SOLUTIONS FOR A NONLOCAL CONSERVATION LAW WITH FADING MEMORY

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Abstract. Global entropy solutions in BV for a scalar nonlocal conservation law with fading memory are constructed as limits of vanishing viscosity approximate solutions. The uniqueness and stability of entropy solutions in BV are established, which also yield the existence of entropy solutions in L^∞ while the initial data is only in L^∞. Moreover, if the memory kernel depends on a relaxation parameter $\varepsilon > 0$ and tends to a delta measure weakly as measures when $\varepsilon \to 0+$, then the global entropy solution sequence in BV converges to an admissible solution in BV for the corresponding local conservation law.

1. Introduction and Main Theorems

We study global entropy solutions to a scalar nonlocal conservation law with fading memory:

$$u_t + f(u)_x + \int_0^t k(t-\tau) f(u(\tau))_x \, d\tau = 0, \qquad x \in \mathbb{R}, \tag{1.1}$$

and initial data

$$u(0,x) = u_0(x), \tag{1.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function and $u_0 \in BV(\mathbb{R})$. For simplicity, we sometimes use the notation $u(t) := u(t,x)$ to emphasize the state at $t > 0$ as in (1.1).

In one-dimensional viscoelasticity, hyperbolic conservation laws

$$U_t + F(U)_x = 0 \tag{1.3}$$

correspond to the constitutive relations of an elastic medium when the value of the flux function F at (t,x) is solely determined by the value of $U(t,x)$. However, this model (1.3) is inadequate when viscosity and relaxation phenomena are present. In that case, the flux function depends also on the past history of the material, i.e., on $U(\tau,x)$ for $\tau < t$. Under these circumstances, we say that the material has memory. An important class of media of this type are materials with fading memory, which correspond to the constitutive relations with flux functions of the form

$$F(U(t,x)) + \int_0^t k(t-\tau) G(U(\tau,x)) \, d\tau, \tag{1.4}$$

where F, G are smooth functions and k is a smooth kernel, integrable over $\mathbb{R}_+ := [0,\infty)$. When the kernel satisfies appropriate conditions motivated by physical

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considerations, the influence of the memory term in (1.4) reflects a damping effect. Consequently, global smooth solutions exist for given small initial data (cf. Renardy-Hrusa-Nohel [15]), in contrast to the situation with elastic media that classical solutions in general break down in finite time even when the initial data is small. However, when the initial data is large, the destabilizing action of nonlinearity of the flux function f prevails over the damping, and solutions break down in a finite time; see Dafermos [3] and Malek-Madani–Nohel [10].

In this paper we first construct global entropy solutions in BV to the nonlocal conservation law (1.1) with fading memory via the following vanishing viscosity approximation:

$$u^\nu_t + f(u^\nu)_x + \int_0^t k(t-\tau) f(u^\nu(\tau))_x \, d\tau = \nu \,(u^\nu + \int_0^t k(t-\tau) u^\nu(\tau)\, d\tau)_{xx}. \tag{1.5}$$

For a scalar local conservation law modeling elastic materials, such a result was first established in [8, 13, 17].

The main motivation for the vanishing viscosity approximation (1.5) is that the conservation law (1.1) can be viewed as a linear Volterra equation, which was first observed by MacCamy [11] and later employed in Dafermos [4] and Nohel–Rogers–Tzavaras [12] (also see [1]). In this way, it is easy to extract the damping character of the memory term. Let $r(t)$ be the resolvent kernel associated with k:

$$r + k * r = -k. \tag{1.6}$$

Then we can write (1.1) as

$$-f(u)_x = u_t + \int_0^t r(t-\tau)\, u'(\tau)\, d\tau.$$

Integrating by parts yields

$$u_t + f(u)_x + r(0)u = r(t)u_0 - \int_0^t r'(t-\tau) u(\tau)\, d\tau, \tag{1.7}$$

which is equivalent to (1.1). The vanishing viscosity approximation (1.5) is equivalent to the following artificial viscosity approximation to (1.7):

$$u^\nu_t + f(u^\nu)_x + r(0)u^\nu = r(t)u_0 - \int_0^t r'(t-\tau) u^\nu(\tau)\, d\tau + \nu\, u^\nu_{xx}. \tag{1.8}$$

The existence of a unique, regular local solution $u^\nu(t,x)$ of (1.8) when the initial data u_0 is smooth can be established through the standard Banach Fixed Point Theorem. The local solution may be extended to a global solution with the help of the apriori L^∞ estimate established in Section 2.1.

A function $u = u(t,x)$ is called an *entropy solution* to the Cauchy problem (1.1)–(1.2) if it satisfies that, for any test function $\varphi \in C^1_0(\mathbb{R}^2_+)$ with $\mathbb{R}^2_+ := \mathbb{R}_+ \times \mathbb{R}$, $\varphi \geq 0$,

$$\begin{aligned} \iint_{\mathbb{R}^2_+} \big(\eta(u)\varphi_t + q(u)\varphi_x + \eta'(u)(r(t)u_0 - r(0)u - \int_0^t r'(t-\tau)u(\tau)d\tau)\varphi\big)\, dt dx \\ + \int_{\mathbb{R}} \eta(u_0(x))\varphi(0,x) dx \geq 0, \end{aligned} \tag{1.9}$$

for any convex entropy $\eta(u)$, where $q(u)$ is the corresponding entropy flux satisfying $q'(u) = \eta'(u) f'(u)$.

Before we state the results, we introduce some notations. Let ϱ be the standard mollifier. We define the mollification of u_0 to be

$$u_0^\nu := (u_0\chi_\nu) * \varrho_\nu, \tag{1.10}$$

where, for each $\nu > 0$, $\varrho_\nu(x) := \frac{1}{\nu}\,\varrho(\frac{x}{\nu})$ and $\chi_\nu(x) := 1$ for $|x| \le 1/\nu$ and 0 otherwise. The main result is the following.

Theorem 1.1. *Consider the Cauchy problem* (1.5) *with Cauchy data:*

$$u^\nu(0,x) = u_0^\nu(x), \tag{1.11}$$

where the initial data u_0^ν is given by (1.10) *and $u_0 \in BV(\mathbb{R})$. Let the resolvent kernel r associated with k as defined in* (1.6) *be a nonnegative, non-increasing function in $L^1(\mathbb{R}_+)$. Then, for each $\nu > 0$, the Cauchy problem* (1.5) *and* (1.11) *has a unique solution u^ν defined globally with a uniform BV bound. Moreover, as $\nu \to 0$, u^ν converges in L^1_{loc} to an entropy solution $u \in BV$ to* (1.1)–(1.2)*, which satisfies*

$$\|u\|_{L^\infty(\mathbb{R}^2_+)} \le 2\|u_0\|_{L^\infty(\mathbb{R})}, \tag{1.12}$$

$$TV\{u(t)\} + \int_0^t r(t-\tau)TV\{u(\tau)\}\,d\tau \le L\,M(u_0), \tag{1.13}$$

$$\|u(t) - u(s)\|_{L^1(\mathbb{R})} \le CM(u_0)|t-s|, \tag{1.14}$$

where $L = 1 + \|r\|_{L^1(\mathbb{R}_+)}$, $M(u_0) = TV\{u_0\} + 2\|u_0\|_{L^\infty(\mathbb{R})}$, and $C > 0$ is independent of ν and $TV\{u_0\}$.

Furthermore, we have

Theorem 1.2. *Let the resolvent kernel $r(t)$ associated with k be a nonnegative and nonincreasing function in $L^1(\mathbb{R}_+)$. Let $u, v \in BV(\mathbb{R}^2_+)$ be entropy solutions to* (1.1) *with initial data $u_0, v_0 \in BV(\mathbb{R})$, respectively. Then*

$$\|u(t) - v(t)\|_{L^1(\mathbb{R})} + \int_0^t r(t-\tau)\|u(\tau) - v(\tau)\|_{L^1(\mathbb{R})}\,d\tau \le L\,\|u_0 - v_0\|_{L^1(\mathbb{R})}. \tag{1.15}$$

That is, any entropy solution in BV to (1.1)–(1.2) *is unique and stable in L^1. As a consequence, if u_0 is only in L^∞, not necessarily in $BV(\mathbb{R})$, there exists a global entropy solution $u \in L^\infty$ to* (1.1)–(1.2)*.*

Having established the above results, we then analyze the case when the kernel in the scalar equation (1.1) is a relaxation kernel k_ε that depends on a small parameter $\varepsilon > 0$ so that $k_\varepsilon(t) \rightharpoonup (\alpha - 1)\,\delta(t)$ weakly as measures when $\varepsilon \to 0+$, where $\delta(t)$ denotes the Dirac mass centered at the origin. That is,

$$u_t^\varepsilon + f(u^\varepsilon)_x + \int_0^t k_\varepsilon(t-\tau) f(u^\varepsilon(\tau))_x\,d\tau = 0, \tag{1.16}$$

$$u^\varepsilon(0,x) = u_0(x) \in BV(\mathbb{R}), \tag{1.17}$$

with $\sup_{\varepsilon>0}\|k_\varepsilon\|_{L^1(\mathbb{R}_+)} < \infty$.

We denote the entropy solution to the above problem by $u^\varepsilon(t,x)$. Let r_ε be the resolvent kernel associated with k_ε via (1.6). Hence, (1.16) reduces to

$$u_t^\varepsilon + f(u^\varepsilon)_x + r_\varepsilon(0)u^\varepsilon = r_\varepsilon(t)u_0 - \int_0^t r_\varepsilon'(t-\tau)u(\tau)\,d\tau \tag{1.18}$$

with $\sup_{\varepsilon>0}\|r_\varepsilon\|_{L^1(\mathbb{R}_+)} < \infty$.

By Theorems 1.1–1.2, we then conclude that the unique entropy solution sequence $u^\varepsilon \in BV$ to (1.16)–(1.17) is uniformly bounded in L^∞ and uniformly L^1–stable, independent of ε. Then the solution sequence $\{u^\varepsilon\}$ is a compact set in L^1_{loc} so that we can extract a subsequence $\{u^{\varepsilon_k}\}$ that converges in L^1_{loc} to an admissible weak solution of the local conservation law:

$$u_t + \alpha f(u)_x = 0, \tag{1.19}$$

with Cauchy data $u_0 \in BV$.

Theorem 1.3. *Consider the Cauchy problem* (1.16)–(1.17) *with* $u_0 \in BV(\mathbb{R})$. *Let the resolvent kernel* r_ε *associated with* k_ε *as defined in* (1.6) *be a nonnegative, nonincreasing function with uniform* L^1*-norm independent of* ε. *Then the entropy solutions* u^ε *to* (1.16)–(1.17) *are uniformly bounded in* L^∞ *and stable in* L^1*:*

$$\begin{aligned}
&\|u^\varepsilon(t)\|_{L^\infty(\mathbb{R}^2_+)} \le 2\|u_0\|_{L^\infty(\mathbb{R})},\\
&TV\{u^\varepsilon(t)\} \le LM(u_0),\\
&\|u^\varepsilon(t) - u^\varepsilon(s)\|_{L^1(\mathbb{R})} \le CM(u_0)|t-s|,\\
&\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^1(\mathbb{R})} \le L\|u_0 - v_0\|_{L^1(\mathbb{R})},
\end{aligned}$$

where $L := 1 + \sup_{\varepsilon>0}\|r_\varepsilon\|_{L^1} < \infty$, $C > 0$ *is independent of* ε *and* $TV\{u_0\}$, *and* $v^\varepsilon(t,x)$ *is the entropy solution to* (1.16)–(1.17) *with initial data* $v_0 \in BV$. *Furthermore, if* $k_\varepsilon(t) \rightharpoonup (\alpha-1)\,\delta(t)$ *weakly as measures when* $\varepsilon \to 0+$, *then* u^ε *converges in* L^1_{loc} *to an admissible weak solution* u *of the Cauchy problem* (1.19) *and* (1.2) *with initial data* $u_0 \in BV$.

In Theorems 1.1–1.3, the assumptions on $r_\varepsilon(t)$, or $r(t)$, can easily be converted to the assumptions on $k_\varepsilon(t)$, or $k_\varepsilon(t)$, because of their symmetry between the kernel and the resolvent through (1.6). For example, such kernels $k_\varepsilon(t)$ especially include the following Set of Kernels (i)–(vi):

(i) $k'_\varepsilon(t) \ge 0$ and $\|k_\varepsilon\|_{L^1(\mathbb{R}_+)} \le K$ for some constant K independent of $\varepsilon > 0$;

(ii) $det(1+\hat{k}_\varepsilon(z)) \ne 0$ for any z with $Re(z) \ge 0$, and $\hat{k}_\varepsilon(t)(1+\hat{k}_\varepsilon(t)) \le 0$ for the Laplace transform $\hat{k}_\varepsilon$ of k_ε;

(iii) $sup_{\omega\in\mathbb{R}}|(1+\hat{k}_\varepsilon(i\omega))^{-1}| \le q$ for some constant q independent of ε;

(vi) There exist positive numbers $T \sim \varepsilon$ and $\tau \sim \varepsilon$ such that

$$\int_{|s|\ge T} |k_\varepsilon(t)| \le \frac{1}{12q}, \qquad \sup_{0<s<\eta}\int_{\mathbb{R}} |k_\varepsilon(t) - k_\varepsilon(t-s)|\,dt \le \frac{1}{4}.$$

The prototype is

$$k_\varepsilon(t) = -\frac{1-\alpha}{\varepsilon}exp(-\frac{t}{\varepsilon}), \qquad 0<\alpha<1, \tag{1.20}$$

for which the corresponding family of resolvent kernels is

$$r_\varepsilon(t) = \frac{1-\alpha}{\varepsilon}exp(-\frac{\alpha t}{\varepsilon}). \tag{1.21}$$

In Section 2, we develop techniques for the nonlocal case, motivated by Vol'pert-Kruzkov's techniques [17, 8] and the L^∞-estimate techniques for the local case, to establish uniform L^∞ and BV estimates of the vanishing viscosity approximate solutions, independent of ν, by using the damping nature of the memory term. As corollaries of these estimates, we establish the convergence of the vanishing viscosity approximate solutions to obtain the existence of entropy solutions in BV.

In Section 3, we show that the entropy solution in BV is unique and stable in L^1 with respect to the initial perturbation. In Section 4, we prove Theorem 1.3 and discuss the hypotheses of the theorems. Finally we give an example and show the relation of the fading memory limit with the zero relaxation limit as first considered systematically in Chen-Levermore-Liu [2]; also see [9, 16, 18] for the model.

2. Proof of Theorem 1.1

In this section, we prove the uniform L^∞ and BV estimates, as well as the dependence on time, which are used not only for the global existence of the vanishing viscosity approximate solutions, but also for their compactness. We also establish the existence and regularity of entropy solutions.

2.1. L^∞ **Estimate.** We first obtain a uniform L^∞ estimate. First note that, by employing the resolvent kernel r, equation (1.5) can be written in form (1.8), i.e., we study the Cauchy problem

$$u^\nu_t + f(u^\nu)_x + r(0)u^\nu = r(t)u^\nu_0 - \int_0^t r'(t-\tau)u^\nu(\tau)\,d\tau + \nu\, u^\nu_{xx}, \tag{2.1}$$

$$u(0,x) = u^\nu_0(x), \qquad x \in \mathbb{R}, \tag{2.2}$$

where $u^\nu_0(x)$ is defined in (1.10). By rescaling the coordinates, $(t,x) \to (s,y) = (r_0 t, r_0 x)$, we rewrite (1.8) as

$$\bar{u}^\nu_s + f(\bar{u}^\nu)_y + \bar{u}^\nu = \frac{1}{r_0} r(\frac{s}{r_0})u^\nu_0 - \frac{1}{r_0^2}\int_0^s r'(\frac{s-\tau}{r_0})\bar{u}^\nu(\tau)\,d\tau + \nu\, r_0\, \bar{u}^\nu_{yy}, \tag{2.3}$$

where $r_0 = r(0) > 0$ and $\bar{u}^\nu(s,y) := u^\nu(\frac{s}{r_0}, \frac{y}{r_0})$. For any even integer p, multiplying (2.3) by $p\,|\bar{u}^\nu|^{p-1}$ and integrating over $[0,S]\times\mathbb{R}$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\bar{u}^\nu(S)|^p\,dy + p\int_0^S\int_{-\infty}^{\infty} |\bar{u}^\nu(s,y)|^p\,dyds \\
&\quad\le \int_{-\infty}^{\infty} |\bar{u}^\nu_0(y)|^p\,dy + p\int_0^S\int_{-\infty}^{\infty} \frac{1}{r_0} r(\frac{s}{r_0})|\bar{u}^\nu_0(y)||\bar{u}^\nu(s,y)|^{p-1}\,dyds \\
&\qquad - p\int_0^S\int_{-\infty}^{\infty} |\bar{u}^\nu(s,y)|^{p-1}\int_0^s \frac{1}{r_0^2} r'(\frac{s-\tau}{r_0})|\bar{u}^\nu(\tau,y)|\,d\tau dyds.
\end{aligned} \tag{2.4}$$

By employing the standard inequality $ab \le \frac{C_0}{\varepsilon_0}\frac{a^p}{p} + \varepsilon_0\frac{b^q}{q}$ for $\varepsilon_0 = \frac{p}{4(p-1)}$, the second term on the right-hand side of (2.4) is estimated as

$$\begin{aligned}
&p\int_0^S\int_{-\infty}^{\infty} \frac{1}{r_0} r(\frac{s}{r_0})\,|\bar{u}^\nu_0(y)|\,|\bar{u}^\nu(s,y)|^{p-1}\,dyds \\
&\quad\le \int_0^S\int_{-\infty}^{\infty} \frac{C_0}{\varepsilon_0}\left(\frac{1}{r_0} r(\frac{s}{r_0})\right)^p |\bar{u}^\nu_0(y)|^p\,dyds + \frac{1}{4}p\int_0^S\int_{-\infty}^{\infty} |\bar{u}^\nu(s,y)|^p\,dy\,ds,
\end{aligned} \tag{2.5}$$

so that the last term in (2.5) is dominated by the damping in (2.4). Similarly, we treat the last term in (2.4) to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} |\bar{u}^\nu(S,y)|^p\,dy \le \int_{-\infty}^{\infty} |\bar{u}^\nu_0(y)|^p\,dy + \frac{C_0}{\varepsilon_0}\int_0^S \left(\frac{1}{r_0} r(\frac{s}{r_0})\right)^p ds \int_{-\infty}^{\infty} |\bar{u}^\nu_0(y)|^p\,dy \\
+ \beta(S)\int_0^S\int_{-\infty}^{\infty} |\bar{u}^\nu(\tau,y)|^p\,dy\,d\tau,
\end{aligned}$$

where $\alpha(s) := \left(\int_0^s \left|\frac{1}{r_0^2} r'(\frac{s-\tau}{r_0})\right|^q d\tau\right)^{p/q}$ and $\beta(S) := \frac{C_0}{\varepsilon_0}\int_0^S \alpha(s)\,ds$. By Gronwall's inequality, we have

$$\int_0^S \int_{-\infty}^{\infty} |\bar{u}^\nu(\tau)|^p\,dy\,d\tau \le e^{\int_0^S \beta(w)\,dw} \int_0^S W(s)\,ds \tag{2.6}$$

for $W(s) := \left(1 + \frac{C_0}{\varepsilon_0}\int_0^s \left(\frac{1}{r_0} r(\frac{\tilde{s}}{r_0})\right)^p d\tilde{s}\right) \int |\bar{u}_0^\nu(y)|^p\,dy$. Let

$$K_S = \sup\left\{ \left(\frac{1}{r_0^2} r'(\frac{\tilde{s}}{r_0})\right)^{(q-1)\frac{p}{q}} : p \ge p_0,\ \tilde{s} \in [0,S]\right\}, \quad q = \frac{p}{p-1}. \tag{2.7}$$

Then, for all $s \in [0,S]$, we have

$$\alpha(s) \le K_S \left(-\frac{1}{r_0^2}\int_0^s r'(\frac{s-\tau}{r_0})\,d\tau\right)^{p/q} \le K_S \tag{2.8}$$

since r is nonincreasing. Hence, $\lim_{p\to\infty} \frac{1}{p}\int_0^S \beta(w)\,dw = 0$. Also,

$$\left(\int_0^S W(s)\,ds\right)^{\frac{1}{p}} \le S^{\frac{1}{p}} \left(1 + \left(\frac{C_0}{\varepsilon_0}\right)^{\frac{1}{p}} \left(\int_0^S \left(\frac{1}{r_0} r(\frac{\tilde{s}}{r_0})\right)^p d\tilde{s}\right)^{\frac{1}{p}}\right) \left(\int |\bar{u}_0^\nu|^p\,dy\right)^{\frac{1}{p}}.$$

Thus, if we raise (2.6) to $1/p$ and take the limit as $p \to \infty$, we conclude

$$\|\bar{u}^\nu\|_{L^\infty([0,S]\times\mathbb{R})} \le \|(1 + \frac{1}{r_0} r(\frac{s}{r_0}))|\bar{u}_0(y)|\|_{L^\infty([0,S]\times\mathbb{R})}. \tag{2.9}$$

Since $r(t)$ is nonincreasing, then $r(t)/r(0) \le 1$ for all $t \ge 0$ and hence

$$\|\bar{u}^\nu\|_{L^\infty(\mathbb{R}^2_+)} \le 2\|u_0\|_{L^\infty(\mathbb{R})}. \tag{2.10}$$

By rescaling the coordinates backwards, we obtain the uniform L^∞–bound on u^ν to (1.8) independent of ν;

$$\|u^\nu\|_{L^\infty(\mathbb{R}^2_+)} \le 2\|u_0\|_{L^\infty(\mathbb{R})}. \tag{2.11}$$

2.2. BV–**Regularity and Estimates.** First note that $u_0^\nu \in C^\infty$ has the following bounds:

$$\|(u_0^\nu)_x\|_{L^1} \le M(u_0), \qquad \|(u_0^\nu)_{xx}\|_{L^1} \le \frac{C_0}{\nu} M(u_0), \tag{2.12}$$

for some $C_0 > 0$ independent of ν, where $M(u_0) := TV\{u_0\} + 2\|u_0\|_{L^\infty}$.

Set $v = u_x^\nu$ and $w = u_t^\nu$. Then the evolution equations of v and w are

$$v_t + (f'(u^\nu)v)_x + r(0)v = r(t)(u_0^\nu)_x - \int_0^t r'(t-\tau)v(\tau)\,d\tau + \nu v_{xx}, \tag{2.13}$$

$$v(0,x) = (u_0^\nu)_x, \tag{2.14}$$

and

$$w_t + (f'(u^\nu)w)_x + r(0)w = -\int_0^t r'(t-\tau)w(\tau)\,d\tau + \nu w_{xx}, \tag{2.15}$$

$$w(0,x) = u_t^\nu(0,x) = (u_0^\nu)_{xx} - f'(u_0^\nu)\,(u_0^\nu)_x. \tag{2.16}$$

Multiplying (2.13) by $sgn(v(t,x))$ and integrating with respect to x, we get

$$\frac{d}{dt}\|v(t)\|_{L^1} + r(0)\|v(t)\|_{L^1} \le r(t)M(u_0) - \int_0^t r'(t-\tau)\|v(\tau)\|_{L^1}\,d\tau$$

since $r(0) > 0$ and r is nonincreasing. Integrating over $t \in [0,T]$ yields

$$\begin{aligned}&\|v(T)\|_{L^1} + r(0)\int_0^T \|v(t)\|_{L^1}\,dt\\ &\le \big(1 + \int_0^T r(t)\,dt\big)M(u_0) - \int_0^T\int_0^t r'(t-\tau)\|v(\tau)\|_{L^1}\,d\tau\,dt. \end{aligned}\tag{2.17}$$

Changing the order of integration in the last term, we arrive at

$$\begin{aligned}&\|v(T)\|_{L^1} + r(0)\int_0^T \|v(t)\|_{L^1}\,dt\\ &\le \big(1 + \int_0^T r(t)\,dt\big)M(u_0) - \int_0^T (r(T-\tau) - r(0))\|v(\tau)\|_{L^1}\,d\tau.\end{aligned}$$

Thus, we have

$$\|v(T)\|_{L^1} + \int_0^T r(T-\tau)\|v(\tau)\|_{L^1}\,d\tau \le (1 + \int_0^T r(\tau)\,d\tau)M(u_0).$$

Because $r(\cdot)$ is bounded in $L^1(\mathbb{R}_+)$, we obtain the following uniform bound on the gradient $v = u_x$,

$$\|u^\nu_x(t)\|_{L^1} + \int_0^t r(t-\tau)\|u^\nu_x(\tau)\|_{L^1}\,d\tau \le L\,M(u_0),\tag{2.18}$$

where $L := 1 + \|r\|_{L^1(\mathbb{R}_+)}$. Similarly, we have

$$\|w(t)\|_{L^1} + \int_0^t r(t-\tau)\|w(\tau)\|_{L^1}\,d\tau \le \|w(0)\|_{L^1}.\tag{2.19}$$

Using (2.1) and the bounds in (2.12) for the initial data, we find from (2.16) that

$$\|w(0)\|_{L^1} \le \nu\|(u^\nu_0)_{xx}\|_{L^1} + \|f'(u^\nu_0)\|_{L^\infty}M(u_0) \le C\,M(u_0).$$

Hence, by (2.19), u^ν_t is uniformly bounded in L^1. Thus, for $0 < s < t$, we get the continuous dependence on time for the solutions to (2.1)–(2.2):

$$\|u^\nu(t) - u^\nu(s)\|_{L^1} \le \int_s^t \|w(\tau)\|\,d\tau \le C\,M(u_0)\,|t-s|.\tag{2.20}$$

2.3. **Existence of Entropy Solutions in** BV **to** (1.1)–(1.2)**.** Using (2.11), (2.18), and (2.20), Helly's Compactness Theorem yields that a convergent subsequence $\{u^{\nu_m}\}$ may be extracted with $\nu_m \downarrow 0$ as $m \to \infty$, whose limit is denoted by u, i.e.,

$$u^{\nu_m}(t) \longrightarrow u(t) \qquad \text{in } L^1_{loc} \quad \text{for all } t > 0.\tag{2.21}$$

The limit $u(t,\cdot)$ is a BV function satisfying (1.12)–(1.14) for all $t,\,s > 0$. By construction, it is easy to check that the limit function $u(t,x)$ is an entropy solution to (1.1)–(1.2).

3. Proof of Theorem 1.2

In this section, we prove the uniqueness of entropy solutions in BV as stated in Theorem 1.2. For any $u \in BV$, the whole space $\mathbb{R}^2_+$ can be decomposed into three parts (see [5, 6, 17]):

$$\mathbb{R}^2_+ = J(u) \cup \mathcal{C}(u) \cup \mathcal{I}(u),$$

where $J(u)$ is the set of points of approximate jump discontinuity, $C(u)$ the set of points of approximate continuity of u, and $I(u)$ is the set of irregular points of u whose one-dimensional Hausdorff measure is zero.

First, the entropy inequality (1.9) implies that, on a shock $x = x(t)$ in $J(u)$,

$$\sigma[\eta(u)] - [q(u)] \geq 0, \tag{3.1}$$

where $[\eta(u)] = \eta(u(t, x(t) + 0)) - \eta(u(t, x(t) - 0))$ and $\sigma = x'(t)$ is the shock speed.

Now assume that $u, v \in BV(\mathbb{R}^2_+)$ are the entropy solutions with initial data $u_0, v_0 \in BV(\mathbb{R})$, respectively. Then it can be easily checked that, on $J(u) \cup J(v)$,

$$\sigma[|u - v|] - [sign(u - v)(f(u) - f(v))] \leq 0.$$

In the continuous region $C(u) \cap C(v)$, since $r'(t) \leq 0$,

$$\begin{aligned}
\mu(t, x) &:= |u(t) - v(t)|_t + q(u, v)_x + r(0)\,|u(t) - v(t)| \\
&\qquad - r(t)|u_0 - v_0| + \int_0^t r'(t - \tau)|u(\tau) - v(\tau)|\,d\tau \\
&= -\int_0^t |r'(t - \tau)|\,(|u(\tau) - v(\tau)| - sgn(u(t) - v(t))(u(\tau) - v(\tau)))\;d\tau \leq 0.
\end{aligned}$$

Therefore, μ as a measure on $\mathbb{R}^2_+$ satisfies

$$\mu(\mathbb{R}^2_+) = -\sum_{J(u)\cup J(v)} (\sigma[\eta] - [q]) + \mu(C(u) \cap C(v)) \leq 0.$$

Then we follow the same steps as for the BV estimates in Section 2.2 to conclude (1.15). When $u_0 \in L^\infty$, let u_0^k be a sequence of initial data in BV for which $u_0^k \to u_0$ as $k \to \infty$. Then the L^1-stability result (1.15) implies that the corresponding entropy solution sequence $u^k \in BV$ to (1.1) with data u_0^k is a Cauchy sequence in L^1 which yields a subsequence converging to $u(t, x) \in L^\infty$. It is easy to check that the limit $u(t, x)$ is an entropy solution.

4. Proof of Theorem 1.3

Let $u^\varepsilon \in BV$ denote the unique entropy solution to (1.18) with initial data $u_0 \in BV$. Then the solution sequence $\{u^\varepsilon\}$ is uniformly bounded and is uniformly stable in L^1 with respect to the initial data since

$$\|u^\varepsilon\|_{L^\infty(\mathbb{R}^2_+)} \leq 2\|u_0\|_{L^\infty(\mathbb{R})}, \qquad \|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^1} \leq L\,\|u_0 - v_0\|_{L^1}$$

and satisfies the following apriori uniform bounds:

$$TV\{u^\varepsilon(t)\} \leq L\,M(u_0), \qquad \|u^\varepsilon(t) - u^\varepsilon(s)\|_{L^1(\mathbb{R})} \leq CM(u_0)\,|t - s|. \tag{4.1}$$

This implies that there exists a convergent subsequence $\{u^{\varepsilon_m}\}$ with $\varepsilon_m \to 0$ as $m \to \infty$, whose limit is denoted by u, i.e., $u^{\varepsilon_m}(t, x) \to u(t, x)$ in L^1_{loc}. Then, since $k_\varepsilon(t) \rightharpoonup (\alpha - 1)\,\delta(t)$ weakly as measures when $\varepsilon \to 0$, we conclude that u is an admissible weak solution of the Cauchy problem (1.19) and (1.2). The proof of Theorem 1.3 is complete.

Finally we discuss some families of kernels $\{k_\varepsilon\}$ that satisfy the assumptions stated in Theorem 1.3.

Suppose that $k_\varepsilon \in L^1(\mathbb{R}_+)$ for all $\varepsilon > 0$. Then, by the Paley-Wiener Theorem [14], the resolvent r_ε of k_ε is in $L^1(\mathbb{R}_+)$ if and only if

$$det(1 + \hat{k}_\varepsilon(z)) \neq 0 \qquad \text{for all } Re(z) \geq 0$$

for the Laplace transform $\hat{k}_\varepsilon$ of k_ε. By extending k_ε as zero to the negative real axis, we choose the number q such that

$$q \geq sup_{\omega\in\mathbb{R}}|(1 + \hat{k}_\varepsilon(i\omega))^{-1}|,$$

and choose positive numbers T and η satisfying

$$\int_{|s|\geq T} |k_\varepsilon(t)|dt \leq \frac{1}{12q}, \qquad \sup_{0<s<\eta} \int_{-\infty}^{\infty} |k_\varepsilon(t) - k_\varepsilon(t-s)|\, dt \leq \frac{1}{4}.$$

Then

$$\|r_\varepsilon\|_{L^1(\mathbb{R}_+)} \leq \big(8\lceil 6qT\|k_\varepsilon\|_{L^1(\mathbb{R}_+)}\rceil \lceil 8\|k_\varepsilon\|_{L^1(\mathbb{R}_+)}/\eta\rceil + 6\big)\, q\|k_\varepsilon\|_{L^1(\mathbb{R}_+)}, \tag{4.2}$$

where $\lceil s\rceil$ denotes the smallest integer $\geq s$.

With this, for each ε, we can take $k_\varepsilon \in L^1$ such that $det(1 + \hat{k}_\varepsilon(z)) \neq 0$ for all $Re(z) \geq 0$ and the numbers defined above to be: q independent of ε, $T \sim \varepsilon$ and $\eta \sim \varepsilon$. Then, by (4.2), r_ε is uniformly bounded in $L^1(\mathbb{R})$. Furthermore, any kernel $k_\varepsilon(t)$ in the Set of Kernels (i)–(vi) satisfies the assumptions in Theorem 1.2.

A prototype is the family of kernels $k_\varepsilon(t)$ in (1.20) that satisfies these assumptions. Then the corresponding family of resolvent kernels is $r_\varepsilon(t)$ in (1.21) which fulfills the assumptions of Theorem 1.3 when $0 < \alpha < 1$. For this example, the scalar nonlocal equation (1.1):

$$u_t + f(u)_x - \frac{1-\alpha}{\varepsilon}\int_0^t e^{-\frac{t-\tau}{\varepsilon}} f(u(\tau))_x\, d\tau = 0$$

can also be written as a system of two equations:

$$\begin{cases} u_t + (f(u) - v)_x = 0, \\ v_t = \dfrac{(1-\alpha)f(u) - v}{\varepsilon}. \end{cases} \tag{4.3}$$

Then the range of $\alpha \in (0, 1)$ is the sub-characteristic condition. Thus, the result of Theorem 1.3 applying this special case is equivalent to establishing the convergence of the relaxation limit (4.3) as considered in [2, 9, 16, 18].

Remark 4.1. *In order to obtain that the resolvent r_ε of k_ε is integrable and $\|r_\varepsilon\|_{L^1(\mathbb{R}_+)} \leq 20$, it also suffices by the Shea-Wainger Theorem* [7] *to require that a family of kernels $\{k_\varepsilon\}$ satisfies that, for each $\varepsilon > 0$, $k_\varepsilon \in L^1_{loc}(\mathbb{R}_+; \mathbb{R})$ and is nonnegative, nonincreasing, and convex on $\mathbb{R}_+$.*

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