Local existence and finite-time blow-up in multidimensional radiation hydrodynamics *

Xinhua Zhong

Department of Basic Courses, Beijing Information Technology Institute,
Beijing 100101, China (zhong_xinhua@iapcm.ac.cn)

Song Jiang

Institute of Applied Physics and Computational Mathematics,
P.O. Box 8009, Beijing 100088, China (jiang@iapcm.ac.cn)

Abstract

We first prove the local existence of smooth solutions to the Cauchy problem for the equations of multidimensional radiation hydrodynamics which are a hyperbolic-Boltzmann coupled system. Then, we show that a smooth solution will blow up in finite time if the initial data are large. Moreover, the property of finite propagation speed is obtained simultaneously.

1 Introduction

This paper is concerned with the local well-posedness and finite-time blow-up of smooth solutions to the Cauchy problem for the general equations arising from radiation hydrodynamics.

The importance of thermal radiation in physical problems increases as the temperature is raised. At moderate temperatures, the role of the radiation is primarily one of transporting energy by radiative process, while at higher temperature, the energy and momentum densities of the radiation field may become comparable to or even dominate the corresponding fluid quantities. In this case, the radiation field significantly affects the dynamics of the fluid. Hydrodynamics with explicit account of the radiation energy and momentum contributions constitutes the charter of "radiation hydrodynamics". The theory of radiation hydrodynamics finds a wide range of application, including such diverse astrophysical phenomena as waves and oscillations in stellar atmospheres and envelopes, nonlinear stellar pulsation, supernova explosions, stellar winds, and many others. It has also direct application in other areas, for instance to the physics of laser fusion and reentry of vehicles. As will be seen below, the general equations of radiation hydrodynamics are a system of the Euler equations (hyperbolic) coupled with a transport equation (Boltzmann equation). Therefore, the study of mathematical theory of radiation hydrodynamics is of great importance both from the mathematical theory and application point of view.

Since in radiation hydrodynamic problems the matter is generally in the gaseous state, one could envision describing the matter by a kinetic (transport) equation similar to the

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equation of radiative transfer, generally referred to as the Boltzmann equation. In radiation hydrodynamic work such a detailed kinetic description of the matter is not used. Rather, one uses hydrodynamics, with a proper accounting of the effects of the radiation field, to describe the motion of the fluid. Actually, the equations of hydrodynamics follow from simple kinetic theory considerations and hence constitute an approximation to the Boltzmann (kinetic) equation.

In the term radiation hydrodynamics, it is necessary to include effects of the radiation field in the hydrodynamic equations for this class of problems. The equations of hydrodynamics result from particle, momentum, and energy balances for a differential volume of space. If a significant radiation field is present, one has to include the radiation momentum and energy in these balances. This gives rise to radiation terms in the equations of hydrodynamics. We first introduce the basic concepts needed to describe the radiation field and its interaction with matter. Consider the contributions of the radiation field to the energy and momentum density and flux. At any time, $2N$ variables are required to specify the position of a photon in phase space, namely N position variables and N momentum variables. We denote the N position variables by the vector x. In radiative transfer work it is conventional to use, rather than the N momentum variables, N equivalent variables, namely the frequency ν and the direction of travel of the photon Ω. In terms of these variables, we introduce the distribution function $f(x,t,\nu,\Omega)$, such that $f dx d\nu d\Omega$ is the number of photons (at time t) at space point x in a differential volume element dx, with frequency ν in a frequency interval $d\nu$, and travelling in a direction Ω in a solid angle element $d\Omega$. The specific intensity of radiation $I(x,t,\nu,\Omega)$ is then defined as

$$I(x,t,\nu,\Omega) = ch\nu f(x,t,\nu,\Omega)$$

with the Plank constant h and the light speed c.

Under the consideration of the three basic interactions between photons and matter, namely absorption, scattering and emission, we find the equation of transfer in the conventional form (cf. [5, 4])

$$\begin{aligned}\frac{1}{c}\frac{\partial I(\nu,\Omega)}{\partial t} &+ \Omega\cdot\nabla I(\nu,\Omega) = S(\nu) - \sigma_a(\nu)I(\nu,\Omega)\\ &+\int_0^\infty d\nu'\int_{S^{N-1}}\left[\frac{\nu}{\nu'}\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)I(\nu',\Omega') - \sigma_s(\nu\to\nu',\Omega\cdot\Omega')I(\nu,\Omega)\right]d\Omega'. \quad (1.1)\end{aligned}$$

Here $I(\nu,\Omega)\equiv I(x,t,\nu,\Omega)$, S^{N-1} is the unit ball in $\mathbb{R}^N$, $S(\nu)\equiv S(x,t,\nu)$ is the rate of energy emission due to spontaneous processes. $\sigma_a(\nu)\equiv\sigma_a(x,t,\nu,\varrho,\theta)$ denotes the absorption coefficient that may also depend on the mass density ϱ and the temperature θ of the matter. The dependence of σ_a upon ϱ and θ can have the form, for example, (cf. [10, 2])

$$\sigma_a = O(\varrho^\alpha\theta^{-\beta}), \quad \alpha,\beta>0.$$

Similar to absorption, a photon can undergo scattering interactions with matter, and the scattering interaction serves to change the photon's characteristics ν' and Ω' to a new set of characteristics ν and Ω. To quantitatively describe the scattering event, one requires a probabilistic statement concerning this change. This leads to the definition of the "differential scattering coefficient" $\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)\equiv\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega,\varrho,\theta)$ that may depend on ϱ and θ (in general, σ_s is independent of θ, cf. Remark 2.3), such that the probability of a photon being scattered from ν' to ν contained in $d\nu$, and from Ω' to Ω contained in $d\Omega$, in travelling a distance ds is given by $\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)d\nu d\Omega ds$. Therefore,

$$\text{outscattering} = \int_0^\infty d\nu'\int_{S^{N-1}}\sigma_s(\nu\to\nu',\Omega\cdot\Omega')I(\nu,\Omega)d\Omega',$$

$$\text{inscattering} = \int_0^\infty d\nu' \int_{S^{N-1}} \sigma_s(\nu' \to \nu, \Omega' \cdot \Omega) I(\nu', \Omega') d\Omega'.$$

In the special case, for example, of scattering of photons from a Maxwellian gas of free electrons at some temperature θ, the scattering kernel σ_s has the property (cf. [1, 5])

$$\sigma_s(\nu' \to \nu, \Omega' \cdot \Omega) W(\nu')/h\nu' = \sigma_s(\nu \to \nu', \Omega \cdot \Omega') W(\nu)/h\nu, \quad W(\nu) = \nu^3 e^{-h\nu/k\theta},$$

and σ_s behaves like $\sigma_s = O(\varrho)$.

In the above, for the sake of simplicity, we have assumed that S and σ_a are independent of Ω and σ_s depends only upon $\Omega \cdot \Omega$. This means no inherent preferred direction in the matter. However, the fact that in radiation hydrodynamic problems the material is in general in motion changes the situation. This motion does introduce a preferred direction in the matter, namely the direction of motion of the fluid, and consequently, introduces an Ω (angular) dependence into S and σ_a, and separate Ω and Ω' dependences into σ_s. These Ω (angular) dependences are not inherent properties of the material, but arise only from the relative motion between the fluid and the observer. It should be pointed out that the rate of energy emission S may also depend on ϱ and θ (see Remark 2.3). Moreover, our local well-posedness Theorem 2.1 in Section 2 does still hold for the angular-dependent σ_a, σ_s, and the angular- and (ϱ, θ)-dependent S (cf. Remark 2.2).

For simplicity of presentation, in what follows, we will suppress the x, t, ϱ and θ dependences unless it is stated, and describe the assumption on theses dependences in Theorem 2.1.

In terms of the specific intensity, we define three quantities, namely, the energy density, the radiative flux and the radiative pressure tensor, by

$$\begin{cases} \mathbf{E}_r = \dfrac{1}{c} \displaystyle\int_0^\infty d\nu \int_{S^{N-1}} I(\nu, \Omega) d\Omega, \\ \mathbf{F}_r = \displaystyle\int_0^\infty d\nu \int_{S^{N-1}} \Omega I(\nu, \Omega) d\Omega, \\ \mathbf{P}_r = \dfrac{1}{c} \displaystyle\int_0^\infty d\nu \int_{S^{N-1}} \Omega \otimes \Omega I(\nu, \Omega) d\Omega. \end{cases} \tag{1.2}$$

Including effects due to the presence of a radiation field, the equations of (nonrelativistic) hydrodynamics, in Eulerian coordinates, are written as

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho u) = 0, \tag{1.3}$$

$$\frac{\partial}{\partial t}\left(\varrho u + \frac{1}{c^2}\mathbf{F}_r\right) + \nabla \mathbf{P}_m + \nabla \cdot (\varrho u \otimes u + \mathbf{P}_r) = 0, \tag{1.4}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\varrho u^2 + \mathbf{E}_m + \mathbf{E}_r\right) + \nabla \cdot \left[\left(\frac{1}{2}\varrho u^2 + \mathbf{E}_m + \mathbf{P}_m\right) u + \mathbf{F}_r\right] = W, \tag{1.5}$$

where $u = (u_1, \cdots, u_N)$ is the fluid velocity, W denotes the external energy, $\mathbf{E}_m$ and $\mathbf{P}_m$ are the fluid energy density and the material pressure, respectively. In this paper we consider only polytropic ideal gases, namely,

$$\mathbf{E}_m := c_\nu \varrho \theta, \qquad \mathbf{P}_m := R \varrho \theta \ \ (\equiv c_\nu (\gamma - 1) \varrho \theta), \tag{1.6}$$

with ϱ and θ denoting the mass density and the temperature of the fluid, respectively, $c_\nu > 0$ being the heat conductivity, $R = c_\nu(\gamma - 1)$, and $\gamma > 1$ being the specific heat ratio.

The equations (1.1)-(1.5) are a hyperbolic-Boltzmann coupled system of first order in which (1.3)-(1.5) describe the conservation laws of mass, momentum and energy. Aside from the radiation terms, these are just the classical (nonrelativistic) equations of hydrodynamics for a compressible, ideal fluid. We refer to [5, 4] for more details on radiation hydrodynamics.

The aim of this paper is first to prove the local existence of smooth solutions to the Cauchy problem for (1.1)-(1.5), and then to show that a smooth solution in general will blow up in finite time if the initial data are large. Moreover, the property of finite propagation speed is also obtained. Roughly speaking, the local existence is obtained by using an iteration and the Banach contraction mapping principle, a standard procedure (see, e.g., [3, 6]), while the blow-up and the finite propagation speed results are proved by adapting and modifying Sideris' arguments [8, 9] for the Euler equations. However, we should point out here that the main difficulties in the proof lie in dealing with the nonlinear and non-local terms in the system, and we shall employ delicate energy estimates to control the terms.

Throughout this paper we denote the usual Sobolev spaces by $H^s(\mathbb{R}^N)$ with norm $\|\cdot\|_s$. $L^p(I,B)$ resp. $\|\cdot\|_{L^p(I,B)}$ denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p=\infty$) functions from I to B resp. its norm, $I\subset\mathbb{R}$ an interval, B a Banach space. For simplicity we also use the following abbreviations:

$$\|\cdot\| \equiv \|\cdot\|_{L^2(\mathbb{R}^N)}, \quad \|\cdot\|_s \equiv \|\cdot\|_{H^s(\mathbb{R}^N)}, \quad |||\cdot|||_{s,T} \equiv \max_{0\le t\le T}\|\cdot\|_s.$$

The same letter C (sometimes used as $C(X,\cdots)$ to emphasize the dependence of C on $X,\cdots$) will denote various positive constants.

2 Reformulation and the local existence

In this section we give a local existence theorem. For this purpose we first rewrite the system (1.1)-(1.5) to a symmetric hyperbolic system of first order. Then, we use the Banach contraction mapping principle and tricky energy estimates to prove the local existence and uniqueness of smooth solutions to the Cauchy problem for (1.1)-(1.5).

Let us consider the equation of transfer, i.e.,

$$\begin{aligned}&\frac{1}{c}\frac{\partial I(\nu,\Omega)}{\partial t}+\Omega\cdot\nabla I(\nu,\Omega)\\&\quad=S(\nu)-\sigma(\nu)I(\nu,\Omega)+\int_0^\infty\int_{S^{N-1}}\frac{\nu}{\nu'}\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)I(\nu',\Omega')d\Omega'd\nu',\end{aligned}\tag{2.1}$$

where $\sigma(\nu)$ is the total interaction coefficient, i.e., $\sigma(\nu)=\sigma_a(\nu)+\sigma_s(\nu)$, here

$$\sigma_s(\nu)=\int_0^\infty\int_{S^{N-1}}\sigma_s(\nu\to\nu',\Omega\cdot\Omega')d\Omega'd\nu',$$

Without loss of generality, we assume $W=0$ in (1.5). Denote the vector and matrix

$$V=(\varrho,u_1,\cdots,u_N,\theta)^t,\quad \widetilde{A_j}(V)=\{\widetilde{a}_{mn}\}_{(N+2)\times(N+2)}$$

where $\widetilde{a}_{ii}=u_j$, $\widetilde{a}_{1(j+1)}=\varrho$, $\widetilde{a}_{(j+1)1}=R\theta/\varrho$, $\widetilde{a}_{(j+1)(N+2)}=R$, $\widetilde{a}_{(N+2)(j+1)}=R\theta/c_\nu$ for $j=1,\cdots,N$, and the rest elements of $\widetilde{a}_{ij}$ are set to 0. Define the vector $G(V,I)$ by

$$G(V,I)=(g_0,g_1,\cdots,g_{N+1})^t$$

with $g_0 = 0$, and for $j = 1, \cdots, N$,

$$g_j = -\frac{1}{c\varrho}\Big\{ \int_0^\infty d\nu \int_{S^{N-1}} \Omega_j \big[S(\nu) - \sigma(\nu) I(\nu,\Omega)\big] d\Omega$$

$$+ \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \Omega_j \frac{\nu}{\nu'} \sigma_s(\nu' \to \nu, \Omega'\cdot\Omega) I(\nu', \Omega') d\Omega' \Big\},$$

$$g_{N+1} = \frac{1}{cc_\nu \varrho}\Big\{ \int_0^\infty d\nu \int_{S^{N-1}} u\cdot\Omega (S(\nu) - \sigma(\nu) I(\nu,\Omega)) d\Omega$$

$$+ \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} u\cdot\Omega \frac{\nu}{\nu'} \sigma_s(\nu' \to \nu, \Omega'\cdot\Omega) I(\nu', \Omega') d\Omega' \Big\}$$

$$-\frac{1}{c_\nu \varrho}\Big\{ \int_0^\infty d\nu \int_{S^{N-1}} (S(\nu) - \sigma(\nu) I(\nu,\Omega)) d\Omega$$

$$+ \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu}{\nu'} \sigma_s(\nu' \to \nu, \Omega'\cdot\Omega) I(\nu', \Omega') d\Omega' \Big\}.$$

Thus, using (1.1) and (1.2) to delete the derivative terms involved with radiation in the system (1.3)-(1.5), we can rewrite (1.3)-(1.5) as

$$\frac{\partial V}{\partial t} + \sum_{j=1}^N \widetilde{A_j}(V) \frac{\partial V}{\partial x_j} = G(V, I). \tag{2.2}$$

We shall study the Cauchy problem for (2.1) and (2.2) together with the initial data

$$I(x, 0, \nu, \Omega) = I_0(x, \nu, \Omega), \quad V(x, 0) = V_0(x), \qquad x \in \mathbb{R}^N. \tag{2.3}$$

Then, the main result of this section reads:

Theorem 2.1 *Let $s > \frac{N}{2} + 1$. Assume that*

(*A1*) $$S \in L^\infty(0, T; L^2(0, \infty; H^s(\mathbb{R}^N)));$$

(*A2*) $$\max_{(\nu,\Omega)\in[0,\infty)\times S^{N-1}} \|\sigma(\cdot, t, \nu, \Omega, \varrho, \theta) - \bar\sigma\|_s \le C(\|\varrho - \bar\varrho\|_s, \|\theta - \bar\theta\|_s), \quad t \ge 0;$$

(*A3*) $$\int_0^\infty d\nu \int_{S^{N-1}} \Big(\int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu^2}{\nu'^2} \|\sigma_s(\cdot, t, \nu' \to \nu, \Omega'\cdot\Omega, \varrho, \theta) - \bar\sigma_s\|_s^2 d\Omega' \Big)^\lambda d\Omega$$

$$\le C(\|\varrho - \bar\varrho\|_s, \|\theta - \bar\theta\|_s), \quad t \ge 0 \qquad (\lambda = \tfrac{1}{2} \quad \text{or} \quad \lambda = 1)$$

for $(\rho - \bar\rho, \theta - \bar\theta) \in H^s(\mathbb{R}^N)$ with $M_1 \le \rho, \theta \le M_2$, and

$$(V_0, I_0) \in G := \Big\{ (V, I) \mid (\varrho - \bar\varrho, \theta - \bar\theta) \in H^s(\mathbb{R}^N),$$

$$I(x, \nu, \Omega) \in L^2((0,\infty) \times S^{N-1}, H^s(\mathbb{R}^N)), \ \ M_3 \le \int_0^\infty \int_{S^{N-1}} g d\Omega d\nu \le M_4 \Big\},$$

where $\bar\varrho$, $\bar\theta$ and M_i ($i = 1, \cdots, 4$) are positive constants, $\bar\sigma \equiv \sigma(x, t, \nu, \Omega, \bar\varrho, \bar\theta)$ and $\bar\sigma_s \equiv \sigma_s(x, t, \nu' \to \nu, \Omega'\cdot\Omega, \bar\varrho, \bar\theta)$. Then, there exists a $T > 0$, such that the problem (2.1)-(2.3) has a unique smooth solution (V, I) on $[0, T]$ satisfying

$$V \in C^1(\mathbb{R}^N \times [0, T]), \ \ I \in C^1(\mathbb{R}^N \times [0, T], (0, \infty) \times S^{N-1}), \ \ \text{and}$$

$$(V, I) \in G_1 \quad \text{for some } \bar G_1 \subset\subset G.$$

Remark 2.1 *We can obtain a similar existence-uniqueness theorem without essential changes in the arguments, if we, instead of (1.1), consider the following equation of transfer which has included the effects of induced processes and is in the local thermodynamic equilibrium:*

$$
\begin{aligned}
\frac{1}{c}\frac{\partial I(\nu,\Omega)}{\partial t} \quad &+ \quad \Omega\cdot\nabla I(\nu,\Omega) = \sigma_a'(\nu)[B(\nu)-I(\nu,\Omega)] \\
&+ \quad \int_0^\infty\int_{S^{N-1}} \frac{\nu}{\nu'}\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)I(\nu',\Omega')[1+c^2I(\nu,\Omega)/2h\nu^3]d\Omega'd\nu' \\
&- \quad \int_0^\infty\int_{S^{N-1}} \sigma_s(\nu\to\nu',\Omega\cdot\Omega')I(\nu,\Omega)[1+c^2I(\nu',\Omega')/2h{\nu'}^3]d\Omega'd\nu',
\end{aligned}
$$

where B is the Plank function, i.e.,

$$B(\nu)=\frac{2h\nu^3}{c^2}(e^{h\nu/k\theta}-1)^{-1}, \quad \textit{and} \quad \sigma_a'(\nu)=\sigma_a(\nu)(1-e^{-h\nu/k\theta}).$$

Remark 2.2 *Theorem 2.1 still remains valid if σ_a, σ_s and S are angular-dependent and satisfy suitable conditions, i.e.,*

$$S\equiv S(\nu,\Omega), \quad \sigma_a\equiv\sigma_a(\nu,\Omega), \quad \sigma_s\equiv\sigma_s(\nu'\to\nu,\Omega'\to\Omega).$$

Remark 2.3 *The conditions (A2) and (A3) are satisfied when the absorption coefficient and the scattering kernel (Compton scattering) are given by, for example (see [5]),*

$$
\begin{aligned}
&\sigma_a(\nu)=C_1\varrho\theta^{-1/2}\exp\Big[-\frac{C_2}{\theta^{1/2}}\Big(\frac{\nu-\nu_0}{\nu_0}\Big)^2\Big], \\
&\sigma_s(\nu\to\nu',\xi) \\
&=\frac{C_3\varrho(1+\xi^2)}{[1+\gamma(1-\xi)]^2}\times\Big\{1+\frac{\gamma^2(1-\xi)^2}{(1+\xi^2)[1+\gamma(1-\xi)]}\Big\}\delta\Big(\nu'-\frac{\nu}{1+\gamma(1-\xi)}\Big),
\end{aligned}
$$

where $\gamma=C_4\nu$, $\xi=\Omega\cdot\Omega'$, C_i $(i=1,\cdots,4)$ are positive constants, ν_0 is the fixed frequency.

PROOF. The proof is based on a classical iteration scheme and (tricky) energy estimates as well as the Banach contraction mapping principle, a standard procedure (cf. [3]). First, we symmetrize (2.2) by multiplying it with the symmetrizing matrix $A_0(V)$ defined by

$$A_0(V)=\begin{pmatrix} \varrho^{-1} & 0 & 0 \\ 0 & \frac{\varrho}{R\theta}\mathrm{id}_{N\times N} & 0 \\ 0 & 0 & \frac{c_\nu\varrho}{R\theta^2}\end{pmatrix}.$$

Therefore, we shall prove Theorem 2.1 for the quasilinear symmetric system:

$$A_0(V)\frac{\partial V}{\partial t}+\sum_{j=1}^N A_j(V)\frac{\partial V}{\partial x_j}=F(V,I), \tag{2.4}$$

where $A_j(V):=A_0(V)\widetilde{A_j}(V)$ is symmetric, and

$$F(V,I):=A_0(V)G(V,I)=(f_0,f_1,\cdots,f_{N+1})^t$$

with $f_0 = 0$, and for $j = 1, \cdots, N$,

$$
\begin{aligned}
f_j = & -\frac{1}{cR\theta}\Big\{ \int_0^\infty d\nu \int_{S^{N-1}} \Omega_j (S(\nu) - \sigma(\nu) I(\nu, \Omega)) d\Omega \\
& + \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \Omega_j \frac{\nu}{\nu'} \sigma_s(\nu' \to \nu, \Omega' \cdot \Omega) I(\nu', \Omega') d\Omega' \Big\}, \\
f_{N+1} = & \frac{1}{cR\theta^2}\Big\{ \int_0^\infty d\nu \int_{S^{N-1}} u \cdot \Omega (S(\nu) - \sigma(\nu) I(\nu, \Omega)) d\Omega \\
& + \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} u \cdot \Omega \frac{\nu}{\nu'} \sigma_s(\nu' \to \nu, \Omega' \cdot \Omega) I(\nu', \Omega') d\Omega' \Big\} \\
& - \frac{1}{R\theta^2}\Big\{ \int_0^\infty d\nu \int_{S^{N-1}} (S(\nu) - \sigma(\nu) I(\nu, \Omega)) d\Omega \\
& + \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu}{\nu'} \sigma_s(\nu' \to \nu, \Omega' \cdot \Omega) I(\nu', \Omega') d\Omega' \Big\}.
\end{aligned}
$$

In the sequel, we construct a smooth solution of (2.1), (2.4) and (2.3). Choose $j(x) \in C_0^\infty(\mathbb{R}^N)$, $\text{supp} j \subseteq \{x : |x| \leq 1\}$, $j \geq 0$ and $\int_{\mathbb{R}^N} j(x)dx = 1$. Set $j_\varepsilon = \varepsilon^{-N} j(\frac{x}{\varepsilon})$. Define $J_\varepsilon u \in C^\infty(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$ by

$$J_\varepsilon u(x) = \int_{\mathbb{R}^N} j_\varepsilon(x - y) u(y) dy.$$

Set $\varepsilon_k = 2^{-k}\varepsilon_0$ $(k = 0, 1, 2, \cdots)$ and

$$V_0^k(x) = J_{\varepsilon_k} V_0(x), \qquad I_0^k(x, \nu, \Omega) = J_{\varepsilon_k} I_0(x, \nu, \Omega),$$

where $\varepsilon_0 > 0$ will be chosen later.

We will construct a solution to (2.1), (2.4) and (2.3) through the following iteration scheme: First, we set

$$V^0(x, t) = V_0^0(x), \quad I^0(x, t, \nu, \Omega) = I_0^0(x, \nu, \Omega), \tag{2.5}$$

and for $k = 0, 1, 2, \cdots$, we define $V^{k+1}(x, t)$ and $I^{k+1}(x, t, \nu, \Omega)$ inductively as the solution of the following linearized equation:

$$
\begin{aligned}
& \frac{1}{c}\frac{\partial I^{k+1}(\nu, \Omega)}{\partial t} + \Omega \cdot \nabla I^{k+1}(\nu, \Omega) + \sigma^k(\nu) I^{k+1}(\nu, \Omega) \\
& \qquad = S(\nu) + \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu}{\nu'} \sigma_s^k(\nu' \to \nu, \Omega' \cdot \Omega) I^k(\nu', \Omega') d\Omega', \\
& A_0(V^k) \frac{\partial V^{k+1}}{\partial t} + \sum_{j=1}^N A_j(V^k) \frac{\partial V^{k+1}}{\partial x_j} = F(V^k, I^k), \\
& I^{k+1}(x, 0, \nu, \Omega) = I_0^{k+1}(x, \nu, \Omega), \quad V^{k+1}(x, 0) = V_0^{k+1}(x),
\end{aligned}
\tag{2.6}
$$

where σ^k and σ_s^k are σ and σ_s with (ϱ, θ) replaced by (ϱ^k, θ^k), respectively.

It follows immediately that

$$V^{k+1} \in C^\infty([0, T_k] \times \mathbb{R}^N), \quad I^{k+1} \in L^2((0, \infty) \times S^{N-1}, C^\infty([0, T_k] \times \mathbb{R}^N))$$

with T_k being the largest time of existence for (2.6) where the estimates

$$\max_{0\le t\le T_k}\int_0^\infty d\nu\int_{S^{N-1}}\|I^k-I_0^0\|_s^2 d\Omega\le R_1\quad\text{and}\quad |||V^k-V_0^0|||_{s,T_k}\le R_1$$

are valid. The following crucial lemma guarantees that there is a $T>0$ such that $T_k\ge T$ for $k=1,2,\cdots$.

Lemma 2.2 *There are constants $R_1>0$, $L>0$ and $T>0$ such that $I^{k+1}(x,t,\nu,\Omega)$, $V^{k+1}(x,t)$ $(k=0,1,\cdots)$ defined by the solution of (2.6) satisfy*

$$\max_{0\le t\le T}\int_0^\infty d\nu\int_{S^{N-1}}\|I^{k+1}-I_0^0\|_s^2 d\Omega\le R_1,\tag{2.7}$$

$$|||V^{k+1}-V_0^0|||_{s,T}\le R_1,\quad |||\frac{\partial V^{k+1}}{\partial t}|||_{s-1,T}\le L.\tag{2.8}$$

Proposition 2.3 *i)* (Moser-type calculus inequality) *For $f,g\in H^s\cap L^\infty$ and $|\alpha|\le s$*

$$\|D^\alpha(fg)\|\le C_s(\|f\|_{L^\infty}\|D^s g\|+\|g\|_{L^\infty}\|D^s f\|).$$

ii) (Sobolev's embedding inequality) For $s>N/2$,

$$\|f\|_{L^\infty}\le C_s||f||_s.$$

The following corollary is the direct consequence of Proposition 2.3.

Corollary 2.4 *If $s>N/2$, then for $f,g\in H^s$ and $|\alpha|\le s$,*

$$\|D^\alpha(fg)\|\le C_s||f||_s||g||_s.$$

Proof of Lemma 2.2. *Step I. Estimate of $\int_0^\infty d\nu\int_{S^{N-1}}\|I^{k+1}-I_0^0\|_s^2 d\Omega$.*

Denoting $W^{k+1}=I^{k+1}-I_0^0$, we find by a straightforward calculation that W^{k+1} satisfies

$$\begin{aligned}&\frac{1}{c}\frac{\partial W^{k+1}(\nu,\Omega)}{\partial t}+\Omega\cdot\nabla W^{k+1}(\nu,\Omega)+\sigma^k(\nu)W^{k+1}(\nu,\Omega)\\&\qquad=S(\nu)+H+\int_0^\infty d\nu'\int_{S^{N-1}}\frac{\nu}{\nu'}\sigma_s^k(\nu'\to\nu,\Omega'\cdot\Omega)I^k(\nu',\Omega')d\Omega',\\&W^{k+1}(x,0,\nu,\Omega)=I_0^{k+1}(x,0,\nu,\Omega)-I_0^0(x,0,\nu,\Omega),\end{aligned}\tag{2.9}$$

where

$$H=-\Omega\cdot\nabla I_0^0-\sigma^k(\nu)I_0^0.$$

For simplicity of presentation, we drop out the superscript k and consider

$$\begin{aligned}&\frac{1}{c}\frac{\partial W(\nu,\Omega)}{\partial t}+\Omega\cdot\nabla W(\nu,\Omega)+\sigma(\nu)W(\nu,\Omega)\\&\qquad=S(\nu)+H+\int_0^\infty d\nu'\int_{S^{N-1}}\frac{\nu}{\nu'}\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)I(\nu',\Omega')d\Omega',\\&W(x,0,\nu,\Omega)=W_0(x,\nu,\Omega).\end{aligned}\tag{2.10}$$

Differentiating (2.10) α-times with respect to x, multiplying the resulting equation by $D^\alpha W$, and then integrating over $\mathbb{R}^N \times (0,\infty) \times S^{N-1}$, we deduce that

$$\begin{aligned}
&\frac{1}{c}\frac{d}{dt}\int_0^\infty d\nu \int_{S^{N-1}} \int_{R^N} |D^\alpha W|^2 dx d\Omega \le C \int_0^\infty d\nu \int_{S^{N-1}} \|D^\alpha W\|^2 d\Omega \\
&+C\int_0^\infty d\nu \int_{S^{N-1}} \|D^\alpha (S(\nu)+H)\|^2 d\Omega + \int_0^\infty d\nu \int_{S^{N-1}} \|D^\alpha(\sigma(\nu)W)\|^2 d\Omega \qquad (2.11)\\
&+\int_{\mathbb{R}^N} dx \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu}{\nu'} D^\alpha(\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)I(\nu',\Omega'))D^\alpha W d\Omega'.
\end{aligned}$$

From Corollary 2.4 and Hölder's inequality we get

$$\begin{aligned}
&\int_0^\infty d\nu \int_{S^{N-1}} \|D^\alpha(\sigma(\nu)W)\|^2 d\Omega \\
&\le C\int_0^\infty d\nu \int_{S^{N-1}} \|D^\alpha((\sigma(\nu)-\bar\sigma)W)\|^2 d\Omega + C\int_0^\infty d\nu \int_{S^{N-1}} \|D^\alpha(\bar\sigma W)\|^2 d\Omega \\
&\le C_s\Big(\max_{[0,\infty)\times S^{N-1}} \|\sigma(\nu)-\bar\sigma\|_s^2 + \max_{[0,\infty)\times S^{N-1}} \|\bar\sigma\|_s^2\Big)\int_0^\infty d\nu \int_{S^{N-1}} \|W\|_s^2 d\Omega,
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathbb{R}^N} dx \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu}{\nu'} D^\alpha(\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)I(\nu',\Omega'))D^\alpha W d\Omega' \\
\le\ & C\int_{\mathbb{R}^N} dx \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu}{\nu'} D^\alpha((\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)-\bar{\sigma_s})I(\nu',\Omega'))D^\alpha W d\Omega' \\
&+C\int_{\mathbb{R}^N} dx \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu}{\nu'} D^\alpha(\bar{\sigma_s} I(\nu',\Omega'))D^\alpha W d\Omega' \\
\le\ & C_s\int_0^\infty d\nu \int_{S^{N-1}} \|D^\alpha W\|^2 d\Omega + C_s\Big(\int_0^\infty d\nu \int_{S^{N-1}} \|I(\nu,\Omega)\|_s^2 d\Omega\Big) \\
&\cdot \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu^2}{\nu'^2}\Big(\|\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)-\bar{\sigma_s}\|_s^2 + \|\bar{\sigma_s}\|_s^2\Big) d\Omega'.
\end{aligned}$$

Thus, from (2.11), it follows that

$$\begin{aligned}
&\frac{1}{c}\frac{d}{dt}\int_0^\infty d\nu \int_{S^{N-1}} \int_{\mathbb{R}^N} |D^\alpha W|^2 dx d\Omega \\
&\le C\int_0^\infty d\nu \int_{S^{N-1}} \|D^\alpha W\|^2 d\Omega + C\int_0^\infty d\nu \int_{S^{N-1}} (\|D^\alpha S(\nu)\|^2 + \|D^\alpha H\|^2) d\Omega \\
&+C_s\Big(\max_{[0,\infty)\times S^{N-1}} \|\sigma(\nu)-\bar\sigma\|_s^2 + \max_{[0,\infty)\times S^{N-1}} \|\bar\sigma\|_s^2\Big)\int_0^\infty d\nu \int_{S^{N-1}} \|W\|_s^2 d\Omega \\
&+C_s\Big(\int_0^\infty d\nu \int_{S^{N-1}} \|I(\nu,\Omega)\|_s^2 d\Omega\Big) \\
&\cdot \int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu^2}{\nu'^2}\Big(\|\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)-\bar{\sigma_s}\|_s^2 + \|\bar{\sigma_s}\|_s^2\Big) d\Omega'.
\end{aligned}$$

Applying Gronwall's inequality to the above inequality and using the assumptions of Theorem 2.1, we obtain (2.7).

Step II. Estimate of $\|D^\alpha F(V^k, I^k)\|$, $|\alpha| \le s$.

For simplicity of presentation, we drop out the superscript k in $F(V^k, I^k)$. By Proposition 2.3 we see that

$$\begin{aligned}
&\|D^\alpha f_j\| \le C_s\Big[\|\frac{1}{\theta}\|_{L^\infty} + \|D^s(\frac{1}{\theta})\|\Big] \\
&\cdot\Big\{\|\int_0^\infty d\nu \int_{S^{N-1}} \Omega_j S(\nu)\|_s + \|\int_0^\infty d\nu \int_{S^{N-1}} \Omega_j \sigma(\nu) I\|_s \\
&\quad + \Big\|\int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \Omega_j \frac{\nu}{\nu'}\sigma_s(\nu'\to\nu, \Omega'\cdot\Omega) I(\nu', \Omega') d\Omega'\Big\|_s\Big\}.
\end{aligned} \tag{2.12}$$

If one uses Minkowski's, Sobolev's and and Hölder's inequalities, one obtains from Corollary 2.4 that for $|\alpha| \le s$,

$$\begin{aligned}
&\|D^\alpha \int_0^\infty d\nu \int_{S^{N-1}} \Omega_j \sigma(\nu) I d\Omega d\nu\| \\
&\le C_s \int_0^\infty d\nu \int_{S^{N-1}} \Big(\|\sigma(\nu) - \bar\sigma\|_s + \|\bar\sigma\|_s\Big)\|I\|_s d\Omega \\
&\le C_s \Big(\int_0^\infty d\nu \int_{S^{N-1}} \|I\|_s^2 d\Omega\Big)^{1/2} \\
&\quad \cdot\Big\{\Big(\int_0^\infty d\nu \int_{S^{N-1}} \|\sigma(\nu) - \bar\sigma\|_s^2 d\Omega\Big)^{1/2} + \Big(\int_0^\infty d\nu \int_{S^{N-1}} \|\bar\sigma\|_s^2 d\Omega\Big)^{1/2}\Big\}
\end{aligned}$$

which implies

$$\begin{aligned}
&\|\int_0^\infty d\nu \int_{S^{N-1}} \Omega_j \sigma(\nu) I d\Omega\|_s \le C_s \Big(\int_0^\infty d\nu \int_{S^{N-1}} \|I\|_s^2 d\Omega\Big)^{1/2} \\
&\quad \cdot\Big\{\Big(\int_0^\infty d\nu \int_{S^{N-1}} \|\sigma(\nu) - \bar\sigma\|_s^2 d\Omega\Big)^{1/2} + \Big(\int_0^\infty d\nu \int_{S^{N-1}} \|\bar\sigma\|_s^2 d\Omega\Big)^{1/2}\Big\}.
\end{aligned} \tag{2.13}$$

Similarly, we have

$$\|\int_0^\infty d\nu \int_{S^{N-1}} \Omega_j S(\nu) d\Omega d\nu\|_s \le \Big(\int_0^\infty \int_{S^{N-1}} \|S(\nu)\|_s^2 d\Omega d\nu\Big)^{1/2} \tag{2.14}$$

and

$$\begin{aligned}
&\|\int_0^\infty d\nu \int_{S^{N-1}} d\Omega \int_0^\infty d\nu' \int_{S^{N-1}} \Omega_j \frac{\nu}{\nu'}\sigma_s(\nu'\to\nu, \Omega'\cdot\Omega) I(\nu', \Omega') d\Omega'\|_s \\
\le\ & C_s \left(\int_0^\infty d\nu \int_{S^{N-1}} \|I(\nu, \Omega)\|_s^2 d\Omega\right)^{1/2} \\
&\cdot\Big\{\int_0^\infty d\nu \int_{S^{N-1}} \left(\int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu^2}{\nu'^2}\|\bar\sigma_s\|_s^2 d\Omega'\right)^{1/2} d\Omega \\
+\ & \int_0^\infty d\nu \int_{S^{N-1}} \left(\int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu^2}{\nu'^2}\|\sigma_s(\nu'\to\nu, \Omega'\cdot\Omega) - \bar\sigma_s\|_s^2 d\Omega'\right)^{1/2} d\Omega\Big\}.
\end{aligned} \tag{2.15}$$

Thus, combining (2.12) with (2.13)-(2.15), we conclude that for $1 \le j \le N$,

$$\begin{aligned}
&\|D^\alpha f_j\| \le C_s \left(\|\frac{1}{\theta}\|_{L^\infty} + \|D^s(\frac{1}{\theta})\|\right)\Big\{\Big(\int_0^\infty \int_{S^{N-1}} \|S(\nu)\|_s^2 d\Omega d\nu\Big)^{1/2} \\
&+ \Big(\int_0^\infty d\nu \int_{S^{N-1}} \|I\|_s^2 d\Omega\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\cdot\Big[\int_0^\infty d\nu\int_{S^{N-1}}\Big(\int_0^\infty d\nu'\int_{S^{N-1}}\frac{\nu^2}{\nu'^2}\|\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)-\bar{\sigma_s}\|_s^2 d\Omega'\Big)^{1/2}d\Omega\\
&+\Big(\int_0^\infty d\nu\int_{S^{N-1}}\|\sigma(\nu)-\bar{\sigma}\|_s^2 d\Omega\Big)^{1/2}+\Big(\int_0^\infty d\nu\int_{S^{N-1}}\|\bar{\sigma}\|_s^2 d\Omega\Big)^{1/2}\\
&+\int_0^\infty d\nu\int_{S^{N-1}}\Big(\int_0^\infty d\nu'\int_{S^{N-1}}\frac{\nu^2}{\nu'^2}\|\bar{\sigma_s}\|_s^2 d\Omega'\Big)^{1/2}d\Omega\Big]\Big\}. \qquad (2.16)
\end{aligned}
$$

In the same manner, we can get

$$
\begin{aligned}
&\|D^\alpha f_{N+1}\|\le C_s\Big(\|\frac{1}{\theta^2}\|_{L^\infty}+\|D^s(\frac{1}{\theta^2})\|+\sum_{i=1}^N\Big[\|\frac{u_i}{\theta^2}\|_{L^\infty}+\|D^s(\frac{u_i}{\theta^2})\|\Big]\Big)\\
&\cdot\Big\{\Big(\int_0^\infty d\nu\int_{S^{N-1}}\|I\|_s^2 d\Omega\Big)^{1/2}\Big[\Big(\int_0^\infty d\nu\int_{S^{N-1}}\|\sigma(\nu)-\bar{\sigma}\|_s^2 d\Omega\Big)^{1/2}\\
&+\int_0^\infty d\nu\int_{S^{N-1}}\Big(\int_0^\infty d\nu'\int_{S^{N-1}}\frac{\nu^2}{\nu'^2}\|\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)-\bar{\sigma_s}\|_s^2 d\Omega'\Big)^{1/2}d\Omega\\
&+\Big(\int_0^\infty d\nu\int_{S^{N-1}}\|S(\nu)\|_s^2 d\Omega\Big)^{1/2}+\Big(\int_0^\infty d\nu\int_{S^{N-1}}\|\bar{\sigma}\|_s^2 d\Omega\Big)^{1/2}\\
&+\int_0^\infty d\nu\int_{S^{N-1}}\Big(\int_0^\infty d\nu'\int_{S^{N-1}}\frac{\nu^2}{\nu'^2}\|\bar{\sigma_s}\|_s^2 d\Omega'\Big)^{1/2}d\Omega\Big]\Big\}. \qquad (2.17)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|D^\alpha F\|\le C_N\sum_{j=1}^{N+1}\|D^\alpha f_j\|\le C_{N,s}\Big\{\|\frac{1}{\theta}\|_{L^\infty}+\|D^s(\frac{1}{\theta})\|+\|\frac{1}{\theta^2}\|_{L^\infty}+\|D^s(\frac{1}{\theta^2})\|\\
&+\sum_{i=1}^N(\|\frac{u_i}{\theta^2}\|_{L^\infty}+\|D^s(\frac{u_i}{\theta^2})\|)\Big\}\Big\{\Big(\int_0^\infty d\nu\int_{S^{N-1}}\|S(\nu)\|_s^2 d\Omega\Big)^{1/2}\\
&+\Big(\int_0^\infty d\nu\int_{S^{N-1}}\|I\|_s^2 d\Omega\Big)^{1/2}\\
&\cdot\Big[\int_0^\infty d\nu\int_{S^{N-1}}\Big(\int_0^\infty d\nu'\int_{S^{N-1}}\frac{\nu^2}{\nu'^2}\|\sigma_s(\nu'\to\nu,\Omega'\cdot\Omega)-\bar{\sigma_s}\|_s^2 d\Omega'\Big)^{1/2}d\Omega\\
&+\Big(\int_0^\infty d\nu\int_{S^{N-1}}\|\sigma(\nu)-\bar{\sigma}\|_s^2 d\Omega\Big)^{1/2}+\Big(\int_0^\infty d\nu\int_{S^{N-1}}\|\bar{\sigma}\|_s^2 d\Omega\Big)^{1/2}\\
&+\int_0^\infty d\nu\int_{S^{N-1}}\Big(\int_0^\infty d\nu'\int_{S^{N-1}}\frac{\nu^2}{\nu'^2}\|\bar{\sigma_s}\|_s^2 d\Omega'\Big)^{1/2}d\Omega\Big]\Big\}. \qquad (2.18)
\end{aligned}
$$

Step III. Derivation of the estimates (2.8).

It is easy to see that U^{k+1} satisfies

$$
\begin{cases}
A_0(V^k)\dfrac{\partial U^{k+1}}{\partial t}+\displaystyle\sum_{j=1}^N A_j(V^k)\frac{\partial U^{k+1}}{\partial x_j}=F(V^k,I^k)+H^k,\\
U^{k+1}(x,0)=V_0^{k+1}(x)-V_0^0(x),
\end{cases} \qquad (2.19)
$$

where

$$
H^k=-\sum_{j=1}^N A_j(V^k)\frac{\partial U_0^0}{\partial x_j}.
$$

Then, with the aid of Steps I and II, we can follow the same procedure as in [3] to obtain the estimates (2.8). Thus, the proof of Lemma 2.2 is complete. □

The following lemma implies that the operator associated with (V^k, I^k) is contracted.

Lemma 2.5 *There exist $T_* \in (0,T]$, $\alpha < 1$, $\{\beta_j\}_{j=1}^\infty$ and $\{\gamma_j\}_{j=1}^\infty$ with $\sum_j |\beta_j| < \infty$, and $\sum_j |\gamma_j| < \infty$, such that for $k = 1, 2, \cdots$,*

$$\begin{aligned} & |||V^{k+1} - V^k|||_{0,T_*} + \max_{[0,T_*]} \Big[\int_0^\infty \int_{S^{N-1}} \|I^{k+1} - I^k\|^2 d\Omega d\nu \Big]^{1/2} \\ \le \;& \alpha \Big\{ |||V^k - V^{k-1}|||_{0,T_*} + \max_{[0,T_*]} \Big[\int_0^\infty d\nu \int_{S^{N-1}} \|I^k - I^{k-1}\|^2 d\Omega \Big]^{1/2} \Big\} + \beta_k + \gamma_k. \end{aligned} \tag{2.20}$$

PROOF. By (2.6) we obtain

$$A_0(V^k) \frac{\partial (V^{k+1} - V^k)}{\partial t} + \sum_{j=1}^N A_j(V^k) \frac{\partial (V^{k+1} - V^k)}{\partial x_j} = F(V^k, I^k) - F(V^{k-1}, I^{k-1}) + G,$$

where

$$G = -(A_0(V^k) - A_0(V^{k-1})) \frac{\partial V^k}{\partial t} - \sum_{j=1}^N (A_j(V^k) - A_j(V^{k-1})) \frac{\partial V^k}{\partial x_j}.$$

From Lemma 2.2, Proposition 2.3 and Taylor's expansion theorem, we deduce that

$$|||G|||_{0,T} \le C |||V^k - V^{k-1}|||_{0,T}. \tag{2.21}$$

We find from the mean value theorem and Hölder's inequality that for $j = 1, 2, \cdots, N+1$,

$$\|f_j(V^k, I^k) - f_j(V^{k-1}, I^{k-1})\| \le C \Big(\|V^k - V^{k-1}\| + (\int_0^\infty d\nu \int_{S^{N-1}} \|I^k - I^{k-1}\|^2 d\Omega)^{1/2} \Big).$$

Thus, one concludes

$$\|F(V^k, I^k) - F(V^{k-1}, I^{k-1})\| \le C \Big\{ \|V^k - V^{k-1}\| + (\int_0^\infty d\nu \int_{S^{N-1}} \|I^k - I^{k-1}\|^2 d\Omega)^{1/2} \Big\},$$

from which, we get (see [3])

$$\begin{aligned} & |||V^{k+1} - V^k|||_{0,T_1} \\ & \le \alpha_1 \Big(|||V^k - V^{k-1}|||_{0,T_1} + \max_{[0,T_1]} (\int_0^\infty d\nu \int_{S^{N-1}} \|I^k - I^{k-1}\|^2 d\Omega)^{1/2} \Big) + \beta_k, \end{aligned} \tag{2.22}$$

where $\alpha_1 < 1/2$, $\sum |\beta_k| < \infty$.

To bound $I^{k+1} - I^k$, we use the first equation of (2.6) to see that

$$\begin{aligned} & \frac{1}{c} \frac{\partial (I^{k+1} - I^k)}{\partial t} + \Omega \cdot \nabla (I^{k+1} - I^k) + \sigma^k(\nu)(I^{k+1} - I^k) = -(\sigma^k(\nu) - \sigma^{k-1}(\nu)) I^k \\ & \quad + \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu}{\nu'} \sigma_s^k(\nu' \to \nu, \Omega' \cdot \Omega)(I^k(\nu', \Omega') - I^{k-1}(\nu', \Omega')) d\Omega' \\ & + \int_0^\infty d\nu' \int_{S^{N-1}} \frac{\nu}{\nu'} (\sigma_s^k(\nu' \to \nu, \Omega' \cdot \Omega) - \sigma_s^{k-1}(\nu' \to \nu, \Omega' \cdot \Omega)) I^{k-1}(\nu', \Omega')) d\Omega'. \end{aligned}$$

Hence, we deduce from Hölder's inequality that

$$
\begin{aligned}
& (\int_0^\infty d\nu \int_{S^{N-1}} \|I^{k+1} - I^k\|^2 d\Omega)^{1/2} \\
\leq\ & (\int_0^\infty d\nu \int_{S^{N-1}} \|I_0^{k+1} - I_0^k\|^2 d\Omega)^{1/2} + CT\Big\{|||\varrho^k - \varrho^{k-1}|||_{0,T} + |||\theta^k - \theta^{k-1}|||_{0,T} \\
& + \max_{[0,T]}(\int_0^\infty d\nu \int_{S^{N-1}} \|I^k - I^{k-1}\|^2 d\Omega)^{1/2}\Big\} \\
\leq\ & \alpha_2\Big\{|||V^k - V^{k-1}|||_{0,T_2} + \max_{[0,T_2]}(\int_0^\infty d\nu \int_{S^{N-1}} \|I^k - I^{k-1}\|^2 d\Omega)^{1/2}\Big\} + \gamma_k,
\end{aligned}
\tag{2.23}
$$

where $\alpha_2 < 1/2$ and $\sum_k |\gamma_k| < \infty$. Finally, taking $T_* = \min\{T_1, T_2\}$, we obtain Lemma 2.5 by using (2.22) and (2.23). This completes the proof. □

We continue to prove Theorem 2.1. Lemmas 2.2, 2.5 and Sobolev's imbedding theorem guarantee the strong convergence of (V^k, I^k) in $C([0,T], H^{s'}(\mathbb{R}^N)) \times C([0,T], L^2((0,\infty) \times S^{N-1}, H^{s'}(\mathbb{R}^N)))$ for any s' with $1 + N/2 < s' < s$. So, the existence of a solution in the function class given in Theorem 2.1 has been shown. Finally, the uniqueness of the solutions is obtained by an application of the standard energy method (see [7]). □

3 Formation of singularities

For the sake of simplicity, in this section we only consider the three-dimensional case $N = 3$. By adapting and modifying the arguments in [8] for the Euler equations, we shall prove in this section that, in general, a smooth solution to (2.1)-(2.3) will break down in finite time when the initial data are large enough. For this purpose, we consider (2.1)-(2.3) with $\sigma_s = 0$ and $S(\nu)$ and $\sigma_a(\nu)$ are replaced by

$$
\sigma_a'(\nu)\bar{B}(\nu)\Big(1 + \frac{c^2 I(\nu,\Omega)}{2h\nu^3}\Big) \quad \text{and} \quad \sigma_a'(\nu)\Big(1 + \frac{c^2 \bar{B}(\nu)}{2h\nu^3}\Big), \quad \text{respectively,} \tag{3.1}
$$

where $\bar{B}$ is a function of ν only, the absorption coefficent $\sigma_a'(\nu) = \sigma_a'(x,t,\nu,\Omega,\varrho,\theta)$. (3.1) is refereed to as resulting from the so-called "reduced processes" describing the manifestation in the equation of transfer of the quantum statistics obeyed by photons (see, e.g., [5] and Remark 2.2).

With (3.1), we can write (2.1)-(2.3) as

$$
\frac{1}{c}\frac{\partial I}{\partial t} + \Omega \cdot \nabla I = \sigma_a'(\nu)(\bar{B}(\nu) - I), \tag{3.2}
$$

$$
A_0(V)\frac{\partial V}{\partial t} + \sum_{j=1}^{3} A_j(V)\frac{\partial V}{\partial x_j} = F(V, I), \tag{3.3}
$$

together with initial data

$$
I(x,0,\nu,\Omega) = I^0(x,\nu,\Omega), \quad V(x,0) = V^0(x) = (\varrho^0(x), u^0(x), \theta^0(x)), \tag{3.4}
$$

where

$$
A_0(V) = \begin{pmatrix} R\varrho^{-1}\theta^2 & 0 & 0 \\ 0 & \varrho\theta I_3 & 0 \\ 0 & 0 & c_\nu\varrho \end{pmatrix}
$$

and the symmetric matrix $A_j(V)=A_0(V)B_j(V)$, $B_j(V)=(b_{mn})_{5\times5}$ with $b_{ii}=u_j$, $b_{1(j+1)}=\varrho$, $b_{(j+1)1}=R\theta/\varrho$, $b_{(j+1)5}=R$, $b_{5(j+1)}=R\theta/c_\nu$ for $j=1,2,3$, and the rest elements are equal to 0, $F(V,I)=A_0(V)G(V,I)=(f_0,f_1,\cdots,f_{N+1})^t$ with $f_0=0$,

$$
\begin{aligned}
f_j &= -\frac{\theta}{c}\int_0^\infty d\nu\int_{S^2}\Omega_j\sigma_a'(\nu)(\bar{B}-I)d\Omega, \quad j=1,2,3,\\
f_4 &= \frac{1}{c}\int_0^\infty d\nu\int_{S^2}u\cdot\Omega\sigma_a'(\nu)(\bar{B}-I)d\Omega-\int_0^\infty d\nu\int_{S^2}\sigma_a'(\nu)(\bar{B}-I)d\Omega.
\end{aligned}
$$

In what follows, let (V,I) denote a smooth solution of (3.2)-(3.4) guaranteed by Theorem 2.1 on $[0,T]$ for some $T>0$. In the sequel, we shall prove that (V,I) will breaks down at some $t_0\geq T$ provided that the initial data are sufficiently large. For this purpose, we assume throughout this section that

$$
\begin{aligned}
&I^0(x,\nu,\Omega)=\bar{B}(\nu), \quad V^0(x)=\bar{V}\equiv(\bar{\varrho},0,\bar{\theta}) \quad \text{for all } |x|\geq R_0,\\
&\sigma_a'(\nu)>0, \ \ \varrho^0(x)>0, \ \ \theta^0(x)\geq 0,
\end{aligned}
\tag{3.5}
$$

where R_0, $\bar{\varrho}$ and $\bar{\theta}$ are positive constants. Then, one has

Lemma 3.1 *Assume $I^0(x,\nu,\Omega)\geq\bar{B}(\nu)$, then $I(x,t,\nu,\Omega)\geq\bar{B}(\nu)$.*

PROOF. Since $\bar{B}$ is independent of x and t, the equation (3.2) can be rewritten as

$$
\frac{1}{c}\frac{\partial(I-\bar{B})}{\partial t}+\Omega\cdot\nabla(I-\bar{B})+\sigma_a'(\nu)(I-\bar{B})=0.
$$

Thus, the lower bound $I(x,t,\nu,\Omega)\geq\bar{B}$ follows from an application of the method of characteristics to the above linear equation. □

Next, we prove the property of finite propagation speed for the system (3.2)-(3.4) which is needed in the proof of the blow-up result later. Denote

$$
\begin{aligned}
&D(t):=\{x\in\mathbb{R}^3:|x|\geq R_0+\beta t\}, \ \ \text{where } \beta=(R^2\bar{\theta}c_\nu^{-1}+R\bar{\theta})^{1/2};\\
&E:=\{(x,t):x\in D(t),0\leq t\leq T\}.
\end{aligned}
$$

Thus, we have the following theorem of finite propagation speed.

Proposition 3.2 (Finite propagation speed) *Let $\beta\geq1$ and (3.5) be satisfied. If (V,I) is a C^1-solution of (3.2)-(3.4), then $(V,I)\equiv(\bar{V},\bar{B})$ in E.*

PROOF. First, as a consequence of the proposition in [8], we easily find that $I\equiv\bar{B}$ in $\tilde{D}(t)=\{x\in\mathbb{R}^3:|x|\geq R_0+\beta t\}$ for $0\leq t\leq T$.

Next, we use and modify the local energy estimate arguments used in [8] for the Euler equations to prove the finite propagation speed. To this end, define $Q(\lambda,\xi)=\lambda I_5-\sum_{i=1}^3\xi_iB_i(\bar{V})$ for $(\lambda,\xi)\in\mathbb{R}\times S^2$. Then, the characteristic equation $\det Q(\lambda,\xi)=0$ has 5 real roots, where $\lambda(\xi)=|\xi|\beta$ and $\mu(\xi)=-|\xi|\beta$ denote the largest and smallest root, respectively. Set $\bar{A}_j=A_j(\bar{V}), j=0,\cdots,3$, and

$$
P=\bar{A}_0\frac{\partial}{\partial t}+\sum_{j=1}^N\bar{A}_j\frac{\partial}{\partial x}.
$$

Because of $D(t)\subset\tilde{D}(t)$ for $0\leq t\leq T$, $I\equiv\bar{B}$ in E. Hence,

$$
A_0(V)\frac{\partial(V-\bar{V})}{\partial t}+\sum_{j=1}^3A_j(V)\frac{\partial(V-\bar{V})}{\partial x_j}=0, \quad x\in D(t), \quad 0\leq t\leq T. \tag{3.6}
$$

With the help of (3.6), we can complete the rest of the proof in the same manner as that in [8] for the Euler equations, and therefore, we omit it here. $\square$

Now, define

$$
\begin{aligned}
m(t) &:= \int_{\mathbb{R}^3} (\varrho - \bar{\varrho}) dx, \\
e(t) &:= \int_{\mathbb{R}^3} \Big(\frac{1}{2} \varrho |u|^2 + E_m - \bar{E}_m + E_r - \int_0^\infty d\nu \int_{S^2} \bar{B} d\Omega \Big) dx, \\
F(t) &:= \int_{\mathbb{R}^3} x \cdot \varrho u dx + \frac{1}{c^2} \int_{\mathbb{R}^3} dx \int_0^\infty d\nu \int_{S^2} x \cdot \Omega (I - \bar{B}) d\Omega.
\end{aligned}
$$

Thus, by (3.2) and (3.3), we obtain by straightforward calculations that

$$
m(t) = m(0), \qquad e(t) = e(0).
$$

Then, the main theorem of this section reads:

Theorem 3.3 (Blow-up) *Let (3.5) hold and (ϱ, u, θ, I) be a C^1-solution of (3.2)-(3.4) for $0 \le t < T$ with T being the maximal existence interval. If*

$$
1 < \gamma \le \frac{5}{3}, \;\; I^0(x, \nu, \Omega) \ge \bar{B}, \;\; m(0) \ge 0, \;\; \int_{\mathbb{R}^3} (E^0_m - \bar{E}_m) dx \ge 0, \tag{3.7}
$$

$$
F(0) \ge \frac{16\pi}{3} \beta R_0^4 \Big\{ \frac{2}{5 - 3\gamma} \max \varrho^0(x) + \frac{1}{c^3} \max \int_0^\infty d\nu \int_{S^2} (I^0 - \bar{B}) d\Omega \Big\}, \tag{3.8}
$$

then T is finite.

To show one way in which (3.7-3.8) can be satisfied, we take the initial conditions as

$$
\varrho^0 \equiv \bar{\varrho}, \qquad \theta^0 \equiv \bar{\theta} \quad \text{and} \quad I^0 \equiv \bar{B}.
$$

Then $m(0) = \int_{\mathbb{R}^3} (E^0_m - \bar{E}_m) dx = 0$ and (3.8) holds if

$$
\int_{\mathbb{R}^3} x \cdot u^0(x) dx \ge \frac{32\pi}{3(5 - 3\gamma)} \beta R_0^4.
$$

Comparing both sides, we find that the inial flow velocity must be supersonic in some region (also cf. [9]).

PROOF. From (1.2) and (1.4), we have

$$
\frac{\partial}{\partial t} (\varrho u) + \nabla \mathbf{P}_m + \nabla \cdot (\varrho u \otimes u) = -\frac{1}{c} \int_0^\infty d\nu \int_{S^2} \Omega (\frac{1}{c} \frac{\partial I}{\partial t} + \Omega \cdot \nabla I),
$$

therefore, by direct computation and integration by parts, and using Lemma 3.1 and recalling the relation $R = c_\nu (\gamma - 1)$, we see that

$$
\begin{aligned}
& F'(t) = \int_{\mathbb{R}^3} x \cdot [-\nabla (P_m - \bar{P}_m) - \nabla \cdot (\varrho u \otimes u)] dx \\
& -\frac{1}{c} \int_{\mathbb{R}^3} dx \int_0^\infty d\nu \int_{S^2} x \cdot \Omega \Omega \cdot \nabla (I - \bar{B}) d\Omega \\
= \;\; & \int_{\mathbb{R}^3} \varrho |u|^2 dx + 3 \int_{\mathbb{R}^3} (P_m - \bar{P}_m) dx + \frac{1}{c} \int_{\mathbb{R}^3} dx \int_0^\infty d\nu \int_{S^2} \Omega \cdot \Omega (I - \bar{B}) d\Omega
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \varrho|u|^2 dx + \frac{3R}{c_\nu}\int_{\mathbb{R}^3}(E_m - \bar{E}_m)dx + \frac{1}{c}\int_{\mathbb{R}^3} dx \int_0^\infty d\nu \int_{S^2} \Omega\cdot\Omega(I-\bar{B})d\Omega \\
&= (1-\frac{3R}{2c_\nu})\int_{\mathbb{R}^3} \varrho|u|^2 dx + \frac{3R}{c_\nu}\Big\{e(0) - \int_{\mathbb{R}^3} dx(E_r - \int_0^\infty d\nu\int_{S^2}\bar{B}d\Omega)\Big\} \\
&\quad +\frac{1}{c}\int_{\mathbb{R}^3} dx \int_0^\infty d\nu \int_{S^2} \Omega\cdot\Omega(I-\bar{B})d\Omega \\
&= \frac{5-3\gamma}{2}\int_{\mathbb{R}^3} \varrho|u|^2 dx + \frac{3R}{2c_\nu}\int_{\mathbb{R}^3}\varrho^0(x)|u^0(x)|^2dx + \frac{3R}{c_\nu}\int_{\mathbb{R}^3}(E_m^0 - \bar{E}_m)dx \\
&\quad +\frac{3R}{c_\nu}\int_{\mathbb{R}^3}(E_r^0 - E_r)dx + \frac{1}{c}\int_{\mathbb{R}^3} dx \int_0^\infty d\nu \int_{S^2} \Omega\cdot\Omega(I-\bar{B})d\Omega \\
&\geq \frac{5-3\gamma}{2}\int_{\mathbb{R}^3} \varrho|u|^2 dx + \frac{1}{c}\int_{\mathbb{R}^3} dx \int_0^\infty d\nu \int_{S^2} \Omega\cdot\Omega(I-\bar{B})d\Omega. \qquad (3.9)
\end{aligned}
$$

Set $B(t) = D(t)^c = \{x \in R^3 : |x| \geq R_0 + \beta t\}$ for $0 \leq t < T$. From (3.9) and Hölder's inequality we get

$$
\begin{aligned}
&F^2(t) \leq 2(\int_{B(t)} x\cdot\varrho u dx)^2 + \frac{2}{c^4}(\int_{B(t)} dx \int_0^\infty d\nu \int_{S^2} x\cdot\Omega(I-\bar{B})d\Omega)^2 \\
&\leq 2(\int_{B(t)}|x|^2\varrho dx)(\int_{B(t)}\varrho|u|^2dx) \\
&\quad +\frac{2}{c^4}(\int_{B(t)} dx \int_0^\infty d\nu \int_{S^2} |x|^2(I-\bar{B})d\Omega)(\int_{B(t)} dx \int_0^\infty d\nu \int_{S^2} \Omega\cdot\Omega(I-\bar{B})d\Omega) \\
&\leq \Big\{\frac{4}{5-3\gamma}\int_{B(t)}|x|^2\varrho dx + \frac{2}{c^3}\int_{B(t)} dx \int_0^\infty d\nu \int_{S^2} |x|^2(I-\bar{B})d\Omega\Big\}F'(t). \qquad (3.10)
\end{aligned}
$$

On the one hand,

$$
\begin{aligned}
\int_{B(t)}|x|^2\varrho dx &\leq (R_0+\beta t)^2\int_{B(t)}\varrho dx \\
&= (R_0+\beta t)^2\int_{B(t)}\varrho^0(x)dx \\
&\leq \omega^{N-1}(R_0+\beta t)^{N+2}\max\varrho^0(x). \qquad (3.11)
\end{aligned}
$$

Integrating (3.2) and using Lemma 3.1, we infer that

$$
\int_{B(t)} dx \int_0^\infty d\nu \int_{S^2}(I-\bar{B})d\Omega \leq \int_{B(t)} dx \int_0^\infty d\nu \int_{S^2}(I^0-\bar{B})d\Omega,
$$

whence,

$$
\begin{aligned}
\int_{B(t)} dx \int_0^\infty d\nu \int_{S^2}|x|^2(I-\bar{B})d\Omega &\leq (R_0+\beta t)^2\int_{B(t)} dx \int_0^\infty d\nu \int_{S^2}(I-\bar{B})d\Omega \\
&\leq (R_0+\beta t)^2\int_{B(t)} dx \int_0^\infty d\nu \int_{S^2}(I^0-\bar{B})d\Omega \\
&\leq \omega^{N-1}(R_0+\beta t)^{N+2}\max\int_0^\infty d\nu \int_{S^2}(I^0-\bar{B})d\Omega. \qquad (3.12)
\end{aligned}
$$

Combining (3.10), (3.11) with (3.12) we obtain

$$
F^2(t) \quad \leq \quad 2\omega^{N-1}(R_0+\beta t)^{N+2} \qquad (3.13)
$$

$$\cdot\Big(\frac{2}{5-3\gamma}\max_x \varrho^0(x) + \frac{1}{c^3}\max_x \int_0^\infty d\nu \int_{S^2}(I^0-\bar{B})d\Omega\Big)F'(t).$$

From (3.13) and the conditions of the theorem, it follows that $\lim_{t\to t_0} F(t) \to \infty$ where $t_0 > 0$ is a constant and can be made smaller than T by choosing $F(0)$ sufficiently large. This completes the proof. □

Remark 3.1 *From the proof it is easy to see that Proposition 3.2 and Theorem 3.3 still hold when the function $\bar{B}$, in addition, depends on Ω. But, due to the technical reasons, the dependence of $\bar{B}$ on t and x is not allowed, this unfortunately excludes the physically interesting case $\bar{B} = 2h\nu^3c^{-2}(e^{h\nu/(k\theta)}-1)^{-1}$ (the Plank function), and the further study is needed for this case.*

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