]$, in view of the considerations in Remark 2.1. □

In the main theorem of this section we shall prove relation (2.9) for solutions to scalar conservation laws (1.1) with general (non uniformly) convex flux. The initial data will be chosen to belong in one of the two subsets of $\mathcal{B}$

$$\mathcal{B}_{fc} = \left\{ u \in L^\infty(\mathbb{R}),\ u \geq 0,\ \begin{array}{l} \text{supp}(u) \text{ compact and} \\ \text{with a finite number of} \\ \text{connected components} \end{array}, \int_{\mathbb{R}} u(x)dx = 1 \right\},$$

$$\mathcal{B}_{BV} := \mathcal{B} \cap BV(\mathbb{R}).$$

Theorem 2.5. *Let us consider solutions u and v to* (1.1) *with initial data $\bar{u}, \bar{v}$ belonging either in $\mathcal{B}_{fc}$ or in $\mathcal{B}_{BV}$ and assume the flux f in* (1.1) *is convex. Then, for any $t > 0$,*

$$d_\infty(u(t), v(t)) \leq d_\infty(\bar{u}, \bar{v}). \tag{2.10}$$

Proof. The result of this theorem is a consequence of the previous one, via an approximation procedure. Consider sequences $\bar{u}_n, \bar{v}_n \in \mathcal{B}_c$ such that

$$\begin{aligned} \bar{u}_n &\to \bar{u} \text{ in } L^1(\mathbb{R}) \\ \bar{v}_n &\to \bar{v} \text{ in } L^1(\mathbb{R}) \end{aligned}$$

and such that $\|\bar{u}_n\|_{L^\infty}$, $\|\bar{v}_n\|_{L^\infty} \leq M$ uniformly in n, with the notation $M = \max\{\|\bar{u}\|_{L^\infty}, \|\bar{v}\|_{L^\infty}\}$. As proven in Section 5 (Theorem 5.4 and Theorem 5.5), given $\delta > 0$ arbitrarily small, it is always possible to choose the sequences $\bar{u}_n, \bar{v}_n \in \mathcal{B}_c$ in such a way that

$$d_\infty(\bar{u}_n, \bar{v}_n) \leq d_\infty(\bar{u}, \bar{v}) + \delta. \tag{2.11}$$

In addition, let us consider a sequence f^ϵ of smooth functions such that $f^\epsilon \to f$ in the (uniform) topology of $C^1([-M, M])$ and such that $(f^\epsilon)''(u) > \epsilon$ for any $u \in [-M, M]$. Note that we can choose the sequence f^ϵ in such a way that $(f^\epsilon(u))' \leq C$ for any $u \in [-M, M]$, C independent on ϵ. Therefore the speed of propagation of

$$u_t + f^\epsilon(u)_x = 0 \tag{2.12}$$

is bounded uniformly with respect to ϵ. Let us denote with u_n^ϵ, v_n^ϵ respectively the solutions of (2.12) with $\bar{u}_n$, $\bar{v}_n$ as initial datum. Then we can apply Proposition 2.4 and conclude

$$d_\infty(u_n^\epsilon(t), v_n^\epsilon(t)) \leq d_\infty(\bar{u}_n, \bar{v}_n), \tag{2.13}$$

which together with the choice (2.11) implies

$$d_p(u_n^\epsilon(t), v_n^\epsilon(t)) \leq d_\infty(\bar{u}, \bar{v}) + \delta, \tag{2.14}$$

for any $p \in [2, \infty)$. As it is well known (see for instance [6]), scalar conservation laws enjoy L^1 contraction property for any $t > 0$, which in particular gives

$$\|u_n^\epsilon(t) - u^\epsilon(t)\|_{L^1(\mathbb{R})} \leq \|\bar{u}_n - \bar{u}\|_{L^1(\mathbb{R})}$$
$$\|v_n^\epsilon(t) - v^\epsilon(t)\|_{L^1(\mathbb{R})} \leq \|\bar{v}_n - \bar{v}\|_{L^1(\mathbb{R})},$$

where we have denoted with u^ϵ, v^ϵ the solutions of (2.12) with $\bar{u}$, $\bar{v}$ as initial data. Hence, for any $t \geq 0$, $u_n^\epsilon(t) \to u^\epsilon(t)$ and $v_n^\epsilon(t) \to v^\epsilon(t)$, as $n \to +\infty$, strongly in $L^1(\mathbb{R})$ and, passing if necessary to subsequences, almost everywhere and bounded. Hence, since all functions involved here have compact support uniformly in n, we can pass to the limit as $n \to +\infty$ in the left hand side of (2.14) due to the continuity of all the p–th moments of $\bar{u}_n$ and $\bar{v}_n$ (see [12]) and obtain

$$d_p(u^\epsilon(t), v^\epsilon(t)) \leq d_\infty(\bar{u}, \bar{v}) + \delta, \quad p \in [2, \infty)$$

and, since the left hand side does not depend on δ, we also have

$$d_p(u^\epsilon(t), v^\epsilon(t)) \leq d_\infty(\bar{u}, \bar{v}), \quad p \in [2, \infty). \tag{2.15}$$

Moreover, we can use once again the contraction in L^1 of (2.12) to prove that the sequences $u^\epsilon(t)$ and $v^\epsilon(t)$ are equibounded and equicontinuous in $L^1(\mathbb{R})$ and therefore we obtain, up to subsequences, the convergence, as $\epsilon \downarrow 0$, $u^\epsilon(t) \to u(t)$, $v^\epsilon(t) \to v(t)$ strongly in $L^1(\mathbb{R})$ and bounded almost everywhere. Finally, we can use the lower semi–continuity of the d_p's with respect to L^1 convergence (see [12]) in order to pass to the limit in (2.15) as $\epsilon \downarrow 0$. Thus, we obtain

$$d_p(u(t), v(t)) \leq d_\infty(\bar{u}, \bar{v})$$

which implies (2.10) by taking the limit as $p \to +\infty$ in the left hand side. □

As it is well known [3], the estimate proved in the previous theorem gives also a control of the speed of propagation of the supports of the two solutions $u(t)$ and $v(t)$. We establish this property in the next corollary.

Corollary 2.6. *Let us consider solutions u and v to* (1.1) *with initial data $\bar{u}, \bar{v}$ belonging either in $\mathcal{B}_{fc}$ or in $\mathcal{B}_{BV}$ and assume the flux f in* (1.1) *is convex. Then*

$$|\inf[\mathrm{supp}(u(t))] - \inf[\mathrm{supp}(v(t))]| \leq d_\infty(\bar{u}, \bar{v}),$$
$$|\sup[\mathrm{supp}(u(t))] - \sup[\mathrm{supp}(v(t))]| \leq d_\infty(\bar{u}, \bar{v}).$$

Remark 2.7. By refining the argument of Theorem 5.4, it is actually possible to enlarge the class of initial data $\mathcal{B}_{fc}$ to the case of an initial datum in $\mathcal{B}$ having an countably infinite number of connected components in its support, which accumulate only in a finite number of points.

We conclude this section with a proof of the optimality of the result of Theorem 2.5, by testing (2.10) for two N–waves, solution of the Burgers' equation.

Remark 2.8. Let us consider the Burgers' equation

$$u_t + \frac{1}{2}(u^2)_x = 0 \tag{2.16}$$

and its solutions $N_1(x,t)$ and $N_2(x,t) = N_1(x - x_0, t)$ emanating from initial data $\bar{u}_1(x)$ and $\bar{u}_2(x) = \bar{u}_1(x - x_0)$ defined by

$$\bar{u}_1(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{elsewhere,} \end{cases} \quad \bar{u}_2(x) = \begin{cases} 1 & \text{if } x_0 \leq x \leq x_0 + 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then, for any $t > 2$ (namely, when all the interactions in the solutions have already occurred), the solutions take the form

$$N_1(x,t) = \begin{cases} \dfrac{x}{t} & \text{if } 0 \leq x \leq \sqrt{2t} \\ 0 & \text{elsewhere} \end{cases}$$

and

$$N_2(x,t) = \begin{cases} \dfrac{x - x_0}{t} & \text{if } x_0 \leq x \leq x_0 + \sqrt{2t} \\ 0 & \text{elsewhere} \end{cases}.$$

Now, let us denote with $\bar{v}_1^{-1} : [0,1] \to [0,1]$ and with $\bar{v}_2^{-1} : [0,1] \to [x_0, x_0 + 1]$ the inverses of the primitive of the initial data $\bar{u}_1$ and $\bar{u}_2$. Then, a direct calculation gives

$$\bar{v}_1^{-1}(\xi) = \xi, \ \bar{v}_2^{-1}(\xi) = x_0 + \xi, \ \text{for any } \xi \in [0,1].$$

Therefore,

$$d_\infty(\bar{u}_1, \bar{u}_2) = \|\bar{v}_1^{-1} - \bar{v}_2^{-1}\|_{L^\infty([0,1])} = x_0.$$

Moreover, denoting with $V_1^{-1}(\cdot, t) : [0,1] \to [0, \sqrt{2t}]$ and with $V_2^{-1}(\cdot, t) : [0,1] \to [x_0, x_0 + \sqrt{2t}]$ the inverses of the primitive of the solutions $N_1(\cdot, t)$ and $N_2(\cdot, t)$, $t > 2$, we have

$$V_1^{-1}(\xi, t) = \sqrt{2\xi t},\ V_2^{-1}(\xi) = x_0 + \sqrt{2\xi t},\ \text{for any } \xi \in [0,1],\ t > 2.$$

Thus,

$$d_\infty(N_1(t), N_2(t)) = \|V_1^{-1}(t) - V_2^{-1}(t)\|_{L^\infty([0,1])} = x_0,\ \text{for any } t > 2.$$

Finally, let us fix a $t^* \in (0, 2]$. Then from (2.10), we get

$$d_\infty(N_1(t^*), N_2(t^*)) \leq d_\infty(\bar{u}_1, \bar{u}_2) = x_0.$$

In addition, since (2.16) is autonomous, for any $t > 2$, $i = 1, 2$, we can consider $N_i(x, t^* + t)$ the solution at time t emanating from the initial datum $N_i(x, t^*)$. Hence, using once again (2.10), one has

$$x_0 = d_\infty(N_1(t^* + t), N_2(t^* + t)) \leq d_\infty(N_1(t^*), N_2(t^*)).$$

Hence, the d_∞–Wasserstein distance between the two solutions considered here, two shifted N–waves, is constant in time. In fact, it is equal to the distance between the infimum of the two supports and therefore the result of Theorem 2.5 is optimal.

3. Intermediate asymptotics

In this section we propose a nonlinear time–dependent scaling of the solutions to (1.1) in order to study their asymptotic behavior. Let $\mathcal{P}(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$. For $\mu \in \mathcal{P}(\mathbb{R})$, we denote the second moment of μ (eventually infinite) by

$$\theta[\mu] = \frac{1}{2} \int_{\mathbb{R}} x^2 d\mu.$$

We define the manifold

$$\mathcal{M} = \{\mu \in \mathcal{P}(\mathbb{R}),\ \theta[\mu] = 1,\ \text{supp}(\mu) \text{ compact}\}. \tag{3.1}$$

It can be easily checked that the metric space $(\mathcal{M}, d_\infty)$ is complete (see [12]). We shall deal with initial data for (1.1) belonging in the following subsets of $\mathcal{M}$:

$$\begin{aligned} \mathcal{M}_b &= \mathcal{M} \cap \mathcal{B} \\ \mathcal{M}_c &= \mathcal{M} \cap \mathcal{B}_c \\ \mathcal{M}_{fc} &= \mathcal{M} \cap \mathcal{B}_{fc} \\ \mathcal{M}_{BV} &= \mathcal{M} \cap \mathcal{B}_{BV} \end{aligned} \tag{3.2}$$

Throughout this section we shall denote by $\theta[u]$ (by abuse of notation) the second moment of a probability measure which is absolutely continuous with respect to Lebesgue measure having u as Radon–Nicodym derivative.

Let $u_0 \in \mathcal{M}_b$ and let $u(x,t)$ be the unique entropy solution of the equation (1.1) with u_0 as initial datum. We observe that, since $u(t)$ has compact support, the second moment of the solution $\theta[u(t)]$ is finite at any time $t \geq 0$. Hence, we can define a *renormalized* flow map $S(t) : \mathcal{M}_b \to \mathcal{M}_b$ in the spirit of [2] as follows. For $\bar{u} \in \mathcal{M}_b$ we set

$$(S(t)\bar{u})(x) := \theta[u(t)]^{1/2} u(\theta[u(t)]^{1/2} x, t) \tag{3.3}$$

where $u(\cdot,\cdot)$ is the solution to (1.1) with initial datum $\bar{u}$. A straightforward computation yields $S(t)\bar{u} \in \mathcal{M}_b$. Following the ideas in [2], we want to establish a strict contraction result for the map $S(t)$ with respect to a suitable metric. Due to the contraction result proven in the previous section, our choice is the Wasserstein distance d_∞.

We first establish one of the main ingredients of our machinery, namely, we show that the second moment of any solution to (1.1) diverges as $t \to +\infty$, provided the flux f satisfies the additional assumption

$$\exists\, \alpha \in (0,1), \quad r \mapsto f(r)^{1-\alpha} \ \text{ is convex on } (0,+\infty). \tag{3.4}$$

Under the requirement (3.4), it has been proven in [11] that the L^∞_x–norm of any solution $u(t)$ to (1.1) with initial data in $\mathcal{M}_b$ decays to zero as $t \to +\infty$. More precisely, using the results in [5], they are able to show [11, Proposition 2.1] that

$$\|u(t)\|_{L^\infty(\mathbb{R})} \leq f^{-1}\left(\frac{C(\alpha)}{t}\|u(0)\|_{L^1(\mathbb{R})}\right) \tag{3.5}$$

for all $t > 0$. The decay in L^∞ provided by (3.5) is crucial in the proof of the next lemma, which is contained in [2, Lemma 2.1] where an analogous result is shown for nonlinear diffusion equations.

Lemma 3.1. *Let $u(t)$ be the entropy solution to* (1.1) *with initial datum $\bar{u} \in \mathcal{M}_b$. Then, $\theta[u(t)] \to +\infty$ as $t \to +\infty$ uniformly in the set of initial data $\mathcal{M}_b$.*

Now we state a general result for Wasserstein distances, which will be useful in the sequel.

Lemma 3.2. *Let u, v be two compactly supported probability densities such that $u \neq v$ on a set of positive Lebesgue measure. Suppose also $\theta[u] = \theta[v] = \theta > 0$. For $a \geq 0$ let*

$$v_a(x) := (1+a)^{1/2} v((1+a)^{1/2} x).$$

Then, for any $p \in [2, +\infty]$, there exists a universal constant $C = C(p)$ not depending on u, v and θ such that

$$d_p(u, v) \leq C d_p(u, v_a) \tag{3.6}$$

for all $a \geq 0$.

Proof. STEP 1. Let u, v be fixed and suppose $\theta = 1$. We proceed by contradiction. Let us first suppose $p < \infty$. Inequality (3.6) being false for any $C > 0$ is equivalent to say that there exists a sequence of positive real numbers $\{a_n\}_{n \in \mathbb{N}}$ such that

$$d_p(u, v) > n d_p(u, v_{a_n}),$$

and this trivially implies

$$\lim_{n \to +\infty} d_p(u, v_{a_n}) = 0. \tag{3.7}$$

By well known properties of the p–Wasserstein distance (see [12]), (3.7) implies $\theta[v_{a_n}] \to \theta[u] = 1$. A direct computation yields $\theta[v_{a_n}] = 1 + a_n$, which implies $a_n \to 0$. But this fact, together with the definition of v_{a_n}, implies

$$v_{a_n} \rightharpoonup v \qquad \text{in the sense of measures,} \qquad \text{as } n \to +\infty.$$

Finally, elementary properties of the d_p (see [12]) imply

$$d_p(u, v) \leq \liminf_{n \to \infty} d_p(u, v_{a_n}) = 0,$$

which implies $u \equiv v$ almost everywhere, and this is in contradiction with the hypotheses on u and v. The assertion for $p = +\infty$ can be obtained in a similar way by sending $p \to +\infty$ in the above statements.

STEP 2. Let u be fixed. We prove that C in (3.6) is independent of v in a similar fashion as in step 1. For some $a > 0$, take a sequence v^n such that

$$d_p(u, v^n) > n d_p(u, v_a^n).$$

This implies $d_p(u, v_a^n) \to 0$ and $\theta[v_a^n] \to 1$ which is in contradiction with $\theta[v_a^n] = 1 + a$ for any n.

STEP 3. The constant C is independent on u, as it can be seen by repeating the same argument of step 2 with a sequence u_n such that $d_p(u_n, v) > n d_p(u_n, v_a)$ for some v and some $a > 0$.

STEP 4. The general case $\theta > 0$ follows by rescaling u and v in order to obtain two densities with unit second moment and by applying the result for $\theta = 1$. In particular, it is clear that the constant C does not depend on the fixed temperature θ. □

We remark that the constant C in (3.6) equals 1 in Euclidean case $p = 2$ (see [2]).

Another important ingredient in our asymptotic analysis is a uniform control of the support of the generalized N–wave after the scaling (3.3).

Proposition 3.3. *Suppose* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is* C^1 *and such that* (3.4) *is satisfied. Let*

$$N(x,t) = \begin{cases} (f')^{-1}\left(\frac{x}{t}\right) & 0 \le x \le b(t) \\ 0 & \textit{otherwise} \end{cases}$$

be the N–wave solution of the scalar conservation law $u_t + f(u)_x = 0$ *having mass* $M > 0$*. Then, given*

$$\theta[N(t)] = \frac{1}{2}\int |x|^2 N(x,t)dx,$$

the support of the function

$$\widehat{N}(x,t) := \theta[N(t)]^{1/2} N(x\theta[N(t)]^{1/2}, t)$$

stays globally bounded in time.

Proof. STEP 1. We start by providing an explicit formula for $b(t)$. The conservation of the mass implies

$$\begin{aligned} M = \int_0^{b(t)} &= (f')^{-1}\left(\frac{x}{t}\right) dx = t\int_0^{(f')^{-1}(b(t)/t)} uf''(u)du \\ &= t\left[(f')^{-1}(b(t)/t)\frac{b(t)}{t} - f\left((f')^{-1}(b(t)/t)\right)\right]. \end{aligned}$$

The convexity of f and its regularity assumptions imply the following explicit expression of the Legendre transform f^*

$$f^*(u) = (f')^{-1}(u)u - f((f')^{-1}(u)),$$

for $u \ge 0$ sufficiently small. Therefore, since $b(t)/t \to 0$ as $t \to +\infty$, for t larger than a certain t^* we have

$$\frac{M}{t} = f^*\left(\frac{b(t)}{t}\right).$$

Hence, we can consider the nonnegative branches of f and f^* in order to have their inverses well–defined. This implies

$$b(t) = t(f^*)^{-1}\left(\frac{M}{t}\right). \tag{3.8}$$

STEP 2. We provide an explicit expression for $\|N(t)\|_{L^\infty}$. Since N is nondecreasing with respect to x on its support, we have

$$\|N(t)\|_{L^\infty} = N\left(\frac{b(t)}{t}, t\right) = (f')^{-1}\left(\frac{b(t)}{t}\right).$$

Thanks to (3.8), we have

$$\|N(t)\|_{L^\infty} = (f')^{-1}\left((f^*)^{-1}\left(\frac{M}{t}\right)\right). \tag{3.9}$$

STEP 3. By means of the identity (3.9), we can estimate the temperature of $N(t)$ from below as in [2, Lemma 2.1]. We obtain the following inequality

$$\theta[N(t)] \geq \frac{M^3}{4^3}\frac{1}{\|N(t)\|^2_{L^\infty}}. \tag{3.10}$$

STEP 4. The support of $\widehat{N}(t)$ at any time $t > 0$ coincides with the interval $[0, l(t)]$, where

$$l(t) := \frac{b(t)}{\theta[N(t)]^{1/2}}.$$

Therefore, from (3.8), (3.9) and (3.10) we obtain the estimate

$$l(t) \leq Ct(f^*)^{-1}\left(\frac{M}{t}\right) \cdot (f')^{-1}\left((f^*)^{-1}\left(\frac{M}{t}\right)\right).$$

Due to the following property of the Legendre transform

$$(f')^{-1} \equiv (f^*)',$$

we recover

$$l(t) \leq CM\frac{t}{M}\frac{(f^*)^{-1}\left(\frac{M}{t}\right)}{((f^*)^{-1})'\left(\frac{M}{t}\right)}. \tag{3.11}$$

STEP 5. Thanks to (3.4) we have

$$f(u) \leq Cu^{\frac{1}{1-\alpha}},$$

for some $\alpha \in (0,1)$. Since the Legendre transform of $u \mapsto Cu^{\frac{1}{1-\alpha}}$ equals $C_1 u^{1/\alpha}$ for some positive C_1, by using the estimate for f in the definition of Legendre transform we obtain

$$f^*(u) \geq C_1 u^{1/\alpha}$$

for some $C_1 > 0$, and this implies

$$(f^*)^{-1}(u) \leq C_2 u^\alpha \tag{3.12}$$

for some $C_2 > 0$. To simplify the notation, let us denote $g := (f^*)^{-1}$ and $\epsilon = M/t$. We want to control the term on the right hand side of

(3.11) uniformly for large times. Such term, with the new notations, reads

$$C\frac{g(\epsilon)}{\epsilon g'(\epsilon)} \quad (\epsilon \to 0).$$

Let us denote

$$h(\epsilon) := g(\epsilon^{1/\alpha}).$$

We have

$$h'(\epsilon) = \frac{1}{\alpha}\epsilon^{\frac{1}{\alpha}-1}g'(\epsilon^{1/\alpha}).$$

Since g' is nonincreasing on a right neighborhood of 0 (g is concave), the limit

$$\lim_{\epsilon\to 0^+} h'(\epsilon)$$

exists. But estimate (3.12) implies

$$\frac{h(\epsilon)}{\epsilon} \leq C,$$

which forces the previous limit to be finite. Therefore,

$$\lim_{\epsilon\to 0^+}\left[\frac{h(\epsilon)}{\epsilon h'(\epsilon)}\right] = 1,$$

which implies

$$\frac{h(\epsilon)}{\epsilon} \leq Ch'(\epsilon)$$

for small ϵ. In terms of g, we have

$$\frac{g(\epsilon^{1/\alpha})}{\epsilon^{1/\alpha}} \leq Cg'(\epsilon^{1/\alpha}),$$

which completes the proof. □

The previous result guarantees the support of an N–wave solution of a conservation law, scaled with respect to its second moment, lies in a compact interval $[-R, R]$, with R depending solely on its mass M. By comparison with the N–wave solution, we generalize this property for any solution of (1.1) with initial datum $\overline{u} \in \mathcal{M}_c$, that is, for $S(t)\overline{u}$, with $S(t)$ given by (3.3) for t sufficiently large.

Proposition 3.4. *Suppose $f : \mathbb{R}_+ \to \mathbb{R}_+$ is C^1 and such that* (3.4) *is satisfied. Let u be a solution of the scalar conservation law* (1.1) *with initial datum $\overline{u} \in \mathcal{M}_c$. Then, given*

$$\theta[u(t)] = \frac{1}{2}\int |x|^2 u(x,t)dx,$$

there exists a time $t^* > 0$ such that the support of the function

$$S(t)\overline{u} = \theta[u(t)]^{1/2}u(x\theta[u(t)]^{1/2}, t) =: \widehat{u}(x,t)$$

lies in $[-K, K]$, with K depending solely on its mass $M > 0$, for any $t > t^*$.

Proof. Let $N(x,t)$ be the (shifted) N–wave of (1.1) with mass $M > 0$ and initial datum $\overline{N} \in \mathcal{M}_c$. Let us define for any $t > 0$

$$\widetilde{\theta}(t) := \min\{\theta[u(t)], \theta[N(t)]\}$$

and consider the mass–preserving scaling

$$\widetilde{u}(x,t) := \alpha(t)^{1/2} u\left(\alpha(t)^{1/2}x, t\right) \qquad \alpha(t) := \frac{\theta[u(t)]}{\widetilde{\theta}(t)},$$

$$\widetilde{N}(x,t) := \beta(t)^{1/2} u\left(\beta(t)^{1/2}x, t\right) \qquad \beta(t) := \frac{\theta[N(t)]}{\widetilde{\theta}(t)}.$$

In this way, we leave unchanged one of the two solution and decrease the second moment of the other to match the second moment of the first one. Moreover

$$S(t)\overline{u} = \widehat{u}(x,t) = \widetilde{\theta}(t)^{1/2}\widetilde{u}\left(\widetilde{\theta}(t)^{1/2}x, t\right)$$

$$S(t)\overline{N} = \widehat{N}(x,t) = \widetilde{\theta}(t)^{1/2}\widetilde{N}\left(\widetilde{\theta}(t)^{1/2}x, t\right).$$

An elementary scaling property of the p–Wasserstein distance for $p \geq 2$ (see [12]) together with the above identities gives

$$d_\infty(S(t)\overline{u}, S(t)\overline{N}) \leq \widetilde{\theta}(t)^{-1/2} d_\infty(\widetilde{u}(t), \widetilde{N}(t)).$$

Moreover, with the aid of Lemma 3.2 we obtain

$$d_\infty(\widetilde{u}(t), \widetilde{N}(t)) \leq C d_\infty(u(t), N(t))$$

and the contraction property of Theorem 2.5 gives

$$d_\infty(u(t), N(t)) \leq d_\infty(\overline{u}, \overline{N}).$$

Hence, we end up with the inequality

$$d_\infty(S(t)\overline{u}, S(t)\overline{N}) \leq C\widetilde{\theta}(t)^{-1/2} d_\infty(\overline{u}, \overline{N}) \leq 1,$$

for any $t \geq t^*$, with $t^* > 0$ sufficiently large, in view of Lemma 3.1. Finally, the result follows from Proposition 3.3 and the well-known control of the distance between the supports of $S(t)\overline{u} = \widehat{u}$ and $S(t)\overline{N} = \widehat{N}$ in terms of the distance d_∞ (see Corollary 2.6). □

We are now ready to state the main theorem of this section. We first introduce the notation

$$\mathcal{M}_0 := \{\mu \in \mathcal{P}(\mathbb{R}),\ \theta[\mu] = 1,\ \text{supp}(\mu) \text{ compact and connected}\}.$$

We remark that the space $(\mathcal{M}_0, d_\infty)$ is complete (in terms of pseudo inverses of the distribution functions, such space is the space of continuous functions on the interval $[0,1]$ endowed with the uniform topology).

Theorem 3.5. *Let $f : [0,+\infty) \to [0,+\infty)$ be a C^1 convex function such that $f(0) = f'(0) = 0$ and such that (3.4) is satisfied. Then, there exist a fixed $t^* > 0$ and a one parameter family of functions $\{U_t^\infty\}_{t\geq t^*} \subset \mathcal{M}_c$ such that, for any u_0 belonging either in $\mathcal{M}_{fc}$ or in $\mathcal{M}_{BV}$ we have*

$$d_\infty(S(t)u_0, U_t^\infty) \to 0, \qquad as \ \ t \to +\infty, \tag{3.13}$$

where the map $S(t)$ is defined in (3.3). Moreover, for any fixed $t > t^*$, U_t^∞ is characterized as the unique fixed point of the map

$$S(t) : \mathcal{M} \to \mathcal{M}.$$

The profiles U_t^∞ have connected compact support uniformly bounded in time.

Proof. We split the proof into three steps. Let us define the complete metric space $(\mathcal{M}_K, d_\infty)$, where $\mathcal{M}_K \subset \mathcal{M}$ is defined by

$$\mathcal{M}_K = \{\mu \in \mathcal{P}(\mathbb{R}),\ \theta[\mu] = 1,\ \mathrm{supp}(\mu) \subset [-K, K]\}\,, \tag{3.14}$$

where $K > 0$ is the constant provided in Proposition 3.4. In the first step we establish a contraction result for the map $S(t)$ (for large enough t) in the subset

$$\mathcal{M}_{c,K} := \mathcal{M}_c \cap \mathcal{M}_K \subset \mathcal{M}_K.$$

In the second step, for t sufficiently large, we extend the map $S(t)$ to the closure $(\mathcal{M}_0 \cap \mathcal{M}_K, d_\infty)$ in order to apply Banach's fixed point Theorem and obtain a family of fixed points in that space. Finally, we repeat the fixed point argument in the whole complete metric space $(\mathcal{M}, d_\infty)$ to get the final result.

Step 1. Thanks to the result in Proposition 3.4, the renormalized flow map $S(t)$ defined by (3.3) is a well–defined operator on the space $\mathcal{M}_{c,K}$ for t larger than a waiting time t^*. Consider then two elements $\overline{u}_1, \overline{u}_2 \in \mathcal{M}_{c,K}$. Let u_1 and u_2 be the entropy solutions having $\overline{u}_1$ and $\overline{u}_2$ as initial data respectively. Proceeding as in the proof of Proposition 3.4, we define for any $t > 0$

$$\widetilde{\theta}(t) := \min\{\theta[u_1(t)], \theta[u_2(t)]\}$$

and we introduce the mass–preserving scaling

$$\widetilde{u}_i(x,t) := \alpha_i(t)^{1/2} u_i\left(\alpha_i(t)^{1/2} x, t\right) \qquad \alpha_i(t) := \frac{\theta[u_i(t)]}{\widetilde{\theta}(t)}, \qquad i = 1, 2.$$

After the above scaling procedure, one of the two between u_1 and u_2 (namely, the one with less second moment) remains unchanged, while the other one is rescaled in such a way that $\widetilde{u}_1$ and $\widetilde{u}_2$ have the same second moment. Moreover, for further use we observe

$$S(t)\overline{u}_i(x) = \widetilde{\theta}(t)^{1/2}\widetilde{u}_i\left(\widetilde{\theta}(t)^{1/2}x, t\right), \quad i = 1, 2. \tag{3.15}$$

Since $\theta[u_i(t)] \geq \widetilde{\theta}(t)$ for $i = 1, 2$, we can apply the result in Lemma 3.2 which implies

$$d_\infty(\widetilde{u}_1(t), \widetilde{u}_2(t)) \leq C d_\infty(u_1(t), u_2(t)). \tag{3.16}$$

Using once again the scaling property of the Wasserstein distances and the identity (3.15) yield

$$d_\infty\left(S(t)\overline{u}_1, S(t)\overline{u}_2\right) \leq \widetilde{\theta}(t)^{-1/2} d_\infty\left(\widetilde{u}_1(t), \widetilde{u}_2(t)\right),$$

which, together with (3.16) implies

$$d_\infty\left(S(t)\overline{u}_1, S(t)\overline{u}_2\right) \leq C\widetilde{\theta}(t)^{-1/2} d_\infty(u_1(t), u_2(t)).$$

Finally, thanks to Lemma 3.1 and to the contraction result of Theorem 2.5 we have, for a sufficiently large t_1^*,

$$d_\infty\left(S(t)\overline{u}_1, S(t)\overline{u}_2\right) \leq \beta(t) d_\infty\left(\overline{u}_1, \overline{u}_2\right) \tag{3.17}$$

for a suitable function $[t_1^*, +\infty) \ni t \to \beta(t) \in (0, +\infty)$ such that $\beta(t) < 1$ for all $t \geq t_1^*$ and such that $\beta(t) \to 0$ as $t \to +\infty$.

Step 2. We now extend the map $S(t)$ to the space of measures $\mathcal{M}_0 \cap \mathcal{M}_K$ for t sufficiently large. Consider $\mu \in \mathcal{M}_0 \cap \mathcal{M}_K$. Thanks to the result in Theorem 5.6, we can construct a sequence $\{u_n\}_n \subset \mathcal{M}_{c,K}$ such that $d_\infty(u_n, \mu) \to 0$ as $n \to +\infty$. Due to (3.17), the sequence $S(t)u_n$ is Cauchy in $\mathcal{M}_0 \cap \mathcal{M}_K$. Hence, by completeness of $(\mathcal{M}_0 \cap \mathcal{M}_K, d_\infty)$, $S(t)u_n$ has a limit ν in $\mathcal{M}_0 \cap \mathcal{M}_K$, and such a limit does not depend on the chosen approximating sequence, due once again to (3.17). We then define $S(t)\mu = \nu$. It easily seen that inequality (3.17) holds true when $\overline{u}_1$ and $\overline{u}_2$ belong to $\mathcal{M}_0 \cap \mathcal{M}_K$. Moreover, in view of Proposition 3.4, $S(t)\mu \in \mathcal{M}_K$ for any $t > t_2^*$. We can then apply Banach's fixed point Theorem for $t > t^* = \max\{t_1^*, t_2^*\}$, which yields the existence of the desired family of fixed points $\{U_t^\infty\} \in \mathcal{M}_0 \cap \mathcal{M}_K$.

Step 3. Let us now consider the whole space of measures $\mathcal{M}$. Proceeding as in Step 2, we can prove the map $S(t)$ to be a contraction in the dense subspace $\mathcal{M}_{fc}$, for any $t > t^*$. By means of the approximation Theorem 5.7, we can extend $S(t)$ to be a contraction on the whole (complete) space of measures $\mathcal{M}$ as in Step 2. Hence we can apply Banach's fixed point Theorem in $(\mathcal{M}, d_\infty)$ and obtain a family of fixed points in this larger space, which indeed coincides with $\{U_t^\infty\}$, due to uniqueness. Finally, the limit (3.13) follows by choosing in (3.17) $\overline{u}_1 = u_0$ and $\overline{u}_2 = U_t^\infty$, namely the fixed point of $S(t)$, because, as shown before, U_t^∞ has a support in $[-K, K]$ for t sufficiently large and therefore $d_\infty(u_0, U_t^\infty)$ is uniformly bounded in t.

At this stage, we only have to prove the additional properties over the fixed points U_t^∞. From the results in [11], there exists a unique entropy solution to (1.1) even in the case of initial datum in $\mathcal{M}$. Hence, the fixed points U_t^∞ must be in L^∞. □

Remark 3.6. We stress that, in the case $f(u) = \frac{1}{\gamma}|u|^\gamma$, the family of fixed points U_t^∞ is independent on time and equals the N–wave $N_{1,0}(x, t_o)$ defined in (1.2) with $P = 0$ and $Q = 1$ evaluated at the time t_o when it has unit second moment. This fact is a consequence of the self–similar structure of the N–wave in the case of a power law flux f. A slightly more general result is contained in Corollary 3.9 below.

Remark 3.7. Just by using the L^∞-decay shown in (3.5) and Lemma 3.1, one can obtain a uniform estimate from below of the divergence in time of the temperature $\theta[u(t)]$ of any solution and thus one obtains an explicit decay rate in time for (3.13) in terms of the inverse of f (see the proof of Lemma 2.1 in [2]).

Remark 3.8. By slightly changing the rescaling in (3.3), we can carry out the previous procedure by working on the manifold

$$\mathcal{M}^{\overline{\theta}} = \big\{\mu \in \mathcal{P}(\mathbb{R}),\ \theta[\mu] = \overline{\theta},\ \mathrm{supp}(\mu) \text{ compact}\big\}$$

for any $\overline{\theta} > 0$. We then obtain a two parameter depending family of fixed points $U_{t,\overline{\theta}}^\infty$ which reduces to U_t^∞ in case of unit second moment $\overline{\theta}$. Moreover, $U_{t,\overline{\theta}}^\infty$ is independent on time when $f(u) = \frac{1}{\gamma}|u|^\gamma$ and it coincides with the N–wave $N_{1,0}(x, \overline{t})$ defined in (1.2) evaluated at the time $\overline{t}$ when it has second moment $\overline{\theta}$.

Finally, we can show that our asymptotic profile converges at $t \to +\infty$ to a universal profile for perturbations of power law fluxes.

Corollary 3.9. *Let $f : [0, +\infty) \to [0, +\infty)$ be a C^1 convex function such that $f(0) = f'(0) = 0$ and such that* (3.4) *is satisfied. Moreover, let us assume that $f(u) = \frac{1}{\gamma}u^\gamma + h(u)$, $\gamma > 1$, with $h(u)u^{-\gamma} \to 0$ as $u \to 0$. Then, the one parameter family of functions $\{U_t^\infty\}_{t \geq t^*}$ constructed in Theorem* 3.5 *verifies*

$$d_\infty(U_t^\infty, N_\gamma(\cdot, t_o)) \to 0, \qquad \text{as } t \to +\infty,$$

where $N_\gamma(x, t_o)$ is the N–wave of the leading behavior $\frac{1}{\gamma}u^\gamma$ at the time t_o in which it has unit second moment. As a consequence, for any u_0

belonging either in $\mathcal{M}_{fc}$ or in $\mathcal{M}_{BV}$ we have

$$d_\infty(S(t)u_0, N_\gamma(\cdot, t_o)) \to 0, \qquad \text{as } t \to +\infty, \tag{3.18}$$

where the map $S(t)$ is defined in (3.3).

Proof. Let us denote by $N_f(x, t+t^*)$ the N–wave solution corresponding to the nonlinearity f and t^* the time at which it has unit second moment. Therefore, taking $u_0 = N_f(x, t^*)$ in (3.13), we have

$$d_\infty(S(t)N_f(\cdot, t^*), U_t^\infty) \to 0, \qquad \text{as } t \to +\infty.$$

Let us denote $\theta_f(t)$ the temperature of $N_f(x, t+t^*)$.

Using that $N_\gamma(x, t_o)$ is a fixed point of the corresponding evolution through the conservation law with $f(u) = \frac{1}{\gamma}u^\gamma$, i.e., the self-similarity of $N_\gamma(x,t)$, we deduce that

$$N_\gamma(x, t_o) = \theta_\gamma(t)^{1/2} N_\gamma(\theta_\gamma(t)^{1/2}x, t+t_o)$$

where $\theta_\gamma(t)$ denotes the temperature of $N_\gamma(x, t+t_o)$. Now, we can estimate

$$d_\infty(S(t)N_f(\cdot, t^*), N_\gamma(\cdot, t_o)) \le \frac{C}{\theta(t)} d_\infty(N_f(\cdot, t+t^*), N_\gamma(\cdot, t+t_o))$$

by Lemma 3.2, where $\theta(t) = \min(\theta_f(t), \theta_\gamma(t))$.

Under the assumption $f(u) = \frac{1}{\gamma}u^\gamma + h(u)$, with $h(u)u^{-\gamma} \to 0$ as $u \to 0$, it is tedious but not difficult to show that

$$\lim_{t\to\infty} \frac{1}{\theta(t)} d_\infty(N_f(\cdot, t+t^*), N_\gamma(\cdot, t+t_o)) = 0$$

that concludes the proof. □

4. Nondecreasing solutions

In this section we turn our attention to solutions to the scalar conservation law (1.1) with initial data in the class

$$\mathcal{I} = \{u \in L^\infty(\mathbb{R}),\ u \text{ non decreasing},\ u(-\infty) = 0,\ u(+\infty) = 1\}.$$

Such a class of functions is invariant under the semigroup induced by (1.1). It is clear that, if $u \in \mathcal{I}$, then its distributional derivative u' is a probability measure. Hence, the p–Wasserstein distances between the space derivatives of any two solutions $u(t)$ and $v(t)$ with initial data belonging in $\mathcal{I}$ can be computed. This issue has been addressed for the first time in [1], where the authors proved the contraction property

$$d_p(u_x(t), v_x(t)) \le d_p(u_x(0), v_x(0)), \quad t \ge 0, \tag{4.1}$$

for any entropy solutions u, v to (1.1) with initial data in $\mathcal{I}$ and f locally Lipschitz function and for all $p \ge 1$. By means of the contraction inequality (4.1), we shall prove that the fixed point approach developed in the previous section can be easily generalized to the space derivatives of the solutions to (1.1) with initial data in $\mathcal{I}$. More precisely, let $\bar{v} \in \mathcal{M}$, where the class of probability measures $\mathcal{M}$ is defined in (3.1). Let us define

$$\bar{u}(x) = \int_{-\infty}^{x} \bar{v}(y)dy,$$

where it is clear that $\bar{u} \in \mathcal{I}$. For fixed $t \geq 0$ we define

$$\left(T(t)\bar{v}\right)(x) := \theta[u_x(t)]^{1/2} u_x(\theta[u_x(t)]^{1/2} x, t), \tag{4.2}$$

where $u(x,t)$ is the unique weak entropy solution to (1.1) with $\bar{u}$ as initial datum. It is worthy to point out that the scaling defined in (4.2) has to be understood as the definition of a measure by duality on how they act on continuous functions and thus the scaling is done accordingly on the test functions.

It is clear that for any $t \geq 0$ the map $T(t) : \mathcal{M} \to \mathcal{M}$ is well defined. We shall analyze the evolution of $T(t)\bar{v}$ for a general $\bar{v} \in \mathcal{M}$ in the next Theorem 4.1.

To this point, no particular assumption regarding the convexity of the flux has been done. Let us assume $f \in C^2$ and $f(0) = 0$ without loss of generality. Under convexity, $f'' > 0$, or concavity, $f'' < 0$, assumptions on the flux, we have some particular explicit solutions.

In the convex case, a special self–similar rarefaction wave solution to (1.1) with initial datum in $\mathcal{I}$ is given by

$$U^\infty(x,t) = \begin{cases} 0 & \text{if } x \leq 0 \\ g\left(\frac{x}{t}\right) & \text{if } 0 \leq x \leq f'(1)t \\ 1 & \text{if } x \geq f'(1)t, \end{cases} \tag{4.3}$$

where g is the inverse function of f'

$$g(\alpha) := \left(f'\right)^{-1}(\alpha), \quad \alpha \geq 0.$$

In the concave case, a special shock solution to (1.1) with initial datum in $\mathcal{I}$ is given by

$$U^\infty(x,t) = \begin{cases} 0 & \text{if } x \leq f(1)t \\ 1 & \text{if } x \geq f(1)t \end{cases}. \tag{4.4}$$

For any $t \geq 0$, the profiles $U^\infty(\cdot,t)$, in each case, have a distributional derivative in the space of probability measures.

Now, let t^* be the time such that the second moment $\theta[U_x^\infty(\cdot,t^*)]$ equals 1. Then, it can be easily checked that the profile $U_x^\infty(\cdot,t^*)$ is a fixed point of the map $T(t)$ defined in (4.2) in each case. We remark that in this case the fixed point is constant on t. We are ready to state the following

Theorem 4.1. *Let $p \geq 2$. Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$ function such that either $f'' > 0$ or $f'' < 0$ and $f(0) = 0$. Then, there exists a unique $V^\infty : \mathbb{R} \to [0, +\infty)$ such that, for any $\bar{v} \in \mathcal{M}$*

$$d_p(T(t)\bar{v}, V^\infty) \to 0 \tag{4.5}$$

as $t \to +\infty$. Moreover,

$$V^\infty(x) = U_x^\infty(x, t^*)$$

where U^∞ is defined by (4.3) for convex fluxes and by (4.4) for concave ones. The time t^* is chosen such that $U_x^\infty(\cdot, t^*)$ has unit second moment. Finally, V^∞ is characterized as the unique fixed point of all the maps $T(t)$, $t > 0$.

Proof. The proof of the present theorem can be carried out by means of the same steps as in Theorem 3.5 or in [2]. Therefore, we shall not perform all of its details. We only point out that the main ingredients are the contraction inequality (4.1), the result in Lemma 3.2, that for the Euclidean Wasserstein distance was obtained in [2], and the fact that the temperature of $u_x(t)$ tends to $+\infty$ as $t \to +\infty$ for any solution u of (1.1) uniformly in the set of initial data $\bar{u}$ such that $\bar{u}_x \in \mathcal{M}$.

To show this last statement, we will make use of our particular explicit solutions: the rarefaction wave or the shock wave. Let us remark that by inequality (4.1), we deduce

$$\begin{aligned} d_2(u_x(t), \delta_0) &\geq d_2(U_x^\infty(t + t^*), \delta_0) - d_2(U_x^\infty(t + t^*), u_x(t)) \\ &\geq d_2(U_x^\infty(t + t^*), \delta_0) - d_2(U_x^\infty(t^*), \bar{u}_x) \\ &\geq d_2(U_x^\infty(t + t^*), \delta_0) - 2, \end{aligned}$$

due to the fact that both $\bar{u}_x, U_x^\infty(t^*) \in \mathcal{M}$. A direct computation shows that

$$\theta[U_x^\infty(\cdot, t)] \to +\infty, \text{as } t \to +\infty$$

in both cases. Taking into account that

$$d_2(u_x(t), \delta_0) = \int_{\mathbb{R}} x^2 u_x(x, t)\, dx \ \text{ and } \ d_2(U_x^\infty(t), \delta_0) = \int_{\mathbb{R}} x^2 U_x^\infty(x, t)\, dx,$$

then the uniform divergence of the second moment in the set of initial data of this theorem is proved. ☐

Remark 4.2. The only point in which the explicit solutions depending on the convexity of the flux were used in the previous theorem was to obtain a uniform bound on the divergence of the second moment of the derivatives, uniform in the set of initial data. We do not see how to prove such a property for neither convex nor concave general fluxes. Being this true, previous theorem will apply giving the existence of an asymptotic profile for general fluxes.

Remark 4.3. Again estimating precisely the divergence of the temperature of $U_x^\infty(t)$, we obtain an explicit uniform estimate from below of the divergence in time of the temperature $\theta[u_x(t)]$ of any solution and thus one obtains an explicit decay rate in time for (4.5).

5. Approximation results

In this section we collect some approximation results we needed in the proofs of some previous theorems.

Given $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$, throughout this section we shall use the notations

$$f(x_0-) := \lim_{x \to x_0^-} f(x), \qquad f(x_0+) := \lim_{x \to x_0^+} f(x).$$

Theorem 5.1. *Let $f : [0,1] \to \mathbb{R}$ be strictly increasing, right–continuous and bounded. Then, there exists a sequence $\{f_n : [0,1] \to \mathbb{R},\ n \in \mathbb{N}\}$ of strictly increasing functions such that*

- *Each f_n has a finite number of discontinuities,*
- *$f_n \to f$ in $L^\infty([0,1])$.*

Proof. Since f is strictly increasing, then f has at most a countable number of jump discontinuities. We denote by $\{\xi_n\}_{n=1}^{+\infty}$ the sequence of all the points of discontinuity of f, and by

$$s_n = f(\xi_n+) - f(\xi_n-)$$

the jump of f at ξ_n, $n \in \mathbb{N}$. Since f is bounded, the following condition must be satisfied

$$\sum_{n=1}^{\infty} s_n < +\infty. \tag{5.1}$$

Let n be fixed. We define the approximating f_n as follows. For any positive integer k we set

$$r_k(\xi) = \begin{cases} 0 & \text{if } 0 \le \xi < \xi_k \\ s_k & \text{if } \xi_k \le \xi \le 1. \end{cases}$$

Hence, for any $\xi \in [0,1]$ we set

$$f_n(\xi) := f(\xi) - \sum_{k>n} r_k(\xi)$$

(note that the previous definition makes sense also for $n = 0$). By definition of f_n it is clear that

$$f_n \to f$$

uniformly on $[0,1]$, because $\|f_n - f\|_\infty$ is controlled by the n–th remainder of the converging series (5.1). We prove that f_n is continuous in any $\xi \in [0,1]$ except for $\xi_1, \dots, \xi_n$ (in particular, f_0 is continuous on the whole $[0,1]$). For any $\xi \in [0,1]$, by definition of r_k we have

$$f_n(\xi+) - f_n(\xi-) = f(\xi+) - \sum_{\xi_k \le \xi} s_k - f(\xi-) + \sum_{\xi_k < \xi} s_k$$
$$= f(\xi+) - f(\xi-) - \sum_{\xi_k = \xi} s_k$$

and the previous expression clearly vanishes if either ξ is a point of continuity of f or $\xi = \xi_k$ for some $k > n$ (we observe that in the previous expression the term denoted by summation consists of one addend at most). □

Lemma 5.2. *Under the same notations and assumptions of Theorem* 5.1*, suppose in addition*

$$f' \ge C > 0$$

almost everywhere on $[0,1]$*. Then, there exists a constant* $K > 0$ *such that*

$$\frac{f_0(\xi) - f_0(\eta)}{\xi - \eta} \ge K \tag{5.2}$$

for any $\eta, \xi \in [0,1]$.

Proof. We start by observing that f_0 is a continuous function. It is also clear that f_0 is nondecreasing, and therefore differentiable almost everywhere. Moreover, the a. e. derivative f_0 is summable on $[0,1]$, and the following estimate is true for all $\xi, \eta \in [0,1]$

$$f_0(\xi) - f_0(\eta) \ge \int_\eta^\xi f_0'(\zeta) d\zeta.$$

By definition of f_0 it is clear that $f' = f_0'$ almost everywhere. Hence, in view of the hypotheses above we can write

$$f_0(\xi) - f_0(\eta) \ge C(\xi - \eta)$$

which concludes the proof. □

Theorem 5.3. *Let* $u \in \mathcal{B}_{BV}$*. Then, there exists a sequence* $u_n \in \mathcal{B}_{fc}$ *such that*

- $d_\infty(u_n, u) \to 0$ *as* $n \to +\infty$,

- $u_n \to u$ *a. e. and in* L^1.

Proof. Since $u \in BV$, then u has a right continuous representative. Let F be the distribution function of u and let F^{-1} be its pseudo inverse. Since F^{-1} is strictly increasing, bounded and right–continuous, we can construct the corresponding approximating sequence f_n as in Theorem 5.1. For any integer n, let $F_n : \mathbb{R} \to [0,1]$ be the pseudo inverse of f_n, namely

$$F_n(x) = \inf\{\xi \ : \ f_n(\xi) > x\}.$$

By definition of the approximating sequence f_n we can easily deduce for all $x \in \mathbb{R}$

$$F^{-1}(F_n(x)) = F_n^{-1}(F_n(x)) + \sum_{k>n} r_k(F_n(x)), \qquad n > 0.$$

By applying F to both member of the above identity we obtain

$$F_n(x) = F\left(F_n^{-1}(F_n(x)) + \sum_{k>n} r_k(F_n(x))\right). \tag{5.3}$$

In order to simplify the expression (5.3), we distinguish between the following two cases. If $x \in \mathbb{R}$ is such that $F_n(x)$ is a point of continuity of f_n, then $f_n(F_n(y)) = x$ for y belonging in a small enough neighborhood of x, and then $F_n^{-1}(F_n(x)) = x$. In case $F_n(x)$ is a point of discontinuity of f_n (we recall that f_n has a finite number of discontinuities), then

$$x \in [f_n(F_n(x)-), f_n(F_n(x)+)] = [f_n(F_n(x)-), f_n(F_n(x))].$$

This clearly implies

$$\begin{aligned} & x + \sum_{k>n} r_k(F_n(x)) \\ & \in [f_n(F_n(x)-) + \sum_{k>n} r_k(F_n(x)), f_n(F_n(x)) + \sum_{k>n} r_k(F_n(x))]. \end{aligned}$$

By (left) continuity of the jump function $\sum_{k>n} r_k(\cdot)$ at the point $F_n(x)$ (because $F_n(x)$ does not belong to the set of points of discontinuity of the jump function) and by using the definition of f_n we have

$$x + \sum_{k>n} r_k(F_n(x)) \in [F^{-1}(F_n(x)-), F^{-1}(F_n(x)+)].$$

Since $F_n(x)$ is also a discontinuity point for F^{-1}, then F is constant on the interval in the above expression. Thus,

$$F\left(x + \sum_{k>n} r_k(F_n(x))\right) = F\left(F_n^{-1}(F_n(x)) + \sum_{k>n} r_k(F_n(x))\right),$$

and (5.3) can be simplified to

$$F_n(x) = F\left(x + \sum_{k>n} r_k(F_n(x))\right) = F(\alpha_n(x)), \tag{5.4}$$

where we have set

$$\alpha_n(x) := x + \sum_{k>n} r_k(F_n(x)). \tag{5.5}$$

Due to the fact that $\sum_{k>n} s_k \to 0$ as $n \to +\infty$, we see that $\alpha_n(x) \to x^+$ as $n \to +\infty$. We now observe that, in view of the result in Lemma 5.2, the approximating function F_0 is globally Lipschitz. This is due to the fact that f_0 is continuous and strictly increasing (and therefore F_0 is the *real* inverse of f_0) and due to the identity

$$f'(\xi) = \frac{1}{u(f(\xi))} \quad \text{for a. e. } \xi \in [0,1],$$

which guarantees $f' \geq C > 0$ almost everywhere (as requested by the previous lemma). Now it is easy to check that, for all $n \geq 1$ and for all $x, y \in \mathbb{R}$,

$$F_n(x) - F_n(y) \leq F_0(x) - F_0(y), \tag{5.6}$$

which implies that all F_n are absolutely continuous functions (because F_0 is globally Lipschitz). Now we can define

$$u_n(x) := \frac{d}{dx} F_n(x),$$

where we clearly have $F_n(x) = \int_{-\infty}^{x} u_n(y)dy$ and $u_n \in L^1$. Moreover, by differentiating with respect to x in (5.4) we easily see that $u_n(x) = u(\alpha_n(x))$ almost everywhere on the set of all points where the function $x \to \alpha_n(x)$ is differentiable. Since such set is the complement of a zero measure set (the jump function in (5.5) has a zero almost everywhere derivative), by right continuity of u and due to $\alpha_n(x) \geq x$, we have

$$u_n(x) = u(\alpha_n(x)) \to u(\alpha_\infty(x)) = u(x) \quad \text{a. e.}$$

By Lebesgue's dominated convergence Theorem we also have

$$u_n \to u \quad \text{in } L^1$$

and since f_n is the pseudo inverse of the distribution function of u_n, thanks to the results in Theorem 5.1 we have

$$d_\infty(u_n, u) \to 0$$

and the proof is complete. □

Theorem 5.4. *Let $\bar u, \bar v \in \mathcal{B}_{fc}$. Then there exist two sequences*

$$\{\bar u_n\}_n,\ \{\bar v_n\}_n \subset \mathcal{B}_c$$

such that

$$\lim_{n\to+\infty} d_p(\bar u_n, \bar v_n) = d_p(\bar u, \bar v), \quad \textit{for any } p < +\infty \tag{5.7}$$

$$d_\infty(\bar u_n, \bar v_n) \le d_\infty(\bar u, \bar v) \tag{5.8}$$

$$\bar u_n \to \bar u,\ \bar v_n \to \bar v \quad \textit{a.e. in } L^1(\mathbb{R}). \tag{5.9}$$

Proof. Let F, G be the distribution functions of $\bar u, \bar v$ respectively, and let F^{-1}, G^{-1} be their pseudo–inverses. The definition of the space $\mathcal{B}_{fc}$ implies that F^{-1} and G^{-1} may not be continuous. By the hypotheses of finite number of connected components stated above, F^{-1} and G^{-1} have at most a finite number of (jump) discontinuities. We shall construct the approximating sequences $\bar u_n$, $\bar v_n$ by means of the corresponding pseudo–inverses F_n^{-1}, G_n^{-1} in such a way that $F_n^{-1} \to F^{-1}$ and $G_n^{-1} \to G^{-1}$ almost everywhere and such that both sequences are uniformly bounded. This fact trivially implies (5.7). Since we want to obtain the inequality (5.8), our approximating sequences has to be carefully constructed according to several cases. In the sequel of the proof we provide the rigorous definition of F^{-1} and G^{-1} and we check that property (5.8) is verified in all cases.

Let us start with the simplest case: let $\xi_0 \in [0,1]$ be such that F^{-1} (G^{-1} resp.) is continuous at ξ_0 and such that the distances between ξ_0 and all the points of discontinuity of F^{-1} (G^{-1} resp.) are larger than $1/n$. Then we fix $F_n^{-1}(\xi) := F^{-1}(\xi)$ ($G_n^{-1}(\xi) := G^{-1}(\xi)$ resp.).

Suppose now that F^{-1} has an isolated discontinuity at ξ_0 and G^{-1} is continuous at ξ_0. Then, in case both $F^{-1}(\xi_0-)$ and $F^{-1}(\xi_0+)$ are larger than $G^{-1}(\xi_0)$ we define F_n^{-1} by modifying the graph of F^{-1} only on the interval $(\xi_0 - 1/n, \xi_0 + 1/n)$ and by setting[1]

$$F_n^{-1}(\xi) = \begin{cases} F^{-1}(\xi_0-) + n[F^{-1}(\xi_0 + 1/n) - F^{-1}(\xi_0-)](\xi - \xi_0) \\ \hfill \text{if } \xi \in [\xi_0, \xi_0 + 1/n] \\ F^{-1}(\xi) \qquad\qquad \text{if } \xi \in [\xi_0 - 1/n, \xi_0]. \end{cases}$$

In case both $F^{-1}(\xi_0-)$ and $F^{-1}(\xi_0+)$ are smaller than $G^{-1}(\xi_0)$, again we define F_n^{-1} by modifying the graph of F^{-1} only on the interval

[1] In this definition and in the following ones, we shall modify the graph of F^{-1} and G^{-1} by linear interpolations on a neighborhood of radius $2/n$ centered at a discontinuity point ξ_0. It could happen the linear part of the graph run the neighborhood of another discontinuity point. This can be avoided by choosing n large enough.

$(\xi_0 - 1/n, \xi_0 + 1/n)$ and by setting

$$F_n^{-1}(\xi) = \begin{cases} F^{-1}(\xi) & \text{if } \xi \in [\xi_0, \xi_0 + 1/n] \\ F^{-1}(\xi_0+) - n[F^{-1}(\xi_0+) - F^{-1}(\xi_0 - 1/n)](\xi_0 - \xi) & \\ & \text{if } \xi \in [\xi_0 - 1/n, \xi_0]. \end{cases}$$

Finally, in case $F^{-1}(\xi_0-) < G^{-1}(\xi_0) < F^{-1}(\xi_0+)$ we set

$$F_n^{-1}(\xi) = G^{-1}(\xi_0) + n[F^{-1}(\xi_0 + 1/n) - F^{-1}(\xi_0 - 1/n)](\xi - \xi_0 + 1/n),$$

for all $\xi \in [\xi_0 - 1/n, \xi_0 + 1/n]$. It easy to verify that in all three cases we have

$$|F_n^{-1}(\xi) - G_n^{-1}(\xi)| \leq |F^{-1}(\xi) - G^{-1}(\xi)|$$

for all $\xi \in [\xi_0 - 1/n, \xi_0 + 1/n]$. Clearly, if G^{-1} is discontinuous at ξ_0 and F^{-1} is not, we can define the two sequences by interchanging the roles of F^{-1} and G^{-1}.

Suppose now ξ_0 is an isolated jump discontinuity for both F^{-1} and G^{-1}. Suppose first

$$F^{-1}(\xi_0-) < G^{-1}(\xi_0-) < F^{-1}(\xi_0+) < G^{-1}(\xi_0+).$$

In this case we set

$$\begin{aligned} F_n^{-1}(\xi) = {} & \frac{G^{-1}(\xi_0-) + F^{-1}(\xi_0+)}{2} \\ & + n[F^{-1}(\xi_0 + 1/n) - F^{-1}(\xi_0 - 1/n)](\xi - \xi_0 + 1/n), \\ G_n^{-1}(\xi) = {} & \frac{G^{-1}(\xi_0-) + F^{-1}(\xi_0+)}{2} \\ & + n[G^{-1}(\xi_0 + 1/n) - G^{-1}(\xi_0 - 1/n)](\xi - \xi_0 + 1/n), \end{aligned}$$

for all $\xi \in [\xi_0 - 1/n, \xi_0 + 1/n]$. Suppose now

$$F^{-1}(\xi_0-) < F^{-1}(\xi_0+) < G^{-1}(\xi_0-) < G^{-1}(\xi_0+).$$

In this case we set

$$G_n^{-1}(\xi) = \begin{cases} G^{-1}(\xi_0-) + n[G^{-1}(\xi_0 + 1/n) - G^{-1}(\xi_0-)](\xi - \xi_0) & \\ & \text{if } \xi \in [\xi_0, \xi_0 + 1/n] \\ F^{-1}(\xi) & \text{if } \xi \in [\xi_0 - 1/n, \xi_0] \end{cases}$$

and

$$F_n^{-1}(\xi) = \begin{cases} F^{-1}(\xi) & \text{if } \xi \in [\xi_0, \xi_0 + 1/n] \\ F^{-1}(\xi_0+) - n[F^{-1}(\xi_0+) - F^{-1}(\xi_0 - 1/n)](\xi_0 - \xi) & \\ & \text{if } \xi \in [\xi_0 - 1/n, \xi_0]. \end{cases}$$

Again, interchanging the roles of F^{-1} and G^{-1} allows to cover all possible cases and we have once again

$$|F_n^{-1}(\xi) - G_n^{-1}(\xi)| \leq |F^{-1}(\xi) - G^{-1}(\xi)|$$

for all $\xi \in [\xi_0 - 1/n, \xi_0 + 1/n]$.

In order to complete the proof, we have to prove (5.9). This can be done by observing that the previous definitions of the approximating sequences F_n^{-1}, G_n^{-1} imply small rearrangements of the mass in the graphs of $\bar{u}$ and $\bar{v}$. Consider for instance the case of a discontinuity ξ_0 for F^{-1} modified on the right neighborhood $(\xi_0, \xi_0 + 1/n)$ as in the first case of the present proof. In this case $\bar{u}$ is modified only on the interval $(F^{-1}(\xi_0-), F^{-1}(\xi_0 + 1/n))$ as follows

$$\bar{u}_n(x) = \left[n\left(F^{-1}(\xi_0 + 1/n) - F^{-1}(\xi_0-)\right)\right]^{-1},$$

for all $x \in (F^{-1}(\xi_0-), F^{-1}(\xi_0 + 1/n))$. Hence $\bar{u}_n \to 0$ on the interval $(F^{-1}(\xi_0-), F^{-1}(\xi_0+))$ and $\bar{u}_n \to \bar{u}$ otherwise. We skip the details of all the cases, which can be proven in a similar way. □

As a trivial consequence of the previous theorems we have the following

Theorem 5.5. *Let $\bar{u}, \bar{v} \in \mathcal{B}_{BV}$ and let $\delta > 0$. Then there exist two sequences $\bar{u}_n, \bar{v}_n \in \mathcal{B}_c$ such that*

$$d_\infty(\bar{u}_n, \bar{v}_n) \leq d_\infty(\bar{u}, \bar{v}) + \delta$$
$$\bar{u}_n \to \bar{u},\ \bar{v}_n \to \bar{v} \quad \textit{a.e. and in } L^1(\mathbb{R}).$$

Proof. Take two sequences $u_n, v_n \in \mathcal{B}_{fc}$ such that

$$u_n \to \bar{u}, \quad v_n \to \bar{v} \quad \text{a.e. and in } L^1$$

and such that

$$d_\infty(u_n, \bar{u}) \to 0, \quad d_\infty(v_n, \bar{v}) \to 0, \quad \text{as } n \to +\infty$$

as guaranteed by Theorem 5.3. Thanks to Theorem 5.4, for any n there exist two sequences $u_{n_k}, v_{n_k} \in B_c$ such that

$$d_\infty(u_{n_k}, v_{n_k}) \leq d_\infty(u_n, v_n)$$
$$u_{n_k} \to u_n, \quad v_{n_k} \to v_n \quad \text{a.e. and in } L^1(\mathbb{R}) \quad \text{as } k \to +\infty.$$

Hence, by choosing

$$\bar{u}_n := u_{n_n}, \quad \bar{v}_n := v_{n_n},$$

we have

$$d_\infty(\bar{u}_n, \bar{v}_n) \leq d_\infty(u_n, v_n) \leq \delta + d_\infty(\bar{u}, \bar{v})$$

for n sufficiently large. □

In the previous theorems the approximating sequences were required to be converging almost everywhere, in L^1 and in the d_∞ topology. In the following theorems we prove some density results of certain subsets of the space of measures $(\mathcal{M}_0, d_\infty)$ without any requirement on the L^1–convergence. Such results are needed in the proof of Theorem 3.5 and the notations concerning the spaces involved are those defined in (3.2). We remark that the measure space $\mathcal{M}_K$ defined in (3.14) is a closed subset of the space $\mathcal{M}$ for any $K > 0$. Hence, some density properties needed in the aforementioned theorem can be obtained via intersection with $\mathcal{M}_K$.

Theorem 5.6. *$(\mathcal{M}_c, d_\infty)$ is dense in $(\mathcal{M}_0, d_\infty)$.*

Proof. For $\mu \in \mathcal{M}_0$, let

$$F(x) := \mu((-\infty, x])$$

be the distribution function of μ. Let $f : [0,1] \to \mathbb{R}$ be the pseudo inverse of F. For $\epsilon > 0$ we define

$$\overline{f}_\epsilon(\xi) := f(\xi) + \epsilon\xi$$
$$f_\epsilon(\xi) := \frac{\overline{f}_\epsilon(\xi)}{\|\overline{f}_\epsilon\|_{L^2([0,1])}}.$$

Since μ has unit second moment, then f has unit L^2 norm, therefore $f_\epsilon \to f$ uniformly on $[0,1]$ and in $L^2([0,1])$. Let F_ϵ be the pseudo inverse of f_ϵ. Due to $(\overline{f}_\epsilon(\xi))' \geq \epsilon$ for any $\xi \in [0,1]$ and in view of $\|\overline{f}_\epsilon\|_{L^2} \to 1$ as $\epsilon \to 0$, we have

$$f_\epsilon(\xi) - f_\epsilon(\eta) \geq 2\epsilon(\xi - \eta)$$

for small enough ϵ. Moreover, since f is strictly invertible on $[0,1]$, the above inequality implies the uniform bound

$$\frac{F_\epsilon(x) - F_\epsilon(y)}{x - y} \leq \frac{1}{2\epsilon}$$

for any $x, y \in \mathbb{R}$. Therefore F_ϵ is absolutely continuous, and its first derivative almost everywhere $u_\epsilon := (F_\epsilon)'$ belongs in $L^1(\mathbb{R})$. The above estimate of the difference quotients implies $u_\epsilon \in L^\infty$. Since F_ϵ is the primitive of u_ϵ, the uniform convergence of f_ϵ to f is equivalent to $d_\infty(u, u_\epsilon) \to 0$ as $\epsilon \to 0$. □

Theorem 5.7. *$(\mathcal{M}_{fc}, d_\infty)$ is dense in $(\mathcal{M}, d_\infty)$.*

Proof. We want to prove that, for any $\mu \in \mathcal{M}$, there exists a sequence of $\{\mu_k\}_k \subset \mathcal{M}_{fc}$ such that

$$d_\infty(\mu, \mu_k) \to 0, \quad \text{as } k \to +\infty.$$

The proof can be performed in the same way as in the proof of Theorem 5.1. In the present case, one has to define the distribution function

$$F(x) := \mu((-\infty, x])$$

and its pseudo inverse $f : [0,1] \to \mathbb{R}$. Let $\{\xi_k\}_k$ be the sequence of the discontinuity points of f. As in the proof of Theorem 5.1, we can define for any positive integer k

$$r_k(\xi) = \begin{cases} 0 & \text{if } 0 \leq \xi < \xi_k \\ f(\xi_k+) - f(\xi_k-) & \text{if } \xi_k \leq \xi \leq 1 \end{cases}$$

and, for any $\xi \in [0,1]$ and $n \geq 0$, the sequence of functions

$$\overline{f}_n(\xi) := f(\xi) - \sum_{k>n} r_k(\xi) + \frac{1}{n}\xi.$$

In order to have the approximating sequence in $\mathcal{M}$, we must normalize it as follows

$$f_n(\xi) := \frac{\overline{f}_n(\xi)}{\|\overline{f}_n\|_{L^2([0,1])}}.$$

It is easily seen that f_n has a finite number of discontinuities and $f_n \to f$ uniformly on $[0,1]$ (we recall that $\|f\|_{L^2} = 1$ because the corresponding measure μ has unit second moment). Once again, as in the proof of the Theorem 5.6, we have to make sure that f_n has a corresponding density $u \in L^\infty$ such that $u \in \mathcal{M}_{fc}$. More precisely, let $\overline{F}_n$ be the pseudo inverse of $\overline{f}_n$ for any $n \geq 0$. We recall (see Theorem 5.1) that the function $\overline{f}_0$ defined before is continuous and satisfies

$$(\overline{f}_0)' = (\overline{f}_n)' \geq \frac{1}{n}$$

almost everywhere for all $n > 0$. Hence, we have

$$\overline{f}_0(\xi) - \overline{f}_0(\eta) \geq \frac{1}{n}(\xi - \eta),$$

which implies a uniform bound (with respect to ξ) for the difference quotients of the pseudo inverse $\overline{F}_0$ as in Theorem 5.6. By means of the same argument leading to (5.6) in the proof of the Theorem 5.3, we can deduce that the difference quotients of $\overline{F}_n$ are uniformly bounded over the real line, which implies that all $\overline{F}_n$ are absolutely continuous. Therefore, so are the pseudo inverses F_n of the approximating functions f_n, and this implies that F_n is the distribution function of a probability density u_n. Moreover, since $(F_n)'$ is essentially bounded, then $u_n \in L^\infty$, and u_n has a finite number of connected components in its support because f_n has a finite number of discontinuities. Finally, $f_n \to f$ uniformly implies $d_\infty(u_n, \mu) \to 0$ as $n \to +\infty$. $\square$

For the sake of completeness, we also prove the following approximation theorem, which is a direct consequence of the previous theorem.

Theorem 5.8. *$(\mathcal{M}_{BV}, d_\infty)$ is dense in $(\mathcal{M}, d_\infty)$.*

Proof. Thanks to the result in the previous Theorem 5.7, it is sufficient to prove the assertion when the measure μ is supported on a finite number of connected subsets of $\mathbb{R}$. Let once again F be the distribution function of μ and let f be its pseudo inverse. Let $\xi_1, \ldots, \xi_n$ be the (finite) discontinuity points of f. As before we can define the jump functions r_k for $k = 1, \ldots, n$. As proven in Theorem 5.1, the function

$$\overline{f}_0(\xi) := f(\xi) - \sum_{k=1}^{n} r_k(\xi)$$

is continuous and $(\overline{f}_0)' = f'$ almost everywhere. We want to regularize the function $\overline{f}_0$ in such a way that the corresponding approximating densities belong in $\mathcal{M}_0$. For a small $\epsilon > 0$, let ρ^ϵ be a standard Friedrichs' mollifier. We set

$$\overline{f}_0^\epsilon(\xi) := \rho^\epsilon * \overline{f}_0(\xi) + \epsilon\xi.$$

It is easily seen that

- $\overline{f}_0^\epsilon \in C^\infty([0,1])$
- $(\overline{f}_0^\epsilon)' \geq \epsilon$.

Let $\overline{F}_0^\epsilon$ be the pseudo inverse of $\overline{f}_0^\epsilon$. We have

$$(\overline{F}_0^\epsilon)''(x) = -\frac{(\overline{f}_0^\epsilon)''(\overline{F}_0^\epsilon(x))}{((\overline{f}_0^\epsilon)'(\overline{F}_0^\epsilon(x)))^2}.$$

Therefore, $\overline{F}_0^\epsilon \in C^2(K_\epsilon)$ for all $\epsilon > 0$, where K_ϵ is the support of $(\overline{F}_0^\epsilon)'$ which is a compact subset of $\mathbb{R}$. In view of that, the probability density

$$\overline{u}_0^\epsilon := (\overline{F}_0^\epsilon)'$$

is compactly supported and BV. We define now

$$\overline{f}^\epsilon(\xi) := \overline{f}_0^\epsilon(\xi) + \sum_{k=0}^{n} r_k(\xi).$$

Let $\overline{F}^\epsilon$ be the pseudo inverse of $\overline{f}^\epsilon$ and let

$$\overline{u}^\epsilon(\xi) := (\overline{F}^\epsilon)'.$$

It is easily seen that $\overline{u}^\epsilon$ is still a BV function. Indeed,

$$TV(\overline{u}^\epsilon) \leq TV(\overline{u}_0^\epsilon) + 2n\|\overline{u}_0^\epsilon\|_{L^\infty(\mathbb{R})}.$$

In order to prove the previous estimate we only need to observe that the support of $\overline{u}^\epsilon$ has n connected components $K_1, \ldots, K_n$ and

$$\overline{u}^\epsilon(x) = \overline{u}_0^\epsilon(x - \alpha_j), \quad \text{for all } x \in K_j,\ j = 1, \ldots, n$$

for certain constants $\alpha_1, \ldots, \alpha_n$. Hence, the increase in the total variation of $\overline{u}^\epsilon$ is just a byproduct of the finite number of jumps occurring at the extremal points of the sets K_j, $j = 1, \ldots, n$. We now consider

$$f^\epsilon(\xi) := \frac{\overline{f}^\epsilon(\xi)}{\|\overline{f}^\epsilon\|_{L^2([0,1])}}.$$

From the previous definitions it is clear that

$$\|f^\epsilon - f\|_{L^\infty([0,1])} \to 0$$

as $\epsilon \to 0$. Finally, let F^ϵ be the pseudo inverse of f^ϵ. Then, the probability density $u^\epsilon := (F^\epsilon)'$ is compactly supported and BV. Hence $u^\epsilon \in \mathcal{M}_0$ and $d_\infty(u^\epsilon, \mu) \to 0$ as $\epsilon \to 0$ which completes the proof. $\square$

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