CONVERGENCE OF FINITE VOLUME SCHEMES FOR TRIANGULAR SYSTEMS OF CONSERVATION LAWS

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Abstract. We consider non-strictly hyperbolic systems of conservation laws in triangular form, which turn up in applications like three-phase flows in porous media flow. We device finite volume schemes of Godunov type for these systems that exploit the triangular structure. We prove that the finite volume schemes converge to weak solutions as the discretization parameters tend to zero. Some numerical examples are presented, one of which is related to flows in porous media.

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1. Introduction

We study 2×2 systems of conservation laws of the form

$$\begin{cases} u_t + f(u)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ v_t + g(u,v)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ (u,v)(x,0) = (u_0(x), v_0(x)), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where u, v are the unknowns, whereas the initial values u_0, v_0 and the flux functions f, g are presribed. Triangular systems of conservation laws occur in a variety of

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models in physics and engineering. As a specific example, we shall discuss a system describing three-phase flow in porous media, see Sections 2 and 6.

System (1.1) is a special case of the followig more general system:

$$\begin{cases} (u^1)_t + (f^1(u^1))_x = 0, \\ (u^2)_t + (f^2(u^1, u^2))_x = 0, \\ \quad \vdots \qquad\qquad \vdots \\ (u^n)_t + (f^n(u^1, u^2, \dots, u^n))_x = 0. \end{cases}$$

This system is called triangular due to its special structure where evolution of u_i does not depend on the succeeding unknowns $(u_{i+1}, \dots, u_n)$.

Using vector notation, (1.1) reads

$$U_t + F(U)_x = 0 \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \tag{1.2}$$

with $U = (u, v)$ and $F(U) = (f, g)$. The Jacobian matrix for (1.1) reads

$$A = \partial F = \begin{pmatrix} f_u & 0 \\ g_u & g_v \end{pmatrix}.$$

The eigenvalues of the above matrix are $\lambda_1 = f_u$ and $\lambda_2 = g_v$. From this it follows that the system (1.1) is hyperbolic but may not be strictly hyperbolic as the wave speeds can coincide. Thus, the key difficulty in analyzing a triangular system like (1.1) is due to this non-strict hyperbolicity or resonance.

Although 2×2 systems are well studied and several rigorous results have been obtained for them, see [19] and the references therein, none of these results can be directly applied to the system (1.1) on the account of its resonant wave structure. Hence, a different approach exploiting the triangular structure of (1.1) is required in order to analyse the system. This paper in an attempt in this direction. Indeed, our aim is to design numerical schemes of finite volume type for computing approximate solutions of (1.1), and to show that these schemes converge to a weak solution of (1.1), thereby proving existence of weak solutions as well.

If we write (1.1) as a system of equations, we can group different (first order accurate) numerical methods into three main categories: (1) Central type schemes like the Lax-Friedrichs or the Local Lax-Friedrichs (Rusanov) schemes, which do not rely on Riemann solvers. (2) Upwind type schemes like the Godunov and Roe schemes, which are based on (approximate) Riemann solvers. (3) Front tracking schemes, which are based on exact Riemann solvers. Exact Riemann solvers may be difficult to design on the account of a complicated wave structure in the solutions of (1.1). If $f \equiv 0$, then front tracking schemes are viable, see, e.g., [24]. For the triangular system (with degenerate diffusion), see [20] for numerical results with a front tracking based operator splitting method. For a comparison of central and upwind type schemes, see [12].

Numerical schemes based on exact or approximate Riemann solvers are difficult to design and analyze due to the complicated structure of the solutions to (1.1), which is a manifestation of the non-strictly hyperbolic (resonance) feature of the system. Because of this, we propose to use a class of "simple" upwind schemes that exploit the triangular structure of the system (1.1). These schemes are based on the close relationship between (1.1) and scalar conservation laws with discontinuous coefficients, i.e., equations of the form

$$\begin{cases} w_t + h(k(x,t), w)_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ w(x,0) = w_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.3}$$

where w is the conserved variable and the coefficient k is allowed to be discontinuous along curves in the (x,t) plane. Indeed, one can see that since the evolution of u is independent of v in (1.1), we can evolve u and treat it is a coefficient in the evolution equation for v, thus reducing (1.1) to an equation of the form (1.3). In this paper, we are going to exploit this close relationship.

Conservation laws with discontinuous coefficients occur in a wide variety of models, such as in two phase flows in heterogeneous porous media (see, e.g., [17]), in models for sedimentation of suspensions in a clarifier-thickener unit (see, e.g., [8]), and in models for traffic flow (see, e.g., [5]). These equations have been studied in several papers. An incomplete list includes [1, 2, 3, 6, 13, 17, 24, 21, 22, 23, 28, 26, 27, 29, 32, 33] (see the references therein for a more complete picture). A key difficulty of the analysis is the nonlinear resonant behavior, which means that one cannot expect to bound the total variation of the conserved quantities directly but only when measured under a certain singular mapping. The singular mapping method, used in most of the literature on these problems, is however difficult to apply if the coefficient k in (1.3) is discontinuous both in space and time and additionally is merely BV regular, i.e., not necessarily piecewise smooth. Herein we will therefore take a different approach avoiding the singular mapping altogether.

The schemes we shall analyze are proposed in Section 4 and we shall refer to them as Semi-Godunov schemes. These schemes are very easy to implement as they do not rely on the full wave structure of the 2×2 system. Their numerical dissipation is much smaller than that of the Lax-Friedrichs scheme, and their numerical performance is comparable with front tracking schemes. Additionally, we show that these schemes converge to a weak solution of (1.1). For the convergence analysis, since BV estimates are not available and the singular mapping approach seems difficult to implement for our system, we employ the Murat-Tartar compensated compactness method [30, 31]. Parts this analysis rely heavily on the particular structure of our system (1.1).

The remainder of this paper is organised this paper as follows: In Section 2 we describe a reduced three-phase flow model where (1.1) arises. The mathematical framework and detailed assumptions on the initial data and the fluxes are described in Section 3. In Section 4 we describe the numerical schemes for (1.1). Convergence analyses of (some of) the numerical schemes are carried out in Section 5. Some numerical results are presented in section 6.

2. A triangular three-phase flow model

Simulation of a variety of oil recovery processes involve models of three-phase flow in porous media. Often the three-phases of interest are oil, gas, and water. As a model we consider incompressible, immiscible three-phase flow in a one-dimensional homogeneous and isotropic reservoir (see, e.g., [9]). The oil, water, and gas saturations are given by S_o, S_w, S_g respectively.

The mass conservation equation for phase $l =, w, o, g$ reads

$$\phi(S_l)_t + (U_l)_x = 0, \tag{2.1}$$

where ϕ is the porosity of the medium and U_l is the Darcy velocity or flow rate corresponding to each phase l. By Darcy's law, the flow rate is given by

$$U_l = -k\lambda_l \left(\frac{\partial P_l}{\partial x} - G \right) \qquad l = w, o, g,$$

where k denotes the absolute permeability of the medium, λ_l is the mobility (relative permeability divided by viscosity) of phase l, P_l is the pressure of phase l, and G is the gravity term. We assume that the flow is incompressible i.e., the total flow rate $q = \sum_{l=w,o,g} U_l$ is a constant. For the sake of simplicity, we assume that the capillary pressures between the phases are zero. This assumption is reasonable when the total flow rate is high (the flow is convection dominated).

By adding the mass conservation equations (2.1) and using the above assumptions, we arrive at the following 2×2 system of conservation laws:

$$\begin{cases} (S_g)_t + (F_g(S_g, S_w, S_o))_x = 0, \\ (S_w)_t + (F_w(S_g, S_w, S_o))_x = 0, \\ S_g + S_w + S_o = 1, \end{cases} \tag{2.2}$$

where the fluxes are given by,

$$\begin{aligned} F_g(S_g, S_w, S_o) &= \frac{q\lambda_g}{\lambda_t} + \frac{k}{\lambda_t}\lambda_w\lambda_g(g_w - g_g) + \frac{k}{\lambda_t}\lambda_o\lambda_g(g_o - g_g), \\ F_w(S_g, S_w, S_o) &= \frac{q\lambda_w}{\lambda_t} + \frac{k}{\lambda_t}\lambda_w\lambda_g(g_g - g_w) + \frac{k}{\lambda_t}\lambda_o\lambda_w(g_o - g_w), \end{aligned}$$

where $\lambda_t = \lambda_o + \lambda_g + \lambda_w$ is the total mobility.

It is well known that (2.2) can be a mixed type system, i.e., contain elliptic regions and thus fail to be hyperbolic. It is outside the scope of this paper to discuss this feature here. Instead we refer to [7] (and the references therein) for a review some of the current views that exist today regarding mathematical and numerical theory for mixed type systems.

In many situations the mobility of the gas phase is much larger than that of the other phases. This means that the flux of gas is largely independent of whether the other phase is oil or water. As a consequence

$$F_g(S_g, S_w, S_o) = \tilde{F}(S_g, 1 - S_g) = \hat{F}(S_g).$$

Assuming this relationship, system (2.2) reduces to the following system

$$\begin{aligned} (S_g)_t + (\hat{F}_g(S_g))_x &= 0 \\ (S_w)_t + (F_w(S_g, S_w))_x &= 0. \end{aligned} \tag{2.3}$$

The above equation is a special case of (1.1). We refer to [20] for the model when capillary forces are included.

It is to be emphasized that that although the assumption of independence of the gas phase is not valid for all fractional flow functions, there exists a large class of fractional flow functions for which this assumption appears to be reasonable. In view of the fact that this assumption makes the model simpler and much more tractable, we can use this "reduced" model in several situations. A careful numerical study of this model (2.3) as an approximation to the full three-phase flow model needs to be carried out. An essential ingredient for this program is the development of efficient numerical schemes for (1.1).

We remark that a one dimensional model like the one that we are using is a good starting point for developing numerical schemes for the full three dimensional model where one can use the one dimensional numerical fluxes in directions normal to volume interfaces or along streamlines.

3. Mathematical Framework

We start by stating precise conditions on the flux functions f, g. Fix real numbers s, S, α, β such that $s \leq S$ and $\alpha \leq \beta$. Then we assume that f and g satisfy
[label=**A**.0] $f \in \mathrm{Lip}([\alpha,\beta])$, $g(u_1, s) = g(u_2, s), g(u_1, S) = g(u_2, S)$ for all $u_1, u_2 \in [\alpha, \beta]$, $u \mapsto g(u,v) \in C^1([\alpha,\beta])$ for each $v \in [s, S]$, $v \mapsto g(u,v) \in C^2([s,S])$ for each $u \in [\alpha,\beta]$, $v \mapsto g(u,v)$ is genuinely nonlinear, for each $u \in [\alpha,\beta]$, that is,

$$g_{vv}(u,v) \neq 0 \text{ for a.e. } v \in [s,S], \text{ for each } u \in [\alpha,\beta].$$

Conditions 1, 4, and 5) are usual smoothness assumptions on the fluxes. Condition 2 is a sufficient condition to obtain L^∞ bounds on v and can be relaxed in several ways. We have chosen to use this form since it holds for the three-phase flow model (2.3). Condition 5 is required to achieve strong convergence of the approximate solutions with the compensated compactness method. All the assumptions are quite general. In particular we require no further assumptions on the shape of the flux functions.

Regarding the initial data we assume the following conditions:
[label=**A**.0,resume] $u_0 \in L^\infty(\mathbb{R})$ with $\alpha \leq u_0 \leq \beta$ for a.e $x \in \mathbb{R}$, $u_0 \in BV(\mathbb{R})$, $s \leq v_0(x) \leq S$ for all $x \in \mathbb{R}$.

A weak solution to (1.2) is a pair of function $U = (u, v)$ such that

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} (U\varphi_t + F(U)\varphi_x)\,dxdt + \int_{\mathbb{R}} U(x,0)\varphi(x,0)\,dx = 0, \quad \forall \varphi \in C_c^\infty(\mathbb{R}\times\mathbb{R}^+).$$

It is well known that the weak solutions of conservation laws are not unique, and therefore the solution concept has to be supplemented with additional admissibility criteria, so called entropy conditions. We refer to [19] for an introduction to entropy conditions and the general theory of conservation laws.

As mentioned in the introduction, one of our aims is to show that weak solutions of (1.1) exist by showing that proposed numerical schemes converge. Our chief tool is the compensated compactness method [30, 31, 14, 10, 11, 25]. As mentioned earlier, We are going to exploit the close local relationship between the system (1.1) and conservation laws with discontinuous coefficients (1.3). For conservation laws with discontinuous coefficients, the compensated compactness method was used as a convergence tool for the Lax-Friedrichs scheme in [23]. We will follow the approach used in [23]. This approach is based on the use of scalar entropy-entropy flux pairs.

A pair of functions $(\eta(u,v), Q(u,v))$, where u is considered as a fixed parameter in $[\alpha,\beta]$, is called a scalar entropy/entropy flux pair for the equation $v_t + g(u,v)_x = 0$ if $v \mapsto \eta(u,v)$ is C^2 regular and

$$Q_v(u,v) = \eta_v(u,v)g_v(u,v), \qquad \text{for all } v \in [s,S].$$

Suitably modified for our purposes, the compensated compactness lemma used in [23] reads as follows:

Lemma 3.1. *Let u be the unique entropy solution u of the single conservation law*

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x) \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}),$$

and let $\{v^\varepsilon\}_{\varepsilon>0}$ be a sequence of functions on $\mathbb{R}\times\mathbb{R}^+$ such that for all $\varepsilon > 0$

(i) $s \le v^\varepsilon \le S$, *and*
(ii) *the sequences*

$$\{\eta_1(v^\varepsilon)_t + Q_1(u,v^\varepsilon)_x\}_{\varepsilon>0}, \qquad \{\eta_2(u,v^\varepsilon)_t + Q_2(u,v^\varepsilon)_x\}_{\varepsilon>0}$$

belong to a compact subset of $W^{-1,2}_{\text{loc}}(\mathbb{R}\times\mathbb{R}^+)$, where for all u, v

$$\eta_1(v) = v - c, \qquad Q_1(u,v) = g(u,v) - g(u,c)$$
$$\eta_2(u,v) = g(u,v) - g(u,c), \qquad Q_2(u,v) = \int_c^v (g_v(u,\xi))^2\, d\xi,$$

for all $c \in \mathbb{R}$.

Then there exists a function subsequence of $\{v^\varepsilon\}_{\varepsilon>0}$ that converges in $L^p_{\text{loc}}(\mathbb{R}\times\mathbb{R}^+)$ $\forall p < \infty$ and a.e. to a bounded function v.

We also need the following technical result (see, e.g., [10, 25]):

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let q, p, r be such that $1 < q \le p < r < \infty$, then*

$$\begin{aligned}&(\textit{compact set of } W^{-1,q}(\Omega)) \cap (\textit{bounded set of } W^{-1,r}(\Omega))\\ &\quad\subset (\textit{compact set of } W^{-1,p}(\Omega)).\end{aligned}$$

4. Description of numerical schemes

In this section we describe the Semi-Godunov type finite volume schemes for (1.1). We start however with the unpacking of some needed notation.

Let Δt, and Δx be the time step and mesh size respectively. For simplicity we use a uniform mesh in both space and time although variable step sizes can be handled in the same manner. We assume that the time step and the mesh size satisfy the following CFL condition:

$$\lambda M \le \frac{1}{2}, \quad \lambda = \frac{\Delta t}{\Delta x}, \quad M = \max\left\{\max_{u\in[\alpha,\beta]} |f_u|, \max_{u\in[\alpha,\beta], v\in[s,S]} |g_v|\right\},$$

Let $t^n = n\Delta t$, and $x_j = j\Delta x$ for integers $n = 0, 1, 2, \dots$ and $j = \dots, -3/2, -1, -1/2, 0, 1/2, 1, 3/2, 2, \dots$. Let I_j and I^n denote the intervals

$$I_j = [x_{j-1/2}, x_{j+1/2}), \qquad I^n = [t^n, t^{n+1}).$$

Set

$$\chi_j^n(x,t) = \chi_{I_j}(x)\chi_{I^n}(t),$$

where χ_Ω denotes the characteristic function of a set Ω.

4.1. **Staggered Semi-Godunov (SSG) scheme.** For this scheme, we are going to stagger the discretization of the two unknowns u and v.

Define $U_j^0 = (u_{j+1/2}^0, v_j^0)$ as

$$u_{j+1/2}^0 = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u_0(x)\,dx, \quad v_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} v_0(x)\,dx.$$

Given given $U_j^n = (u_{j+1/2}^n, v_j^n)$, we shall determine $U_j^{n+1} = (u_{j+1/2}^{n+1}, v_j^{n+1})$. We start by updating $u_{j+1/2}^n$. This is straightforward as the evolution of u does not depend on the unknown v. We use the standard Godunov scheme, see [18]:

$$u_{j+1/2}^{n+1} = u_{j+1/2}^n - \lambda(f^{\mathrm{G}}(u_{j+1/2}^n, u_{j+3/2}^n) - f^{\mathrm{G}}(u_{j-1/2}^n, u_{j+1/2}^n)), \tag{4.1}$$

where f^{G} is the standard scalar Godunov numerical flux:

$$f^{\mathrm{G}}(a,b) = \begin{cases} \min_{\theta\in[a,b]} f(\theta), & \text{if } a \le b, \\ \max_{\theta\in[b,a]} f(\theta), & \text{otherwise.} \end{cases} \tag{4.2}$$

We define an approximate solution $u^{\Delta x}$ on $\mathbb{R}\times\mathbb{R}^+$ by

$$u^{\Delta x}(x,t) = \sum_{n,j+1/2} \chi_{j+1/2}^n(x,t) u_{j+1/2}^n. \tag{4.3}$$

Next, we are going to use the function u^{Δ} to define a Riemann solver for v. At any time level t^n, we will substitute $u^{\Delta x}(x,t^n)$ instead of u in the conservation law for v. To this end, for $(x,t) \in I_{j+1/2}\times I^n$ we define $v^{\Delta x}$ to be the solution of the following conservation law with the discontinuous coefficient $u^{\Delta x}$:

$$v_t^{\Delta x} + g\left(u_{j+1/2}^n, v^{\Delta x}\right)_x = 0, \qquad v^{\Delta x}(x,t^n) = \begin{cases} v_j^n, & x < x_{j+1/2} \\ v_{j+1}^n, & x > x_{j+1/2}. \end{cases} \tag{4.4}$$

Since waves from different Riemann problems at t^n do not interact by the CFL-condition, we can use a Godunov scheme to determine v_j^{n+1}. We evolve the solution of the Riemann problem until $t = t^{n+1}$. At time $t = t^{n+1}$, we define v_j^{n+1} by averaging over grid cells I_j:

$$v_j^{n+1} = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} v^{\Delta x}(x, t^{n+1}-)\,dx. \tag{4.5}$$

This gives the formula,

$$v_j^{n+1} = v_j^n - \lambda\left(g^{\mathrm{G}}\left(u_{j+1/2}^n, v_j^n, v_{j+1}^n\right) - g^{\mathrm{G}}\left(u_{j-1/2}^n, v_{j-1}^n, v_j^n\right)\right),$$

where $g^{\mathrm{G}}(u,a,b)$ is the standard Godunov flux (4.2) corresponding to the flux function $v \mapsto g(u,v)$. Collecting the updates for u and v, we get the following finite volume scheme:

$$\begin{aligned} u_{j+1/2}^{n+1} &= u_{j+1/2}^n - \lambda\left(f^{\mathrm{G}}\left(u_{j+1/2}^n, u_{j+3/2}^n\right) - f^{\mathrm{G}}\left(w_{j-1/2}^n, u_{j+1/2}^n\right)\right), \\ v_j^{n+1} &= v_j^n - \lambda\left(g^{\mathrm{G}}\left(u_{j+1/2}^n, v_j^n, v_{j+1}^n\right) - g^{\mathrm{G}}\left(u_{j-1/2}^n, v_{j-1}^n, v_j^n\right)\right) \end{aligned} \tag{4.6}$$

We coin (4.6) the Staggered Semi-Godunov (SSG) scheme due to the staggered discretizations of the coefficient and the unknown. This scheme is similar in spirit to the schemes in [32, 33, 21] for conservation laws with discontinuous coefficients.

For the purpose of analysis, we define an approximate solution $U^{\Delta x} = (u^{\Delta x}, v^{\Delta x})$ on $\mathbb{R} \times \mathbb{R}^+$ by (4.3) and (4.4), (4.5).

4.2. **Aligned Semi-Godunov (ASG) scheme.** Unlike the SSG-scheme (4.6), for the ASG-scheme we align the discretizations of both the unknowns. As a result the ASG-scheme becomes more complicated to implement and analyze than the SSG-scheme, but we have found that it gives somewhat better results.

Define $U_j^0 = (u_j^0, v_j^0)$ by

$$u_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,dx, \quad v_j^0 = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} v_0(x)\,dx.$$

As in the SSG-scheme, we first update for u using the standard Godunov scheme:

$$u_j^{n+1} = u_j^n - \lambda(f^{\mathrm{G}}(u_j^n, u_{j+1}^n) - f^{\mathrm{G}}(u_{j-1}^n, u_j^n)),$$

where the numerical flux f^{G} is defined in (4.2). Equipped with (4.1), we define an approximate solution $u^{\Delta x}$ on $\mathbb{R} \times \mathbb{R}^+$ by

$$u^{\Delta x}(x,t) = \sum_{n,j} \chi_j^n(x,t) u_j^n. \tag{4.7}$$

As in the SSG-scheme, we use $u^{\Delta x}(x,t)$ and define the evolution of v by the solution of the conservation law (4.4) with $u^{\Delta x}$ defined by (4.7). Hence, at the time level t^n, we solve the following local Riemann problem at each interface $x_{j+1/2}$:

$$\begin{aligned} &v_t^{\Delta x} + g\left(u_j^n, v^{\Delta x}\right)_x = 0, \quad \text{if } x < x_{j+1/2}, \\ &v_t^{\Delta x} + g\left(u_{j+1}^n, v^{\Delta x}\right)_x = 0, \quad \text{if } x > x_{j+1/2}, \\ &v^{\Delta x}(x,t^n) = \begin{cases} v_j^n, & x < x_{j+1/2}, \\ v_{j+1}^n, & x > x_{j+1/2}. \end{cases} \end{aligned} \tag{4.8}$$

As we have aligned the discretization of both the unknowns, we end up with a Riemann problem corresponding to a single conservation law with a discontinuous coefficient. The Riemann problem (4.8) can be solved (see, e.g., [13, 16, 17]), and an explicit formula for the (Godunov type) numerical flux has been obtained in [3, 4, 27] for a large class of flux functions.

We define v_j^{n+1} by averaging, cf. (4.5), and obtain

$$v_j^{n+1} = v_j^n - \lambda\left(g_{\mathrm{A}}^{\mathrm{R}}\left((u_j^n, u_{j+1}^n), (v_j^n, v_{j+1}^n)\right) - g_{\mathrm{A}}^{\mathrm{R}}\left((u_{j-1}^n, u_j^n), (v_{j-1}^n, v_j^n)\right)\right), \tag{4.9}$$

where $g_{\mathrm{A}}^{\mathrm{R}}(k,l)(a,b)$ is the Godunov numerical flux corresponding to the Riemann problem with left flux function $g(k,.)$, right flux function $g(l,.)$ and Riemann data a (left) and b (right). As mentioned above, explicit formulas for $g_{\mathrm{A}}^{\mathrm{R}}$ can be given in many cases. For example, if $v \mapsto g(u,v)$ has at most one minimum and no maxima for every u, then the explicit formula is

$$g_{\mathrm{A}}^{\mathrm{R}}((k,l),(a,b)) = \max\left\{g(k, \max(a,\theta_k)), g(l, \min(\theta_l, b)\right\},$$

where θ_k, θ_l are the minimum points of $g(k,\cdot)$ and $g(l,\cdot)$ respectively. Explicit formulas in other (non-convex) cases are given in [4],[27].

Finally, we define an approximate solution $U^{\Delta x} = (u^{\Delta x}, v^{\Delta x})$ on $\mathbb{R} \times \mathbb{R}^+$ via formulas (4.7) and (4.8).

5. Convergence Analysis

In this section we prove that the approximate solutions generated by the SSG-scheme (4.6) and the ASG-scheme (4.9) converge to weak solutions of (1.1). We start by analyzing the SSG-scheme (4.6) and then detail the differences for the ASG-scheme . In what follows and without loss of generality we will assume that the approximate solutions (and their initial data) have compact support, that is, there exist constants $X, T > 0$ independent of Δx such that

$$\operatorname{supp}(u^{\Delta x}), \operatorname{supp}(v^{\Delta x}) \subset [-X, X] \times [0, T]. \tag{5.1}$$

5.1. **The SSG-scheme.** We will carry out the above steps for the SSG-scheme (4.6). Using standard theory for scalar conservation laws, it is straightforward to establish the following facts:

Lemma 5.1. *Let* $u^n_{j+1/2}$, $u^{\Delta x}$ *be defined by* (4.1) *and* (4.3). *Then*

$$\alpha \le u^n_{j+1/2} \le \beta, \quad \text{for } j \in \mathbb{Z},\ n = 0, 1, 2, \dots,$$
$$\sum_j \left| u^{n+1}_{j+1/2} - u^n_{j+1/2} \right| \le \sum_j \left| u^n_{j+1/2} - u^{n-1}_{j+1/2} \right|, \quad n \ge 1,$$
$$\sum_j \left| u^n_{j+1/2} - u^n_{j-1/2} \right| \le \sum_j \left| u^{n-1}_{j+1/2} - u^{n-1}_{j-1/2} \right|, \quad n \ge 1.$$

Furthermore, the sequence $\left\{u^{\Delta x}\right\}_{\Delta x > 0}$ *converges to the unique entropy solution* $u \in L^\infty(\mathbb{R} \times \mathbb{R}^+) \cap BV(\mathbb{R} \times \mathbb{R}^+)$ *of the first equation in* (1.1). *The convergence takes place in* $L^p_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+))$ $\forall p < \infty$ *and a.e. in* $\mathbb{R} \times \mathbb{R}^+$.

The boundedness of $v^{\Delta x}$ is a standard result, and follows from Assumption 2 and the monotonicity properties of g^{G}, see [21]. We state the result in a lemma.

Lemma 5.2. *Let* v^n_j *be generated by the SSG-scheme (cf. Subsection 4.1), and suppose* $s \le v^0_j \le S$ *for all* j. *Then*

$$s \le v^n_j \le S \text{ for all } j \text{ and } n \ge 0.$$

We remark that our particular L^∞ bound holds under Assumption 2, but one can show L^∞ bounds under more general conditions, see [33].

The next step in the convergence analysis is to provide an entropy dissipation estimate for the SSG-scheme (4.6). A entropy estimate for the Lax-Friedrichs scheme for (1.3) was proved in [23]. On the other hand, the approach of [23] is based on cell entropy inequalities and does not extend to three point schemes like the Godunov scheme based on Riemann solvers, i.e., the ASG-scheme . Therefore we will employ the original framework of DiPerna [14], of viewing finite volume schemes as layered integral average methods. For other applications based on this approach, see [15, 11] and the references therein.

Here and in what follows, we make use of the notation

$$[\![A]\!](x,t) = \lim_{h \to 0} A(x+h,t) - A(x-h,t),$$

for any quantity $A = A(x,t)$.

Let (η, Q) be a scalar entropy/entropy flux pair. Pick a test function φ having compact support in $\mathbb{R} \times [0, T]$ with $T = N\Delta t$ for some integer N. By the

Gauss-Green formula and the local structure of the approximations $v^{\Delta x}$ ($v^{\Delta x}$ is the solution of a Riemann problem at each interface $x_{j+1/2}$),

$$\begin{aligned}&\iint_{\mathbb{R}\times\mathbb{R}^+} \eta\left(u^{\Delta x}, v^{\Delta x}\right)\varphi_t + Q\left(u^{\Delta x}, v^{\Delta x}\right)\varphi_x\,dxdt \\ &\qquad = I_1(\varphi) + I_2(\varphi) + I_3(\varphi) + I_4(\varphi),\end{aligned} \tag{5.2}$$

where

$$I_1(\varphi) = \int_{\mathbb{R}} \eta\left(u^{\Delta x}(x,t), v^{\Delta x}(x,t)\right)\varphi(x,t)\,\Big|_{t=0}^{t=T}\,dx,$$

$$\begin{aligned}I_2(\varphi) = \sum_n \int_{\mathbb{R}} \Big[&\eta\left(u^{\Delta x}(x,t^n-), v^{\Delta x}(x,t^n-)\right) \\ &- \eta\left(u^{\Delta x}(x,t^n+), v^{\Delta x}(x,t^n+)\right)\Big]\varphi(x,t^n)\,dx,\end{aligned} \tag{5.3}$$

$$I_3(\varphi) = \sum_{n,j}\sum_{\sigma}\int_{t^n}^{t^{n+1}} \Big[(\sigma[\![S]\!] - [\![Q]\!])\varphi\Big](x_{j+1/2}+\sigma t, t)\,dt, \tag{5.4}$$

$$I_4(\varphi) = \sum_{n,j}\int_{t^n}^{t^{n+1}} \left[Q\left(u^n_{j-1/2}, v^n_j\right) - Q\left(u^n_{j+1/2}, v^n_j\right)\right]\varphi(x_j,t)\,dt, \tag{5.5}$$

where the summation over σ extends to all shocks with speed σ in the solution of the Riemann problem at the interface $x_{j+1/2}$.

Lemma 5.3. *Let v^n_j, $v^{\Delta x}$ be generated by the SSG-scheme (cf. Subsection 4.1). We have*

$$\sum_{j,n}\int_{x_{j-1/2}}^{x_{j+1/2}} \left|v^n_j - v^{\Delta x}(x,t^n-)\right|^2 dx \le C, \tag{5.6}$$

where $C = C(X,T)$ is constant independent of Δx and X, T are defined in (5.1). *Additionally,*

$$\sum_{n,j}\sum_{\sigma}\int_{t^n}^{t^{n+1}} \Big[\sigma[\![S]\!] - [\![Q]\!]\Big](x_{j+1/2}+\sigma t, t)\,dt \le C. \tag{5.7}$$

Proof. In this proof we use the convex entropy/entropy flux pair (η, Q) defined by $\eta(u,v) = S(v) = \frac{1}{2}v^2$, $Q_v(u,v) = vg_v(u,v)$.

Since the approximate solutions (and their initial data) have compact support, we can take $\varphi \equiv 1$ in (5.2). Hence

$$I_2(1) + I_3(1) \le \int_{\mathbb{R}} \frac{(v^{\Delta x}(x,0))^2}{2}\,dx - I_4(1) \le C - I_4(1), \tag{5.8}$$

for a constant C independent of Δx (but dependent on X, T).

Let us estimate I_2. Here and elsewhere, we write $v^n_\pm$ for $v^{\Delta x}(\cdot, t^n\pm)$. Equipped with this notation, we find

$$\begin{aligned}I_2(\varphi) &= \frac{1}{2}\sum_{n,j}\int_{x_{j-1/2}}^{x_{j+1/2}} \left(v^n_-\right)^2 - \left(v^n_j\right)^2\,dx \\ &= \frac{1}{2}\sum_{n,j}\int_{x_{j-1/2}}^{x_{j+1/2}} \left(v^n_- - v^n_j\right)^2\,dx - \sum_{n,j}\int_{x_{j-1/2}}^{x_{j+1/2}} v^n_j\left(v^n_- - v^n_j\right)\,dx\end{aligned}$$

$$= \frac{1}{2} \sum_{n,j} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(v_-^n - v_j^n\right)^2 \, dx,$$

where the last equality is a consequence of (4.5).

Regarding the term $I_3(\varphi)$, we have that $v^{\Delta x}$ is locally the exact solution of a scalar Riemann problem with a "constant coefficient". Thus

$$\sigma [\![\eta \left(v^{\Delta x}, u^{\Delta x}\right)]\!] - [\![Q \left(v^{\Delta x}, u^{\Delta x}\right)]\!] \geq 0,$$

and consequently $I_3(\varphi) \geq 0$.

It remains to estimate I_4, which is done using Lemma 5.1 as follows:

$$|I_4(\varphi)| \leq \Delta t \sum_{n,j} \left| Q\left(u_{j-1/2}^n, v_j^n\right) - Q\left(u_{j+1/2}^n, v_j^n\right) \right| \leq C \left|u_0\right|_{BV} T.$$

Collecting the above bounds the estimates (5.6) and (5.7) follow from (5.8). □

Lemma 5.3 gives a bound on the variation of the approximate solution across the discrete time levels. This estimate can be converted to an estimate on the spatial variation as in [15] (we omit the details).

Lemma 5.4. *Let v_j^n be defined by the SSG-scheme (cf. Subsection 4.1). There exists a constant $C = C(X,T)$ independent of Δx such that*

$$\Delta x \sum_{n,j} \left| v_{j+1}^n - v_j^n \right|^2 \leq C.$$

We carry on by proving the $W_{\mathrm{loc}}^{-1,2}$ compactness required by Lemma 3.1.

Lemma 5.5. *Let $v^{\Delta x}$ be generated by the SSG-scheme (cf. Subsection 4.1). Let u be the unique entropy solution to the first equation in* (1.1). *Equipped with any scalar entropy/entropy flux pair (η, Q), form the distribution*

$$\mu^{\Delta x} := \eta(u, v^{\Delta x})_t + Q(u, v^{\Delta x})_x.$$

Then $\left\{\mu^{\Delta x}\right\}_{\Delta x>0}$ belongs to a compact subset of $W_{\mathrm{loc}}^{-1,2}(\mathbb{R} \times \mathbb{R}^+)$.

Proof. In what follows we fix a bounded open set $\Omega \subset \mathbb{R} \times \mathbb{R}^+$, which we can assume is of the form $(-X, X) \times (0, T)$ with $X > 0$ and $T = N\Delta t$ for some integer N. We break apart $\mu^{\Delta x}$ as $\mu_1^{\Delta x} + \mu_2^{\Delta x}$, where

$$\mu_1^{\Delta x} := \left[\eta\left(u, v^{\Delta x}\right) - \eta\left(u^{\Delta x}, v^{\Delta x}\right)\right]_t + \left[Q\left(u, v^{\Delta x}\right) - Q\left(u^{\Delta x}, v^{\Delta x}\right)\right]_x,$$
$$\mu_2^{\Delta x} := \eta\left(u^{\Delta x}, v^{\Delta x}\right)_t + Q\left(u^{\Delta x}, v^{\Delta x}\right)_x.$$

In view of Lemma 5.1, fixing any $q_1 \in (1, 2]$, the sequence $\left\{\mu_1^{\Delta x}\right\}$ is clearly compact in $W^{-1,q_1}(\Omega)$. It remains to estimate $\mu_2^{\Delta x}$.

Let $\phi \in C_c(\Omega)$. Proceeding as in the proof of Lemma 5.3 we find

$$\left| \iint_{\Omega} \mu_2^{\Delta x} \varphi \, dx \right| \leq |I_2(\varphi)| + |I_3(\varphi)| + |I_4(\varphi)|,$$

where I_2, I_3, I_4 are defined in (5.3), (5.4), (5.5) respectively.

Thanks to (5.7) and the spatial BV regularity part of Lemma 5.1,

$$|I_3(\varphi)|, |I_4(\varphi)| \leq C \left\|\varphi\right\|_{L^\infty(\Omega)}, \quad \text{for some constant } C \text{ independent of } \Delta x,$$

and hence I_3, I_4 are bounded (independently of Δx) in the space $\mathcal{M}(\Omega)$ of bounded Radon measures on Ω. Recalling that $\mathcal{M}(\Omega)$ is compactly embedded in $W^{-1,p}(\Omega)$

for any $p \in [1,2)$, we conclude that by a fixing any $q_2 \in (1,2)$ the sequences $\{I_2\}_{\Delta x>0}$, $\{I_4\}_{\Delta x>0}$ are compact in $W^{-1,q_2}(\Omega)$.

It remains to estimate the term I_2, which we decompose as

$$I_2(\varphi) = I_{2,1}(\varphi) + I_{2,2}(\varphi) + I_{2,3}(\varphi) + I_{2,4}(\varphi),$$

where (we still utilize v_-^n as short-hand notation for $v^{\Delta x}(\cdot, t^n-)$)

$$I_{2,1}(\varphi) = \sum_{n,j} \int_{x_{j-1/2}}^{x_j} \left(\eta\left(u_{j-1/2}^{n-1}, v_-^n\right) - \eta\left(u_{j-1/2}^n, v_-^n\right)\right)\varphi(x,t^n)\,dx$$

$$I_{2,2}(\varphi) = \sum_{n,j} \int_{x_j}^{x_{j+1/2}} \left(\eta\left(u_{j+1/2}^{n-1}, v_-^n\right) - \eta\left(u_{j+1/2}^n, v_-^n\right)\right)\varphi(x,t^n)\,dx$$

$$I_{2,3}(\varphi) = \sum_{n,j} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\eta\left(u_{j+1/2}^{n}, v_-^n\right) - \eta\left(u_{j+1/2}^n, v_j^n\right)\right)\varphi(x,t^n)\,dx$$

$$I_{2,4}(\varphi) = \sum_{n,j} \int_{x_{j-1/2}}^{x_j} \Bigg[\left(\eta\left(u_{j+1/2}^{n}, v_j^n\right) - \eta\left(u_{j-1/2}^n, v_j^n\right)\right) - \left(\eta\left(u_{j+1/2}^{n}, v_-^n\right) - \eta\left(u_{j-1/2}^n, v_-^n\right)\right)\Bigg]\varphi(x,t^n)\,dx,$$

By Lemmas 5.1 and 5.2 and

$$|I_{2,1}(\varphi)|, |I_{2,2}(\varphi)|, |I_{2,4}(\varphi)| \le C\,\|\varphi\|_{L^\infty(\Omega)},$$

and accordingly $\{I_{2,1}\}_{\Delta x>0}$, $\{I_{2,2}\}_{\Delta x>0}$, $\{I_{2,4}\}_{\Delta x>0}$ are compact sequences in $W^{-1,q_2}(\Omega)$, where $q_2 \in [1,2)$ has been fixed before.

To continue we write

$$I_{2,3}(\varphi) = I_{2,3,1}(\varphi) + I_{2,3,2}(\varphi),$$

where

$$I_{2,3,1}(\varphi) = \sum_{n,j} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\eta\left(u_{j+1/2}^{n}, v_-^n\right) - \eta\left(u_{j+1/2}^n, v_j^n\right)\right)\varphi_j^n\,dx, \quad \varphi_j^n = \varphi(x_j,t^n),$$

$$I_{2,3,2}(\varphi) = \sum_{n,j} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\eta\left(u_{j+1/2}^{n}, v_-^n\right) - \eta\left(u_{j+1/2}^n, v_j^n\right)\right)\left(\varphi(x,t^n) - \varphi_j^n\right)\,dx.$$

Next

$$\eta\left(u_{j+1/2}^n, v_-^n\right) - \eta\left(u_{j+1/2}^n, v_j^n\right) = \eta_v\left(u_{j+1/2}^n, v_j^n\right)\left(v_-^n - v_j^n\right) + \frac{1}{2}\eta_{vv}\left(u_{j+1/2}^n, \theta_j^n\right)\left(v_-^n - v_j^n\right)^2,$$

for some intermediate value $\theta_j^n(x)$. Taking into account the definition of v_j^n, see (4.5), and Lemma 5.3 we obtain therefore

$$|I_{2,3,1}(\varphi)| = \left|\sum_{n,j} \varphi_j^n \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{1}{2}\eta_{vv}\left(u_{j+1/2}^n, \theta_j^n\right)\left(v_-^n - v_j^n\right)^2\,dx\right| \le C\,\|\varphi\|_{L^\infty(\Omega)},$$

where C is independent of Δx, and so $\{I_{2,3,1}\}_{\Delta x>0}$ is compact in $W^{-1,q_2}(\Omega)$.

To proceed we assume additionally that φ is Hölder continuous with some exponent $\alpha \in (1/2, 1)$. Applying Hölder's inequality and Lemma 5.3 yield

$$|I_{2,3,2}(\varphi)| \leq \left\{ \sum_{n,j} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\eta \left(u^n_{j+1/2}, v^n_- \right) - \eta \left(u^n_{j+1/2}, v^n_j \right) \right)^2 dx \right\}^{\frac{1}{2}}$$

$$\times \left\{ \sum_{n,j} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\varphi(x, t^n) - \varphi^n_j \right)^2 dx \right\}^{\frac{1}{2}}$$

$$\leq C \left\| \phi \right\|_{C^{0,\alpha}(\Omega)} (\Delta x)^{\alpha - \frac{1}{2}} \leq \tilde{C} \left\| \phi \right\|_{W^{1,p}(\Omega)} (\Delta x)^{\alpha - \frac{1}{2}},$$

for some constants $C, \tilde{C}$ that are independent of Δx. To derive the last inequality we used that $W^{1,p}(\Omega) \subset C^{0,\alpha}(\Omega)$ for $p = 2/(1-\alpha)$, where our α lies in $(1, 1/2)$. Hence $\|I_{2,3,2}\|_{W^{-1,q_3}}$, with $q_3 = 2/(1+\alpha)$ and $\alpha \in (1/2, 1)$, tends to zero as $\Delta x \to 0$.

Let us now summarize our findings. We have proved that $\left\{\mu^{\Delta x}\right\}_{\Delta x > 0}$ is compact in $W^{-1,q}(\Omega)$ for $q = \min(q_1, q_2, q_3) < 2/(1+\alpha) < 2$. Additonally, by the L^∞ bounds on $u, u^{\Delta x}, v^{\Delta x}$ the sequence $\left\{\mu^{\Delta x}\right\}_{\Delta x > 0}$ is bounded in $W^{-1,r}(\Omega)$ for any $r \in (2, \infty]$. That being the case, an application of Lemma 3.2 concludes the proof. □

In view of the compensated compactness theory (Lemma 3.1), the foregoing lemma allows us to prove convergence of the SSG-scheme.

Theorem 5.1. *Let $u^{\Delta x}, v^{\Delta x}$ be generated by the SSG-scheme (cf. Subsection 4.1). Then there exist limit functions $u \in L^\infty(\mathbb{R}\times\mathbb{R}^+) \cap BV(\mathbb{R}\times\mathbb{R}^+)$ and $v \in L^\infty(\mathbb{R}\times\mathbb{R}^+)$ such that, along a subsequence as $\Delta x \to 0$*

$$u^{\Delta x} \to u,\ v^{\Delta x} \to v \ \text{in } L^p_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)\ \forall p < \infty \ \text{and a.e. in } \mathbb{R} \times \mathbb{R}^+,$$

The limit pair (u, v) constitutes a weak solution of (1.1).

Proof. We refer to Lemma 5.1 for the convergence of $u^{\Delta x}$. The L^p_{loc} convergence of $v^{\Delta x}$ is a direct consequence of Lemmas 5.5 and 3.1.

It remains to show that the limit pair (u, v) is a weak solution. To this end, let $\tilde{v}^{\Delta x}$ be the piecewise constant function defined as

$$\tilde{v}^{\Delta x}(x, t) = \sum_{n,j} v^n_j \chi^n_j(x, t).$$

We claim that

$$\lim_{\Delta x \to 0} \left\| \tilde{v}^{\Delta x} - v \right\|_{L^2_{\text{loc}}(\mathbb{R}\times\mathbb{R}^+)} = 0. \tag{5.9}$$

For $(x, t) \in [x_{j-1/2}, x_j] \times I^n$ we have

$$\left| \tilde{v}^{\Delta x} - v^{\Delta x} \right| = \left| v^n_j - v^{\Delta x} \right| \leq \left| v^n_j - v^n_{j-1} \right|,$$

since $v^{\Delta x}$ is the solution of a scalar Riemann problem with left state v^n_{j-1} and right state v^n_j. Therefore

$$\left\| \tilde{v}^{\Delta x} - v^{\Delta x} \right\|^2_{L^2 \text{loc}(\mathbb{R}\times\mathbb{R}^+)} = \sum_{n,j} \iint_{I^n_{j-1/2}} \left(\tilde{v}^{\Delta x} - v^{\Delta x} \right)^2 dxdt$$

$$\leq 2\Delta x \Delta t \sum_{n,j} \left(v^n_j - v^n_{j-1} \right)^2 \leq C\Delta t,$$

by Lemma 5.4. Hence (5.9) follows.

Pick a test fuction φ having compact support in $\mathbb{R}\times[0,T)$ with $N\Delta t = T$ for some integer N. Multiplying the scheme (4.6) by $\varphi_j^n = \varphi(x_j,t^n)$ and doing partial summations, we get

$$0 = \underbrace{\Delta x \sum_j \left(v^N \varphi_j^{N-1} - v_j^0\varphi_j^0\right)}_{I_1} - \underbrace{\Delta x\Delta t \sum_{n,j} v_j^n \frac{\varphi_j^n - \varphi_j^{n-1}}{\Delta t}}_{I_2}$$
$$- \underbrace{\Delta x\Delta t \sum_{n,j} g^{\mathrm{G}}\left(u_{j-1/2}^n, v_{j-1}^n, v_j^n\right)\frac{\varphi_j^n - \varphi_{j-1}^n}{\Delta x}}_{I_3}.$$

It is straightforward to show that

$$\lim_{\Delta x\to 0} I_1 = -\int_{\mathbb{R}} v_0\varphi(x,0)\,dx, \qquad \lim_{\Delta x\to 0} I_2 = \iint_{\mathbb{R}\times\mathbb{R}^+} v\varphi_t\,dxdt.$$

Next we study the term I_3, which we rewrite as follows:

$$\begin{aligned}
I_3 &= \frac{\Delta x\Delta t}{2}\sum_{n,j} g^{\mathrm{G}}\left(u_{j-1/2}^n, v_{j-1}^n, v_j^n\right)\frac{\varphi_j^n - \varphi_{j-1}^n}{\Delta x} \\
&\quad + \frac{\Delta x\Delta t}{2}\sum_{n,j} g^{\mathrm{G}}\left(u_{j+1/2}^n, v_j^n, v_{j+1}^n\right)\frac{\varphi_{j+1}^n - \varphi_j^n}{\Delta x} \\
&= \frac{\Delta x\Delta t}{2}\sum_{n,j} g\left(u_{j-1/2}^n, v_j^n\right)\frac{\varphi_j^n - \varphi_{j-1}^n}{\Delta x} \\
&\quad + \frac{\Delta x\Delta t}{2}\sum_{n,j} g\left(u_{j+1/2}^n, v_j^n\right)\frac{\varphi_{j+1}^n - \varphi_j^n}{\Delta x} + E_1^{\Delta x} + E_2^{\Delta x} \\
&= \iint_{\mathbb{R}\times\mathbb{R}^+} g\left(u^{\Delta x}, \tilde{v}^{\Delta x}\right)\varphi_x\,dxdt + \mathcal{O}(\Delta x) + E_1^{\Delta x} + E_2^{\Delta x},
\end{aligned}$$

where

$$E_1^{\Delta x} = \frac{\Delta x\Delta t}{2}\sum_{n,j}\left[g^{\mathrm{G}}\left(u_{j-1/2}^n, v_{j-1}^n, v_j^n\right) - g\left(u_{j-1/2}^n, v_j^n\right)\right]\frac{\varphi_j^n - \varphi_{j-1}^n}{\Delta x},$$
$$E_2^{\Delta x} = \frac{\Delta x\Delta t}{2}\sum_{n,j}\left[g^{\mathrm{G}}\left(u_{j+1/2}^n, v_j^n, v_{j+1}^n\right) - g\left(u_{j+1/2}^n, v_j^n\right)\right]\frac{\varphi_{j+1}^n - \varphi_j^n}{\Delta x}$$

By consistency/Lipschitz properties of the Godunov flux, Hölder's inequality, and Lemma 5.4,

$$\left|E_1^{\Delta x}\right|, \left|E_2^{\Delta x}\right| \le C_\varphi\sqrt{\Delta x},$$

where C_φ depends on φ but not Δx. Hence

$$\lim_{\Delta x\to 0} I_3 = \iint_{\mathbb{R}\times\mathbb{R}^+} g(u,v)\varphi_x\,dxdt.$$

Therefore v is a weak solution of (1.1). □

5.2. **The ASG-scheme.** We begin by pointing out that Lemmas 5.1 and 5.2 for the SSG-scheme continue to hold for the ASG-scheme (with the notation porperly adjusted).

To carry out the convergence analysis for the ASG-scheme we make a digression and present some general results on entropy estimates for conservation laws with discontinuous coefficients. To this end, consider the Riemann problem [13, 16, 17]

$$\begin{cases} v_t + g\left(u_l, v\right)_x &= 0, \quad v(x,0) = v_l \quad x < 0, \\ v_t + g\left(u_r, v\right)_x &= 0, \quad v(x,0) = v_r \quad x > 0, \end{cases}$$

where $u_{l,r}$ and $v_{l,r}$ are constants. The Rankine-Hugoniot condition states that the values

$$v'_{l,r} = \lim_{x\to 0-,+} v(x,t),$$

are such that

$$g_0 := g\left(u_l, v'_l\right) = g\left(u_r, v'_r\right).$$

In general, this does not determine $v'_{l,r}$ uniquely, and we need additional conditions. We choose to use the so called *minimal jump entropy condition* which states that among the possible choices, we select v'_l and v'_r such that $|v'_l - v'_r|$ is minimal. This implies the following

$$v'_l \le v'_r \quad \Longrightarrow \quad \begin{cases} g\left(u_l, v\right) \ge g\left(u_l, v'_l\right), & \text{for all } v \in [v'_l, v'_r], \text{ or} \\ g\left(u_r, v\right) \ge g\left(u_r, v'_r\right), & \text{for all } v \in [v'_l, v'_r], \end{cases} \tag{5.10}$$

$$v'_r \le v'_l \quad \Longrightarrow \quad \begin{cases} g\left(u_l, v\right) \le g\left(u_l, v'_l\right), & \text{for all } v \in [v'_r, v'_l], \text{ or} \\ g\left(u_r, v\right) \le g\left(u_r, v'_r\right), & \text{for all } v \in [v'_r, v'_l]. \end{cases} \tag{5.11}$$

Lemma 5.6. *If the values v'_l and v'_r are chosen according to the minimal jump entropy condition, then, for any constant $c \in [s, S]$,*

$$Q_r\left(v'_r, c\right) - Q_l\left(v'_l, c\right) \le \left|g\left(u_r, c\right) - g\left(u_l, c\right)\right|, \tag{5.12}$$

where Q_l and Q_r are the Kružkov entropy fluxes,

$$Q_l(v,c) = \operatorname{sign}\left(v - c\right)\left(g\left(u_l, v\right) - g\left(u_l, c\right)\right),$$
$$Q_r(v,c) = \operatorname{sign}\left(v - c\right)\left(g\left(u_r, v\right) - g\left(u_r, c\right)\right).$$

Proof. If $\operatorname{sign}\left(v'_l - c\right) = \operatorname{sign}\left(v'_r - c\right)$ then the right-hand side of (5.12) equals

$$\operatorname{sign}\left(v'_l - c\right)\big(g\left(u_r, v'_r\right) - g\left(u_r, c\right) - g\left(u_l, v'_l\right) + g\left(u_l, c\right)\big) = \operatorname{sign}\left(v'_l - c\right)\left(g\left(u_l, c\right) - g\left(u_r, c\right)\right),$$

and the inequality clearly holds. If $v'_l \le v'_r$ then (5.12) reads

$$2g_0 - g\left(u_l, c\right) - g\left(u_r, c\right) \le \left|g\left(u_r, c\right) - g\left(u_l, c\right)\right|$$

or

$$2g_0 - \max\left\{g\left(u_l, c\right), g\left(u_r, c\right)\right\} - \min\left\{g\left(u_l, c\right), g\left(u_r, c\right)\right\} \le \max\left\{g\left(u_l, c\right), g\left(u_r, c\right)\right\} - \min\left\{g\left(u_l, c\right), g\left(u_r, c\right)\right\}.$$

In other words (5.12) is the same as

$$g_0 \le \max\left\{g\left(u_l, c\right), g\left(u_r, c\right)\right\},$$

and it is immediate that (5.10) implies this. If $v_r' \le v_l'$ then (5.12) reads

$$g_0 \ge \min\left\{g\left(u_l, c\right), g\left(u_r, c\right)\right\},$$

which is implied by (5.11). ☐

Let $u_j^n, v_j^n, v^{\Delta x}(x,t)$ be defined by the ASG-scheme (cf. Subsection 4.2), and set

$$v_{j+1/2}^{n,\pm} = \lim_{x \to x_{j+1/2}\pm} v^{\Delta x}(x,t), \qquad t \in I^n. \tag{5.13}$$

With $Q_c(u,v) = \operatorname{sign}(v-c)\left(g(u,v) - g(u,c)\right)$, Lemma 5.6 implies that the quantity $Q\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right) - Q\left(u_j^n, v_{j+1/2}^{n,-}\right)$ is upper bounded by $\left|u_{j+1}^n - u_j^n\right|$. To establish the $W_{\mathrm{loc}}^{-1,2}$ compactness for the ASG-scheme we need a similar upper bound for any smooth scalar entropy/entropy flux pair $(\eta(u,v), Q(u,v))$.

Lemma 5.7. *Let $\eta(u,v)$ be a smooth function defined on the rectangle $[\alpha,\beta]\times[s,S]$ and define $Q : [\alpha,\beta]\times[s,S] \to \mathbb{R}$ by $Q_v(u,v) = \eta_v(u,v) g_v(u,v)$. If η_{vvv} is uniformly bounded, then for all j and n*

$$Q\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right) - Q\left(u_j^n, v_{j+1/2}^{n,-}\right) \le C\left|u_{j+1}^n - u_j^n\right|, \tag{5.14}$$

for some constant C independent of Δx (but dependent on the smoothness of η).

Proof. Set $h = (S-s)/M$ for some positive integer M, and let

$$c_i = s + ih, \quad i = 0, \dots, M.$$

For $u \in [\alpha,\beta]$ and $v \in [s,S]$, define the function

$$\eta^M(u,v) = \sum_{i=1} k_i(u)\left|v - c_i\right| + \eta(u,s) + \frac{\eta(u,S) - \eta(u,s)}{S-s}(v-s),$$

where

$$k_i(u) = \frac{1}{2h}\left(g\left(u, c_{i+1}\right) - 2g\left(u, c_i\right) + g\left(u, c_{i-1}\right)\right) = \frac{1}{2} h \eta_{vv}\left(u, \theta_j\right) \ge 0,$$

for some θ_j in (c_{i-1}, c_{i+1}). Since $v \mapsto \eta^M(u,v)$ is the piecewise linear interpolation to $v \mapsto \eta(u,v)$ between the points c_i, we have that $Ch \ge \left|\eta^M(u,v) - \eta(u,v)\right|$ for $(u,v) \in [\alpha,\beta]\times[s,S]$ and some non negative constant C. Next define the function

$$Q^M(u,v) = \sum_i^M k_i(u) Q_i(u,v), \quad Q_i(u,v) = \operatorname{sign}\left(v - c_i\right)\left(g(u,v) - g(u,c_i)\right).$$

Now

$$\eta_v^M(u,v) = \sum_i k_i(u) \operatorname{sign}\left(v - c_i\right) \quad \text{and}$$

$$Q_v^M(u,v) = \sum_i k_i(u) \operatorname{sign}\left(v - c_i\right) g_v(u,v),$$

so that $Q_v^M(u,v) = \eta_v^M(u,v) g_v(u,v)$.

Now we can use Lemma 5.6 to show that

$$\begin{aligned}
&k_i\left(u_{j+1}^n\right) Q_i\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right) - k_i\left(u_j^n\right) Q_i\left(u_j^n, v_{j+1/2}^{n,-}\right) \\
&\le \left|k_i\left(u_{j+1}^n\right)\right| \left|Q_i\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right) - Q_i\left(u_j^n, v_{j+1/2}^{n,-}\right)\right| \\
&\quad + \left|Q_i\left(u_j^n, v_{j+1/2}^{n,-}\right)\right| \left|k_i\left(u_{j+1}^n\right) - k_i\left(u_j^n\right)\right|
\end{aligned}$$

$$\begin{aligned}&\leq C_1 h \left|\eta_{vv}\left(u_{j+1}^n,\theta_i\right)\right|\left|u_{j+1}^n-u_{j-1}^n\right| \\ &\qquad + C_2 h \left|\eta_{vvu}\left(\omega_{j+1/2}^n,\theta_i\right)\right|\left|u_{j+1}^n-u_j^n\right| \\ &\leq Ch\left|u_{j+1}^n-u_j^n\right|,\end{aligned}$$

where $\omega_{j+1/2}^n$ is between u_j^n and u_{j+1}^n. From this it follows that

$$Q^M\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right) - Q^M\left(u_j^n, v_{j+1/2}^{n,-}\right) \leq C(S-s)\left|u_{j+1}^n-u_j^n\right|.$$

Now we can let $M\to\infty$ and conclude that (5.14) holds. □

Now that the preliminaries are out of the way, we set out to prove convergence of the ASG-scheme, following the route laid out for the SSG-scheme.

Lemma 5.8. *Let $v^{\Delta x}$ be defined by the ASG-scheme (cf. Subsection 4.2). There exists a constant $C = C(X,T)$ independent of Δx such that*

$$\sum_{j,n} \int_{x_{j-1/2}}^{x_{j+1/2}} \left|v^{\Delta x}\left(x,t^n-\right)-v_j^n\right|^2 \, dx \leq C.$$

Additionally,

$$\Delta x \sum_{n,j}\left|v_{j+1}^n - v_j^n\right|^2 \leq C. \tag{5.15}$$

Proof. The proof of this lemma is virtually identical to the proof of Lemma 5.4 (see also Lemma 5.4). Indeed for ASG-scheme we still have the decomposition (5.2) except that the term $I_4(\varphi)$ now reads

$$I_4(\varphi) = \sum_{n,j}\int_{I^n}\left[Q\left(u_j^n, v_{j+1/2}^{n,-}\right) - Q\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right)\right]\varphi(x_{j+1/2},t)\,dt, \tag{5.16}$$

where $v_{j+1/2}^{n,-}, v_{j+1/2}^{n,+}$ are defined in (5.13). Proceeding as before we choose $S = v^2/2$ and $\varphi = 1$, by that means obtaining (5.8). The only new ingredient compared to the SSG-scheme lies in the treatment of $I_4(\varphi)$, which can now only be bounded from below. As it happens, Lemma 5.7 yields

$$I_4(\varphi) \geq -C\left|u_0\right|_{BV(\mathbb{R})},$$

for some constant C independent of Δx. This concludes the proof. □

The upcoming lemma is a version of Lemma 5.5 for the ASG-scheme. We remark that here the $W^{-1,2}_{\mathrm{loc}}$ compactness are not proved for all scalar entropy-entropy flux pairs but merely for the pairs defined in Lemma 3.1. It appears to be essential for the analysis of the ASG-scheme that we can restrict the $W^{1,2}_{\mathrm{loc}}$ compactness analysis to merely two entropy/entropy flux pairs. This is a crucial difference with the compactness proof for the SSG-scheme.

Lemma 5.9. *Let $v^{\Delta x}$ be generated by the ASG-scheme (cf. Subsection 4.2). Let u be the unique entropy solution to the first equation in* (1.1). *Equipped with any of the specific scalar entropy/entropy flux pairs (η_i, Q_i), $i = 1,2$, introduced in Lemma 3.1, form the distribution*

$$\mu^{\Delta x} := \eta_i(u, v^{\Delta x})_t + Q_i(u, v^{\Delta x})_x.$$

Then $\left\{\mu^{\Delta x}\right\}_{\Delta x>0}$ belongs to a compact subset of $W^{-1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)$.

Proof. Since the proof is similar to the proof of Lemma 5.5, we shall only outline the maindifferences. We using the notation of that proof and additionally write $u^n_\pm$ instead of $u^{\Delta x}(\cdot, t^n\pm)$.

For any scalar entropy-entropy flux pair (η, Q) we decompose the term I_2 as

$$\begin{aligned} I_2(\varphi) &:= \sum_n \int_{\mathbb{R}} \left(\eta\left(u^n_-, v^n_-\right) - \eta\left(u^n_+, v^n_+\right)\right) \varphi(x, t^n)\, dx \\ &= \underbrace{\sum_n \int_{\mathbb{R}} \left(\eta\left(u^n_-, v^n_-\right) - \eta\left(u^n_+, v^n_-\right)\right) \varphi(x, t^n)\, dx}_{I_{2,1}(\varphi)} \\ &\quad + \underbrace{\sum_n \int_{\mathbb{R}} \left(\eta\left(u^n_+, v^n_-\right) - \eta\left(u^n_+, v^n_+\right)\right) \varphi(x, t^n)\, dx}_{I_{2,2}(\varphi)} . \end{aligned}$$

Now

$$|I_{2,1}(\varphi)| \le C \left\|\varphi\right\|_{L^\infty(\Omega)} .$$

We continue by splitting up $I_{2,2}$ as follows:

$$\begin{aligned} I_{2,2}(\varphi) = &\underbrace{\sum_{n,j} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\eta\left(u^n_j, v^n_-\right) - \eta\left(u^n_j, v^n_j\right)\right) \varphi^n_j\, dx}_{I_{2,2,1}(\varphi)} \\ &+ \underbrace{\sum_{n,j} \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\eta\left(u^n_j, v^n_-\right) - \eta\left(u^n_j, v^n_j\right)\right) \left(\varphi(x, t^n) - \varphi^n_j\right)\, dx}_{I_{2,2,2}(\varphi)}, \end{aligned}$$

where $\varphi^n_j = \varphi(x_j, t^n)$.

As before we write $I_{2,2,1}$ as

$$I_{2,2,1}(\varphi) = \frac{1}{2} \sum_{n,j} \varphi^n_j \int_{x_{j-1/2}}^{x_{j+1/2}} \eta_{vv}\left(u^n_j, \theta^n_j\right) \left(v^n_- - v^n_j\right)\, dx,$$

where $\theta^n_j(x)$ is some value between v^n_j and v^n_-. By Lemma 5.8

$$|I_{2,2,1}(\varphi)| \le C \left\|\varphi\right\|_{L^\infty(\Omega)} .$$

Arguing as before we deduce

$$|I_{2,2,2}(\varphi)| \le C \left\|\varphi\right\|_{C^{0,\alpha}(\Omega)} (\Delta x)^{\alpha - 1/2}, \qquad \alpha \in (1/2, 1).$$

Compared to the proof of Lemma 5.5, the key difference lies in the handling of the term $I_4(\varphi)$ (defined in (5.16)). As alluded to earlier we are going to estimate this term for the specific scalar entropy-entropy flux pairs (η_1, Q_2) and (η_1, Q_2) set forth in Lemma 3.1. It is only in the estimation of $I_4(\varphi)$ where we need the special properties of these entropy-entropy flux pairs.

In what follows we utilize the following notation ($v^{n,\pm}_{j+1/2}$ are defined in (5.13)):

$$[\![Q]\!]^n_{j+1/2} = Q\left(u^n_j, v^{n,-}_{j+1/2}\right) - Q\left(u^n_{j+1}, v^{n,+}_{j+1/2}\right).$$

The remaining part of the proof is divided into two cases.

Case 1 - (η_1, Q_1). Considering the entropy flux $Q_1(u,v) = g(u,v) - g(u,c)$ and using the Rankine-Hugoniot condition at the interface $x = x_{j+1/2}$, which ensures $g\left(u_j^n, v_{j+1/2}^{n,-}\right) = g\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right)$, we obtain

$$\begin{aligned}
[\![Q_1]\!]_{j+1/2}^n &= g\left(u_j^n, v_{j+1/2}^{n,-}\right) - g\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right) + g\left(u_{j+1}^n, c\right) - g\left(u_j^n, c\right) \\
&= g\left(u_{j+1}^n, c\right) - g\left(u_j^n, c\right).
\end{aligned}$$

Consequently,

$$\left|[\![Q_1]\!]_{j+1/2}^n\right| \le C\left|u_{j+1}^n - u_j^n\right|,$$

Hence, summing over j, n and exploiting the BV regularity of u_j^n, we can deliver the desired estimate

$$|I_4(\varphi)| \le \Delta t \sum_{j,n}\left|[\![Q_1]\!]_{j+1/2}^n\right| \le C\,\|\varphi\|_{L^\infty(\Omega)}.$$

Case 2 - (η_2, Q_2). Next we consider the entropy flux pair $Q_2(u,v) = \int_c^v (g_v(u,\xi))^2\,d\xi$. We have

$$\begin{aligned}
[\![Q_2]\!]_{j+1/2}^n &= \int_c^{v_{j+1/2}^{n,-}} \left(g_v\left(u_j^n,\xi\right)\right)^2\,d\xi - \int_c^{v_{j+1/2}^{n,+}} (g_v(u_{j+1}^n,\xi))^2\,d\xi \\
&= \underbrace{\int_c^{v_{j+1/2}^{n,-}} \left(g_v\left(u_j^n,\xi\right)\right)^2 - \left(g_v\left(u_{j+1}^n,\xi\right)\right)^2\,d\xi}_{[\![Q_{2,1}]\!]_{j+1/2}^n} + \underbrace{\int_{v_{j+1/2}^{n,+}}^{v_{j+1/2}^{n,-}} \left(g_v\left(u_{j+1}^n,\xi\right)\right)^2\,d\xi}_{[\![Q_{2,2}]\!]_{j+1/2}^n}
\end{aligned}$$

Clearly, since the numerical solutions are bounded, we can produce a constant C, independent of Δx, such that

$$\left|[\![Q_{2,1}]\!]_{j+1/2}^n\right| \le C\left|u_j^n - u_{j+1}^n\right|$$

and

$$\left|[\![Q_{2,2}]\!]_{j+1/2}^n\right| \le C[\![Q_{2,3}]\!]_{j+1/2}^n, \quad [\![Q_{2,3}]\!]_{j+1/2}^n := \int_{v_{j+1/2}^{n,+}}^{v_{j+1/2}^{n,-}} |g_v(u_{j+1}^n,\xi)|\,d\xi.$$

To estimate $[\![Q_{2,3}]\!]_{j+1/2}^n$, we are going to make a simplifying assumption, namely that $g(u,\cdot)$ is a function having at most one local minimum (respectively maximum) and no local maxima (respectively minima) in $[s,S]$ for all $u \in [\alpha,\beta]$. This assumption is quite general as it includes the fluxes for the triangular three-phase flow model in Section 2. Although the more general case of finitely many points of extrema can also be handled, we will not do so here since it is (only) notationally more cumbersome. In what follows we present the details only for the case of local minima; The case of local maxima follows along the same lines.

For fixed n, j let θ_j^n and θ_{j+1}^n be the local minima of the fluxes $g(u_j^n,\cdot)$ and $g(u_{j+1}^n,\cdot)$ respectively. Without loss of generality assume that $\theta_{j+1}^n \le \theta_j^n$ and $g(u_j^n,\theta_j^n) \le g(u_{j+1}^n,\theta_{j+1}^n)$. The solution of the Riemann problem (4.8) with data (v_j^n, v_{j+1}^n) can be grouped into the four cases, each of which will be detailed below.

Case 2.1 $[g_v(u_j^n, v_{j+1/2}^{n,-}) \ge 0$ and $g_v(u_{j+1}^n, v_{j+1/2}^{n,+}) \ge 0]$. In this case we have $v_{j+1/2}^{n,-} \ge \theta_j^n \ge \theta_{j+1}^n$ and $v_{j+1/2}^{n,+} \ge \theta_{j+1}^n$. As $g(u_j^n, v_{j+1/2}^{n,-}) = g(u_{j+1}^n, v_{j+1/2}^{n,+})$ by the

Rankine-Hugoniot condition at the interface $x_{j+1/2}$, it follows that

$$\begin{aligned}
[\![Q_{2,3}]\!]_{j+1/2}^n &= g\left(u_{j+1}^n, v_{j+1/2}^{n,-}\right) - g\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right) \\
&= g\left(u_{j+1}^n, v_{j+1/2}^{n,-}\right) - g\left(u_j^n, v_{j+1/2}^{n,-}\right) \\
&\quad + g\left(u_j^n, v_{j+1/2}^{n,-}\right) - g\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right) \\
&= g\left(u_{j+1}^n, v_{j+1/2}^{n,-}\right) - g\left(u_j^n, v_{j+1/2}^{n,-}\right),
\end{aligned}$$

from which we conclude there is a constant independent of Δx such that

$$\left|[\![Q_{2,3}]\!]_{j+1/2}^n\right| \leq C\left|u_j^n - u_{j+1}^n\right|. \tag{5.17}$$

Case 2.2 $[g_v(u_j^n, v_{j+1/2}^{n,-}) \leq 0$ and $g_v(u_{j+1}^n, v_{j+1/2}^{n,+}) \leq 0]$. Proceeding as in Case 2.1, we obtain again (5.17).

Case 2.3 $[g_v(u_j^n, v_{j+1/2}^{n,-}) \leq 0$ and $g_v(u_{j+1}^n, v_{j+1/2}^{n,+}) \geq 0]$. This is the under-compressive case. In this case, the *minimal jump entropy condition* [16] implies that either $v_{j+1/2}^{n,-} = v_{j+1/2}^{n,+} = \alpha_{j,j+1}^n$ (if the adjacent fluxes intersect at a point $\alpha_{j,j+1}^n$ with $g_v(u_j^n, \alpha_{j,j+1}^n) < 0$ and $g_v(u_{j+1}^n, \alpha_{j,j+1}^n) > 0$) or $v_{j+1/2}^{n,+} = \theta_{j+1}^n$ (otherwise). In either case, following exactly the proof of (5.17), we obtain

$$\left|[\![Q_{2,3}]\!]_{j+1/2}^n\right| \leq \left|g\left(u_{j+1}^n, v_{j+1/2}^{n,-}\right) - g\left(u_{j+1}^n, v_{j+1/2}^{n,+}\right)\right| \leq C\left|u_j^n - u_{j+1}^n\right|.$$

Case 2.4 $[g_v(u_j^n, v_{j+1/2}^{n,-}) \geq 0$ and $g_v(u_{j+1}^n, v_{j+1/2}^{n,+}) \leq 0]$. In this case the Riemann solution is of the form $v_{j+1/2}^{n,-} = v_j^n$ and $v_{j+1/2}^{n,+} = v_{j+1}^n$ (a steady shock is formed at the interface $x_{j+1/2}$). Additionally, there holds $v_{j+1}^n \leq \theta_{j+1}^n \leq \theta_j^n \leq v_j^n$. Therefore

$$[\![Q_{2,3}]\!]_{j+1/2}^n = g\left(u_{j+1}^n, v_j^n\right) - g\left(u_{j+1}^n, \theta_{j+1}^n\right) + g\left(u_{j+1}^n, v_{j+1}^n\right) - g\left(u_{j+1}^n, \theta_{j+1}^n\right).$$

Expanding in Taylor series up to second order around θ_{j+1}^n, keeping in mind that $g_v(u_{j+1}^n, \theta_{j+1}^n) = 0$ and $v_{j+1}^n \leq \theta_{j+1}^n \leq v_j^n$, we extract

$$\left|[\![Q_{2,3}]\!]_{j+1/2}^n\right| \leq C\left[\left(v_j^n - \theta_{j+1}^n\right)^2\left(v_{j+1}^n - \theta_{j+1}^n\right)^2\right] \leq C\left(v_j^n - v_{j+1}^n\right)^2,$$

where the constant C is independent of Δx.

Collecting our findings so far, there is a constant C independent of Δx such that

$$\left|[\![Q_i]\!]_{j+1/2}^n\right| \leq C\left[\left|u_j^n - u_{j+1}^n\right| + (v_j^n - v_{j+1}^n)^2\right], \quad i = 1, 2.$$

Summing this bound over j, n yields

$$|I_4(\varphi)| \leq C\Delta t \left\|\varphi\right\|_{L^\infty(\Omega)} \sum_{j,n}\left|[\![Q_i]\!]_{j+1/2}^n\right| \leq C\left\|\varphi\right\|_{L^\infty(\Omega)}, \quad i = 1, 2,$$

where we have exploited the BV regularity of u_j^n and (5.15) to produce a final constant C that is independent of Δx.

Now we can finish the proof as we did with Lemma 5.5. □

We are now in a position to prove convergence of the ASG-scheme.

Theorem 5.2. *Let $u^{\Delta x}, v^{\Delta x}$ be generated by the ASG-scheme (cf. Subsection 4.2). Then there exist limit functions $u \in L^\infty(\mathbb{R}\times\mathbb{R}^+)\cap BV(\mathbb{R}\times\mathbb{R}^+)$ and $v \in L^\infty(\mathbb{R}\times\mathbb{R}^+)$ such that, along a subsequence as $\Delta x \to 0$*

$$u^{\Delta x} \to u,\ v^{\Delta x} \to v \ \text{in}\ L^p_{\text{loc}}(\mathbb{R}\times\mathbb{R}^+)\ \forall p < \infty \ \text{and a.e. in}\ \mathbb{R}\times\mathbb{R}^+,$$

The limit pair (u, v) constitutes a weak solution of (1.1).

Proof. The convergence statement for $u^{\Delta x}$ is clear, while the convergence of $v^{\Delta x}$ is a direct consequence of Lemmas 5.9 and 3.1.

What remains to be shown is that v is a weak solution. Pick a test fuction φ having compact support in $\mathbb{R}\times[0, T)$ with $N\Delta t = T$ for some integer N, and set

$$\varphi^{\Delta x}(x,t) = \sum_j \varphi(x_j, t)\chi_{I_j}(x), \qquad v^n_\pm(x) = v^{\Delta x}\left(x, t^n\pm\right).$$

Since $v^{\Delta x}$ is a weak solution in each strip $\mathbb{R}\times I^n$, cf. (4.8), we can work out the details as follows:

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}^+} v^{\Delta x}\varphi_t + g\left(u^{\Delta x}, v^{\Delta x}\right)\varphi_x \, dxdt + \int_{\mathbb{R}} v^{\Delta x}(x,0)\varphi(x,0)\,dx \\
&= \sum_n \int_{\mathbb{R}} \left(v^n_- - v^n_+\right)\varphi(x,t^n)\,dx \\
&= \sum_n \int_{\mathbb{R}} \left(v^n_- - v^n_+\right)\left(\varphi(x,t^n) - \varphi^{\Delta x}(x,t^n)\right)\,dx \\
&\le \left[\sum_n \int_{\mathbb{R}} \left(v^n_- - v^n_+\right)^2 dx\right]^{1/2} \left[\sum_n \int_{\mathbb{R}} \left(\varphi(x,t^n) - \varphi^{\Delta x}(x,t^n)\right)^2 dx\right]^{1/2} \le C\sqrt{\Delta x},
\end{aligned}$$

where we have used Hölder's inequality and Lemma 5.8. Since C is independent of Δx, sending $\Delta x \to 0$ shows that the limit pair (u, v) is a weak solution of (1.1). □

6. Numerical Experiments

We have tested the different schemes designed in this paper and described in Section 4 on a wide variety of test problems and have found the results to be in accordance with the theory. In particular, the SSG-scheme the ASG-scheme have been tested extensively. We report two experiments here.

6.0.1. *Example 1.* We start with simple model problem in order to illustrate the numerical schemes. We choose the following flux functions pair of fluxes

$$f(u) = \frac{1}{2}u^2, \quad \text{and} \quad g(u,v) = 4uv(v-1),$$

and Riemann initial data,

$$u(x,0) = \begin{cases} 0.75 & x < 0 \\ 0.25 & x \ge 0 \end{cases} \qquad v(x,0) = 0.5.$$

This has the exact solution given by

$$u(x,t) = \begin{cases} 3/4 & x < t/2, \\ 1/4 & x \ge t/2, \end{cases} \qquad v(x,t) = \begin{cases} 1/2 & x < -t, \\ 5/6 & -t \le x < t/2, \\ 1/2 & x \ge t/2. \end{cases}$$

In Figure 1 we show the approximations at $t = 0.75$ using the SSG- and the ASG-scheme using $\Delta x = 1/20$ in the interval $[-1, 1]$. From this it seems that the both schemes perform equally well, and this impression is confirmed by other computations. Since we have a formula for the exact solution in this case, we have computed

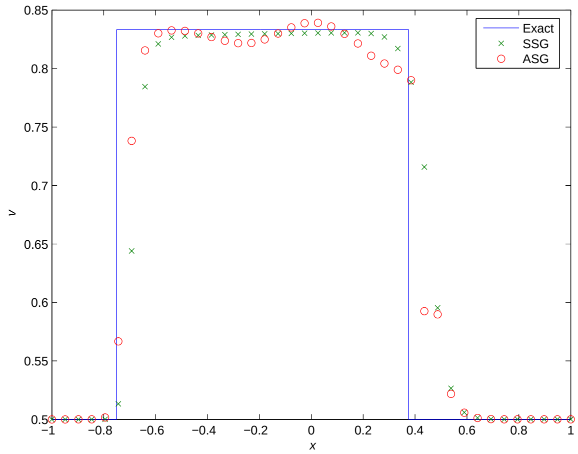

FIGURE 1. Example 1, the SSG- and the ASG-scheme with $\Delta x = 1/20$ and $t = 0.75$.

the relative errors in the L^1 norm for various Δx. The relative errors are defined as

$$e = \frac{\sum_j \left|v^{\Delta x}(x_j, t) - v_{\text{ex}}(x_j, t)\right|}{\sum_j |v_{\text{ex}}(x_j, t)|},$$

where v_{ex} denotes the exact solution, and $t = 0.75$. These errors are reported in Table 1. From this table it seems that both schemes are first order convergent.

n	3	4	5	6	7	8	9
SSG	8.6	5.1	2.8	1.4	0.7	0.4	0.018
ASG	7.6	3.7	2.1	1.1	0.6	0.3	0.015

TABLE 1. $100\times$ Relative L^1 error for the SSG- and ASG-scheme s. We used $\Delta x = 2^{-n}$ in the interval $[-1, 1]$.

6.0.2. *Example 2.* In order to test the applicability of the triangular model as a model of three-phase flow in porous media, we have compared the results obtained by the triangular and the full model on a water flooding problem. We use the relative permeabilities

$$\lambda_{g,w,o} = \frac{1}{\nu_{g,w,o}} S_{g,w,o}^2,$$

with S_i denoting the saturation of phase i, and ν_i the viscosity. We have used the following viscosities

$$\nu_g = 1, \quad \nu_w = 80 \quad \text{and} \quad \nu_o = 100.$$

In addition we have set

$$\rho_g = 1/20, \quad \rho_w = 1 \quad \text{and} \quad \rho_o = 9/10,$$

and have set the gravitational constant and the absolute permeability to unity. We make no claim for these values to be realistic. This gives the flux functions

$$F_g(u,v) = \frac{u^2}{u^2 + v^2/100 + (1-v-u)^2/80}\left(1 - \frac{17v^2}{200} - \frac{19(1-v-u)^2}{160}\right)$$

$$F_o(u,v) = \frac{v^2}{100u^2 + v^2 + (5/4)(1-v-u)^2}\left(1 + \frac{v^2}{10} + \frac{19(1-v-u)^2}{20}\right), \tag{6.1}$$

where we have set $u = S_g$ and $v = S_o$. This is the "full" model, and we see that F_g is not very dependent on v. In order to define a triangular method we set

$$v = \frac{1-u}{2} \quad \text{and} \quad (1-u-v) = \frac{1-u}{2},$$

which gives the flux function

$$F_g(u) = \frac{u^2}{u^2 + (9/1600)(1-u^2)}\left(1 - \frac{163}{3200}(1-u)^2\right). \tag{6.2}$$

We have used the initial values

$$v(x,0) = \begin{cases} 0 & x < 0, \\ \frac{1}{2} + \frac{1}{4}\sin(2\pi x) & x \geq 0, \end{cases} \qquad u(x,0) = \begin{cases} 0 & x < 0, \\ 1 - v(x,0) & x > 0. \end{cases} \tag{6.3}$$

This is meant to model the situation where one has a mixture of oil and gas in the reservoir, and one attempts to inject water in order to force out the oil and the gas. We have used (6.2) and (6.1) as f and g and the ASG-scheme (which in this case coincides with the upwind scheme) to calculate approximate solutions. In Figure 2 we show contour plots of the gas, oil, and water saturations in as functions of x and t for $-0.05 \leq x \leq 5$ and $0 \leq t \leq 2$. This looks very similar to results obtained with the full model, although a more thorough justification for using the triangular model is beyond the scope of this paper.

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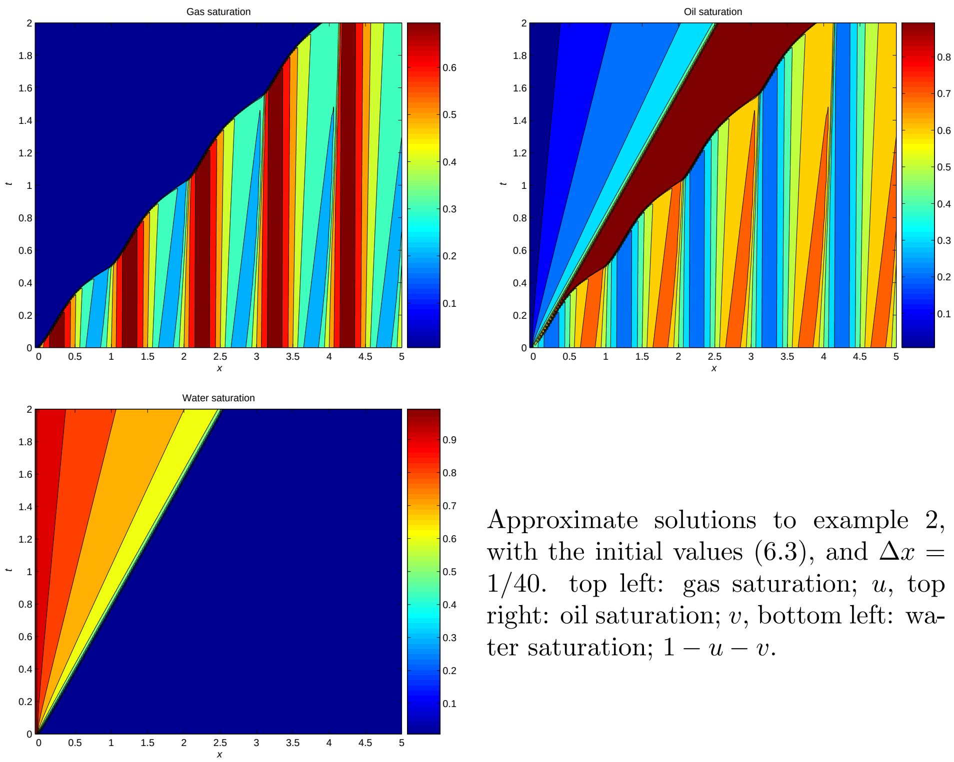

Approximate solutions to example 2, with the initial values (6.3), and $\Delta x = 1/40$. top left: gas saturation; u, top right: oil saturation; v, bottom left: water saturation; $1 - u - v$.

FIGURE 2

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