GLOBAL CONSERVATIVE SOLUTIONS OF THE GENERALIZED HYPERELASTIC-ROD WAVE EQUATION

HELGE HOLDEN AND XAVIER RAYNAUD

Abstract. We prove existence of global and conservative solutions of the Cauchy problem for the nonlinear partial differential equation $u_t - u_{xxt} + f(u)_x - f(u)_{xxx} + (g(u) + \frac{1}{2}f''(u)(u_x)^2)_x = 0$ where f is strictly convex or concave and g is locally uniformly Lipschitz. This includes the Camassa–Holm equation ($f(u) = u^2/2$ and $g(u) = \kappa u + u^2$) as well as the hyperelastic-rod wave equation ($f(u) = \gamma u^2/2$ and $g(u) = (3-\gamma)u^2/2$) as special cases. It is shown that the problem is well-posed for initial data in $H^1(\mathbb{R})$ if one includes a Radon measure that corresponds to the energy of the system with the initial data. The solution is energy preserving. Stability is proved both with respect to initial data and the functions f and g. The proof uses an equivalent reformulation of the equation in terms of Lagrangian coordinates.

1. Introduction

We solve the Cauchy problem on the line for the equation

$$u_t - u_{xxt} + f(u)_x - f(u)_{xxx} + (g(u) + \frac{1}{2}f''(u)(u_x)^2)_x = 0 \tag{1.1}$$

for strictly convex or concave functions f and locally uniformly Lipschitz functions g with initial data in $H^1(\mathbb{R})$. This equation includes the Camassa–Holm equation [4], the hyperelastic-rod wave equation [11] and its generalization [6, 7] as special cases.

For $f(u) = \frac{u^2}{2}$ and $g(u) = \kappa u + u^2$, we obtain the Camassa–Holm equation:

$$u_t - u_{xxt} + \kappa u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad u|_{t=0} = \bar{u}, \tag{1.2}$$

which has been extensively studied the last decade [4, 5]. It was first introduced as a model describing propagation of unidirectional gravitational waves in a shallow water approximation, with u representing the fluid velocity, see [18]. The Camassa–Holm equation has a bi-Hamiltonian structure, it is completely integrable, and it has infinitely many conserved quantities.

For $f(u) = \frac{\gamma u^2}{2}$ and $g(u) = \frac{3-\gamma}{2}u^2$, we obtain the hyperelastic-rod wave equation:

$$u_t - u_{txx} + 3uu_x - \gamma(2u_x u_{xx} + uu_{xxx}) = 0,$$

which was introduced by Dai [11, 10, 12] in 1998. It describes far-field, finite length, finite amplitude radial deformation waves in cylindrical compressible hyperelastic rods, and u represents the radial stretch relative to a pre-stressed state.

Furthermore, for $f(u) = \frac{\gamma u^2}{2}$ we find the generalized hyperelastic-rod equation

$$u_t - u_{xxt} + \frac{1}{2}g(u)_x - \gamma(2u_x u_{xx} + uu_{xxx}) = 0, \tag{1.3}$$

Date: May 16, 2006.

2000 *Mathematics Subject Classification.* Primary: 65M06, 65M12; Secondary: 35B10, 35Q53.

Key words and phrases. Generalized hyperelastic-rod wave equation, Camassa–Holm equation, conservative solutions.

Supported in part by the Research Council of Norway and the Mittag-Leffler Institute, Sweden.

which was recently studied by Coclite, Holden, and Karlsen [6, 7], extending earlier results for the Camassa–Holm equation by Xin and Zhang, see [20]. They analyzed the initial value problem for this equation, using an approach based on a certain viscous regularization. By carefully studying the behavior of the limit of vanishing viscosity they derived the existence of a solution of (1.3). This solution could be called a diffusive solution, and will be distinct from the solutions studied here. We will discuss this in more detail below.

We will not try to cover the extensive body of results regarding various aspects of the Camassa–Holm equation. Suffice it here to note that in the case with $\kappa = 0$ the solution may experience wave breaking in finite time in the sense that the function remains bounded while the spatial derivative becomes unbounded with finite H^1-norm. Various mechanisms and conditions are known as to when and if wave breaking occurs. Specifically we mention that Constantin, Escher, and Molinet [8, 9] showed the following result: If the initial data $u|_{t=0} = \bar{u} \in H^1(\mathbb{R})$ and $\bar{m} := \bar{u} - \bar{u}''$ is a positive Radon measure, then equation (1.2) with $\kappa = 0$ has a unique global weak solution $u \in C([0,T], H^1(\mathbb{R}))$, for any T positive, with initial data $\bar{u}$. However, any solution with odd initial data $\bar{u}$ in $H^3(\mathbb{R})$ such that $\bar{u}_x(0) < 0$ blows up in a finite time.

The problem of continuation of the solution beyond wave breaking is intricate. It can be illustrated in the context of a peakon–antipeakon solution. The one peakon is given by $u(t,x) = c\exp(-|x-ct|)$. If c is positive, the solution is called a peakon, and with c negative it is called an antipeakon. One can construct solutions that consist of finitely many peakons and antipeakons. Peakons move to the right, antipeakons to the left. If initial data are given appropriately, one can have a peakon colliding with an antipeakon. In a particular symmetric case they exactly annihilate each other at collision time t^*, thus $u(t^*, x) = 0$. This immediately raises the question about well-posedness of the equation and allows for several distinct ways to continue the solution beyond collision time. For an extensive discussion of this case, we refer to [17] and references therein. We here consider solutions, called conservative, that preserve the energy. In the example just mentioned this corresponds to the peakon and antipeakon passing through each other, and the energy accumulating as a Dirac delta-function at the origin at the time of collision. Thus the problem cannot be well-posed by considering the solution u only. Our approach for the general equation (1.1) is based on the inclusion of the energy, in the form of a (non-negative Radon) measure, together with the function u as initial data. We have seen that singularities occur in these variables. Therefore we transform to a different set of variables, which corresponds to a Lagrangian formulation of the flow, where the singularities do not occur.

Let us comment on the approach in [6, 7]. The equation (1.3) is rewritten as

$$u_t + \gamma u u_x + P_x = 0, \qquad P - P_{xx} = \frac{1}{2}(g(u) - \gamma u^2) + \gamma (u_x)^2. \tag{1.4}$$

By adding the term ϵu_{xx} to the first equation, it is first shown that the modified system has a unique solution.[1] Subsequently, it is proved that the vanishing viscosity limit $\epsilon \to 0$ exists. The limit is shown to be weak solution of (1.3). In particular, that means that $\|u(t,\cdot)\|_{H^1} \leq \|u(0,\cdot)\|_{H^1}$ and that the solution satisfies an entropy condition $u_x(t,x) \leq K + 2/(\gamma t)$ for some constant K. The solution described above with a peakon and an antipeakon "passing through" each other will not satisfy this entropy condition. Thus the solution concept is different in the two approaches.

[1] In fact, it it is proved that a more general parabolic-elliptic system, allowing, e.g., for explicit spatial and temporal dependence in the various functions, has a solution.

Here we take a rather different approach. Based on recent techniques developed for the Camassa–Holm equation, see [1, 2, 15, 16], we prove that (1.1) possesses a global weak and conservative solution. Furthermore, we show that the problem is well-posed. In particular we show stability with respect to both perturbations in the initial data and the functions f and g in a suitable topology.

The present approach is based on the fact that the equation can be reformulated as a system of ordinary differential equations taking values in a Banach space. It turns out to be advantageous first to rewrite the equation as

$$u_t + f(u)_x + P_x = 0, \tag{1.5a}$$

$$P - P_{xx} = g(u) + \frac{1}{2}f''(u)u_x^2 \tag{1.5b}$$

where we assume[2]

$$\begin{cases} f \in W^{3,\infty}_{\text{loc}}(\mathbb{R}),\ f''(u) \neq 0,\ u \in \mathbb{R}, \\ g \in W^{1,\infty}_{\text{loc}}(\mathbb{R}),\ g(0) = 0. \end{cases} \tag{1.6}$$

We will use this assumption throughout the paper.

Specifically, the characteristics are given by

$$y_t(t,\xi) = f'(u(t, y(t,\xi))).$$

Define subsequently

$$U(t,\xi) = u(t, y(t,\xi)),$$
$$H(t,\xi) = \int_{-\infty}^{y(t,\xi)} \left(u^2 + u_x^2\right)\, dx,$$

where U and H correspond to the Lagrangian velocity and the Lagrangian cumulative energy distribution, respectively. It turns out that one can derive the following system of ordinary differential equations taking values in an appropriately chosen Banach space, viz.

$$\begin{cases} y_t = U, \\ U_t = -Q, \\ H_t = G(U) - 2PU, \end{cases}$$

where the quantities G, Q, and P can be expressed in terms of the unknowns (y, U, H). Short-term existence is derived by a contraction argument. Global existence as well as stability with respect to both initial data and functions f and g, is obtained for a class of initial data that includes initial data $u|_{t=0} = \bar{u}$ in $H^1(\mathbb{R})$, see Theorem 2.8. The transition of this result back to Eulerian variables is complicated by several factors, one being the reduction of three Lagrangian variables to two Eulerian variables. There is a certain redundancy in the Lagragian formulation which is identified, and we rather study equivalence classes that correspond to relabeling of the same Eulerian flow. The main existence result is Theorem 2.9. It is shown that the flow is well-posed on this space of equivalence classes in the Lagrangian variables, see Theorem 3.6. A bijection is constructed between Lagrangian and Eulerian variables, see Theorems 3.8–3.11. On the set $\mathcal{D}$ of Eulerian variables we introduce a metric that turns $\mathcal{D}$ into a complete metric space, see Theorem 3.12.

The main result, Theorem 3.13, states the following: There exists a continuous semigroup on $\mathcal{D}$ which to any initial data $(\bar{u}, \bar{\mu}) \in \mathcal{D}$ associates the pair $(u(t), \mu(t)) \in \mathcal{D}$ such that $u(t)$ is a weak solution of (1.5) and the measure $\mu = \mu(t)$ with $\mu(0) = \bar{\mu}$, evolves according to the linear transport equation $\mu_t + (u\mu)_x = (G(u) - 2Pu)_x$ where the functions G and P are explicitly given. Continuity with respect to all variables,

[2]Without loss of generality we may and will assume that $g(0) = 0$. Otherwise, (1.5b) should be replaced by $P - P_{xx} = g(u) - g(0) + \frac{1}{2}f''(u)u_x^2$

including f and g, is proved. The total energy as measured by μ is preserved, i.e., $\mu(t)(\mathbb{R}) = \bar{\mu}(\mathbb{R})$ for all t.

The abstract construction is illustrated on the one and two peakon solutions for the Camassa–Holm equation.

The paper is organized as follows. In Section 2 the equation is reformulated in terms of Lagrangian variables, and existence is first proved in Lagrangian variables before the results are transformed back to the original Eulerian variables. Stability of the semigroup is provided in Section 3, and the construction is illustrated on concrete examples in Section 4.

2. Existence of solutions

2.1. **Transport equation for the energy density and reformulation in terms of Lagrangian variables.** In (1.5b), P can be written in explicit form:

$$P(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}\big(g\circ u + \frac{1}{2}f''\circ u u_x^2\big)(t,z)\,dz. \tag{2.1}$$

We will derive a transport equation for the energy density $u^2+u_x^2$. Assuming that u is smooth, we get, after differentiating (1.5a) with respect to x and using (1.5b), that

$$u_{xt} + \frac{1}{2}f''(u)u_x^2 + f'(u)u_{xx} + P - g(u) = 0. \tag{2.2}$$

Multiply (1.5a) by u, (2.2) by u_x, add the two to find the following equation

$$(u^2+u_x^2)_t + (f'(u)(u^2+u_x^2))_x = -2(Pu)_x + (2g(u)+f''(u)u^2)u_x. \tag{2.3}$$

Define $G(v)$ as

$$G(v) = \int_0^v (2g(z)+f''(z)z^2)\,dz, \tag{2.4}$$

then (2.3) can be rewritten as

$$(u^2+u_x^2)_t + (f'(u)(u^2+u_x^2))_x = (G(u)-2Pu)_x, \tag{2.5}$$

which is transport equation for the energy density $u^2+u_x^2$.

Let us introduce the characteristics $y(t,\xi)$ defined as the solutions of

$$y_t(t,\xi) = f'(u(t,y(t,\xi))) \tag{2.6}$$

with $y(0,\xi)$ given. Equation (2.5) gives us information about the evolution of the amount of energy contained between two characteristics. Indeed, given ξ_1, ξ_2 in $\mathbb{R}$, let

$$H(t) = \int_{y(t,\xi_1)}^{y(t,\xi_2)} \left(u^2+u_x^2\right)\,dx$$

be the energy contained between the two characteristic curves $y(t,\xi_1)$ and $y(t,\xi_2)$. Then, we have

$$\frac{dH}{dt} = \big[y_t(t,\xi)(u^2+u_x^2)\circ y(t,\xi)\big]_{\xi_1}^{\xi_2} + \int_{y(t,\xi_1)}^{y(t,\xi_2)} (u^2+u_x^2)_t\,dx. \tag{2.7}$$

We use (2.5) and integrate by parts. The first term on the right-hand side of (2.7) cancels because of (2.6) and we end up with

$$\frac{dH}{dt} = [(G(u)-2Pu)\circ y]_{\xi_1}^{\xi_2}. \tag{2.8}$$

We now derive a system equivalent to (1.5). The calculations here are formal and will be justified later. Let y still denote the characteristics. We introduce two other variables, the Lagrangian velocity U and cumulative energy distribution H defined by

$$U(t,\xi) = u(t, y(t,\xi)), \tag{2.9}$$

$$H(t,\xi) = \int_{-\infty}^{y(t,\xi)} \left(u^2 + u_x^2\right)\, dx, \tag{2.10}$$

respectively. From the definition of the characteristics, it follows from (1.5a) that

$$\begin{aligned} U_t(t,\xi) &= u_t(t,y) + y_t(t,\xi)u_x(t,y) \\ &= (u_t + f'(u)u_x)\circ y(t,\xi) \\ &= -P_x\circ y\,(t,\xi). \end{aligned} \tag{2.11}$$

This last term can be expressed uniquely in term of U, y, and H. Namely, we have

$$P_x\circ y\,(t,\xi) = -\frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(y(t,\xi)-z)e^{-|y(t,\xi)-z|}(g\circ u + \frac{1}{2}f''\circ u\, u_x^2)(t,z)\, dz$$

and, after the change of variables $z = y(t,\eta)$,

$$\begin{aligned} P_x\circ y(t,\xi) = -\frac{1}{2}\int_{\mathbb{R}} \Big[&\operatorname{sgn}(y(t,\xi)-y(t,\eta))e^{-|y(t,\xi)-y(t,\eta)|} \\ &\times \Big(g\circ u + \frac{1}{2}f''\circ uu_x^2\Big)(t,y(t,\eta))y_\xi(t,\eta)\Big]\, d\eta. \end{aligned}$$

Finally, since $H_\xi = (u^2+u_x^2)\circ y\, y_\xi$,

$$\begin{aligned} P_x\circ y\,(\xi) = -\frac{1}{2}\int_{\mathbb{R}} &\operatorname{sgn}(y(\xi)-y(\eta))e^{-|y(\xi)-y(\eta)|} \\ &\times \Big(\big(g(U) - \frac{1}{2}f''(U)U^2\big)y_\xi + \frac{1}{2}f''(U)H_\xi\Big)(\eta)\, d\eta \end{aligned} \tag{2.12}$$

where the t variable has been dropped to simplify the notation. Later we will prove that y is an increasing function for any fixed time t. If, for the moment, we take this for granted, then $P_x\circ y$ is equivalent to Q where

$$\begin{aligned} Q(t,\xi) = -\frac{1}{2}\int_{\mathbb{R}} &\operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big) \\ &\times \Big(\big(g(U) - \frac{1}{2}f''(U)U^2\big)y_\xi + \frac{1}{2}f''(U)H_\xi\Big)(\eta)\, d\eta, \end{aligned} \tag{2.13}$$

and, slightly abusing the notation, we write

$$\begin{aligned} P(t,\xi) = \frac{1}{2}\int_{\mathbb{R}} &\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big) \\ &\times \Big(\big(g(U) - \frac{1}{2}f''(U)U^2\big)y_\xi + \frac{1}{2}f''(U)H_\xi\Big)(\eta)\, d\eta. \end{aligned} \tag{2.14}$$

Thus $P_x\circ y$ and $P\circ y$ can be replaced by equivalent expressions given by (2.13) and (2.14) which only depend on our new variables U, H, and y. We introduce yet another variable, $\zeta(t,\xi)$, simply defined as

$$\zeta(t,\xi) = y(t,\xi) - \xi.$$

It will turn out that $\zeta\in L^\infty(\mathbb{R})$. We have now derived a new system of equations, formally equivalent (1.5). Equations (2.11), (2.8) and (2.6) give us

$$\begin{cases} \zeta_t = U, \\ U_t = -Q, \\ H_t = G(U) - 2PU. \end{cases} \tag{2.15}$$

Detailed analysis will reveal that the system (2.15) of ordinary differential equations for $(\zeta, U, H)\colon [0,T] \to E$ is well-posed, where E is a Banach space to be defined in the next section. We have

$$Q_\xi = -\frac{1}{2}f''(U)H_\xi + \big(P + \frac{1}{2}f''(U)U^2 - g(U)\big)y_\xi,$$

and $P_\xi = Qy_\xi$. Hence, differentiating (2.15) yields

$$\begin{cases} \zeta_{\xi t} = f''(U)U_\xi \text{ (or } y_{\xi t} = f''(U)U_\xi),\\ U_{\xi t} = \frac{1}{2}f''(U)H_\xi - \big(P + \frac{1}{2}f''(U)U^2 - g(U)\big)y_\xi,\\ H_{\xi t} = -2Q\,Uy_\xi + (2g(U) - f''(U)U^2 - 2P)U_\xi. \end{cases} \tag{2.16}$$

The system (2.16) is semilinear with respect to the variables y_ξ, U_ξ and H_ξ.

2.2. **Existence and uniqueness of solutions in Lagrangian variables.** In this section, we focus our attention on the system of equations (2.15) and prove, by a contraction argument, that it admits a unique solution. Let V be the Banach space defined by

$$V = \{f \in C_b(\mathbb{R}) \mid f_\xi \in L^2(\mathbb{R})\}$$

where $C_b(\mathbb{R}) = C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and the norm of V is given by $\|f\|_V = \|f\|_{L^\infty(\mathbb{R})} + \|f_\xi\|_{L^2(\mathbb{R})}$. Of course $H^1(\mathbb{R}) \subset V$ but the converse is not true as V contains functions that do not vanish at infinity. We will employ the Banach space E defined by

$$E = V \times H^1(\mathbb{R}) \times V$$

to carry out the contraction map argument. For any $X = (\zeta, U, H) \in E$, the norm on E is given by

$$\|X\|_E = \|\zeta\|_V + \|U\|_{H^1(\mathbb{R})} + \|H\|_V\,.$$

The following lemma gives the Lipschitz bounds we need on Q and P.

Lemma 2.1. *For any $X = (\zeta, U, H)$ in E, we define the maps $\mathcal{Q}$ and $\mathcal{P}$ as $\mathcal{Q}(X) = Q$ and $\mathcal{P}(X) = P$ where Q and P are given by (2.13) and (2.14), respectively. Then, $\mathcal{P}$ and $\mathcal{Q}$ are Lipschitz maps on bounded sets from E to $H^1(\mathbb{R})$. Moreover, we have*

$$Q_\xi = -\frac{1}{2}f''(U)H_\xi + \big(P + \frac{1}{2}f''(U)U^2 - g(U)\big)(1+\zeta_\xi), \tag{2.17}$$

$$P_\xi = Q(1+\zeta_\xi). \tag{2.18}$$

Proof. We rewrite $\mathcal{Q}$ as

$$\begin{aligned} \mathcal{Q}(X)(\xi) = &-\frac{e^{-\zeta(\xi)}}{2}\int_{\mathbb{R}} \chi_{\{\eta<\xi\}}(\eta)e^{-(\xi-\eta)}e^{\zeta(\eta)} \\ &\qquad \times \Big(\big(g(U) - \frac{1}{2}f''(U)U^2\big)(1+\zeta_\xi) + \frac{1}{2}f''(U)H_\xi\Big)(\eta)\,d\eta \\ &+ \frac{e^{\zeta(\xi)}}{2}\int_{\mathbb{R}} \chi_{\{\eta>\xi\}}(\eta)e^{(\xi-\eta)}e^{-\zeta(\eta)} \\ &\qquad \times \Big(\big(g(U) - \frac{1}{2}f''(U)U^2\big)(1+\zeta_\xi) + \frac{1}{2}f''(U)H_\xi\Big)(\eta)\,d\eta, \end{aligned} \tag{2.19}$$

where χ_B denotes the indicator function of a given set B. We decompose $\mathcal{Q}$ into the sum $\mathcal{Q}_1 + \mathcal{Q}_2$ where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the operators corresponding to the two terms on the right-hand side of (2.19). Let $h(\xi) = \chi_{\{\xi>0\}}(\xi)e^{-\xi}$ and A be the map defined by $A\colon v \mapsto h \star v$. Then, Q_1 can be rewritten as

$$\mathcal{Q}_1(X)(\xi) = -\frac{e^{-\zeta(\xi)}}{2}A \circ R(\zeta, U, H)(\xi) \tag{2.20}$$

where R is the operator from E to $L^2(\mathbb{R})$ given by

$$R(\zeta,U,H)(\xi)=e^{\zeta(\xi)}\big((g(U)-\frac{1}{2}f'(U)U^2)(1+\zeta_\xi)+\frac{1}{2}f''(U)H_\xi\big)(\xi). \tag{2.21}$$

We claim that A is continuous from $L^2(\mathbb{R})$ into $H^1(\mathbb{R})$. The Fourier transform of h can easily be computed, and we obtain

$$\hat{h}(\eta)=\int_{\mathbb{R}}h(\xi)e^{-2i\pi\eta\xi}\,d\xi=\frac{1}{1+2i\pi\eta}. \tag{2.22}$$

The $H^1(\mathbb{R})$ norm can be expressed in term of the Fourier transform as follows, see, e.g., [14],

$$\|h\star v\|_{H^1(\mathbb{R})}=\left\|(1+\eta^2)^{\frac{1}{2}}\widehat{h\star v}\right\|_{L^2(\mathbb{R})}.$$

Since $\widehat{h\star v}=\hat{h}\hat{v}$, we have

$$\begin{aligned}\|h\star v\|_{H^1(\mathbb{R})}&=\left\|(1+\eta^2)^{\frac{1}{2}}\hat{h}\hat{v}\right\|_{L^2(\mathbb{R})}\\&\leq C\,\|\hat{v}\|_{L^2(\mathbb{R})}\qquad\text{by (2.22)}\\&=C\,\|v\|_{L^2(\mathbb{R})}\qquad\text{by Plancherel equality}\end{aligned}$$

for some constant C. Hence, $A\colon L^2(\mathbb{R})\to H^1(\mathbb{R})$ is continuous. We prove that $R(\zeta,U,H)$ belongs to $L^2(\mathbb{R})$ by using the assumption that $g(0)=0$. Then, $A\circ R(\zeta,U,H)$ belongs to H^1. Let us prove that $R\colon E\to L^2(\mathbb{R})$ is locally Lipschitz. For that purpose we will use the following short lemma.

Lemma 2.2. *Let $B_M=\{X\in E\mid \|X\|_E\leq M\}$ be a bounded set of E.*
(i) If g_1 is Lipschitz from B_M to $L^\infty(\mathbb{R})$ and g_2 Lipschitz from B_M to $L^2(\mathbb{R})$, then the product g_1g_2 is Lipschitz from B_M to $L^2(\mathbb{R})$.
(ii) If g_1,g_2 are two Lipschitz maps from B_M to $L^\infty(\mathbb{R})$, then the product g_1g_2 is Lipschitz from B_M to $L^\infty(\mathbb{R})$.

Proof of Lemma 2.2. Let X and $\bar{X}$ be in B_M, and assume that g_1 and g_2 satisfy the assumptions of (i). We denote by L_1 and L_2, the Lipschitz constants of g_1 and g_2, respectively. We have

$$\begin{aligned}&\left\|g_1(X)g_2(X)-g_1(\bar{X})g_2(\bar{X})\right\|_{L^2(\mathbb{R})}\\&\quad\leq\left\|g_1(X)-g_1(\bar{X})\right\|_{L^\infty(\mathbb{R})}\|g_2(X)\|_{L^2(\mathbb{R})}+\left\|g_1(\bar{X})\right\|_{L^\infty(\mathbb{R})}\left\|g_2(X)-g_2(\bar{X})\right\|_{L^2(\mathbb{R})}\\&\quad\leq\left[2L_1L_2M+L_1\|g_2(0)\|_{L^2(\mathbb{R})}+L_2\|g_1(0)\|_{L^\infty(\mathbb{R})}\right]\left\|X-\bar{X}\right\|_E\end{aligned}$$

and (i) is proved. One proves (ii) the same way. □

Let us consider a bounded set $B_M=\{X\in E\mid \|X\|_E\leq M\}$ of E. For $X=(\zeta,U,H)$ and $\bar{X}=(\bar{\zeta},\bar{U},\bar{H})$ in B_M, we have $\|U\|_{L^\infty(\mathbb{R})}\leq\frac{1}{\sqrt{2}}\|U\|_{H^1(\mathbb{R})}\leq\frac{1}{\sqrt{2}}M$, because $\frac{1}{\sqrt{2}}$ is the constant of the Sobolev embedding from $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$, and, similarly, $\left\|\bar{U}\right\|_{L^\infty(\mathbb{R})}\leq\frac{1}{\sqrt{2}}M$. Let $I_M=[-\frac{1}{\sqrt{2}}M,\frac{1}{\sqrt{2}}M]$ and

$$L_M=\|f\|_{W^{3,\infty}(I_M)}+\|g\|_{W^{1,\infty}(I_M)}<\infty. \tag{2.23}$$

Then

$$\left\|g(\bar{U})-g(U)\right\|_{L^2(\mathbb{R})}\leq\|g\|_{W^{1,\infty}(I_M)}\left\|\bar{U}-U\right\|_{L^2(\mathbb{R})}\leq L_M\left\|\bar{U}-U\right\|_{L^2(\mathbb{R})}.$$

Hence, $g_1\colon X\to g(U)$ is Lipschitz from B_M to $L^2(\mathbb{R})$. For $X,\bar{X}$ in B_M, we have $\|\zeta\|_{L^\infty(\mathbb{R})}\leq M$ and $\left\|\bar{\zeta}\right\|_{L^\infty(\mathbb{R})}\leq M$. The function $x\mapsto e^x$ is Lipschitz on $\{x\in\mathbb{R}\mid |x|\leq M\}$. Hence, $g_2\colon X\mapsto e^\zeta$ is Lipschitz from B_M to $L^\infty(\mathbb{R})$. Thus, the first term in (2.21), $g_1(X)g_2(X) = e^{\zeta}g(U)$, is, by Lemma 2.2, Lipschitz from B_M to $L^2(\mathbb{R})$. We look at the second term, that is, $e^{\zeta}g(U)\zeta_\xi$. For $X, \bar{X}$ in B_M, we have

$$\left\|g(U) - g(\bar{U})\right\|_{L^\infty(\mathbb{R})} \leq \|g\|_{W^{1,\infty}(I_M)} \left\|U - \bar{U}\right\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} L_M \left\|U - \bar{U}\right\|_{H^1(\mathbb{R})}.$$

Hence, $X \mapsto g(U)$ is Lipschitz from B_M to $L^\infty(\mathbb{R})$ and, by Lemma 2.2, as $X \mapsto e^{\zeta}$ is also Lipschitz from B_M to $L^\infty(\mathbb{R})$, we have that the product $X \mapsto e^{\zeta}g(U)$ is Lipschitz from B_M to $L^\infty(\mathbb{R})$. After using again Lemma 2.2, since $X \mapsto \zeta_\xi$, being linear, is obviously Lipschitz from B_M to $L^2(\mathbb{R})$, we obtain, as claimed, that the second term in (2.21), $e^{\zeta}g(U)\zeta_\xi$, is Lipschitz from B_M to $L^2(\mathbb{R})$. We can handle the other terms in (2.21) similarly and prove that R is Lipschitz from B_M to $L^2(\mathbb{R})$. Since $A\colon L^2(\mathbb{R}) \to H^1(\mathbb{R})$ is linear and continuous, $A \circ R$ is Lipschitz from B_M to $H^1(\mathbb{R})$. Then, we use the following lemma whose proof is basically the same as the proof of Lemma 2.2.

Lemma 2.3. *Let $\mathcal{R}_1\colon B_M \to V$, $\mathcal{R}_2\colon B_M \to H^1(\mathbb{R})$, and $\mathcal{R}_3\colon B_M \to V$ be three Lipschitz maps. Then, the products $X \mapsto R_1(X)R_2(X)$ and $X \mapsto R_1(X)R_3(X)$ are also Lipschitz maps from B_M to $H^1(\mathbb{R})$ and B_M to V, respectively.*

Since the map $X \mapsto e^{-\zeta}$ is Lipschitz from B_M to V, Q_1 is the product of two Lipschitz maps, one from B_M to $H^1(\mathbb{R})$ and the other from B_M to V, and therefore it is Lipschitz map from B_M to $H^1(\mathbb{R})$. Similarly, one proves that $\mathcal{Q}_2$ and therefore $\mathcal{Q}$ are Lipschitz on B_M. Furthermore, $\mathcal{P}$ is Lipschitz on B_M. The formulas (2.17) and (2.18) are obtained by direct computation using the product rule, see [13, p. 129]. $\square$

In the next theorem, by using a contraction argument, we prove the short-time existence of solutions to (2.15).

Theorem 2.4. *Given $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{H})$ in E, there exists a time T depending only on $\left\|\bar{X}\right\|_E$ such that the system (2.15) admits a unique solution in $C^1([0,T], E)$ with initial data $\bar{X}$.*

Proof. Solutions of (2.15) can be rewritten as

$$X(t) = \bar{X} + \int_0^t F(X(\tau))\, d\tau \tag{2.24}$$

where $F\colon E \to E$ is given by $F(X) = (f'(U), -\mathcal{Q}(X), G(U) - 2\mathcal{P}(X)U)$ where $X = (\zeta, U, H)$. The integrals are defined as Riemann integrals of continuous functions on the Banach space E. Let B_M and L_M be defined as in the proof of Lemma 2.1, see (2.23). We claim that $X = (\zeta, U, H) \mapsto f'(U)$ and $X = (\zeta, U, H) \mapsto G(U)$ are Lipschitz from B_M to V. Then, using Lemma 2.1, we can check that each component of $F(X)$ is a product of functions that satisfy one of the assumptions of Lemma 2.3 and using this same lemma, we obtain that $F(X)$ is Lipchitz on B_M. Thus, F is Lipschitz on any bounded set of E. Since E is a Banach space, we use the standard contraction argument to show the existence of short-time solutions and the theorem is proved. For any $X = (\zeta, U, H)$ and $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{H})$ in B_M, we have $\|U\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}}M$ and $\left\|\bar{U}\right\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}}M$. Then,

$$\left\|f'(U) - f'(\bar{U})\right\|_{L^\infty(\mathbb{R})} \leq \|f'\|_{W^{1,\infty}(I_M)} \left\|U - \bar{U}\right\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2}} L_M \left\|X - \bar{X}\right\|_E$$

and $X \mapsto f'(U)$ is Lipschitz from B_M into $L^\infty(\mathbb{R})$. Since f' is C^1 and $U \in H^1(\mathbb{R})$, using [19, Appendix A.1], we obtain that $f'(U)_\xi \in L^2(\mathbb{R})$ and

$$f'(U)_\xi = f''(U)U_\xi.$$

As before, it is not hard to prove that $X \mapsto f''(U)$ is Lipschitz from B_M into $L^\infty(\mathbb{R})$. It is clear that $X \mapsto U_\xi$ is Lipschitz from B_M into $L^2(\mathbb{R})$. Hence, it follows from Lemma 2.2 that $X \mapsto f''(U)U_\xi$ is Lipschitz from B_M into $L^2(\mathbb{R})$. Therefore, $X \mapsto f'(U)$ is Lipschitz from B_M into V. Similarly, one proves that $X \mapsto G(U)$ is Lipschitz from B_M into V and our previous claim is proved. $\square$

We now turn to the proof of existence of global solutions of (2.15). We are interested in a particular class of initial data that we are going to make precise later, see Definition 2.5. In particular, we will only consider initial data that belong to $E \cap \left[W^{1,\infty}(\mathbb{R})\right]^3$ where

$$W^{1,\infty}(\mathbb{R}) = \{f \in C_b(\mathbb{R}) \mid f_\xi \in L^\infty(\mathbb{R})\}.$$

Given $(\bar\zeta, \bar U, \bar H) \in E \cap [W^{1,\infty}(\mathbb{R})]^3$, we consider the short-time solution $(\zeta, U, H) \in C([0,T], E)$ of (2.15) given by Theorem 2.4. Using the fact that $\mathcal{Q}$ and $\mathcal{P}$ are Lipschitz on bounded sets (Lemma 2.1) and, since $X \in C([0,T], E)$, we can prove that P and Q belongs to $C([0,T], H^1(\mathbb{R}))$. We now consider U, P and Q as given function in $C([0,T], H^1(\mathbb{R}))$. Then, for any *fixed* $\xi \in \mathbb{R}$, we can solve the system of ordinary differential equations

$$\begin{cases} \dfrac{d}{dt}\alpha(t,\xi) = f''(U)\beta(t,\xi), \\ \dfrac{d}{dt}\beta(t,\xi) = \dfrac{1}{2}f''(U)\gamma(t,\xi) + \big(-\dfrac{1}{2}f''(U)U^2 + g(U) - P\big)(1+\alpha)(t,\xi), \\ \dfrac{d}{dt}\gamma(t,\xi) = -2(Q\,U)(1+\alpha)(t,\xi) + (2g(U) + f''(U)U^2 - 2P)\beta(t,\xi), \end{cases} \tag{2.25}$$

which is obtained by substituting ζ_ξ, U_ξ and H_ξ in (2.16) by the unknowns α, β, and γ, respectively. Concerning the initial data, we set $(\alpha(0,\xi), \beta(0,\xi), \gamma(0,\xi)) = (\bar\zeta_\xi, \bar U_\xi, \bar H_\xi)$ if $\left|\bar\zeta_\xi(\xi)\right| + \left|\bar U_\xi(\xi)\right| + \left|\bar H_\xi(\xi)\right| < \infty$ and $(\alpha(0,\xi), \beta(0,\xi), \gamma(0,\xi)) = (0,0,0)$ otherwise. In the same way as in [16, Lemma 2.4], see also Lemma 2.6 below, we can prove that solutions of (2.25) exist in $[0,T]$ and that, for all time $t \in [0,T]$,

$$(\alpha(t,\xi), \beta(t,\xi), \gamma(t,\xi)) = (\zeta_\xi(t,\xi), U_\xi(t,\xi), H_\xi(t,\xi))$$

for almost every $\xi \in \mathbb{R}$. Thus, we can select a special representative for $(\zeta_\xi, U_\xi, H_\xi)$ given by (α, β, γ), which is defined for all $\xi \in \mathbb{R}$ and which, for any given ξ, satisfies the ordinary differential equation (2.25) in $\mathbb{R}^3$. From now on we will of course identify the two and set $(\zeta_\xi, U_\xi, H_\xi)$ equal to (α, β, γ).

Our goal is to find solutions of (1.5) with initial data $\bar u$ in H^1 because H^1 is the natural space for the equation. However, Theorem 2.4 gives us the existence of solutions to (2.15) for initial data in E. Therefore we have to find initial conditions that match the initial data $\bar u$ and belong to E. A natural choice would be to use $\bar y(\xi) = y(0,\xi) = \xi$ and $\bar U(\xi) = u(\xi)$. Then $y(t,\xi)$ gives the position of the particle which is at ξ at time $t = 0$. But, if we make this choice, then $\bar H_\xi = \bar u^2 + \bar u_x^2$ and $\bar H_\xi$ does not belong to $L^2(\mathbb{R})$ in general. We consider instead $\bar y$ implicitly given by

$$\xi = \int_{-\infty}^{\bar y(\xi)} (\bar u^2 + \bar u_x^2)\,dx + \bar y(\xi) \tag{2.26a}$$

and

$$\bar U = \bar u \circ \bar y\,, \tag{2.26b}$$

$$\bar H = \int_{-\infty}^{\bar y} \left(\bar u^2 + \bar u_x^2\right)\,dx. \tag{2.26c}$$

In the next lemma we prove that $(\bar y, \bar U, \bar H)$ belongs to the set $\mathcal{G}$ where $\mathcal{G}$ is defined as follows.

Definition 2.5. The set $\mathcal{G}$ is consists of all $(\zeta, U, H) \in E$ such that

$$(\zeta, U, H) \in \left[W^{1,\infty}(\mathbb{R})\right]^3, \tag{2.27a}$$

$$y_\xi \geq 0, H_\xi \geq 0, y_\xi + H_\xi > 0 \text{ almost everywhere, and } \lim_{\xi \to -\infty} H(\xi) = 0, \tag{2.27b}$$

$$y_\xi H_\xi = y_\xi^2 U^2 + U_\xi^2 \text{ almost everywhere,} \tag{2.27c}$$

where we denote $y(\xi) = \zeta(\xi) + \xi$.

Lemma 2.6. *Given $\bar{u} \in H^1(\mathbb{R})$, then $(\bar{y}, \bar{U}, \bar{H})$ as defined in (2.26) belongs to $\mathcal{G}$.*

Proof. The function $k \colon z \mapsto \int_0^z (\bar{u}^2 + \bar{u}_x^2)(x)\,dx + z$ is a strictly increasing continuous function with $\lim_{z\to\pm\infty} k(z) = \pm\infty$. Hence, k is invertible and $\bar{y}(\xi) = k^{-1}(\xi)$ is well-defined. We have to check that $(\bar{\zeta}, \bar{U}, \bar{H})$ belongs to E. It follows directly from the definition that $\bar{y}$ is a strictly increasing function. We have

$$\bar{\zeta}(\xi) = -\int_{-\infty}^{\bar{y}(\xi)} \left(\bar{u}^2 + \bar{u}_x^2\right)\,dx, \tag{2.28}$$

and therefore, since $\bar{u} \in H^1$, $\bar{\zeta}$ is bounded. For any $(\xi, \xi') \in \mathbb{R}^2$, we have

$$\begin{aligned} |\xi - \xi'| &= \left| \int_{\bar{y}(\xi')}^{\bar{y}(\xi)} (\bar{u}^2 + \bar{u}_x^2)\,dx + \bar{y}(\xi) - \bar{y}(\xi') \right| \\ &= \left| \int_{\bar{y}(\xi')}^{\bar{y}(\xi)} (\bar{u}^2 + \bar{u}_x^2)\,dx \right| + |\bar{y}(\xi) - \bar{y}(\xi')| \end{aligned} \tag{2.29}$$

because the two quantities inside the absolute values have the same sign. It follows from (2.29) that $\bar{y}$ is Lipschitz (with Lipschitz constant at most 1) and therefore almost everywhere differentiable. From (2.28), we get that, for almost every $\xi \in \mathbb{R}$,

$$\bar{\zeta}_\xi = -(\bar{u}^2 + \bar{u}_x^2)\circ\bar{y}\,\bar{y}_\xi$$

Since $\bar{y}_\xi = 1 + \bar{\zeta}_\xi$, it implies

$$\bar{\zeta}_\xi(\xi) = -\frac{\bar{u}^2 + \bar{u}_x^2}{1 + \bar{u}^2 + \bar{u}_x^2}\circ\bar{y}(\xi)\,. \tag{2.30}$$

Therefore $\bar{\zeta}_\xi$ is bounded almost everywhere and $\bar{\zeta}$ belongs to $W^{1,\infty}(\mathbb{R})$. We also have

$$\bar{y}_\xi = \frac{1}{1 + \bar{u}^2 + \bar{u}_x^2}\circ\bar{y} \tag{2.31}$$

which implies that $\bar{y}_\xi > 0$ almost everywhere. From (2.28), we see that $\bar{H} = -\bar{\zeta}$ and therefore $\bar{H}$ belongs to $W^{1,\infty}(\mathbb{R})$. Since $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, $\bar{U} = \bar{u}\circ\bar{y}$ is bounded. We have, for almost every $\xi \in \mathbb{R}$,

$$\bar{H}_\xi = (\bar{u}^2 + \bar{u}_x^2)\circ\bar{y}\,\bar{y}_\xi \tag{2.32}$$

which, since $\bar{U}_\xi = \bar{u}_x\circ\bar{y}\,y_\xi$ almost everywhere, gives us

$$\bar{y}_\xi \bar{H}_\xi = \bar{y}_\xi^2 \bar{U}^2 + \bar{U}_\xi^2. \tag{2.33}$$

Hence, $\bar{U}_\xi^2 \leq \bar{y}_\xi \bar{H}_\xi$ and $\bar{U}_\xi$ is bounded and $\bar{U}$ belongs to $W^{1,\infty}(\mathbb{R})$. We have $(\bar{\zeta}, \bar{U}, \bar{H}) \in [W^{1,\infty}(\mathbb{R})]^3$. It remains to prove that $\bar{U}$, $\bar{\zeta}_\xi$, $\bar{U}_\xi$ and $\bar{H}_\xi$ belong to $L^2(\mathbb{R})$. By making the change of variable $x = \bar{y}(\xi)$ and using (2.31), we obtain

$$\begin{aligned} \left\|\bar{U}\right\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \bar{u}^2(x)(1 + \bar{u}^2 + \bar{u}_x^2)(x)\,dx \\ &\leq \|\bar{u}\|_{L^2(\mathbb{R})}^2 + \|\bar{u}\|_{L^\infty(\mathbb{R})}^2 \|\bar{u}\|_{H^1(\mathbb{R})}^2\,. \end{aligned}$$

Hence, $\bar{U} \in L^2(\mathbb{R})$. Since $0 \le \bar{H}_\xi \le 1$, $\bar{H}$ is monotone and

$$\left\|\bar{H}_\xi\right\|_{L^2(\mathbb{R})}^2 \le \left\|\bar{H}_\xi\right\|_{L^\infty(\mathbb{R})} \left\|\bar{H}_\xi\right\|_{L^1(\mathbb{R})} \le \lim_{\xi\to\infty} \bar{H}(\xi) = \|\bar{u}\|_{H^1(\mathbb{R})}^2 .$$

Hence, $\bar{H}_\xi$, and therefore $\bar{\zeta}_\xi$, belong to $L^2(\mathbb{R})$. From (2.33) we get

$$\left\|\bar{U}_\xi\right\|_{L^2(\mathbb{R})}^2 \le \left\|\bar{y}_\xi \bar{H}_\xi\right\|_{L^1(\mathbb{R})} \le (1 + \left\|\bar{\zeta}_\xi\right\|_{L^\infty(\mathbb{R})}) \left\|\bar{H}\right\|_{L^\infty(\mathbb{R})}$$

and $\bar{U}_\xi \in L^2(\mathbb{R})$. □

For initial data in $\mathcal{G}$, the solution of (2.15) exists globally. To prove that we will use the following lemma.

Lemma 2.7. *Given initial data $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{H})$ in $\mathcal{G}$, let $X(t) = (\zeta(t), U(t), H(t))$ be the short-time solution of* (2.15) *in $C([0,T], E)$ for some $T > 0$ with initial data $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{H})$. Then,*
(*i*) *$X(t)$ belongs to $\mathcal{G}$ for all $t \in [0,T]$, that is, $\mathcal{G}$ is preserved by the flow.*
(*ii*) *for almost every $t \in [0,T]$, $y_\xi(t,\xi) > 0$ for almost every $\xi \in \mathbb{R}$,*
(*iii*) *For all $t \in [0,T]$, $\lim_{\xi\to\pm\infty} H(t,\xi)$ exists and is independent of time.*

We denote by $\mathcal{A}$ the set of all $\xi \in \mathbb{R}$ for which $\left|\bar{\zeta}_\xi(\xi)\right| + \left|\bar{U}_\xi(\xi)\right| + \left|\bar{H}_\xi(\xi)\right| < \infty$ and the relations in (2.27b) and (2.27c) are fulfilled for $\bar{y}_\xi$, $\bar{U}_\xi$ and $\bar{H}_\xi$. Since by assumption $\bar{X} \in \mathcal{G}$, we have $\operatorname{meas}(\mathcal{A}^c) = 0$, and we set $(\bar{U}_\xi, \bar{H}_\xi, \bar{\zeta}_\xi)$ equal to zero on $\mathcal{A}^c$. Then, as we explained earlier, we choose a special representative for $(\zeta(t,\xi), U(t,\xi), H(t,\xi))$ which satisfies (2.16) as an ordinary differential equation in $\mathbb{R}^3$ for every $\xi \in \mathbb{R}$.

Proof. (*i*) The fact that $W^{1,\infty}(\mathbb{R})$ is preserved by the equation can be proved in the same way as in [16, Lemma 2.4] and we now give only a sketch of this proof. We look at (2.15) as a system of ordinay differential equations in $E \cap W^{1,\infty}(\mathbb{R})$. We have already established the short-time existence of solutions in E, and, since (2.16) is semilinear with respect to y_ξ, U_ξ and H_ξ (and affine with respect to ζ_ξ, U_ξ and H_ξ), it is not hard to establish, by a contraction argument, the short-time existence of solutions in $E \cap W^{1,\infty}(\mathbb{R})$. Let $C_1 = \sup_{t\in[0,T]} \big(\|U(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|P(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|Q(t,\cdot)\|_{L^\infty(\mathbb{R})} \big)$ and $Z(t) = \|\zeta_\xi(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|U_\xi(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|H_\xi(t,\cdot)\|_{L^\infty(\mathbb{R})}$. We have $C_1 \le \sup_{t\in[0,T]} \|X(t,\cdot)\|_E < \infty$. Using again the semi-linearity of (2.16), we get that

$$Z(t) \le Z(0) + CT + C\int_0^t Z(\tau)\,d\tau$$

for a constant C that only depends on C_1. Hence, it follows from Gronwall's lemma that $\sup_{t\in[0,T)} Z(t) < \infty$, which proves that the space $W^{1,\infty}(\mathbb{R})$ is preserved by the flow of (2.15). Let us prove that (2.27c) and the inequalities in (2.27b) hold for any $\xi \in \mathcal{A}$ and therefore almost everywhere. We consider a fixed ξ in $\mathcal{A}$ and drop it in the notations when there is no ambiguity. From (2.16), we have, on the one hand,

$$\begin{aligned}(y_\xi H_\xi)_t &= y_{\xi t} H_\xi + H_{\xi t} y_\xi \\ &= f''(U) U_\xi H_\xi + \big(G'(U) U_\xi - 2QU y_\xi - 2PU_\xi\big) y_\xi \\ &= f''(U) U_\xi H_\xi + \big(2g(U) + f''(U) U^2\big) U_\xi y_\xi - 2Q y_\xi^2 U - 2P U_\xi y_\xi\end{aligned}$$

and, on the other hand,

$$\begin{aligned}(y_\xi^2 U^2 + U_\xi^2)_t &= 2y_{\xi t} y_\xi U^2 + 2y_\xi^2 U_t U + 2U_{\xi t} U_\xi \\ &= 2f''(U) U_\xi y_\xi U^2 - 2y_\xi^2 QU \\ &\quad + 2U_\xi\big(\frac{1}{2} f''(U) H_\xi - \frac{1}{2} f''(U) U^2 y_\xi + g(U) y_\xi - P y_\xi\big).\end{aligned}$$

Thus, $(y_\xi H_\xi - y_\xi^2 U^2 - U_\xi^2)_t = 0$, and since $y_\xi H_\xi(0) = (y_\xi^2 U^2 + U_\xi^2)(0)$, we have $y_\xi H_\xi(t) = (y_\xi^2 U^2 + U_\xi^2)(t)$ for all $t \in [0,T]$. We have proved (2.27c). Let us introduce t^* given by

$$t^* = \sup\{t \in [0,T] \mid y_\xi(t') \geq 0 \text{ for all } t' \in [0,t]\}.$$

Recall that we consider a fixed $\xi \in \mathcal{A}$, and drop it in the notation. Assume that $t^* < T$. Since $y_\xi(t)$ is continuous with respect to time, we have

$$y_\xi(t^*) = 0. \tag{2.34}$$

Hence, from (2.27c) that we just proved, $U_\xi(t^*) = 0$ and, by (2.16),

$$y_{\xi t}(t^*) = f''(U)U_\xi(t^*) = 0. \tag{2.35}$$

From (2.16), since $y_\xi(t^*) = U_\xi(t^*) = 0$, we get

$$y_{\xi tt}(t^*) = f''(U)U_{\xi t}(t^*) = \frac{1}{2}f''(U)^2 H_\xi(t^*). \tag{2.36}$$

If $H_\xi(t^*) = 0$, then $(y_\xi, U_\xi, H_\xi)(t^*) = (0,0,0)$ and, by the uniqueness of the solution of (2.16), seen as a system of ordinary differential equations, we must have $(y_\xi, U_\xi, H_\xi)(t) = 0$ for all $t \in [0,T]$. This contradicts the fact that $y_\xi(0)$ and $H_\xi(0)$ cannot vanish at the same time ($\bar{y}_\xi + \bar{H}_\xi > 0$ for all $\xi \in \mathcal{A}$). If $H_\xi(t^*) < 0$, then $y_{\xi tt}(t^*) < 0$ because f does not vanish and, because of (2.34) and (2.35), there exists a neighborhood $\mathcal{U}$ of t^* such that $y(t) < 0$ for all $t \in \mathcal{U} \setminus \{t^*\}$. This contradicts the definition of t^*. Hence,

$$H_\xi(t^*) > 0, \tag{2.37}$$

and, since we now have $y_\xi(t^*) = y_{\xi t}(t^*) = 0$ and $y_{\xi tt}(t^*) > 0$, there exists a neighborhood of t^* that we again denote by $\mathcal{U}$ such that $y_\xi(t) > 0$ for all $t \in \mathcal{U} \setminus \{t^*\}$. This contradicts the fact that $t^* < T$, and we have proved the first inequality in (2.27b), namely that $y_\xi(t) \geq 0$ for all $t \in [0,T]$. Let us prove that $H_\xi(t) \geq 0$ for all $t \in [0,T]$. This follows from (2.27c) when $y_\xi(t) > 0$. Now, if $y_\xi(t) = 0$, then $U_\xi(t) = 0$ from (2.27c), and we have seen that $H_\xi(t) < 0$ would imply that $y_\xi(t') < 0$ for some t' in a punctured neighborhood of t, which is impossible. Hence, $H_\xi(t) \geq 0$, and we have proved the second inequality in (2.27b). Assume that the third inequality in (2.27c) does not hold. Then, by continuity, there exists a time $t \in [0,T]$ such that $(y_\xi + H_\xi)(t) = 0$. Since y_ξ and H_ξ are positive, we must have $y_\xi(t) = H_\xi(t) = 0$ and, by (2.27c), $U_\xi(t) = 0$. Since zero is a solution of (2.16), this implies that $y_\xi(0) = U_\xi(0) = H_\xi(0)$, which contradicts $(y_\xi + H_\xi)(0) > 0$. The fact that $\lim_{\xi \to -\infty} H(t,\xi) = 0$ will be proved below in (*iii*).

(*ii*) We define the set

$$\mathcal{N} = \{(t,\xi) \in [0,T] \times \mathbb{R} \mid y_\xi(t,\xi) = 0\}.$$

Fubini's theorem gives us

$$\operatorname{meas}(\mathcal{N}) = \int_{\mathbb{R}} \operatorname{meas}(\mathcal{N}_\xi)\, d\xi = \int_{[0,T]} \operatorname{meas}(\mathcal{N}_t)\, dt \tag{2.38}$$

where $\mathcal{N}_\xi$ and $\mathcal{N}_t$ are the ξ-section and t-section of $\mathcal{N}$, respectively, that is,

$$\mathcal{N}_\xi = \{t \in [0,T] \mid y_\xi(t,\xi) = 0\}$$

and

$$\mathcal{N}_t = \{\xi \in \mathbb{R} \mid y_\xi(t,\xi) = 0\}.$$

Let us prove that, for all $\xi \in \mathcal{A}$, $\operatorname{meas}(\mathcal{N}_\xi) = 0$. If we consider the sets $\mathcal{N}_\xi^n$ defined as

$$\mathcal{N}_\xi^n = \{t \in [0,T] \mid y_\xi(t,\xi) = 0 \text{ and } y_\xi(t',\xi) > 0 \text{ for all } t' \in [t - 1/n, t + 1/n] \setminus \{t\}\},$$

then

$$\mathcal{N}_\xi = \bigcup_{n\in\mathbb{N}} \mathcal{N}_\xi^n. \tag{2.39}$$

Indeed, for all $t \in \mathcal{N}_\xi$, we have $y_\xi(t,\xi) = 0$, $y_{\xi t}(t,\xi) = 0$ from (2.27c) and (2.16) and $y_{\xi tt}(t,\xi) = \frac{1}{2}f''(U)^2 H_\xi(t,\xi) > 0$ from (2.16) and (2.27b) (y_ξ and H_ξ cannot vanish at the same time for $\xi \in \mathcal{A}$). Since f'' does not vanish, this implies that, on a small punctured neighborhood of t, y_ξ is strictly positive. Hence, t belongs to some $\mathcal{N}_\xi^n$ for n large enough. This proves (2.39). The set $\mathcal{N}_\xi^n$ consists of isolated points that are countable since, by definition, they are separated by a distance larger than $1/n$ from one another. This means that $\operatorname{meas}(\mathcal{N}_\xi^n) = 0$ and, by the subadditivity of the measure, $\operatorname{meas}(\mathcal{N}_\xi) = 0$. It follows from (2.38) and since $\operatorname{meas}(\mathcal{A}^c) = 0$ that

$$\operatorname{meas}(\mathcal{N}_t) = 0 \text{ for almost every } t \in [0,T]. \tag{2.40}$$

We denote by $\mathcal{K}$ the set of times such that $\operatorname{meas}(\mathcal{N}_t) > 0$, i.e.,

$$\mathcal{K} = \{t \in \mathbb{R}_+ \mid \operatorname{meas}(\mathcal{N}_t) > 0\}. \tag{2.41}$$

By (2.40), $\operatorname{meas}(\mathcal{K}) = 0$. For all $t \in \mathcal{K}^c$, $y_\xi > 0$ almost everywhere and, therefore, $y(t,\xi)$ is strictly increasing and invertible (with respect to ξ).

(*iii*) For any given $t \in [0,T]$, since $H_\xi(t,\xi) \geq 0$, $H(t,\xi)$ is an increasing function with respect to ξ and therefore, as $H(t,\cdot\,) \in L^\infty(\mathbb{R})$, $H(t,\xi)$ has a limit when $\xi \to \pm\infty$. We denote those limits $H(t,\pm\infty)$. Since $U(t,\cdot\,) \in H^1(\mathbb{R})$, we have $\lim_{\xi\to\pm\infty} U(t,\xi) = 0$ for all $t \in [0,T]$. We have

$$H(t,\xi) = H(0,\xi) + \int_0^t [G(U) - 2PU](\tau,\xi)\, d\tau \tag{2.42}$$

and $\lim_{\xi\to\pm\infty} G(U(t,\xi)) = 0$ because $\lim_{\xi\to\pm\infty} U(t,\xi) = 0$, $G(0) = 0$ and G is continuous. As U, $G(U)$ and P are bounded in $L^\infty([0,T]\times\mathbb{R})$, we can let ξ tend to $\pm\infty$ and apply the Lebesgue dominated convergence theorem. We get $H(t,\pm\infty) = H(0,\pm\infty)$ for all $t \in [0,T]$. Since $\bar{X} \in \mathcal{G}$, $H(0,-\infty) = 0$ and therefore $H(t,-\infty) = 0$ for all $t \in [0,T]$. □

In the next theorem, we prove global existence of solutions to (2.15). We also state that the solutions are continuous with respect to the functions $(f,g) \in \mathcal{E}$ (cf. (1.6)) that appear in (1.5). Therefore we need to specify the topology we use on $\mathcal{E}$. The space $L^\infty_{\mathrm{loc}}(\mathbb{R})$ is a locally convex linear topological space. Let K_j be a given increasing sequence of compact sets such that $\mathbb{R} = \cup_{j\in\mathbb{N}} K_j$, then the topology of $L^\infty_{\mathrm{loc}}(\mathbb{R})$ is defined by the sequence of semi-norms $h \mapsto \|h\|_{L^\infty(K_n)}$. The space $L^\infty_{\mathrm{loc}}(\mathbb{R})$ is metrizable, see [14, Proposition 5.16]. A subset B of $L^\infty_{\mathrm{loc}}(\mathbb{R})$ is bounded if, for all $n \geq 1$, there exists $C_n > 0$ such that $\|f\|_{L^\infty(K_n)} \leq C_n$ for all $f \in B$, see [21, I.7] for the general definition of bounded sets in a linear topological space. The topologies of $W^{k,\infty}_{\mathrm{loc}}(\mathbb{R})$ follows naturally from the topology of $L^\infty_{\mathrm{loc}}(\mathbb{R})$ applied to the k first derivatives. We equip $\mathcal{E}$ with the topology induced $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}) \times L^\infty_{\mathrm{loc}}(\mathbb{R})$. We will also consider bounded subsets of $\mathcal{E}$ in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}) \times W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$. A subset $\mathcal{E}'$ of $\mathcal{E}$ is bounded in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}) \times W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ if for all $n \geq 1$, there exists C_n such that $\|f\|_{W^{2,\infty}(K_n)} + \|g\|_{W^{1,\infty}(K_n)} \leq C_n$ for all $(f,g) \in \mathcal{E}'$. In the remaining, by bounded sets of $\mathcal{E}$ we will always implicitly mean bounded sets of $\mathcal{E}$ in $W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}) \times W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$.

Theorem 2.8. *Assume* (1.6). *For any* $\bar{X} = (\bar{y},\bar{U},\bar{H}) \in \mathcal{G}$, *the system* (2.15) *admits a unique global solution* $X(t) = (y(t),U(t),H(t))$ *in* $C^1(\mathbb{R}_+,E)$ *with initial data* $\bar{X} = (\bar{y},\bar{U},\bar{H})$. *We have* $X(t) \in \mathcal{G}$ *for all times. If we equip* $\mathcal{G}$ *with the topology induced by the* E*-norm, then the map* $S\colon \mathcal{G}\times\mathcal{E}\times\mathbb{R}_+ \to \mathcal{G}$ *defined as*

$$S_t(\bar{X},f,g) = X(t)$$

is a semigroup which is continuous with respect to all variables, on any bounded set of $\mathcal{E}$.

Proof. The solution has a finite time of existence T only if $\|X(t,\cdot)\|_E$ blows up when t tends to T because, otherwise, by Theorem 2.4, the solution can be extended by a small time interval beyond T. Thus, We want to prove that

$$\sup_{t\in[0,T)} \|X(t,\cdot)\|_E < \infty.$$

Since $X(t) \in \mathcal{G}$, $H_\xi \geq 0$, from (2.27b), and $H(t,\xi)$ is an increasing function in ξ for all t and, from Lemma 2.7, we have $\lim_{\xi\to\infty} H(t,\xi) = \lim_{\xi\to\infty} H(0,\xi)$. Hence, $\sup_{t\in[0,T)} \|H(t,\cdot)\|_{L^\infty(\mathbb{R})} = \|\bar{H}\|_{L^\infty(\mathbb{R})}$ and therefore $\sup_{t\in[0,T)} \|H(t,\cdot)\|_{L^\infty(\mathbb{R})}$ is finite. To simplify the notation we suppress the dependence in t for the moment and denote $h = \|\bar{H}\|_{L^\infty(\mathbb{R})}$. We have

$$U^2(\xi) = 2\int_{-\infty}^{\xi} U(\eta)U_\xi(\eta)\,d\eta = 2\int_{\{\eta\leq\xi \mid y_\xi(\eta)>0\}} U(\eta)U_\xi(\eta)\,d\eta \tag{2.43}$$

since, from (2.27c), $U_\xi(\eta) = 0$ when $y_\xi(\eta) = 0$. For almost every ξ such that $y_\xi(\xi) > 0$, we have

$$|U(\xi)U_\xi(\xi)| = \left|\sqrt{y_\xi}U(\xi)\frac{U_\xi(\xi)}{\sqrt{y_\xi(\xi)}}\right| \leq \frac{1}{2}\left(U(\xi)^2 y_\xi(\xi) + \frac{U_\xi^2(\xi)}{y_\xi(\xi)}\right) = \frac{1}{2}H_\xi(\xi),$$

from (2.27c). Inserting this inequality in (2.43), we obtain $U^2(\xi) \leq H(\xi)$ and we have

$$U(t,\xi) \in I := [-\sqrt{h}, \sqrt{h}] \tag{2.44}$$

for all $t \in [0,T)$ and $\xi \in \mathbb{R}$. Hence, $\sup_{t\in[0,T)} \|U(t,\cdot)\|_{L^\infty(\mathbb{R})} < \infty$. The property (2.44) is important as it says that the $L^\infty(\mathbb{R})$-norm of U is bounded by a constant which does not depend on time. We set

$$\kappa = \|f\|_{W^{2,\infty}(I)} + \|g\|_{W^{1,\infty}(I)}\,.$$

By using (2.44), we obtain

$$\|f'(U)\|_{L^\infty(\mathbb{R})} \leq \|f'\|_{L^\infty(I)} \leq \kappa. \tag{2.45}$$

Hence, from the governing equation (2.15), it follows that

$$|\zeta(t,\xi)| \leq |\zeta(0,\xi)| + \kappa T,$$

and $\sup_{t\in[0,T)} \|\zeta(t,\cdot)\|_{L^\infty(\mathbb{R})}$ is bounded. Next we prove that $\sup_{t\in[0,T)} \|Q(t,\cdot)\|_{L^\infty(\mathbb{R})}$ is finite. We decompose Q into a sum of two integrals that we denote Q_a and Q_b, respectively,

$$\begin{aligned} Q(t,\xi) &= -\frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)e^{-|y(\xi)-y(\eta)|}y_\xi(\eta)\big(g(U) - \frac{1}{2}f''(U)U^2\big)\,d\eta \\ &\quad - \frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)e^{-|y(\xi)-y(\eta)|}f''(U)H_\xi\,d\eta \\ &:= Q_a + Q_b. \end{aligned}$$

Since $y_\xi \geq 0$, we have

$$|Q_a(t,\xi)| \leq C_1\int_{\mathbb{R}} e^{-|y(\xi)-y(\eta)|}y_\xi(\eta)\,d\eta = C_1\int_{\mathbb{R}} e^{-|y(\xi)-x|}\,dx = 2C_1$$

where the constant C_1 depends only on κ and h. Since $H_\xi \geq 0$, we have

$$\begin{aligned}
|Q_b(t,\xi)| &\leq \frac{\kappa}{4}\int_{\mathbb{R}} e^{-|y(\xi)-y(\eta)|}H_\xi(\eta)\,d\eta \\
&= \frac{\kappa}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)e^{-|y(\xi)-y(\eta)|}H(\eta)y_\xi(\eta)\,d\eta \quad \text{(after integrating by parts)} \\
&\leq \frac{\kappa h}{4}\int_{\mathbb{R}} e^{-|y(\xi)-y(\eta)|}y_\xi(\eta)\,d\eta \quad \text{(as } y_\xi \geq 0) \\
&= \frac{\kappa h}{2} \quad \text{(after changing variables).}
\end{aligned}$$

Hence, Q_b and therefore Q are bounded by a constant that depends only on κ and h. Similarly, one proves that $\sup_{t\in[0,T)}\|P(t,\cdot)\|_{L^\infty(\mathbb{R})}$ is bounded by such constant. We denote

$$C_2 = \sup_{t\in[0,T)}\{\|U(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|H(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|\zeta(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|P(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|Q(t,\cdot)\|_{L^\infty(\mathbb{R})}\}. \tag{2.46}$$

We have just proved that C_2 is finite and only depends on $\|\bar{X}\|_E$, T and κ. Let $t\in[0,T)$. We have, as $g(0)=0$,

$$\|g(U(t,\cdot))\|_{L^2(\mathbb{R})} \leq \|g\|_{W^{1,\infty}(I)}\|U\|_{L^2(\mathbb{R})} \leq \kappa\|U\|_{L^2(\mathbb{R})}. \tag{2.47}$$

We use (2.47) and, from (2.21), we obtain that

$$\|R(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(\|U(t,\cdot)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\cdot)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\cdot)\|_{L^2(\mathbb{R})})$$

for some constant C depending only on C_2, h and κ. From now on, we denote generically by C such constants that are increasing functions of $\|\bar{X}\|_E$, T and κ. Since A is a continuous linear map from $L^2(\mathbb{R})$ to $H^1(\mathbb{R})$, it is a fortiori continuous from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, and we get

$$\|A\circ R(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(\|U(t,\cdot)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\cdot)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\cdot)\|_{L^2(\mathbb{R})}).$$

From (2.20), as $\|e^{-\zeta(t,\cdot)}\|_{L^\infty(\mathbb{R})} \leq C$, we obtain that

$$\|Q_1(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(\|U(t,\cdot)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\cdot)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\cdot)\|_{L^2(\mathbb{R})}).$$

The same bound holds for Q_2 and therefore

$$\|Q(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(\|U(t,\cdot)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\cdot)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\cdot)\|_{L^2(\mathbb{R})}). \tag{2.48}$$

Similarly, one proves

$$\|P(t,\cdot)\|_{L^2(\mathbb{R})} \leq C(\|U(t,\cdot)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\cdot)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\cdot)\|_{L^2(\mathbb{R})}). \tag{2.49}$$

Let $Z(t) = \|U(t,\cdot)\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t,\cdot)\|_{L^2(\mathbb{R})} + \|U_\xi(t,\cdot)\|_{L^2(\mathbb{R})} + \|H_\xi(t,\cdot)\|_{L^2(\mathbb{R})}$, then the theorem will be proved once we have established that $\sup_{t\in[0,T)} Z(t) < \infty$. From the integrated version of (2.15) and (2.16), after taking the $L^2(\mathbb{R})$-norms on both sides and adding the relevant terms, we use (2.49), (2.47) and obtain

$$Z(t) \leq Z(0) + C\int_0^t Z(\tau)\,d\tau.$$

Hence, Gronwall's lemma gives us that $\sup_{t\in[0,T)} Z(t) < \infty$. Thus the solution exists globally in time. Moreover we have that

$$\sup_{t\in[0,T]}\|X(t,\cdot)\|_E \leq C(\|\bar{X}\|_E, T, \kappa) \tag{2.50}$$

where C is an increasing function and $\kappa = \|f\|_{W^{3,\infty}(I)} + \|g\|_{W^{1,\infty}(I)}$ with $I = [-\|\bar{H}\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}, \|\bar{H}\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}}]$. Note that in order to obtain (2.47), we needed a bound on g in $W^{1,\infty}_{\rm loc}(\mathbb{R})$, which explains why we are working on a bounded subset of $\mathcal{E}$ in $W^{2,\infty}_{\rm loc}(\mathbb{R})\times W^{1,\infty}_{\rm loc}(\mathbb{R})$.

We now turn to the proof of the continuity of the semigroup. Let $\mathcal{E}'$ be a bounded set of $\mathcal{E}$ in $W^{2,\infty}_{\rm loc}(\mathbb{R})\times W^{1,\infty}_{\rm loc}(\mathbb{R})$. Since $\mathcal{G}$ and $\mathcal{E}$ are metrizable, it is enough to prove sequential continuity. Let $\bar X_n=(\bar y_n,\bar U_n,\bar H_n)\in\mathcal{G}$ and $(f_n,g_n)\in\mathcal{E}'$ be sequences that converge to $\bar X=(\bar y,\bar U,\bar H)\in\mathcal{G}$ and $(f,g)\in\mathcal{E}'$. We denote $X_n(t)=S_t(\bar X_n,f_n,g_n)$ and $X(t)=S_t(\bar X,f,g)$. Let $M=\sup_{n\geq1}\left\|\bar X_n\right\|_E$, we have $\left\|\bar H_n\right\|_{L^\infty(\mathbb{R})}\leq M$ for all $n\geq1$. Hence, from (2.44), it follows that

$$U_n(t,\xi)\in I:=[-\sqrt{M},\sqrt{M}] \tag{2.51}$$

for all $n\geq1$ and $(t,\xi)\in\mathbb{R}_+\times\mathbb{R}$. Since $\mathcal{E}'$ is a bounded set of $\mathcal{E}$ in $W^{2,\infty}_{\rm loc}(\mathbb{R})\times W^{1,\infty}_{\rm loc}(\mathbb{R})$ and $(f_n,g_n)\in\mathcal{E}'$, there exists a constant $\kappa>0$ such that

$$\|f_n\|_{W^{2,\infty}(I)}+\|g_n\|_{W^{1,\infty}(I)}\leq\kappa$$

for all $n\geq1$. Hence, as $I_n:=[-\left\|\bar H_n\right\|^{\frac12}_{L^\infty(\mathbb{R})},\left\|\bar H_n\right\|^{\frac12}_{L^\infty(\mathbb{R})}]\subset I$,

$$\kappa_n=\|f_n\|_{W^{2,\infty}(I_n)}+\|g_n\|_{W^{1,\infty}(I_n)}\leq\kappa \tag{2.52}$$

for all $n\geq1$. Given $T>0$, it follows from (2.50) and (2.52) that, for all $n\geq1$,

$$\sup_{t\in[0,T]}\|X_n(t,\,\cdot\,)\|_E\leq C(\left\|\bar X_n\right\|_E,T,\kappa_n)\leq C(M,T,\kappa)=C' \tag{2.53}$$

and $\sup_{t\in[0,T]}\|X_n(t,\,\cdot\,)\|_E$ is bounded uniformly with respect to n. We have

$$\|X_n(t)-X(t)\|_E\leq\left\|\bar X_n-\bar X\right\|_E+\int_0^t\|F(X_n,f_n,g_n)-F(X,f,g)\|_E(s)\,ds \tag{2.54}$$

where F denotes the right-hand side of (2.15). We consider a fixed time $t\in[0,T]$ and drop the time dependence in the notation for the moment. We have

$$\begin{aligned}\|F(X_n,f_n,g_n)-F(X,f,g)\|_E\leq{}&\|F(X_n,f_n,g_n)-F(X_n,f,g)\|_E\\&+\|F(X_n,f,g)-F(X,f,g)\|_E\,.\end{aligned} \tag{2.55}$$

The map $X\mapsto F(X,f,g)$ is Lipschitz on any bounded set of E, see the proof of Theorem 2.4. Hence, after denoting L the Lipschitz function of this map on the ball $\{X\in E\mid\|X\|_E\leq C'\}$, we get from (2.53) that

$$\|F(X_n,f,g)-F(X,f,g)\|_E\leq L\|X_n-X\|_E\,. \tag{2.56}$$

Denote by Q_n and $\tilde Q_n$ the expressions given by the definition (2.13) of Q where we replace X, f, g by X_n, f_n, g_n and X_n, f, g, respectively. We use the same notations and define P_n and $\tilde P_n$ from (2.14), and G_n and $\tilde G_n$ from (2.4). For example, we have

$$G_n=\int_0^{U_n}(2g_n(z)+f_n''(z)z^2)\,dz\text{ and }\tilde G_n=\int_0^{U_n}(2g(z)+f''(z)z^2)\,dz.$$

Still using the same notations, we have, from (2.21),

$$\begin{aligned}(R_n-\tilde R_n)(\xi)={}&e^{\zeta_n}\big(g_n(U_n)-g(U_n)-\frac12(f_n'(U_n)-f'(U_n))U_n^2)\big)(1+\zeta_{n,\xi})\\&+\frac12e^{\zeta_n}(f_n''(U_n)-f''(U_n))H_{n,\xi}.\end{aligned} \tag{2.57}$$

Let $\delta_n = \|f_n - f\|_{W^{2,\infty}(I)} + \|g_n - g\|_{L^\infty(I)}$. Since $(f_n, g_n) \to (f, g)$ in $\mathcal{E}$, $\delta_n \to 0$ as $n \to \infty$. Hence, from (2.57) and (2.51), we get

$$\begin{aligned}\left\|R_n - \tilde{R}_n\right\|_{L^2(\mathbb{R})} \leq e^{C'}\big(&\|g_n(U_n) - g(U_n)\|_{L^2(\mathbb{R})} + C'\|g_n(U_n) - g(U_n)\|_{L^\infty(\mathbb{R})}\\ &+ \frac{1}{2}C'\|f_n'(U_n) - f'(U_n)\|_{L^\infty(\mathbb{R})}(M + \sqrt{M})\\ &+ \frac{1}{2}C'\|f_n''(U_n) - f''(U_n)\|_{L^\infty(\mathbb{R})}\big).\end{aligned} \tag{2.58}$$

Let $\delta_n' = \|g_n(U) - g(U)\|_{L^2(\mathbb{R})}$, we then have

$$\begin{aligned}\|g(U_n) - g_n(U_n)\|_{L^2(\mathbb{R})} &\leq \|g_n(U_n) - g_n(U)\|_{L^2(\mathbb{R})} + \delta_n' + \|g(U_n) - g(U)\|_{L^2(\mathbb{R})}\\ &\leq 2\kappa\|U_n - U\|_{L^2(\mathbb{R})} + \delta_n'.\end{aligned} \tag{2.59}$$

Since $g_n \to g$ in $L^\infty(I)$, $g_n(U) \to g_n(U)$ in $L^\infty(\mathbb{R})$. As $|g_n(U) - g(U)| \leq 2\kappa|U|$ (because $g(0) = 0$ and $\|g\|_{W^{1,\infty}(I)} \leq \kappa$), we can apply the Lebesgue dominated convergence theorem and obtain that $\lim_{n\to\infty}\delta_n' = 0$. From (2.58) and (2.59), we obtain that

$$\left\|R_n - \tilde{R}_n\right\|_{L^2(\mathbb{R})} \leq C\big(\delta_n + \delta_n' + \|U_n - U\|_{L^2(\mathbb{R})}\big)$$

for some constant C which depends on M, T and κ. Again, we denote generically by C such constants that are increasing functions of M, T and κ, and are independent on n. Since A in (2.20) is continuous from $L^2(\mathbb{R})$ to $H^1(\mathbb{R})$, it follows that $\left\|Q_n - \tilde{Q}_n\right\|_{H^1(\mathbb{R})} \leq C\big(\delta_n + \delta_n' + \|U_n - U\|_{L^2(\mathbb{R})}\big)$. Similarly, one proves that $\left\|P_n - \tilde{P}_n\right\|_{H^1(\mathbb{R})} \leq C\big(\delta_n + \delta_n' + \|U_n - U\|_{L^2(\mathbb{R})}\big)$. We have

$$\begin{aligned}\left\|G_n - \tilde{G}_n\right\|_V = &\left\|G_n - \tilde{G}_n\right\|_{L^\infty(\mathbb{R})}\\ &+ \left\|\big(2(g_n(U_n) - g(U_n)) + (f_n''(U_n) - f''(U_n))U_n^2\big)U_{n,\xi}\right\|_{L^2(\mathbb{R})}\\ \leq &\sqrt{M}\big(2\|g_n(U_n) - g(U_n)\|_{L^\infty(\mathbb{R})} + M\|f_n''(U_n) - f''(U_n)\|_{L^\infty(\mathbb{R})}\big)\\ &+ 2C'\|g_n(U_n) - g(U_n)\|_{L^\infty(\mathbb{R})} + C'M\|f_n''(U_n) - f''(U_n)\|_{L^\infty(\mathbb{R})}\\ \leq &C\delta_n\end{aligned}$$

by (2.59). Finally, we have

$$\|F(X_n, f_n, g_n) - F(X_n, f, g)\|_E \leq C\big(\delta_n + \delta_n' + \|U_n - U\|_{L^2(\mathbb{R})}\big). \tag{2.60}$$

Gathering (2.54), (2.55), (2.56) and (2.60), we end up with

$$\|X_n(t) - X(t)\|_E \leq \left\|\bar{X}_n - \bar{X}\right\|_E + CT(\delta_n + \delta_n') + (L + C)\int_0^t \|X_n - X\|_E(s)\,ds$$

and Gronwall's lemma yields

$$\|X_n(t) - X(t)\|_E \leq \big(\left\|\bar{X}_n - \bar{X}\right\|_E + CT(\delta_n + \delta_n')\big)e^{(L+C)T}.$$

Hence, $X_n \to X$ in E uniformly in $[0, T]$. □

The solutions are well-defined in our new sets of coordinates. Now we want to go back to the original variable u. We define $u(t, x)$ as

$$u(x, t) = U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi). \tag{2.61}$$

Let us prove that this definition is well-posed. Given $x \in \mathbb{R}$, since y is increasing, continuous and $\lim_{\xi\to\pm\infty} y = \pm\infty$, y is surjective and there exists ξ such that $x = y(\xi)$. Suppose we have $\xi_1 < \xi_2$ with $x = y(\xi_1) = y(\xi_2)$. Then, since y is monotone, $y(\xi) = y(\xi_1) = y(\xi_2)$ for all $\xi \in (\xi_1, \xi_2)$ and $y_\xi = 0$ in this interval. From (2.27c), it follows that $U_\xi = 0$ on (ξ_1, ξ_2) and therefore $U(\xi_1) = U(\xi_2)$.

Theorem 2.9 (Existence of weak solutions)**.** *For initial data $\bar{u} \in H^1(\mathbb{R})$, let $(\bar{y}, \bar{U}, \bar{H})$ be as given by* (2.26) *and (y, U, H) be the solution of* (2.15) *with initial data $(\bar{y}, \bar{U}, \bar{H})$. Then u as defined in* (2.61) *belongs to $C(\mathbb{R}_+, L^\infty(\mathbb{R})) \cap L^\infty(\mathbb{R}_+, H^1(\mathbb{R}))$ and is a weak solution of* (1.1).

Proof. Let us prove that $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}))$. We consider a fix time t and drop it in the notation when there is no ambiguity. For any smooth function ϕ, after using the change of variable $x = y(\xi)$, we obtain

$$\int_{\mathbb{R}} u\phi\, dx = \int_{\mathbb{R}} U(\phi \circ y) y_\xi\, d\xi = \int_{\mathbb{R}} U\sqrt{y_\xi}(\phi \circ y)\sqrt{y_\xi}\, d\xi.$$

Hence, by Cauchy–Schwarz,

$$\left|\int_{\mathbb{R}} u\phi\, dx\right| \leq \|\phi\|_{L^2(\mathbb{R})} \sqrt{\int_{\mathbb{R}} U^2 y_\xi\, d\xi} \leq \sqrt{H(\infty)}\, \|\phi\|_{L^2(\mathbb{R})}$$

as $U^2 y_\xi \leq H_\xi$ from (2.27c). Therefore, $u \in L^2(\mathbb{R})$ and $\|u(t,\cdot)\|_{L^2(\mathbb{R})} \leq \sqrt{H(t,\infty)} = \sqrt{H(0,\infty)} = \|\bar{u}\|_{H^1(\mathbb{R})}$. For any smooth function ϕ, we have, after using the change of variable $x = y(\xi)$,

$$\int_{\mathbb{R}} u(x)\phi_x(x)\, dx = \int_{\mathbb{R}} U(\xi)\phi_x(y(\xi)) y_\xi(\xi)\, d\xi = -\int_{\mathbb{R}} U_\xi(\xi)(\phi \circ y)(\xi)\, d\xi. \tag{2.62}$$

Let $B = \{\xi \in \mathbb{R} \mid y_\xi(\xi) > 0\}$. Because of (2.27c), and since $y_\xi \geq 0$ almost everywhere, we have $U_\xi = 0$ almost everywhere on B^c. Hence, we can restrict the integration domain in (2.62) to B. We divide and multiply by $\sqrt{y_\xi}$ the integrand in (2.62) and obtain, after using the Cauchy–Schwarz inequality,

$$\left|\int_{\mathbb{R}} u\phi_x\, dx\right| = \left|\int_B \frac{U_\xi}{\sqrt{y_\xi}}(\phi \circ y)\sqrt{y_\xi}\, d\xi\right| \leq \sqrt{\int_B \frac{U_\xi^2}{y_\xi}\, d\xi}\sqrt{\int_B (\phi \circ y)^2 y_\xi\, d\xi}.$$

By (2.27c), we have $\frac{U_\xi^2}{y_\xi} \leq H_\xi$. Hence, after another change of variables, we get

$$\left|\int_{\mathbb{R}} u\phi_x\, dx\right| \leq \sqrt{H(\infty)}\, \|\phi\|_{L^2(\mathbb{R})},$$

which implies that $u_x \in L^2(\mathbb{R})$ and $\|u_x(t,\cdot)\|_{L^2(\mathbb{R})} \leq \|\bar{u}\|_{H^1(\mathbb{R})}$. Hence, $u \in L^\infty(\mathbb{R}, H^1(\mathbb{R}))$ and

$$\|u\|_{L^\infty(\mathbb{R},H^1(\mathbb{R}))} \leq 2\,\|\bar{u}\|_{H^1(\mathbb{R})}. \tag{2.63}$$

Let us prove sequential convergence in $C(\mathbb{R}_+, L^\infty(\mathbb{R}))$. Given $t \in \mathbb{R}_+$ and a sequence $t_n \in \mathbb{R}_+$ with $t_n \to t$, we set $(y_n(\xi), U_n(\xi), H_n(\xi)) = (y(t_n,\xi), U(t_n,\xi), H(t_n,\xi))$ and, slightly abusing notation, $(y(\xi), U(\xi), H(\xi)) = (y(t,\xi), U(t,\xi), H(t,\xi))$. For any $x \in \mathbb{R}$, there exist ξ_n and ξ, which may not be unique, such that $x = y_n(\xi_n)$ and $x = y(\xi)$. We set $x_n = y_n(\xi)$. We have

$$u(t_n,x) - u(t,x) = u(t_n,x) - u(t_n,x_n) + U_n(\xi) - U(\xi) \tag{2.64}$$

and

$$\begin{aligned}
|u(t_n,x) - u(t_n,x_n)| &= \left|\int_{x_n}^{x} u_x(t_n,x')\, dx'\right| \\
&\leq \sqrt{|x_n - x|}\left(\int_{x_n}^{x} u_x(t_n,x')^2\, dx'\right)^{1/2} \quad \text{(Cauchy–Schwarz)} \\
&\leq \sqrt{|y_n(\xi) - y(\xi)|}\, \|u\|_{L^\infty(\mathbb{R},H^1(\mathbb{R}))} \\
&\leq 2\,\|\bar{u}\|_{H^1(\mathbb{R})}\, \|y - y_n\|_{L^\infty(\mathbb{R})}^{1/2},
\end{aligned} \tag{2.65}$$

by (2.63). Since $y_n \to y$ and $U_n \to U$ in $L^\infty(\mathbb{R})$, it follows from (2.64) and (2.65) that $u_n \to u$ in $L^\infty(\mathbb{R})$.

Since $u \in L^\infty(\mathbb{R}_+, H^1(\mathbb{R}))$, $g(u) + \frac{1}{2}f''(u)u_x^2 \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}))$ and, since $v \mapsto (1-\partial_{xx})^{-1}v$ is continuous from $H^{-1}(\mathbb{R})$ to $H^1(\mathbb{R})$, $P \in L^\infty(\mathbb{R}_+, H^1(\mathbb{R}))$. We say that u is a weak solution of (1.5) if

$$\int_{\mathbb{R}_+\times\mathbb{R}} \left(-u\phi_t + f'(u)u_x\phi + P_x\phi\right)(t,x)\,dtdx = 0 \tag{2.66}$$

for all $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ with compact support. For $t \in \mathcal{K}^c$, that is for almost every t (see (2.41) in Lemma 2.7), $y_\xi(t,\xi) > 0$ for almost every $\xi \in \mathbb{R}$ and $y(t,\cdot)$ is invertible, we have $U_\xi = u_x \circ y y_\xi$ and, after using the change of variables $x = y(t,\xi)$, we get

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}} &[-u(t,x)\phi_t(t,x) + f'(u(t,x))u_x(t,x)\phi(t,x)]\,dxdt\\ &= \int_{\mathbb{R}_+\times\mathbb{R}} [-U(t,\xi)y_\xi(t,\xi)\phi_t(t,y(t,\xi)) + f'(U(t,\xi))U_\xi(t,\xi)\phi(t,y(t,\xi))]\,d\xi dt.\end{aligned} \tag{2.67}$$

Using the fact that $y_t = f'(U)$ and $y_{\xi t} = f''(U)U_\xi$, one easily check that

$$(Uy_\xi\phi\circ y)_t - (f'(U)U\phi\circ y)_\xi = Uy_\xi\phi_t\circ y - f'(U)U_\xi\phi\circ y + U_t y_\xi\phi\circ y. \tag{2.68}$$

After integrating (2.68) over $\mathbb{R}_+\times\mathbb{R}$, the left-hand side of (2.68) vanishes and we obtain

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}} &\Big[-Uy_\xi\,\phi_t\circ y + f'(U)U_\xi\,\phi\circ y\Big]\,d\xi dt\\ &= \frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{R}^2}\Big[\operatorname{sgn}(\xi-\eta)e^{-|y(\xi)-y(\eta)|}\\ &\quad\times\big((g(U) - \frac{1}{2}f''(U)U^2))y_\xi + \frac{1}{2}f''(U)H_\xi\big)(\eta)y_\xi(\xi)\phi\circ y(\xi)\Big]\,d\eta d\xi dt\end{aligned} \tag{2.69}$$

by (2.15). Again, to simplify the notation, we deliberately omitted the t variable. On the other hand, by using the change of variables $x = y(t,\xi)$ and $z = y(t,\eta)$ when $t \in \mathcal{K}^c$, we have

$$\begin{aligned}-\int_{\mathbb{R}_+\times\mathbb{R}} P_x(t,x)\phi(t,x)\,dxdt &= \frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{R}^2}\Big[\operatorname{sgn}(y(\xi)-y(\eta))e^{-|y(\xi)-y(\eta)|}\\ &\quad\times\big(g\circ u + \frac{1}{2}f''\circ u u_x^2\big)(t,y(\eta))\phi(t,y(\xi))y_\xi(\eta)y_\xi(\xi)\Big]\,d\eta d\xi dt.\end{aligned}$$

For $t \in \mathcal{K}^c$, that is, for almost every t, $y_\xi(t,\xi)$ is strictly positive for almost every ξ, and we can replace $u_x(t,y(t,\eta))$ by $U_\xi(t,\eta)/y_\xi(t,\eta)$ in the equation above. Using (2.27c), we obtain

$$-\int_{\mathbb{R}_+\times\mathbb{R}} P_x(t,x)\phi(t,x)\,dxdt \tag{2.70}$$

$$\begin{aligned}&= \frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{R}^2}\Big[\operatorname{sgn}(\xi-\eta)e^{-|y(\xi)-y(\eta)|}\\ &\quad\times\big((g(U) - \frac{1}{2}f''(U)U^2))y_\xi + \frac{1}{2}f''(U)H_\xi\big)(\eta)y_\xi(\xi)\phi\circ y(\xi)\Big]\,d\eta d\xi dt\end{aligned} \tag{2.71}$$

Thus, comparing (2.69) and (2.71), we get

$$\int_{\mathbb{R}_+\times\mathbb{R}} [-Uy_\xi\,\phi_t(t,y) + f'(U)U_\xi\,\phi]\,d\xi dt = -\int_{\mathbb{R}_+\times\mathbb{R}} P_x(t,x)\phi(t,x)\,dxdt$$

and (2.66) follows from (2.67). □

3. Continuous semi-group of solutions

We denote by G the subgroup of the group of homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$ such that

$$f - \mathrm{Id} \text{ and } f^{-1} - \mathrm{Id} \text{ both belong to } W^{1,\infty}(\mathbb{R}) \tag{3.1}$$

where Id denotes the identity function. The set G can be interpreted as the set of relabeling functions. For any $\alpha > 1$, we introduce the subsets G_α of G defined by

$$G_\alpha = \{f \in G \mid \|f - \mathrm{Id}\|_{W^{1,\infty}(\mathbb{R})} + \left\|f^{-1} - \mathrm{Id}\right\|_{W^{1,\infty}(\mathbb{R})} \leq \alpha\}.$$

The subsets G_α do not possess the group structure of G. We have the following characterization of G_α:

Lemma 3.1. [16, Lemma 3.2] *Let $\alpha \geq 0$. If f belongs to G_α, then $1/(1+\alpha) \leq f_\xi \leq 1+\alpha$ almost everywhere. Conversely, if f is absolutely continuous, $f - \mathrm{Id} \in L^\infty(\mathbb{R})$ and there exists $c \geq 1$ such that $1/c \leq f_\xi \leq c$ almost everywhere, then $f \in G_\alpha$ for some α depending only on c and $\|f - \mathrm{Id}\|_{L^\infty(\mathbb{R})}$.*

We define the subsets $\mathcal{F}_\alpha$ and $\mathcal{F}$ of $\mathcal{G}$ as follows

$$\mathcal{F}_\alpha = \{X = (y, U, H) \in \mathcal{G} \mid y + H \in G_\alpha\},$$

and

$$\mathcal{F} = \{X = (y, U, H) \in \mathcal{G} \mid y + H \in G\}.$$

For $\alpha = 0$, we have $G_0 = \{\mathrm{Id}\}$. As we will see, the space $\mathcal{F}_0$ will play a special role. These sets are relevant only because they are in some sense preserved by the governing equation (2.15) as the next lemma shows.

Lemma 3.2. *The space $\mathcal{F}$ is preserved by the governing equation* (2.15)*. More precisely, given $\alpha, T \geq 0$, a bounded set $B_M = \{X \in E \mid \|X\|_E \leq M\}$ of E and a bounded set $\mathcal{E}'$ of $\mathcal{E}$, we have, for any $t \in [0, T]$, $\bar{X} \in \mathcal{F}_\alpha \cap B_M$ and $(f, g) \in \mathcal{E}'$,*

$$S_t(\bar{X}, f, g) \in \mathcal{F}_{\alpha'}$$

where α' only depends on T, α, M and $\mathcal{E}'$.

Proof. Let $\bar{X} = (\bar{y}, \bar{U}, \bar{H}) \in \mathcal{F}_\alpha$, we denote $X(t) = (y(t), U(t), H(t))$ the solution of (2.15) with initial data $\bar{X}$. By definition, we have $\bar{y} + \bar{H} \in G_\alpha$ and, from Lemma 3.1, $1/c \leq \bar{y}_\xi + \bar{H}_\xi \leq c$ almost everywhere, for some constant $c > 1$ depending only α. Let $h = \bar{H}(\infty) = H(t, \infty)$. We have $h \leq M$ and, from (2.44), $\|U\|_{L^\infty(\mathbb{R}_+\times\mathbb{R})} \leq \sqrt{h} \leq \sqrt{M}$. Let $I = [-\sqrt{M}, \sqrt{M}]$. Since $\mathcal{E}'$ is bounded, there exists $\kappa > 0$ such that $\|f\|_{W^{2,\infty}(I)} + \|g\|_{W^{1,\infty}(I)} \leq \kappa$ for all $(f, g) \in \mathcal{E}'$. We consider a fixed ξ and drop it in the notation. Applying Gronwall's inequality to (2.16) to the function $X(t - \tau)$, we obtain

$$|y_\xi(0)| + |H_\xi(0)| + |U_\xi(0)| \leq e^{CT}\left(|y_\xi(t)| + |H_\xi(t)| + |U_\xi(t)|\right) \tag{3.2}$$

for some constant C which depends on $\|f''(U)\|_{L^\infty(\mathbb{R})}$, $\|P\|_{L^\infty(\mathbb{R})}$, $\|g(U)\|_{L^\infty(\mathbb{R})}$, $\|Q\|_{L^\infty(\mathbb{R})}$, $\|U\|_{L^\infty(\mathbb{R})}$ and $\|G'(U)\|_{L^\infty(\mathbb{R})}$. In (2.46), we proved that $\|P\|_{L^\infty(\mathbb{R})}$ and $\|Q\|_{L^\infty(\mathbb{R})}$ only depend on M, κ, T. Hence, the constant C in (3.2) also only depends on M, T and κ. From (2.27c), we have

$$|U_\xi(t)| \leq \sqrt{y_\xi(t) H_\xi(t)} \leq \frac{1}{2}(y_\xi(t) + H_\xi(t)).$$

Hence, since y_ξ and H_ξ are positive, (3.2) gives us

$$\frac{1}{c} \leq \bar{y}_\xi + \bar{H}_\xi \leq \frac{3}{2} e^{CT}(y_\xi(t) + H_\xi(t)),$$

and $y_\xi(t) + H_\xi(t) \geq \frac{2}{3c}e^{-CT}$. Similarly, by applying Gronwall's lemma, we obtain $y_\xi(t) + H_\xi(t) \leq \frac{3}{2}ce^{CT}$. We have $\|(y+H)(t) - \xi\|_{L^\infty(\mathbb{R})} \leq \|X(t)\|_{C([0,T],E)} \leq C(M,T,\kappa)$, see (2.50). Hence, applying Lemma 3.1, we obtain that $y(t,\cdot\,) + H(t,\cdot\,) \in G_{\alpha'}$ and therefore $X(t) \in \mathcal{F}_{\alpha'}$ for some α' depending only on α, T, M and $\mathcal{E}'$. $\square$

For the sake of simplicity, for any $X = (y,U,H) \in \mathcal{F}$ and any function $r \in G$, we denote $(y \circ r, U \circ r, H \circ r)$ by $X \circ r$.

Proposition 3.3. [16, Proposition 3.4] *The map from $G \times \mathcal{F}$ to $\mathcal{F}$ given by $(r,X) \mapsto X \circ r$ defines an action of the group G on $\mathcal{F}$.*

Since G is acting on $\mathcal{F}$, we can consider the quotient space $\mathcal{F}/G$ of $\mathcal{F}$ with respect to the action of the group G. The equivalence relation on $\mathcal{F}$ is defined as follows: For any $X, X' \in \mathcal{F}$, X and X' are equivalent if there exists $r \in G$ such that $X' = X \circ r$. Heuristically it means that X' and X are equivalent up to a relabeling function. We denote by $\Pi(X) = [X]$ the projection of $\mathcal{F}$ into the quotient space $\mathcal{F}/G$. We introduce the map $\Gamma\colon \mathcal{F} \to \mathcal{F}_0$ given by

$$\Gamma(X) = X \circ (y+H)^{-1}$$

for any $X = (y,U,H) \in \mathcal{F}$. We have $\Gamma(X) = X$ when $X \in \mathcal{F}_0$. It is not hard to prove that Γ is invariant under the G action, that is, $\Gamma(X \circ r) = \Gamma(X)$ for any $X \in \mathcal{F}$ and $r \in G$. Hence, there corresponds to Γ a map $\tilde{\Gamma}$ from the quotient space $\mathcal{F}/G$ to $\mathcal{F}_0$ given by $\tilde{\Gamma}([X]) = \Gamma(X)$ where $[X] \in \mathcal{F}/G$ denotes the equivalence class of $X \in \mathcal{F}$. For any $X \in \mathcal{F}_0$, we have $\tilde{\Gamma} \circ \Pi(X) = \Gamma(X) = X$. Hence, $\tilde{\Gamma} \circ \Pi|_{\mathcal{F}_0} = \mathrm{Id}|_{\mathcal{F}_0}$. Any topology defined on $\mathcal{F}_0$ is naturally transported into $\mathcal{F}/G$ by this isomorphism. We equip $\mathcal{F}_0$ with the metric induced by the E-norm, i.e., $d_{\mathcal{F}_0}(X,X') = \|X - X'\|_E$ for all $X, X' \in \mathcal{F}_0$. Since $\mathcal{F}_0$ is closed in E, this metric is complete. We define the metric on $\mathcal{F}/G$ as

$$d_{\mathcal{F}/G}([X],[X']) = \|\Gamma(X) - \Gamma(X')\|_E\,,$$

for any $[X], [X'] \in \mathcal{F}/G$. Then, $\mathcal{F}/G$ is isometrically isomorphic with $\mathcal{F}_0$ and the metric $d_{\mathcal{F}/G}$ is complete.

Lemma 3.4. [16, Lemma 3.5] *Given $\alpha \geq 0$. The restriction of Γ to $\mathcal{F}_\alpha$ is a continuous map from $\mathcal{F}_\alpha$ to $\mathcal{F}_0$.*

Remark 3.5. The map Γ is not continuous from $\mathcal{F}$ to $\mathcal{F}_0$. The spaces $\mathcal{F}_\alpha$ were precisely introduced in order to make the map Γ continuous.

We denote by $S\colon \mathcal{F} \times \mathcal{E} \times \mathbb{R}_+ \to \mathcal{F}$ the continuous semigroup which to any initial data $\bar{X} \in \mathcal{F}$ associates the solution $X(t)$ of the system of differential equation (2.15) at time t as defined in Theorem 2.9. As we indicated earlier, the generalized hyperelastic-rod wave equation is invariant with respect to relabeling, more precisely, using our terminology, we have the following result.

Theorem 3.6. *For any $t > 0$, the map $S_t\colon \mathcal{F} \to \mathcal{F}$ is G-equivariant (for f and g given), that is,*

$$S_t(X \circ r) = S_t(X) \circ r \tag{3.3}$$

for any $X \in \mathcal{F}$ and $r \in G$. Hence, the map $\tilde{S} : \mathcal{F}/G \times \mathcal{E} \times \mathbb{R}_+ \to \mathcal{F}/G$ given by

$$\tilde{S}_t([X], f, g) = [S_t(X, f, g)]$$

is well-defined. It generates a continuous semigroup with respect to all variables, on any bounded set of $\mathcal{E}$.

Proof. For any $X_0 = (y_0, U_0, H_0) \in \mathcal{F}$ and $r \in G$, we denote $\bar{X}_0 = (\bar{y}_0, \bar{U}_0, \bar{H}_0) = X_0 \circ r$, $X(t) = S_t(X_0)$ and $\bar{X}(t) = S_t(\bar{X}_0)$. We claim that $X(t) \circ r$ satisfies (2.15) and therefore, since $X(t) \circ r$ and $\bar{X}(t)$ satisfy the same system of differential equation with the same initial data, they are equal. We denote $\hat{X}(t) = (\hat{y}(t), \hat{U}(t), \hat{H}(t)) = X(t) \circ r$. We have

$$\hat{U}_t = \frac{1}{2} \int_{\mathbb{R}} \operatorname{sgn}(\xi - \eta) \exp\big(- \operatorname{sgn}(\xi - \eta)(\hat{y}(\xi) - y(\eta))\big) \\ \times \Big((g(U) - \frac{1}{2} f''(U) U^2) y_\xi + \frac{1}{2} f''(U) H_\xi \Big)(\eta)\, d\eta. \tag{3.4}$$

We have $\hat{y}_\xi(\xi) = y_\xi(r(\xi)) r_\xi(\xi)$ and $\hat{H}_\xi(\xi) = H_\xi(r(\xi)) r_\xi(\xi)$ for almost every $\xi \in \mathbb{R}$. Hence, after the change of variables $\eta = r(\eta')$, we get from (3.4) that

$$\hat{U}_t = \frac{1}{4} \int_{\mathbb{R}} \operatorname{sgn}(\xi - \eta) \exp\big(- \operatorname{sgn}(\xi - \eta)(\hat{y}(\xi) - \hat{y}(\eta))\big) \\ \times \Big((g(\hat{U}) - \frac{1}{2} f''(\hat{U}) \hat{U}^2) \hat{y}_\xi + \frac{1}{2} f''(\hat{U}) \hat{H}_\xi \Big)(\eta)\, d\eta.$$

We treat the other terms in (2.15) similarly, and it follows that $(\hat{y}, \hat{U}, \hat{H})$ is a solution of (2.15). Since $(\hat{y}, \hat{U}, \hat{H})$ and $(\bar{y}, \bar{U}, \bar{H})$ satisfy the same system of ordinary differential equations with the same initial data, they are equal, i.e.,

$$\bar{X}(t) = X(t) \circ r\,,$$

and (3.3) is proved. Let $\mathcal{E}'$ be a bounded set of $\mathcal{E}$ and $T > 0$. For $t \in [0, T]$, we have the following diagram

$$\begin{tikzcd}
\mathcal{F}_0 \arrow[r, "\Pi"] & \mathcal{F}/G \\
\mathcal{F}_\alpha \arrow[u, "\Gamma"] & \\
\mathcal{F}_0 \times \mathcal{E}' \arrow[u, "S_t"] \arrow[r, "\Pi"] & \mathcal{F}/G \times \mathcal{E}' \arrow[uu, "\tilde{S}_t"']
\end{tikzcd} \tag{3.5}$$

on a bounded domain of $\mathcal{F}_0$ whose diameter together with T and $\mathcal{E}'$ determines the constant α, see Lemma 3.2. By the definition of the metric on $\mathcal{F}/G$, the map $\tilde{\Gamma}$ is an isometry from $\mathcal{F}/G$ to $\mathcal{F}_0$. Hence, from the diagram (3.5), we see that $\tilde{S}_t \colon \mathcal{F}/G \times \mathcal{E}' \to \mathcal{F}/G$ is continuous if and only if $\Gamma \circ S_t \colon \mathcal{F}_0 \times \mathcal{E}' \to \mathcal{F}_0$ is continuous. Let us prove that $\Gamma \circ S_t \colon \mathcal{F}_0 \times \mathcal{E}' \to \mathcal{F}_0$ is sequentially continuous. We consider a sequence $X_n \in \mathcal{F}_0$ that converges to $X \in \mathcal{F}_0$ in $\mathcal{F}_0$, that is, $\lim_{n\to\infty} \|X_n - X\|_E = 0$ and a sequence $(f_n, g_n) \in \mathcal{E}'$ that converges to $(f, g) \in \mathcal{E}'$ in $\mathcal{E}$. From Theorem 2.8, we get that $\lim_{n\to\infty} \|S_t(X_n, f_n, g_n) - S_t(X, f, g)\|_E = 0$. Since $X_n \to X$ in E, there exists a constant $C \geq 0$ such that $\|X_n\| \leq C$ for all n. Lemma 3.2 gives us that for $t \in [0, T]$, $S_t(X_n, f_n, g_n) \in \mathcal{F}_\alpha$ for some α which depends on C, T and $\mathcal{E}'$ but is independent of n. Hence, $S_t(X_n, f_n, g_n) \to S_t(X, f, g)$ in $\mathcal{F}_\alpha$. Then, by Lemma 3.4, we obtain that $\Gamma \circ S_t(X_n, f_n, g_n) \to \Gamma \circ S_t(X, f, g)$ in $\mathcal{F}_0$ and uniformly in $[0, T]$. ☐

3.1. **From Eulerian to Lagrangian coordinates and vice versa.** As noted in [1] in the case of the Camassa-Holm equation, even if $H^1(\mathbb{R})$ is a natural space for the equation, there is no hope to obtain a semigroup of solutions by only considering $H^1(\mathbb{R})$. Thus, we introduce the following space $\mathcal{D}$, which characterizes the solutions in *Eulerian coordinates*:

Definition 3.7. The set $\mathcal{D}$ is composed of all pairs (u,μ) such that u belongs to $H^1(\mathbb{R})$ and μ is a positive finite Radon measure whose absolute continuous part, μ_{ac}, satisfies

$$\mu_{\mathrm{ac}} = (u^2+u_x^2)\,dx. \tag{3.6}$$

There exists a bijection between Eulerian coordinates (functions in $\mathcal{D}$) and Lagrangian coordinates (functions in $\mathcal{F}/G$). Earlier we considered initial data in $\mathcal{D}$ with a special structure: The energy density μ was given by $(u^2+u_x^2)\,dx$ and therefore μ did not have any singular part. The set $\mathcal{D}$ however allows the energy density to have a singular part and a positive amount of energy can concentrate on a set of Lebesgue measure zero. We constructed corresponding initial data in $\mathcal{F}_0$ by the means of (2.26a), (2.26b), and (2.26c). This construction can be generalized in the following way. Let us denote by $L\colon \mathcal{D}\to\mathcal{F}/G$ the map transforming Eulerian coordinates into Lagrangian coordinates whose definition is contained in the following theorem.

Theorem 3.8. [16, Theorem 3.8] *For any (u,μ) in $\mathcal{D}$, let*

$$y(\xi) = \sup\left\{y \mid \mu((-\infty,y))+y<\xi\right\}, \tag{3.7a}$$

$$H(\xi) = \xi - y(\xi), \tag{3.7b}$$

$$U(\xi) = u\circ y(\xi)\,. \tag{3.7c}$$

Then $(y,U,H)\in\mathcal{F}_0$. We define $L(u,\mu)\in\mathcal{F}/G$ to be the equivalence class of (y,U,H).

Remark 3.9. If μ is absolutely continuous, then $\mu=(u^2+u_x^2)dx$ and the function $y\mapsto\mu((-\infty,y))$ is continuous. From the definition (3.7a), we know that there exist an increasing sequence x_i and a decreasing sequence x_i' which both converge to $y(\xi)$ and such that

$$\mu((-\infty,x_i))+x_i<\xi \text{ and } \mu((-\infty,x_i'))+x_i'\geq\xi.$$

Since $y\mapsto\mu((-\infty,y))$ is continuous, it implies, after letting i go to infinity, that $\mu((-\infty,y(\xi)))+y(\xi)=\xi$. Hence,

$$\int_{-\infty}^{y(\xi)}(u^2+u_x^2)\,dx+y(\xi)=\xi$$

for all $\xi\in\mathbb{R}$ and we recover definition (2.26a).

At the very beginning, $H(t,\xi)$ was introduced as the energy contained in a strip between $-\infty$ and $y(t,\xi)$, see (2.10). This interpretation still holds. We obtain μ, the energy density in Eulerian coordinates, by pushing forward by y the energy density in Lagrangian coordinates, $H_\xi\,d\xi$. Recall that the push-forward of a measure ν by a measurable function f is the measure $f_\#\nu$ defined as

$$f_\#\nu(B)=\nu(f^{-1}(B))$$

for all Borel sets B. We are led to the map M which transforms Lagrangian coordinates into Eulerian coordinates and whose definition is contained in the following theorem.

Theorem 3.10. [16, Theorem 3.11] *Given any element $[X]$ in $\mathcal{F}/G$. Then, (u,μ) defined as follows*

$$u(x)=U(\xi) \text{ for any } \xi \text{ such that } x=y(\xi), \tag{3.8a}$$

$$\mu=y_\#(H_\xi\,d\xi) \tag{3.8b}$$

belongs to $\mathcal{D}$ and is independent of the representative $X=(y,U,H)\in\mathcal{F}$ we choose for $[X]$. We denote by $M\colon\mathcal{F}/G\to\mathcal{D}$ the map which to any $[X]$ in $\mathcal{F}/G$ associates (u,μ) as given by (3.8).

Of course, the definition of u coincides with the one given previously in (2.61). The transformation from Eulerian to Lagrangian coordinates is a bijection, as stated in the next theorem.

Theorem 3.11. [16, Theorem 3.12] *The map M and L are invertible. We have*

$$L \circ M = \mathrm{Id}_{\mathcal{F}/G} \text{ and } M \circ L = \mathrm{Id}_{\mathcal{D}}\,.$$

3.2. **Continuous semigroup of solutions on $\mathcal{D}$.** On $\mathcal{D}$ we define the distance $d_{\mathcal{D}}$ which makes the bijection L between $\mathcal{D}$ and $\mathcal{F}/G$ into an isometry:

$$d_{\mathcal{D}}((u,\mu),(\bar{u},\bar{\mu})) = d_{\mathcal{F}/G}(L(u,\mu), L(\bar{u},\bar{\mu})).$$

Since $\mathcal{F}/G$ equipped with $d_{\mathcal{F}/G}$ is a complete metric space, we have the following theorem.

Theorem 3.12. *$\mathcal{D}$ equipped with the metric d_D is a complete metric space.*

For each $t \in \mathbb{R}$, we define the map T_t from $\mathcal{D} \times \mathcal{E}$ to $\mathcal{D}$ as

$$T_t(\,\cdot\,,f,g) = M\tilde{S}_t(\,\cdot\,,f,g)L,$$

for any $(f,g) \in \mathcal{E}$. For a given pair $(f,g) \in \mathcal{E}$, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{D} & \mathcal{F}/G \arrow[l, "M"'] \\ \mathcal{D} \arrow[u, "T_t"] \arrow[r, "L"] & \mathcal{F}/G \arrow[u, "\tilde{S}_t"'] \end{tikzcd} \tag{3.9}$$

Our main theorem reads as follows.

Theorem 3.13. *Assume* (1.6). *$T\colon \mathcal{D} \times \mathcal{E} \times \mathbb{R}_+ \to \mathcal{D}$ (where $\mathcal{D}$ is defined by Definition* 3.7*) defines a continuous semigroup of solutions of* (1.5)*, that is, given $(\bar{u},\bar{\mu}) \in \mathcal{D}$, if we denote $t \mapsto (u(t),\mu(t)) = T_t(\bar{u},\bar{\mu})$ the corresponding trajectory, then u is a weak solution of* (1.5)*. Moreover μ is a weak solution of the following transport equation for the energy density*

$$\mu_t + (u\mu)_x = (G(u) - 2Pu)_x. \tag{3.10}$$

The map T is continuous with respect to all the variables, on any bounded set of $\mathcal{E}$. Furthermore, we have that

$$\mu(t)(\mathbb{R}) = \mu(0)(\mathbb{R}) \text{ for all } t \tag{3.11}$$

and

$$\mu(t)(\mathbb{R}) = \mu_{\mathrm{ac}}(t)(\mathbb{R}) = \|u(t)\|^2_{H^1} = \mu(0)(\mathbb{R}) \text{ for almost all } t. \tag{3.12}$$

Remark 3.14. We denote the unique solution described in the theorem as a *conservative* weak solution of (1.5).

Proof. From (3.12), it follows that $u \in L^\infty(\mathbb{R}_+, H^1(\mathbb{R}))$. The function u is a weak solution of (1.5) if it satisfies (2.66) and μ is a weak solution of (3.10) if

$$\int_{\mathbb{R}_+\times\mathbb{R}} (\phi_t + u\phi_x)(t,x)\,\mu(t,dx)dt = \int_{\mathbb{R}_+\times\mathbb{R}} ((G(u) - 2Pu)\phi_x)(t,x)\,dtdx \tag{3.13}$$

for all $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ with compact support. We already proved in Theorem 2.9 that $u(t)$ satisfies (2.66). We proceed the same way to prove that μ satisfies (3.13). We recall the proof of (3.11) and (3.12), which is the same as in [16]. From (3.8a), we obtain

$$\mu(t)(\mathbb{R}) = \int_{\mathbb{R}} H_\xi\,d\xi = H(t,\infty)$$

which is constant in time, see Lemma 2.7 (*iii*). Hence, (3.11) is proved. We know from Lemma 2.7 (*ii*) that, for $t \in \mathcal{K}^c$, $y_\xi(t,\xi) > 0$ for almost every $\xi \in \mathbb{R}$ (see (2.41) for the definition of $\mathcal{K}$, in particular, we have $\operatorname{meas}(\mathcal{K}) = 0$). Given $t \in \mathcal{K}^c$ (the time variable is suppressed in the notation when there is no ambiguity), we have, for any Borel set B,

$$\mu(t)(B) = \int_{y^{-1}(B)} H_\xi \, d\xi = \int_{y^{-1}(B)} \left(U^2 + \frac{U_\xi^2}{y_\xi^2} \right) y_\xi \, d\xi \tag{3.14}$$

from (2.27c). Since y is one-to-one when $t \in \mathcal{K}^c$ and $u_x \circ y y_\xi = U_\xi$ almost everywhere, we obtain from (3.14) that

$$\mu(t)(B) = \int_B (u^2 + u_x^2)(t,x) \, dx,$$

which, as $\operatorname{meas}(\mathcal{K}) = 0$, proves (3.12). □

3.3. **The topology on $\mathcal{D}$.** The metric $d_{\mathcal{D}}$ gives to $\mathcal{D}$ the structure of a complete metric space while it makes continuous the semigroup T_t of conservative solutions for the Camassa–Holm equation as defined in Theorem 3.13. In that respect, it is a suitable metric for the equation. However, as the definition of $d_{\mathcal{D}}$ is not straightforward, this metric is not so easy to manipulate. That is why we recall the results obtained in [16] where we compare the topology induced by $d_{\mathcal{D}}$ with more standard topologies. We have that convergence in $H^1(\mathbb{R})$ implies convergence in $(\mathcal{D}, d_{\mathcal{D}})$, which itself implies convergence in $L^\infty(\mathbb{R})$. More precisely, we have the following result.

Proposition 3.15. [16, Proposition 5.1] *The map*

$$u \mapsto (u, (u^2 + u_x^2)dx)$$

is continuous from $H^1(\mathbb{R})$ into $\mathcal{D}$. In other words, given a sequence $u_n \in H^1(\mathbb{R})$ converging to u in $H^1(\mathbb{R})$, then $(u_n, (u_n^2 + u_{nx}^2)dx)$ converges to $(u, (u^2 + u_x^2)dx)$ in $\mathcal{D}$.

Proposition 3.16. [16, Proposition 5.2] *Let (u_n, μ_n) be a sequence in $\mathcal{D}$ that converges to (u, μ) in $\mathcal{D}$. Then*

$$u_n \to u \text{ in } L^\infty(\mathbb{R}) \quad \text{and} \quad \mu_n \overset{*}{\rightharpoonup} \mu.$$

4. Examples

We include two examples for the Camassa–Holm equation where $f(u) = \frac{1}{2}u^2$ and $g(u) = u^2$.

(i) For initial data $\bar{u}(x) = ce^{-|x|}$, we have

$$u(t,x) = ce^{-|x-ct|}, \tag{4.1}$$

which is the familiar one peakon solution of the Camassa–Holm equation. The characteristics are the solutions of

$$y_t(t,\xi) = u(t, y(t,\xi)), \tag{4.2}$$

which can be integrated and, for initial data $\bar{y}(\xi) = \xi$, yields

$$y(t,\xi) = \operatorname{sgn}(\xi) \ln\left(e^{(\operatorname{sgn}(\xi)ct)} + e^{|\xi|} - 1\right).$$

Some characteristics are plotted in Figure 1. We have $U(t,\xi) = u(t, y(t,\xi)) = ce^{-|y(t,\xi)-ct|}$. It is easily checked that $y_\xi > 0$ almost everywhere. In this case y is invertible, there is no concentration of energy on a singular set, and we have

$$H(t,\xi) = \int_{y^{-1}((-\infty, y(t,\xi)))} H_\xi(\eta) \, d\eta = \int_{-\infty}^{y(t,\xi)} (u^2 + u_x^2) \, dx,$$

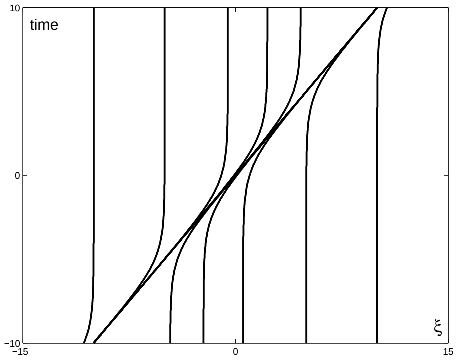

FIGURE 1. Characteristics in the single peakon case.

from (3.14).

(ii) The case with a peakon–antipeakon collision for the Camassa–Holm equation is considerably more complicated. In [17], we prove that the structure of the multipeakons is preserved, even through collisions. In particular, for an n-peakon u, it means that for almost all time the solution $u(t,x)$ can be written as

$$u(t,x) = \sum_{i=1}^{n} p_i(t) e^{-|x-q_i(t)|} \tag{4.3}$$

for some functions p_i and q_i that satisfy a system of ordinary differential equation that however experiences singularities at collisions. In [17] we also present a system of ordinary differential equation satisfied by $y(t,\xi_i)$, $U(t,\xi_i)$ and $H(t,\xi_i)$ with $i = 1,\dots,n$ where $y(t,\xi_i)$ and $U(t,\xi_i)$ correspond to the position and the height of the ith peak, respectively, while $H(t,\xi_i)$ represents the energy contained between $-\infty$ and the ith peak. In the antisymmetric case, this system can be solved explicitly, see [17], and we obtain

$$\begin{aligned}
y(t,\xi_2) &= -y(t,\xi_1) = \ln(\cosh(\frac{Et}{2})),\\
U(t,\xi_2) &= -U(t,\xi_1) = \frac{E}{2}\tanh(\frac{Et}{2}),\\
H(t,\xi_2) - H(t,\xi_1) &= -\frac{E^2}{2}\tanh^2(\frac{Et}{2}) + E^2.
\end{aligned} \tag{4.4}$$

The initial conditions were chosen so that the two peaks collide at time $t = 0$. From (4.3) and (4.4), we obtain

$$u(t,x) = \begin{cases} -\frac{E}{2}\sinh(\frac{Et}{2})e^{x}, & \text{for } x < -\ln\big(\cosh(\frac{Et}{2})\big),\\ \frac{E\sinh(x)}{\sinh(\frac{Et}{2})}, & \text{for } |x| < \ln\big(\cosh(\frac{Et}{2})\big),\\ \frac{E}{2}\sinh(\frac{Et}{2})e^{-x}, & \text{for } x > \ln\big(\cosh(\frac{Et}{2})\big). \end{cases} \tag{4.5}$$

See Figure 2. The formula holds for all $x \in \mathbb{R}$ and t nonzero. For $t = 0$ we find formally $u(0,x) = 0$. Here E denotes the total energy of the system, i.e.,

$$H(t,\infty) = \int_{\mathbb{R}} (u^2 + u_x^2)\,dx = E^2, \quad t \neq 0. \tag{4.6}$$

For all $t \neq 0$ we find

$$\mu((-\infty,y)) = \mu_{\rm ac}((-\infty,y)) = \int_{-\infty}^{y} (u^2 + u_x^2)\,dx. \tag{4.7}$$

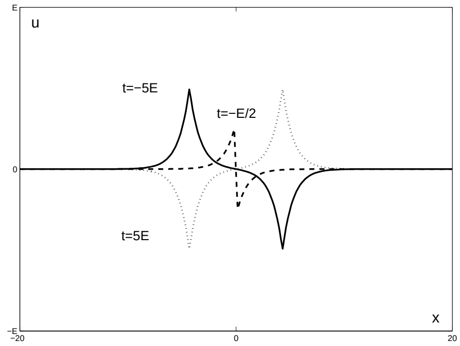

FIGURE 2. The colliding peakons case. Plot of the solution at different times.

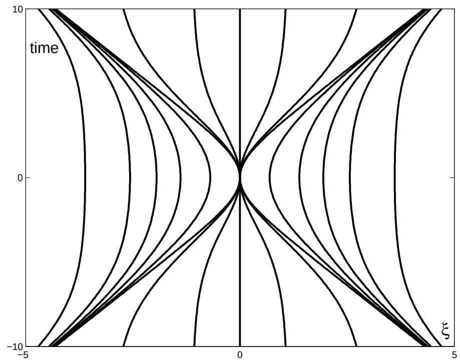

FIGURE 3. Characteristics in the colliding peakons case.

For $t = 0$, $H(0, \xi_2) - H(0, \xi_1) = E^2$, all the energy accumulates at the origin, and we find

$$\mu(x) = E^2\delta(x)\,dx, \quad \mu_{\mathrm{ac}}(x) = 0. \tag{4.8}$$

The function $u(t,x)$ is no longer Lipschitz in x, and (4.2) does not necessarily admit a unique solution. Indeed, given $T > 0$ and x_0 such that $|x_0| < \ln\big(\cosh(\frac{-TE}{2})\big)$, the characteristic arising from $(x_0, -T)$ can be continued past the origin by any characteristic that goes through (x, T) where x satisfies $|x| < \ln\big(\cosh(\frac{TE}{2})\big)$, and still be a solution of (4.2). However by taking into account the energy, the system (2.15) selects one characteristic, and in that sense the characteristics are uniquely defined. We can compute them analytically and obtain

$$y(t) = 2\tanh^{-1}\big(C\tanh^2(\frac{Et}{4})\big)$$

with $|C| < 1$, for the characteristics that collide, and

$$y(t) = \varepsilon\ln\big(C + \cosh(\frac{Et}{2})\big)$$

with $\varepsilon = \pm 1$, $C \geq 1$, for the others. See Figure 3.

Acknowledgments. The authors gratefully acknowledge the hospitality of the Mittag-Leffler Institute, Sweden, creating a great working environment for research, during the Fall of 2005.

References

[1] A. Bressan and A. Constantin. Global conservative solutions of the Camassa–Holm equation. *Preprint.* Submitted, 2005.

[2] A. Bressan and M. Fonte. An optimal transportation metric for solutions of the Camassa–Holm equation. *Methods Appl. Anal.*, to appear.

[3] H. Brezis. *Analyse Fonctionnelle.* Masson, Paris, 1983.

[4] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71(11):1661–1664, 1993.

[5] R. Camassa, D. D. Holm, and J. Hyman. A new integrable shallow water equation. *Adv. Appl. Mech.*, 31:1–33, 1994.

[6] G. M. Coclite, and H. Holden, and K. H, Karlsen. Well-posedness for a parabolic-elliptic system. *Discrete Cont. Dynam. Systems* 13:659–682, 2005.

[7] G. M. Coclite, and H. Holden, and K. H, Karlsen. Global weak solutions to a generalized hyperelastic-rod wave equation. *SIAM J. Math. Anal.*, 37:1044–1069, 2005.

[8] A. Constantin and J. Escher. Global existence and blow-up for a shallow water equation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(2):303–328, 1998.

[9] A. Constantin and L. Molinet. Global weak solutions for a shallow water equation. *Comm. Math. Phys.*, 211(1):45–61, 2000.

[10] H.-H. Dai. Exact travelling-wave solutions of an integrable equation arising in hyperelastic rods. *Wave Motion*, 28(4):367–381, 1998.

[11] H.-H. Dai. Model equations for nonlinear dispersive waves in a compressible Mooney–Rivlin rod. *Acta Mech.*, 127(1-4):193–207, 1998.

[12] H.-H. Dai and Y. Huo. Solitary shock waves and other travelling waves in a general compressible hyperelastic rod. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 456(1994):331–363, 2000.

[13] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[14] G. B. Folland, Real Analysis, Wiley, New York, second edition, 1999.

[15] M. Fonte. Conservative solution of the Camassa Holm equation on the real line. *arXiv:math.AP/0511549*, 3, 2005.

[16] H. Holden and X. Raynaud. Global conservative solutions of the Camassa–Holm equation—a Lagrangian point of view. *Preprint*, 2006, submitted.

[17] H. Holden and X. Raynaud. Global conservative multipeakon solutions of the Camassa–Holm equation. *J. Hyperbolic Differ. Equ.*, to appear.

[18] R. S. Johnson. Camassa–Holm, Korteweg–de Vries and related models for water waves. *J. Fluid Mech.*, 455:63–82, 2002.

[19] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and Measure-Valued Solutions to Evolutionary PDEs*, Chapman & Hall, London, 1996.

[20] Z. Xin and P. Zhang. On the uniqueness and large time behavior of the weak solutions to a shallow water equation. *Comm. Partial Differential Equations*, 27:1815–1844, 2002.

[21] K. Yosida. *Functional Analysis.* Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

(Holden)
Department of Mathematical Sciences, Norwegian University of Science and Technology, NO–7491 Trondheim, Norway, and
Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, NO–0316 Oslo, Norway

E-mail address: `holden@math.ntnu.no`
URL: `www.math.ntnu.no/~holden/`

(Raynaud)
Department of Mathematical Sciences, Norwegian University of Science and Technology, NO–7491 Trondheim, Norway

E-mail address: `raynaud@math.ntnu.no`
URL: `www.math.ntnu.no/~raynaud/`