UNIQUENESS AND SHARP ESTIMATES ON SOLUTIONS TO HYPERBOLIC SYSTEMS WITH DISSIPATIVE SOURCE

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Abstract. Global weak solutions of a strictly hyperbolic system of balance laws in one-space dimension were constructed (cf. Christoforou [C]) via the vanishing viscosity method under the assumption that the source term g is dissipative. In this article, we establish sharp estimates on the uniformly Lipschitz semigroup $\mathcal{P}$ generated by the vanishing viscosity limit in the general case which includes also non-conservative systems. Furthermore, we prove uniqueness of solutions by means of local integral estimates and show that every *viscosity solution* can be constructed as a limit of vanishing viscosity approximations.

1. Introduction

The objective of this work is to study the global solution to the Cauchy problem for hyperbolic systems

$$u_t + A(u)u_x + g(u) = 0, \tag{1.1}$$

$$u(0,x) = u_0(x), \tag{1.2}$$

which is obtained via the method of vanishing viscosity in [C]. Here $x \in \mathbb{R}$, $u(t,x) \in \mathbb{R}^n$, A is $n \times n$ matrix and $g : \mathbb{R}^n \to \mathbb{R}^n$. We assume that the system is strictly hyperbolic, i.e. $A(u)$ has n real distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u), \tag{1.3}$$

and thereby n linearly independent right eigenvectors $r_i(u)$, $i = 1, \ldots, n$.

Over the years, four different techniques have been developed for constructing weak solutions, namely the random choice method of Glimm, the front tracking method, the vanishing viscosity method and the functional analytic method of compensated compactness. Expositions of the current state of the theory together with relevant bibliography may be found in the books [B, D, S, Sm].

For systems of balance laws, the existence of local in time BV solutions was first established by Dafermos and Hsiao [DH], by the random choice method of Glimm [G]. Because of the presence of the production term $g(u)$, small oscillations in the solution may amplify in time, hence in general one does not have long term stability in BV. Global existence was established in [DH] under a suitable dissipativeness assumption on g. (See also [L, AGG]). Recently, global weak solutions to

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(1.1) were constructed via the vanishing viscosity method [C], namely, as the $\varepsilon \downarrow 0+$ limit of a family $\{u^\varepsilon\}$ of functions that satisfy the parabolic system

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x + g(u^\varepsilon) = \varepsilon u^\varepsilon_{xx}. \tag{1.4}$$

This was achieved (cf. Christoforou [C]) by extending the fundamental analysis of Bianchini and Bressan [BiB] to systems of balance laws.

As presented in [C], one should not expect global existence unless the source $g(u)$ is dissipative. Let u^* be an equilibrium solution to (1.1) and consider $n \times n$ matrix

$$B(u) = [r_1(u), ..., r_n(u)]^{-1} Dg(u)[r_1(u), ..., r_n(u)]. \tag{1.5}$$

Under the hypothesis that $B(u^*)$ is *strictly column diagonally dominant*, i.e.

$$B_{ii}(u^*) - \sum_{j \neq i} |B_{ji}(u^*)| > \mu > 0 \qquad\qquad i = 1, ..., n. \tag{1.6}$$

we obtain global BV solutions for the system (1.1). For the sake of completeness we state the principal result of [C]:

Theorem 1.1. *Consider the Cauchy problem*

$$u^\varepsilon_t + A(u^\varepsilon)\, u^\varepsilon_x + g(u^\varepsilon) = \varepsilon u^\varepsilon_{xx} \tag{1.7}$$

$$u^\varepsilon(0, x) = u_0(x). \tag{1.8}$$

Assume that the matrices $A(u)$ have real distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u)$ and thereby n linearly independent eigenvectors $r_1(u), r_2(u), \ldots, r_n(u)$. Under the assumption that the matrix $B(u^*)$ defined by (1.5) is strictly diagonally dominant, there exists a constant $\delta_0 > 0$ such that if $u_0 - u^* \in L^1$ and

$$TV\{u_0\} < \delta_0, \tag{1.9}$$

then for each $\varepsilon > 0$ the Cauchy problem (1.7)-(1.8) has a unique solution u^ε, defined for all $t \geq 0$. Moreover,

$$TV\{u^\varepsilon(t, \cdot)\} \leq C\, e^{-\mu\, t} TV\{u_0\}, \tag{1.10}$$

$$\|u^\varepsilon(t) - u^\varepsilon(s)\|_{L^1} \leq L' \left(|t-s| + \sqrt{\varepsilon} |\sqrt{t} - \sqrt{s}| \right) e^{-\mu s}, \qquad \text{for } t > s, \tag{1.11}$$

where μ is a positive constant that depends on $B(u^*)$. Furthermore, if v^ε is another solution of (1.7) with initial data v_0, then

$$\|u^\varepsilon(t) - v^\varepsilon(t)\|_{L^1} \leq L\, e^{-\mu t} \|u_0 - v_0\|_{L^1}. \tag{1.12}$$

Finally, as $\varepsilon \to 0$, u^ε converges in L^1_{loc} to a function u, which is the admissible weak solution u of (1.1)-(1.2), when the system is in conservation form, $A = Df$.

In the conservative case, $A = Df$, every vanishing viscosity limit is an admissible weak solution to $u_t + (f(u))_x + g(u) = 0$ and the stability estimate (1.12) implies the uniqueness within the family of solutions obtained via the vanishing viscosity method (cf. Theorem 1.1).

The objective of this work is to establish the uniqueness of solutions to (1.1) by means of local integral estimates. More precisely, our goal is twofold, namely treating the general case (1.1) in which the system is not necessarily conservative as well as establishing uniqueness within a broader class of solutions. The main result of this paper is the following:

Theorem 1.2. *Suppose that the hypotheses of Theorem 1.1 hold. Let $\mathcal{P} : \mathcal{D} \times [0, \infty) \mapsto \mathcal{D}$ be the semigroup of vanishing viscosity solutions constructed as limit of the vanishing viscosity approximations via* (1.7) *(as defined in Section 2.1). Then every trajectory $u(t) = \mathcal{P}_t u(0)$, $u(t, \cdot) \in \mathcal{D}$ satisfies the following conditions:*

i. *At every point (τ, ξ), for every $\beta' > 0$ one has*

$$\lim_{h\to 0+} \frac{1}{h} \int_{\xi-\beta' h}^{\xi+\beta' h} \left| u(\tau + h, x) - U^{\sharp}_{(u;\tau,\xi)}(h, x - \xi) \right| dx = 0, \tag{1.13}$$

where $U^{\sharp}_{(u;\tau,\xi)}$ is defined in (3.1)-(3.3).

ii. *There exist constants C, $\beta > 0$ such that for every $\tau \geq 0$ and $\xi \in (a, b)$, one has*

$$\begin{aligned} \limsup_{h\to 0+} \frac{1}{h} \int_{a+\beta h}^{b-\beta h} \left| u(\tau + h, x) - U^{\flat}_{(u;\tau,\xi)}(h, x) \right| dx \leq C \Big[& \left(TV\{u(\tau) : (a, b)\}\right)^2 \\ & + (b - a) \cdot TV\{u(\tau) : (a, b)\} \Big], \end{aligned} \tag{1.14}$$

where $U^{\flat}_{(u;\tau,\xi)}$ is defined in (3.4).

Conversely, let $u : [0, T] \mapsto \mathcal{D}$ be Lipschitz continuous map with values in $L^1(\mathbb{R}, \mathbb{R}^n)$ and assume that the conditions (i) and (ii) hold at almost every time τ. Then $u(t)$ coincides with a trajectory of the semigroup $\mathcal{P}$.

The above result implies the convergence of $\mathcal{P}^{\varepsilon} u_0$ as $\varepsilon \downarrow 0+$ (as a whole sequence and not in the context of a subsequence $\{\varepsilon_m\}$) to a unique limit $\mathcal{P} u_0$. Furthermore, it characterizes the trajectories of the semigroup by means of local integral estimates (1.13) and (1.14). More precisely, if we call a *viscosity solution* a Lipschitz function that satisfies (1.13) and (1.14) (def. is given in Section 3), then the limit $\mathcal{P}_t u_0$ is a *viscosity solution* and every *viscosity solution* can be constructed as a limit of vanishing viscosity approximations. Roughly speaking, in view of the above result, a solution u to the hyperbolic system with dissipative source (1.1) can be approximated by the self-similar solution of a Riemann problem to $u_t + A(u)u_x = 0$ in a neighborhood of (τ, ξ). Also, it relates u to the solution of the corresponding linear hyperbolic system $u_t + \hat{A} u_x + \hat{g} = 0$ with constant coefficients in terms of the total variation over the interval (a, b) and the length $b - a$ of the interval. It should be noted that this result is established for the case $g \equiv 0$ (cf. [BiB]). See also [AGG] for a related work on front tracking approximation.

In [AG2], a Lipschitz semigroup $\tilde{\mathcal{P}}$ is constructed for the conservative system (1.1), ($A = Df$), that satisfies the estimate

$$\|\tilde{\mathcal{P}}_h v - \mathcal{S}_h u - hg(u)\|_{L^1} = \mathcal{O}(1)h^2, \qquad h \to 0 \tag{1.15}$$

where $\mathcal{S}_h$ converges to the corresponding semigroup of (1.1) for $g \equiv 0$. By employing (1.15), and a general uniqueness argument (cf. [B2]) for quasi-differential equations in metric spaces, one can derive that $\tilde{\mathcal{P}}$ is unique and satisfies the integral estimates (1.13)-(1.14) in Theorem 1.2. Hence, our operator $\mathcal{P}$ coincides with $\tilde{\mathcal{P}}$ as constructed in [AG2]. It is possible to prove Theorem 1.2 by following the same strategy as in [AG2], i.e. establishing an estimate of the form (1.15) for the semigroup $\mathcal{P}$ constructed via the vanishing viscosity approximations. However, in this project our goal is to employ the techniques presented [BiB] and generalize them in this setting.

We note that following [AG1] it is possible to recover the same results under a more general assumption on the dissipation of the source term g independent of the choice of the right eigenvectors r_i, $i = 1, \dots, n$.

The outline of this article is as follows: In Section 2, we show that the bulk of a perturbation z^ε to the vanishing viscosity approximations u^ε propagates at a finite speed. Hence as $\varepsilon \to 0$, we get that the values of the vanishing viscosity solution $u(t)$ on $[a, b]$ can be determined by the values of the initial data on $[a - \beta t, b + \beta t]$; in particular

$$\int_a^b |(u(t,x) - v(t,x)| \, dx = L\, e^{-\mu t} \int_{a-\beta t}^{b+\beta t} |u_0 - v_0| \, dx, \tag{1.16}$$

for every two solutions u and v to (1.1) with initial data u_0 and v_0. It should be noted that the exponential decay is induced by the dissipative source term. Note the improvement on the continuous dependence estimate compared with (1.12). Moreover, we consider a uniformly Lipschitz semigroup $\mathcal{P}$ generated by the vanishing viscosity limit: $\mathcal{P} = \lim \mathcal{P}^{\varepsilon_m}$ and establish the finite propagation speed of solutions and tame oscillation property. Be employing these estimates, we prove Theorem 1.2 in Section 3 and naturally extend the definition of a *viscosity solution* in this framework. Finally, in Section 4, we derive an estimate on the dependence of the limit semigroup $\mathcal{P}$ on the matrix A and the vector g.

In the author's opinion, this project completes the work on the vanishing viscosity solutions to hyperbolic systems with dissipative source of the form (1.1) satisfying the dissipativeness assumption (1.6).

2. Propagation speed

Here, we study the properties of the semigroup generated by the vanishing viscosity method. First, we establish the propagation speed on the linearized perturbation z^ε to u^ε . Having this estimate, we can show a finite propagation speed on the vanishing viscosity limit and further establish the tame oscillation. The proof of the next lemma follows closely the one in [BiB]. Here, we present the treatment of the source $g(u)$. For that purpose, additional estimates are established to those already devised in [BiB].

By rescaling the coordinates, we can write (1.7) as

$$u_t + A(u)\, u_x + \varepsilon g(u) = u_{xx}. \tag{2.1}$$

Lemma 2.1. *Let u, v be solutions to* (2.1) *with*

$$u(0,x) = v(0,x), \qquad x \notin [a,b],$$

then,

$$|u(t,x) - v(t,x)| \le \|u(0) - v(0)\|_{L^\infty}\, min\{\alpha e^{\beta t-(x-b)}, \alpha e^{\beta t+(x-a)}\}, \qquad \forall x \in \mathbb{R},\ t > 0, \tag{2.2}$$

for some constants α, $\beta > 0$. On the other hand, if

$$u(0,x) = v(0,x), \qquad x \in [a,b],$$

then,

$$|u(t,x) - v(t,x)| \le \|u(0) - v(0)\|_{L^\infty}\{\alpha e^{\beta t-(x-a)} + \alpha e^{\beta t+(x-b)}\}, \qquad \forall x \in \mathbb{R},\ t > 0, \tag{2.3}$$

for some constants α, $\beta > 0$.

Proof. Consider the infinitesimal perturbation z to (2.1) that satisfies the linearized equation

$$z_t + (A(u)z)_x + \varepsilon Dg(u)z = z_{xx} + (DA(u)\cdot u_x)z - (DA(u)\cdot z)u_x \tag{2.4}$$

with initial data satisfying

$$|z(0,x)| \le 1, \qquad x \le 0$$

$$z(0,x) = 0, \qquad x > 0.$$

Let $B(t)$ be a continuous increasing function that satisfies

$$B(t) \ge 1 + 2\|A\|_\infty \int_0^t \left(\frac{1}{\sqrt{t-s}} + \sqrt{\pi}\right) B(s)\, ds, \qquad\qquad B(0) = 1.$$

This function is introduced in [BiB] and one can check that $B(t) \le 2e^{Ct}$ for some large enough constant C. Consider the function $E(t,x)$

$$E(t,x) \doteq B(t)e^{t-x}\exp\{4\|DA\|_\infty \int_0^t \|u_x(\sigma)\|_\infty\, d\sigma + 2\varepsilon\|Dg\|_\infty\, t\}.$$

Note the presence of the source term g in the definition of $E(t,x)$. We claim

$$|z(t,x)| \le E(t,x), \qquad\qquad \forall x \in \mathbb{R}, \quad t \ge 0. \tag{2.5}$$

Indeed, if $G(t,x)$ is the heat kernel, then we can write

$$\begin{aligned} z(t,x) &= G(t) * z(0) - \int_0^t G_x(t-s) * [A(u)\, z](s)\, ds \\ &\quad + \int_0^t G(t-s) * [(u_x \bullet A(u)) z(s) - (z \bullet A(u)) u_x(s)]\, ds \\ &\quad -\varepsilon \int_0^t G(t-s) * Dg(u) z(s)\, ds, \end{aligned} \tag{2.6}$$

where $*$ denotes convolution and $\alpha \bullet \beta$ is the derivative of β in the direction of α, i.e. $\nabla_\alpha \beta$. This notation is consistent with the one used in [C]. Assume that $\tau > 0$ is the first time at which (2.5) holds as an equality, then we estimate $|z(t,x)|$ via (2.6). It yields

$$\int G(t, x-y) |z(0,y)|\, dy < \int \frac{e^{\frac{(x-y)^2}{4t}}}{2\sqrt{\pi t}} e^{-y}\, dy = e^{t-x}, \tag{2.7}$$

$$\begin{aligned} &\|A\|_\infty \int_0^t \int |G_x(t-s, x-y)| E(s,y)\, dy\, ds = \\ &= \|A\|_\infty \int_0^t \int \frac{|x-y|}{4(t-s)\sqrt{\pi(t-s)}} e^{-\frac{(x-y)^2}{4(t-s)}} B(s) e^{s-y} \cdot \\ &\qquad \cdot \exp\left\{ 4\|DA\|_\infty \int_0^s \|u_x(\sigma)\|\, d\sigma + 2\varepsilon \|Dg\| s \right\} dy\, ds \\ &\leq \|A\|_\infty \exp\left\{ 4\|DA\|_\infty \int_0^t \|u_x(\sigma)\|\, d\sigma + 2\varepsilon \|Dg\| t \right\} e^{t-x} \cdot \\ &\qquad \cdot \int_0^t \frac{B(s)}{4(t-s)\sqrt{\pi(t-s)}} \left[\int |x-y| exp\left\{ -\frac{(y+2(t-s)-x)^2}{4(t-s)} \right\} dy \right] ds \\ &= \exp\left\{ 4\|DA\|_\infty \int_0^t \|u_x(\sigma)\|\, d\sigma + 2\varepsilon \|Dg\| t \right\} e^{t-x} \cdot \\ &\qquad \cdot \int_0^t \frac{\|A\|_\infty B(s)}{\sqrt{\pi(t-s)}} \left(\int |\zeta - \sqrt{t-s}| e^{-\zeta^2}\, d\zeta \right) ds \\ &\leq \exp\left\{ 4\|DA\|_\infty \int_0^t \|u_x(\sigma)\|\, d\sigma + 2\varepsilon \|Dg\| t + t - x \right\} \int_0^t \|A\|_\infty B(s) \left(\frac{1}{\sqrt{t-s}} + \sqrt{\pi} \right) ds \\ &\leq \exp\left\{ 4\|DA\|_\infty \int_0^t \|u_x(\sigma)\|\, d\sigma + 2\varepsilon \|Dg\| t + t - x \right\} \left(\frac{B(t)}{2} - \frac{1}{2} \right) \\ &\quad = \frac{1}{2} E(t,x) - \frac{1}{2} \exp\left\{ 4\|DA\|_\infty \int_0^t \|u_x(\sigma)\|\, d\sigma + 2\varepsilon \|Dg\| t + t - x \right\} \\ &\quad \leq \frac{1}{2} E(t,x) - \frac{1}{2} e^{t-x} \end{aligned} \tag{2.8}$$

and

$$\int_0^t (2\|DA\|_\infty \|u_x(s)\|_\infty + \varepsilon\|Dg\|_\infty)(\int G(t-s,x-y)E(s,y)\,dy)\,ds \leq$$

$$= \int_0^t (2\|DA\|_\infty \|u_x(s)\|_\infty + \varepsilon\|Dg\|_\infty) B(s) e^{t-x} \exp\{4\|DA\|_\infty \int_0^s \|u_x(\sigma)\|_\infty\,d\sigma + \varepsilon\|Dg\|_\infty s\} ds$$

$$\leq B(t)e^{t-x} \int_0^t (2\|DA\|_\infty \|u_x(s)\|_\infty + \varepsilon\|Dg\|_\infty) \cdot$$

$$\cdot \exp\{4\|DA\|_\infty \int_0^s \|u_x(\sigma)\|_\infty\,d\sigma + \varepsilon\|Dg\|_\infty s\} ds$$

$$= B(t)e^{t-x}\frac{1}{2}\left(\exp\left\{4\|DA\|_\infty \int_0^t \|u_x(\sigma)\|\,d\sigma + 2\varepsilon\|Dg\|t\right\} - 1\right)$$

(2.9)

$$\leq \frac{1}{2}E(t,x) - \frac{1}{2}e^{t-x},$$

since $B(s)$ is an increasing function and $B(0)=1$. Hence, for all $t \in [0,\tau]$, we get

$$|z(t,x)| < e^{t-x} + E(t,x) - e^{t-x} = E(t,x),$$

which contradicts the choice of τ. Thus (2.5) holds for all $t \geq 0$. Applying the estimate

$$\|u_x(s)\|_\infty \leq \max\left\{\frac{2\kappa\delta_0}{\sqrt{s}}, \frac{2\kappa\delta_0}{\sqrt{\hat{t}}}\right\} e^{-\varepsilon\mu s} \tag{2.10}$$

derived in Section 8 of [C], we deduce

$$|z(t,x)| \leq E(t,x) \leq 2e^{Ct} \exp\{4\|DA\|_\infty 2\kappa\delta_0(\sqrt{t} + \frac{t}{\sqrt{\hat{t}}}) + \varepsilon\|Dg\|_\infty t\} e^{t-x} \leq \alpha e^{\beta t - x}. \tag{2.11}$$

From this point and on, the proof follows easily if one adjusts the arguments in the proof of Lemma 12.1, [BiB] to this setting. ∎

By rescaling the coordinates backwards, for every $u_0(x) = v_0(x)$ in $x \in [a,b]$, (2.3) implies

$$|u^\varepsilon(t,x) - v^\varepsilon(t,x)| \leq \|u(0) - v(0)\|_{L^\infty} \min\left(\alpha\exp\left\{\frac{\beta t - (x-a)}{\varepsilon}\right\}, \alpha\exp\left\{\frac{\beta t + (x-b)}{\varepsilon}\right\}\right), \tag{2.12}$$

for all $x \in \mathbb{R}$ and $t > 0$, where u^ε and v^ε are solutions to (1.7).

The following subsections complete the survey on the solution u constructed via the vanishing viscosity method and establish the essential estimates and properties needed to prove Theorem 1.2 in the next section. In Section 2.1, we define the semigroup $\mathcal{P}$ generated by the vanishing viscosity solutions. In Section 2.2, we prove the finite propagation speed and obtain a sharper estimate than the stability (1.12) w.r.t. initial data. Last, in Section 2.3, we establish the tame oscillation property of the trajectories of $\mathcal{P}$ which plays a crucial role in the proof of Theorem 1.2. (See [BG].)

2.1. **Semigroup.** Let the domain $\mathcal{D} \subset L^1_{loc}$ be the set of all functions u_0 with $u_0 - u^* \in L^1$ and small total variarion. Then for every $t \geq 0$ we define $\mathcal{P}^\varepsilon_t u_0 \doteq u^\varepsilon(t)$. By (1.10) and proceeding as in [BiB], we define

$$\mathcal{P}_t u_0 \doteq \lim_{m\to+\infty} \mathcal{P}^{\varepsilon_m}_t u_0, \tag{2.13}$$

for some subsequence $\{\varepsilon_m \downarrow 0\}$. Below, we show that $\mathcal{P}$ defines a semigroup and in the next section we prove that it does not depend on the choice of the subsequence $\{\varepsilon_m\}$.

First, one can easily derive the continuity properties:

$$\|\mathcal{P}_t u_0 - \mathcal{P}_s v_0\|_{L^1} \leq L'\, e^{-\mu s}|t-s|\,, \tag{2.14}$$

for $t > s$ and

$$\|\mathcal{P}_t u_0 - \mathcal{P}_t v_0\|_{L^1} \leq L\, e^{-\mu t}\|u_0 - v_0\|_{L^1}, \tag{2.15}$$

by employing the estimates (1.12) and (1.11). Now, $\mathcal{P}_0 u_0 = u_0$ is an immediate consequence of the definition of $\mathcal{P}$. To prove $\mathcal{P}_{s+t}u_0 = \mathcal{P}_s\mathcal{P}_t u_0$, given $r > 0$, we consider the initial data

$$\tilde{u}_m(x) = \begin{cases} \mathcal{P}_t u_0(x) & if \quad |x| > r + 2\beta s, \\ \mathcal{P}^{\varepsilon_m}_t u_0(x) & if \quad |x| \leq r + 2\beta s. \end{cases} \tag{2.16}$$

Assuming that $s > 0$, by (2.12) and (2.15), it follows

$$\begin{aligned} \limsup_{m\to+\infty} \int_{-r}^{r} &|(\mathcal{P}^{\varepsilon_m}_s\mathcal{P}^{\varepsilon_m}_t u_0)(x) - (\mathcal{P}^{\varepsilon_m}_s\mathcal{P}_t u_0)(x)|\,dx \\ &\leq \lim_{m\to+\infty} 2r \sup_{|x|<r} |(\mathcal{P}^{\varepsilon_m}_s\mathcal{P}^{\varepsilon_m}_t u_0)(x) - (\mathcal{P}^{\varepsilon_m}_s\tilde{u}_m)(x)| + \lim_{m\to+\infty} \|\mathcal{P}^{\varepsilon_m}_s\tilde{u}_m - \mathcal{P}^{\varepsilon_m}_s\mathcal{P}_t u_0\|_{L^1} \\ &\leq \lim_{m\to+\infty} 2r\|\mathcal{P}^{\varepsilon_m}_t u_0 - \tilde{u}_m\|_{L^\infty} 2\alpha e^{-\frac{\beta s}{\varepsilon_m}} + \lim_{m\to+\infty} L\|\tilde{u}_m - \mathcal{P}_t u_0\|_{L^1} = 0. \end{aligned} \tag{2.17}$$

If we observe that

$$\mathcal{P}_{s+t}u_0 = \lim_{m\to+\infty} \mathcal{P}^{\varepsilon_m}_s\mathcal{P}^{\varepsilon_m}_t u_0, \qquad \mathcal{P}_s\mathcal{P}_t u_0 = \lim_{m\to+\infty} \mathcal{P}^{\varepsilon_m}_s\mathcal{P}_t u_0, \tag{2.18}$$

then (2.17) implies the identity $\mathcal{P}_{s+t}u_0 = \mathcal{P}_s\mathcal{P}_t u_0$. Thus $\mathcal{P}$ is a semigroup.

2.2. **Finite Propagation Speed.** Consider an interval $[a, b]$ and two initial data u_0, v_0 such that

$$u_0(x) = v_0(x), \qquad x \in [a, b].$$

Hence, by (2.12) it follows

$$|(P_t u_0)(x) - (P_t v_0)(x)| \leq \limsup_{m\to+\infty} |(\mathcal{P}^{\varepsilon_m}_t u_0)(x) - (\mathcal{P}^{\varepsilon_m}_t v_0)(x)| = 0, \qquad x \in (a+\beta t, b-\beta t). \tag{2.19}$$

Thus,

$$(\mathcal{P}_t u_0)(x) \equiv (\mathcal{P}_t v_0)(x), \qquad x \in (a+\beta t, b-\beta t). \tag{2.20}$$

Moreover for every initial data u_0, v_0, take

$$w_0 = \begin{cases} u_0(x) & if \quad x \in [a-\beta t, b+\beta t], \\ v_0(x) & if \quad x \notin [a-\beta t, b+\beta t]. \end{cases} \tag{2.21}$$

In view of the finite propagation speed (2.20) and (2.15), we get

$$\begin{aligned} \int_a^b |(\mathcal{P}_t u_0)(x) - (\mathcal{P}_t v_0)(x)|\, dx &= \int_a^b |(\mathcal{P}_t w_0)(x) - (\mathcal{P}_t v_0)(x)|\, dx \\ &\le \|\mathcal{P}_t w_0 - \mathcal{P}_t v_0\|_{L^1} \le L\|w_0 - v_0\|_{L^1} e^{-\mu t} \\ &= Le^{-\mu t} \int_{a-\beta t}^{b+\beta t} |u_0 - v_0|\, dx, \end{aligned} \tag{2.22}$$

which is a sharper estimate than (2.15).

The following lemma states two estimates that are satisfied by every semigroup which is continuous with respect to time and initial data. We employ these estimates in the next section to prove uniqueness.

Lemma 2.2. *For every Lipschitz continuous map $\omega(t)$, $t \in [0,T]$ taking values in the domain of $\mathcal{P}$, it follows*

$$\|\omega(T) - \mathcal{P}_T\omega(0)\|_{L^1} \le L \int_0^T \liminf_{r\to 0+} \frac{\|\omega(t+r) - \mathcal{P}_r\omega(t)\|_{L^1}}{r}\, dt \tag{2.23}$$

and in particular, given any interval $[a,b]$, then

$$\|\omega(T) - \mathcal{P}_T\omega(0)\|_{L^1(a+\beta T, b-\beta T)} \le L \int_0^T \liminf_{r\to 0+} \frac{\|\omega(t+r) - \mathcal{P}_r\omega(t)\|_{L^1(a+\beta(t+r), b-\beta(t+r))}}{r}\, dt, \tag{2.24}$$

where L is the Lipschitz constant of the semigroup $\mathcal{P}$.

Proof. The proof of (2.23) can be found in [B]. To prove (2.24), let

$$\psi(t) = \|\mathcal{P}_{T-t}\omega(t) - \mathcal{P}_T\omega(0)\|_{L^1(a+\beta T, b-\beta T)}, \tag{2.25}$$

$$\phi(t) = \liminf_{r\to 0+} \frac{1}{r}\|\omega(t+r) - \mathcal{P}_r\omega(t)\|_{L^1(a+\beta(t+r), b-\beta(t+r))} \tag{2.26}$$

and

$$\chi(t) = \psi(t) - L\int_0^t \phi(s)\, ds. \tag{2.27}$$

Since, $\chi(0) = \psi(0) = 0$, it suffices to prove that $\dot\chi(s) \le 0$ for all $s \in [0,T]$. The proof follows by retracing the arguments of the proof of (2.23). ■

2.3. **Tame Oscillation.** We denote by $TV\{u(t);\,(a,b)\}$ the total variation of $u(t,\cdot)$ over the interval $x \in (a,b)$. Consider the triangle $\Delta^{\tau}_{a,b}$ on the $t-x$ plane:

$$\Delta^{\tau}_{a,b} \doteq \{(t,x) : t > \tau, a + \beta t < x < b - \beta t\}. \tag{2.28}$$

Lemma 2.3. *For every $a < b$ and $\tau \geq 0$, then there exists a positive constant C' such that*

$$Osc.\{u; \Delta^{\tau}_{a,b}\} \leq C' \cdot TV\{u(\tau);\,(a,b)\}, \tag{2.29}$$

for every trajectory $u(t) = \mathcal{P}_t u_0$, where $Osc.\{u; \Delta^{\tau}_{a,b}\}$ denotes the oscillation of u over $\Delta^{\tau}_{a,b}$;

$$Osc.\{u; \Delta^{\tau}_{a,b}\} \doteq \sup\{|u(t,x) - u(t',x')|; (t,x),\,(t',x') \in \Delta^{\tau}_{a,b}\}. \tag{2.30}$$

Proof. Without loss of generality assume that $\tau = 0$ and consider the initial data

$$\bar{v}(x) \doteq \begin{cases} u(\tau, a-) & if \quad x \leq a \\ u(\tau, x) & if \quad a < x < b \\ u(\tau, b+) & if \quad x \geq b \end{cases}. \tag{2.31}$$

Let $v(t) = \mathcal{P}_t \bar{v}$. Then the finite speed of propagation (2.20) implies

$$\text{Osc.}\{u; \Delta^{\tau}_{a,b}\} = \text{Osc.}\{v; \Delta^{\tau}_{a,b}\} \leq 2 \sup_t TV\{v(t)\} \leq 2C\, TV\{\bar{v}\}, \tag{2.32}$$

and the proof follows easily in the same way as in [BiB] for the case $g = 0$, since $TV\{u(\tau);\,(a,b)\} = TV\{\bar{v}\}$. ■

As we will see in the next section, this result is essential to establish the uniqueness as stated in Theorem 1.2.

3. Uniqueness of the semigroup

In this section, we first define the functions $U^{\sharp}_{(u;\tau,\xi)}$, $U^{\flat}_{(u;\tau,\xi)}$ stated in Theorem 1.2 and then present the proof.

Given a function $u = u(t,x)$ and a point (τ, ξ), let $U^{\sharp}_{(u;\tau,\xi)}$ be the solution to the Riemann problem

$$w_t + A(w) w_x = 0, \tag{3.1}$$

$$w(0,x) = \begin{cases} u^- & x < 0 \\ u^+ & x \geq 0 \end{cases}, \tag{3.2}$$

where

$$u^- = \lim_{x \to \xi-} u(\tau, x), \qquad u^+ = \lim_{x \to \xi+} u(\tau, x). \tag{3.3}$$

Let $\mathcal{S}$ denote the semigroup that is generated by the vanishing viscosity limits to the hyperbolic system (3.1). In [BiB], it is shown that $\mathcal{S}$ is a well-defined semigroup and it satisfies the corresponding error estimate if $\mathcal{P}$ is replaced by $\mathcal{S}$ in (2.24). This fact is employed in the proof of Theorem

1.2. We refer the reader to [BiB] (in Section 14) for a construction of self-similar solutions to the non-conservative Riemann problem (3.1)-(3.2).

Moreover, we define $U^{\flat}_{(u;\tau,\xi)}$ to be the solution to the linear hyperbolic problem with constant coefficients:

$$w_t + \hat{A}w_x + \hat{g} = 0, \qquad w(0,x) = u(\tau,x), \tag{3.4}$$

where $\hat{A} = A(u(\tau,\xi)$ and $\hat{g} = g(u(\tau,\xi))$.

Proof of Theorem 1.2. Necessity: Let $u(t) = \mathcal{P}_t u(0)$ be a trajectory of the semigroup of vanishing viscosity solutions. To prove (i), given $\beta' > 0$ and (τ,ξ), take a function $\tilde{u}(x)$ such that $\tilde{u}(x) - u^* \in L^1$ and $\tilde{u}(x) = w(0,x), \quad x \in (-\beta'-\beta, \beta'+\beta)$, where $w(0,x)$ is given in (3.2)-(3.3). Let $\tilde{U}_{(u;\tau,\xi)}$ be the solution to (1.1) with initial data $\tilde{u}(x)$, i.e. $\tilde{U}_{(u;\tau,\xi)}(t) = \mathcal{P}_t\tilde{u}$. Now, fix $h > 0$ and small. Hence,

$$\begin{aligned}\frac{1}{h}\int_{\xi-\beta'h}^{\xi+\beta'h} |u(\tau+h,x) - U^{\sharp}_{(u;\tau,\xi)}(h,x-\xi)|\,dx \le{}& \frac{1}{h}\int_{\xi-\beta'h}^{\xi+\beta'h} |u(\tau+h,x) - \tilde{U}_{(u;\tau,\xi)}(h,x-\xi)|\,dx \\ &+ \frac{1}{h}\int_{\xi-\beta'h}^{\xi+\beta'h} |\tilde{U}_{(u;\tau,\xi)}(h,x-\xi)) - U^{\sharp}_{(u;\tau,\xi)}(h,x-\xi)|\,dx.\end{aligned} \tag{3.5}$$

The first term can be estimated as follows: By the continuous dependence property (2.22) of $\mathcal{P}$ and (3.3):

$$\begin{aligned}&\frac{1}{h}\int_{\xi-\beta'h}^{\xi+\beta'h} |u(\tau+h,x) - \tilde{U}_{(u;\tau,\xi)}(h,x-\xi)|\,dx \\ &\qquad \le \frac{L}{h}\left\{\int_{\xi-(\beta'+\beta)h}^{\xi+(\beta'+\beta)h} |u(\tau,x) - \tilde{U}_{(u;\tau,\xi)}(0,x-\xi)|\,dx\right\} \\ &\qquad \le \frac{L}{h}\left\{\int_{\xi-(\beta'+\beta)h}^{\xi} |u(\tau,x) - u(\tau,\xi-)|\,dx + \int_{\xi}^{\xi+(\beta'+\beta)h} |u(\tau,x) - u(\tau,\xi+)|\,dx\right\} \\ &\qquad \le L(\beta'+\beta)\left\{\sup_{\xi-(\beta'+\beta)h<x<\xi} |u(\tau,x) - u(\tau,\xi-)| + \sup_{\xi<x<\xi+(\beta'+\beta)h} |u(\tau,x) - u(\tau,\xi+)|\right\}.\end{aligned} \tag{3.6}$$

Moreover, by the error estimate (2.24) applied on $\mathcal{S}$, we get

$$\begin{aligned}\frac{1}{h}\int_{-\beta'h}^{\beta'h} |\tilde{U}_{(u;\tau,\xi)}(h,x) - U^{\sharp}_{(u;\tau,\xi)}(h,x)|\,dx ={}& \frac{1}{h}\int_{-\beta'h}^{\beta'h} |\mathcal{P}_h w(0) - \mathcal{S}_h w(0)|\,dx \le \\ &\le \frac{L}{h}\int_0^h \liminf_{r\to 0+} \int_{\beta(s+r)-(\beta'+\beta)h}^{-\beta(s+r)+(\beta'+\beta)h} \frac{1}{r}|\tilde{U}_{(u;\tau,\xi)}(s+r,x) - \mathcal{S}_r\tilde{U}_{(u;\tau,\xi)}(s)|\,dx\,ds \\ &\le \frac{L}{h}\int_0^h \int_{\beta s-(\beta'+\beta)h}^{-\beta s+(\beta'+\beta)h} |g(\tilde{U}_{(u;\tau,\xi)}(s,x))|dx\,ds \le L'(\beta'+\beta)h.\end{aligned} \tag{3.7}$$

If we let $h \to 0+$, (3.6), (3.7) together with (3.5) establish (1.13).

To prove (ii), let

$$\bar{v}(x) = \begin{cases} u(\tau, a+) & if \quad x \leq a \\ u(\tau, x) & if \quad a < x < b \\ u(\tau, b-) & if \quad x \geq b \end{cases}, \tag{3.8}$$

and

$$v_t^\varepsilon + A(v^\varepsilon)v_x^\varepsilon + g(v^\varepsilon) = \varepsilon v_{xx}^\varepsilon, \qquad w_t^\varepsilon + \hat{A}w_x^\varepsilon + \hat{g} = \varepsilon w_{xx}^\varepsilon,$$
$$v^\varepsilon(0,x) = w^\varepsilon(0,x) = \bar{v}.$$

Then, by the error estimate (2.24) and the tame oscillation property (2.29), we estimate

$$\begin{aligned}
\frac{1}{h}\int_{a+\beta h}^{b-\beta h} |u(\tau+h,x) - U^\flat_{(u;\tau,\xi)}(h,x)|\,dx &\leq \frac{1}{h}\lim_{\varepsilon\to 0}\int_{a+\beta h}^{b-\beta h} |v^\varepsilon(h,x) - w^\varepsilon(h,x)|\,dx \\
&= \frac{1}{h}\lim_{\varepsilon\to 0}\|\mathcal{P}_h^\varepsilon \bar{v} - w^\varepsilon(h)\|_{L^1(a+\beta h, b-\beta h)} \\
&\leq \frac{1}{h}\lim_{\varepsilon\to 0} L\int_0^h \liminf_{r\to 0}\frac{1}{r}\|w^\varepsilon(t+r) - \mathcal{P}_r^\varepsilon w^\varepsilon(t)\|_{L^1(a+\beta(r+t), b-\beta(r+t))}\,dt \\
&\leq \frac{L}{h}\lim_{\varepsilon\to 0}\int_0^h\int_{a+\beta t}^{b-\beta t} |A(w^\varepsilon(0,\xi))w_x^\varepsilon(t,x) + g(w^\varepsilon(0,\xi)) \\
&\qquad\qquad - A(w^\varepsilon(t,x))w_x^\varepsilon(t,x) - g(w^\varepsilon(t,x))|\,dx\,dt \\
&\leq \frac{L}{h}\left(h\sup_{t,x}[A(w^\varepsilon(0,\xi)) - A(w^\varepsilon(t,x))]\|w_x^\varepsilon(t)\|_{L^1} + h\,C'\,TV\{\bar{v}\}(b-a)\right) \\
&\leq L\left(C(TV\bar{v})^2 + C'(b-a)\cdot TV\bar{v}\right).
\end{aligned}$$

This completes the proof of (1.14).

Sufficiency: Suppose that u satisfies conditions (i) and (ii), then first observe that u is Lipschitz continuous in time and hence, employ the error estimate (2.24). Given $\tau \in [0,T]$, the function $\mathcal{P}_{t-\tau}u_0$ is a trajectory of vanishing viscosity solutions and therefore it satisfies (1.13)-(1.14) according to the necessity part of the theorem, which is already proven. Now, one can proceed by the same arguments in the proof of Lemma 15.2 in [BiB] introducing a partition of $[a+\tau\beta, b-\tau\beta]$ so that the total variation of u is arbitrarily small over each subinterval and then employ $U^\flat_{(u;\tau,x_j)}$ and $U^\sharp_{(u;\tau,y_j)}$. ■

As in [BiB, B1], the definition of *viscosity solution* can be naturally extended to the hyperbolic system with source:

$$u_t + A(u)u_x + g(u) = 0, \tag{3.9}$$

$$u(0,x) = u_0(x). \tag{3.10}$$

Definition 3.1. *A function $u(t,x)$ is a viscosity solution to* (1.1) *if $t \mapsto u(t,\cdot)$ is continuous with values in L^1_{loc} and* (1.13)-(1.14) *hold.*

Remark 3.2. *In view of Theorem 1.2, the family of vanishing viscosity approximations $\mathcal{P}^{\varepsilon}u_0$ converge to a unique limit as $\varepsilon \to 0+$, and hence the definition of $\mathcal{P}u_0$ is independent of the extracted subsequence $\{\varepsilon_m\}$.*

Indeed, suppose that there are $s > 0$ and $u_0 \in \mathcal{D}$ such that

$$\lim_{m\to+\infty} \mathcal{P}_s^{\varepsilon_m} u_0 \neq \lim_{m\to+\infty} \mathcal{P}_s^{\varepsilon'_m} u_0 \tag{3.11}$$

for two different subsequences ε_m, $\varepsilon'_m \to 0$, and let $\mathcal{P}_t u_0$ and $\mathcal{P}'_t u_0$ be the corresponding limits as $m \to +\infty$ that exist L^1_{loc} for all $t \geq 0$. We know, by Section 2.1, that $\mathcal{P}_t u_0$ and $\mathcal{P}'_t u_0$ are semigroups generated by vanishing viscosity limits to (1.1)*. Then, by Theorem 1.2, $u(t) \doteq \mathcal{P}_t u_0$ is a viscosity solution to* (3.9)*. Since it is a viscosity solution, the sufficiency part of Theorem 1.2 implies that $u(t) = \mathcal{P}'_t u_0$ for all $t \geq 0$. This contradicts the assumption* (3.11) *for $t = s$ and proves the statement of this remark.*

4. Dependence on parameters

In this section, we derive an estimate on the dependence of solutions on parameters of the matrix A and the vector g. Let u^* be a constant equilibrium solution to both systems considered below.

Corollary 4.1. *Consider the two hyperbolic systems*

$$u_t + A(u)u_x + g(u) = 0, \tag{4.1}$$

$$u_t + \hat{A}(u)u_x + \hat{g}(u) = 0, \tag{4.2}$$

having initial data u_0 with small total variation and $u_0 - u^* \in L^1$. Suppose that the hypotheses of Theorem 1.1 hold. Call $\mathcal{P}$, $\hat{\mathcal{P}}$ the corresponding semigroups of vanishing viscosity solutions, then

$$\begin{aligned} \|\hat{\mathcal{P}}_t\bar{u} - \mathcal{P}_t\bar{u}\|_{L^1} \quad \leq \quad & M\left(\frac{1-e^{-\mu t}}{\mu}\right)\Big\{\sup_u |\hat{A}(u) - A(u)| TV\{u_0\} \\ & \qquad + \sup_u |D\hat{g}(u) - Dg(u)| \|u_0 - u^*\|_{L^1}\Big\}, \end{aligned} \tag{4.3}$$

for some positive constant M.

Proof. Let $\mathcal{P}^{\varepsilon}$ and $\hat{\mathcal{P}}^{\varepsilon}$ be the corresponding semigroups solutions to the parabolic problems

$$u_t + A(u)u_x + g(u) = \varepsilon u_{xx}, \qquad u_t + \hat{A}(u)u_x + \hat{g}(u) = \varepsilon u_{xx},$$

respectively. Then by (2.23), we get

$$\|\hat{\mathcal{P}}^{\varepsilon}_t u_0 - \mathcal{P}^{\varepsilon}_t u_0\|_{L^1} \leq L \int_0^t \liminf_{r\to 0+} \frac{1}{r} \|w^{\varepsilon}(s+r) - \mathcal{P}^{\varepsilon}_r w^{\varepsilon}(s)\|_{L^1}\, ds, \tag{4.4}$$

where $w^{\varepsilon}(t) \doteq \hat{\mathcal{P}}^{\varepsilon}_t u_0$. Hence,

$$
\begin{aligned}
\|\hat{\mathcal{P}}_t^\varepsilon u_0 - \mathcal{P}_t^\varepsilon u_0\|_{L^1} &\le L\int_0^t \Big[\int |\hat{A}(w^\varepsilon(s,x)) - A(w^\varepsilon(s,x))|\, |w_x^\varepsilon(s,x)|\, dx \\
&\quad + \int |\hat{g}(w^\varepsilon(s,x)) - g(w^\varepsilon(s,x))|\, dx\Big]\, ds \\
&\le L\left(\sup_u |\hat{A}(u) - A(u)|\right)\int_0^t TV\{w^\varepsilon(s)\}\, ds \\
&\quad + L\left(\sup_u |D\hat{g}(u) - Dg(u)|\right)\int_0^t \|w^\varepsilon(s) - u^*\|_{L^1}\, ds. \qquad (4.5)
\end{aligned}
$$

Let $\mu > 0$ be the constant that satisfies the dissipativeness hypotheses (1.6) of both g and $\hat{g}$. Using the bounds

$$
TV\{w^\varepsilon(s)\} \le Ce^{-\mu s} TV\{u_0\}, \qquad \|w^\varepsilon(s) - u^*\|_{L^1} \le Le^{-\mu s}\|u_0 - u^*\|_{L^1},
$$

established in [C], we conclude

$$
\begin{aligned}
\|\hat{\mathcal{P}}_t^\varepsilon u_0 - \mathcal{P}_t^\varepsilon u_0\|_{L^1} &\le M\left(\frac{1-e^{-\mu t}}{\mu}\right)\sup_u |\hat{A}(u) - A(u)| \cdot TV\{u_0\} \\
&\quad + M\left(\frac{1-e^{-\mu t}}{\mu}\right)\sup_u |D\hat{g}(u) - Dg(u)| \cdot \|u_0 - u^*\|_{L^1}. \qquad (4.6)
\end{aligned}
$$

By letting $\varepsilon \to 0$, the above estimate completes the proof. ■

Remark 4.2. *Observe the presence of the exponential decay term in* (4.3) *that is induced by the dissipativeness assumptions on the source g and $\hat{g}$. Notice that as g and $\hat{g}$ tend to* 0*, then*

$$
\lim_{\mu \to 0+} \frac{1-e^{-\mu t}}{\mu} = t
$$

and the above error estimate reduces to the one obtained in [BiB] *for the case of hyperbolic systems* (3.1) *with $g \equiv 0$.*

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