# DEPENDENCE OF ENTROPY SOLUTIONS IN THE LARGE FOR THE EULER EQUATIONS ON NONLINEAR FLUX FUNCTIONS

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Abstract. We study the dependence of entropy solutions in the large for hyperbolic systems of conservation laws whose flux functions depend on a parameter vector $\mu$. We first formulate an effective approach for establishing the $L^1$-estimate pointwise in time between entropy solutions for $\mu \neq 0$ and $\mu = 0$, respectively, with respect to the flux parameter vector $\mu$. Then we employ this approach and successfully establish the $L^1$-estimate between entropy solutions in the large for several important nonlinear physical systems including the isentropic and relativistic Euler equations and for the isothermal Euler equations, respectively, for which the parameters are the adiabatic exponent $\gamma > 1$ and the speed of light $c < \infty$.

## 1. Introduction

Consider the Cauchy problem for a hyperbolic system of conservation laws in one-space dimension

$$\begin{cases} \partial_t W^\mu(U) + \partial_x F^\mu(U) = 0, \qquad x \in \mathbb{R}, \\ U|_{t=0} = U_0(x), \end{cases} \tag{1.1}$$

where $W^\mu, F^\mu : \mathbb{R}^n \to \mathbb{R}^n$ are smooth functions that depend on a parameter vector $\mu = (\mu_1, \dots, \mu_k)$ with $\mu_i \in [0, \mu_0]$ $(i = 1, \dots, k)$ for some $\mu_0 > 0$, and $\nabla W^\mu(U)$ is uniformly invertible when $\mu_i \in [0, \mu_0]$ with $W^0(U) = U$ when $\mu = 0$.

We are interested in the dependence of entropy solutions $U^\mu$ in the large to (1.1) on the parameter vector $\mu$. System (1.1) for $\mu = 0$ is assumed to be uniformly stable in $L^1$ with the Lipschitz Standard Riemann Semigroup $\mathcal{S}$ that generates the entropy solution $U$. We denote by $U^\mu$ the entropy solution to (1.1) for $\mu \neq 0$ which is constructed by the front-tracking method. In this paper, we first formulate an effective approach for establishing the $L^1$-estimates pointwise in time of the following type:

$$\|U^\mu(t) - U(t)\|_{L^1} \leq C\, TV\{U_0\}\, t\, \|\mu\|, \tag{1.2}$$

under the convergence assumption of the front-tracking approximations to system (1.1) for $\mu \neq 0$, even when the initial data function $U_0$ has large variation, where $C > 0$ is a constant independent of the parameter vector $\mu$ and $\|\mu\|$ denotes the magnitude of the vector $\mu$. Then we employ this approach and successfully establish the $L^1$-estimate between entropy solutions in the large for several important nonlinear physical systems including the isentropic and relativistic Euler equations and for the isothermal Euler equations, respectively, for which the parameters are the adiabatic exponent $\gamma > 1$ and the speed of light $c < \infty$.

Our approach utilizes the wave front-tracking method. More precisely, for every $\delta > 0$, let $U^{\delta,\mu}$ be the $\delta$-approximate solution to (1.1) for $\mu \neq 0$ constructed by the wave front-tracking method so that the approximations $U^{\delta,\mu}$ are globally defined piecewise constant functions with finite number of discontinuities, as constructed as follows: First, choose a piecewise constant function $U_0^\delta(x)$ such that

$$\|U_0^\delta - U_0\|_{L^1} < \delta. \tag{1.3}$$

Let $\varrho$ be a constant given at the outset of the construction algorithm. At each discontinuity point of $U_0^\delta(x)$, a Riemann problem arises and the solution consists of shocks, contact discontinuities, and rarefaction waves. We approximate the rarefaction waves by a centered rarefaction fan containing several small jumps traveling at a speed close to the characteristic speed and with strength of each of these fronts less than $\delta$. The piecewise constant approximate solution can be prolonged until when the first set of interactions take place. Depending

*Date*: June 3, 2006.

1991 *Mathematics Subject Classification.* Primary: 35L65,35L67,35B30, 76N15, 76Y05; Secondary: 35B35,83A05.

*Key words and phrases.* Dependence, nonlinear flux functions, Euler equations, entropy solutions, adiabatic exponent, light speed, isothermal, isentropic, relativistic, $L^1$-estimate, error estimate, front-tracking.

on the interaction, either the Accurate Riemann Solver (ARS) or the Simplified Riemann Solver (SRS) is used to solve the Riemann problem that arises at the interaction point. ARS is basically the exact Riemann solution except that the rarefaction waves are approximated by rarefaction fans as mentioned above, while SRS introduces a non-physical wave front with constant speed $\hat{\lambda}$. ASR is employed at $t = 0$ and at every interaction between two physical waves when the product of the strengths of the incoming waves is $|\alpha\,\alpha'| \geq \varrho$, and SRS is used at every interaction involving a non-physical incoming wave-front and also at interactions where $|\alpha\,\alpha'| \leq \varrho$. Hence, the algorithm involves three parameters: A fixed speed $\hat{\lambda}$, strictly larger than all characteristic speeds; a threshold $\varrho > 0$ determining whether ASR or SRS is applied; and the maximum strength of rarefaction fronts which is less than the front-tracking parameter $\delta$.

To complete the construction, one needs to show that, for suitable $\nu$ and $\rho$, the above algorithm produces a $\delta$-*approximate solution* $U^{\delta,\mu}(t,x)$ defined for all $t \geq 0$: $U^{\delta,\mu}(t,x)$ is piecewise constant with discontinuities occurring along finitely many lines in the $(t,x)$-plane. The jumps can be of three types: shock fronts, rarefaction fronts, and non-physical fronts. The shock fronts travel with the Rankine-Hugoniot speed, while the rarefaction fronts travel with the characteristic speed of the right state and have strength less than $\delta$. All non-physical fronts have the same speed $\hat{\lambda}$, where $\hat{\lambda}$ is a fixed constant strictly greater than all the characteristic speeds. The total strength of all non-physical fronts is uniformly small, i.e., $\sum|\alpha| < \delta$. For more details for the front-tracking method, we refer the reader to Bressan [3], Dafermos [10], and Holden-Risebro [14].

Our approach for the $L^1$-estimate of dependence of entropy solutions in $\mu$ can be formulated in the following fashion. Assume that $U^\mu$ can be constructed by the front-tracking method as described above:

$$U^{\delta,\mu} \to U^\mu \qquad \text{in } L^1_{loc} \quad \text{as } \delta \to 0+\,.$$

First, from the approximate initial data $U_0^\delta$ and the corresponding $\delta$-approximate solution $U^{\delta,\mu}$, we employ the following standard formula in semigroup theory:

$$\|\mathcal{S}_t U_0^\delta - U^{\delta,\mu}(t)\|_{L^1} \leq L \int_0^t \liminf_{h\to 0+} \frac{\|\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)\|_{L^1}}{h} d\tau, \tag{1.4}$$

where $L$ is the Lipschitz constant of the semigroup $\mathcal{S}$. Observe that the limit in the integral of (1.4) is invariant for time $t$ between two interaction points. Then the essential ingredient for this approach is to estimate

$$\|\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)\|_{L^1}, \tag{1.5}$$

which is equivalent to solving first the Riemann problem of (1.1) when $\mu = 0$ for $\tau \leq t \leq \tau + h$ with data

$$(U_L, U_R) = \begin{cases} U^{\delta,\mu}(\tau,x), & x < \bar{x}, \\ U^{\delta,\mu}(\tau,x), & x > \bar{x} \end{cases} \tag{1.6}$$

over each front $x = \bar{x}$ at time $\tau$ and to comparing then the Riemann solution with $U^{\delta,\mu}(\tau+h)$. We solve the Riemann problem only for the non-interaction times $\tau$ of the $\delta$–approximate solution $U^{\delta,\mu}$ since there is only a finite number of interactions. For each front at $(\tau,\bar{x})$, there exists $a > 0$ so that the interval $(\bar{x}-a, \bar{x}+a)$ at $t = \tau$ contains only one front, i.e., only the front passing through $(\tau,\bar{x})$. Thus, it suffices to compare the jump of the front of $U^{\delta,\mu}$ at $(\tau,\bar{x})$ as given by (1.6) with the Riemann solution generated by the semigroup $\mathcal{S}$ at time $t = \tau + h$ for $x \in (\bar{x}-a, \bar{x}+a)$. If we can establish that, for sufficiently small $h/a$,

$$\frac{1}{h}\int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\, dx = \mathcal{O}(1)(|U_L - U_R|\,\|\mu\| + \delta), \tag{1.7}$$

then summing over all fronts yields

$$\begin{aligned} \|\mathcal{S}_t U_0^\delta - U^{\delta,\mu}(t)\|_{L^1} &\leq L \int_0^t \sum_{\text{fronts } x=\bar{x}(\tau)} \frac{1}{h}\int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\, dx d\tau \\ &= \mathcal{O}(1)\left(\int_0^t TV\{U^{\delta,\mu}\}(\tau)\, d\tau\, \|\mu\| + \delta\, t\right) \\ &= \mathcal{O}(1)(TV\{U_0\}\, t\, \|\mu\| + t\,\delta), \end{aligned} \tag{1.8}$$

where we use $TV\{U^{\delta,\mu}\}(t) = \mathcal{O}(1)\,TV\{U_0\}$. As $\delta \to 0+$, we obtain

$$\|\mathcal{S}_t U_0 - U^\mu(t)\|_{L^1} = \mathcal{O}(1) TV\{U_0\}\, t\, \|\mu\|. \tag{1.9}$$

Note that $U(t,x) := \mathcal{S}_t U_0(x)$ is unique within the class of viscosity solutions (cf. [3]). Thus the whole sequence $U^\mu$ converges in $L^1$ to the entropy solution $U$ of the limit equation $\mu = 0$, as $\mu \to 0$.

In our applications of this approach to several nonlinear physical systems of Euler equations in this paper, we use the Standard Riemann Semigroup $\mathcal{S}$ to the isothermal Euler equations:

$$\begin{cases} \partial_t \rho + \partial_x(\rho\, u) = 0, \\ \partial_t(\rho\, u) + \partial_x(\rho\, u^2 + \kappa^2 \rho) = 0, \end{cases} \tag{1.10}$$

with large initial data

$$(\rho, \rho u)|_{t=0} = (\rho_0, \rho_0 u_0)(x), \tag{1.11}$$

where $\rho > 0$ and $u$ is the density and velocity of the fluid, respectively. System (1.10) can be rewritten in a conservation form as (1.1) with $\mu = 0$ by setting

$$U = (\rho, \rho u), \qquad W^0(U) = U, \qquad F^0(U) = (\rho u, \rho u^2 + \kappa^2 \rho)^\top.$$

The existence of entropy solutions in $BV$ of (1.10)–(1.11) was first established by Nishida [23] via the Glimm scheme; and the construction of the Standard Riemann Semigroup to (1.10) is presented in Colombo-Risebro [9] when the total variation of the initial data is not necessarily small. More precisely, for fixed two states $U_-$ and $U_+$ in $\Omega$ and a constant $M > 0$, there exist a domain $\mathcal{D} \subseteq BV(\mathbb{R})$, a semigroup $\mathcal{S} :\, [0,\infty) \times \mathcal{D} \mapsto \mathcal{D}$, and a constant $L > 0$ such that

(i) $\mathcal{D}$ contains the $L^1$−closure of the set of those functions $U : \mathbb{R} \to \Omega$ such that $U - U_- \in L^1(-\infty, 0)$, $U - U_+ \in L^1(0,\infty)$ and $TV\{U\} \leq M$;

(ii) $\mathcal{S}$ is $L^1$-Lipschitz continuous with constant $L$, i.e.,

$$\|\mathcal{S}_t U_0 - \mathcal{S}_s W_0\|_{L^1} \leq L\left(\|U_0 - W_0\|_{L^1} + |t - s|\right);$$

(iii) If $U_0$ is piecewise constant, then, for small $t$, the map $(t,x) \mapsto \mathcal{S}_t U_0(x)$ coincides with the solution to (1.10)–(1.11) obtained by piecing together the Lax solutions of the Riemann problems determined at the jumps of $U_0$.

Then the Standard Riemann Semigroup is unique and the trajectory $\mathcal{S}_t U_0$ is an entropy solution to (1.10)–(1.11), which is unique in the class of *viscosity solutions* with interaction potential locally uniformly bounded. These are exactly what we require for our approach for $\mathcal{S}_t U_0$.

For the most relevant work in the literature, we refer the reader to Temple [29] and Bianchini-Colombo [2].

Temple [29] considered systems of conservation laws when the flux depends on a parameter $\mu$ and established the existence of entropy solutions by observing that the nonlinear functional employed in the Glimm scheme [13] depends only on the properties of the equations at $\mu = 0$, for which the dependence of the interaction estimates on the parameter $\mu$ was studied. Our approach here is to establish similar bounds on the $L^1$–difference between entropy solutions to such systems for $\mu \neq 0$ and $\mu = 0$, respectively.

Bianchini-Colombo [2] studied the dependence of entropy solutions on the flux functions. They considered the systems with smooth flux functions $F : \Omega \mapsto \mathbb{R}^n$ with $F \in Hyp(\Omega)$, i.e., $F$ that generates a Standard Riemann Semigroup $\mathcal{S}^F$ with domain $\mathcal{D}^F$, compared two systems with flux functions $F$, $G \in Hyp(\Omega)$ with $\mathcal{D}^G \subset \mathcal{D}^F$, by introducing a *distance* function between $F$ and $G$, and showed that the Standard Riemann Semigroup is a Lipschitz function with respect to the $C^0$–norm of derivative of $F$, i.e. $DF$. Note that, in [2], both systems are required to generate a Standard Riemann Semigroup, which has been established if the initial data is small (cf. [3]) or if the initial data is large for the isothermal Euler equations ($\gamma = 1$) and the relativistic Euler equations ($\gamma = 1$, $c < \infty$) (cf. [9]). Thus, they apply their result to establish the convergence of the entropy solutions of the relativistic Euler equations for the case $\gamma = 1$ and $c < \infty$ to the classical limit (the isothermal Euler equations, $\gamma = 1$). In this paper, we formulate a different approach to study the dependence of entropy solutions on the parameters of the flux function. To compare two systems of conservations laws, we require only that one of them generates a Standard Riemann Semigroup and the second system has a global entropy solution obtained by the front-tracking method. In this way, we can apply our approach to hyperbolic systems even if the initial data is not necessarily small and treat other models of Euler equations for which the stability of entropy solutions has not been established yet.

In Sections 2-4, we present four applications of this effective approach and establish the $L^1$-estimate pointwise in time between entropy solutions for the isothermal Euler equations and those for (i) the isentropic Euler equations, (ii) the relativistic Euler equations for conservation of momentum, and (iii) isentropic relativistic Euler equations, as well as between entropy solutions for the relativistic Euler equations for conservation of momentum for $\gamma = 1$ and those for the general relativistic Euler equations for conservation of momentum for $\gamma > 1$. Furthermore, the approach presented in this paper will be further applied to solving more complicated physical systems in our forthcoming papers.

We remark that the results in this paper show that entropy solutions away from the vacuum of these physical systems are $L^1$-stable when the adiabatic exponent $\gamma > 1$ tends to 1 and the light speed $c$ tends to $\infty$. It would be interesting to study the stability of entropy solutions containing vacuum states of these physical systems, especially when the adiabatic exponent $\gamma > 1$ tends to 1 for the isentropic Euler equations. The existence of such entropy solutions containing vacuum states in $L^\infty$ has been established for the isentropic Euler equations in [4, 11, 12, 22, 21] (also see [5]) and the isothermal Euler equations in [15] (also see [16]).

## 2. Isentropic Euler Equations

In this section, we study the isentropic Euler equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho\, u) + \partial_x(\rho u^2 + p(\rho)) = 0 \end{cases} \tag{2.1}$$

for a perfect polytropic fluid with the pressure-density relation: $p(\rho) = \kappa^2 \rho^\gamma$, $\gamma \geq 1$, where $u$ is the velocity of the fluid. Set

$$\varepsilon = \frac{\gamma - 1}{2}.$$

System (2.1) can be rewritten in a conservation form by setting $U = (U_1, U_2)^\top = (\rho, \rho u)^\top$ and

$$W^\varepsilon(U) = U, \qquad F^\varepsilon(U) = (\rho, \rho u^2 + \kappa^2 \rho^{2\varepsilon+1})^\top$$

with Cauchy initial data

$$U|_{t=0} = U_0(x) := (\rho_0(x), \rho_0(x) u_0(x))^\top. \tag{2.2}$$

System (2.1) is strictly hyperbolic with two distinct eigenvalues when $\rho > 0$:

$$\lambda_1 = u - \kappa\sqrt{\gamma}\rho^{\frac{\gamma+1}{2}}, \qquad \lambda_2 = u + \kappa\sqrt{\gamma}\rho^{\frac{\gamma+1}{2}}. \tag{2.3}$$

Denote the entropy solution to (2.1)–(2.2) by $U^\varepsilon$ obtained as a limit of the front-tracking approximations. Our aim is to first establish the front-tracking method and then investigate the dependence of the entropy solutions $U^\varepsilon$ for $\varepsilon \geq 0$ by employing the approach described in §1.

The Riemann invariants of (2.1) are

$$r = u + \kappa\sqrt{2\varepsilon + 1}\,\frac{\rho^\varepsilon - 1}{\varepsilon}, \qquad s = u - \kappa\sqrt{2\varepsilon + 1}\,\frac{\rho^\varepsilon - 1}{\varepsilon}. \tag{2.4}$$

Note that, as $\varepsilon \to 0+$, the corresponding Riemann invariants are

$$r = u + \kappa \ln \rho, \qquad s = u - \kappa \ln \rho \tag{2.5}$$

to the isothermal Euler equations (1.10).

From now on, we measure the strength of a front that joins the states $U_L$ and $U_R$ by $\alpha = \rho_R/\rho_L$. The following lemma presents the well known relations of shock and rarefaction curves on the $(r, s)$-plane:

**Lemma 2.1.** *The shock curves are*

$$S_1: \begin{cases} r_L - r_R = \kappa\rho_L^\varepsilon\Big(\sqrt{\dfrac{(\alpha-1)(\alpha^{2\varepsilon+1}-1)}{\alpha}} - \sqrt{2\varepsilon+1}\,\dfrac{\alpha^\varepsilon-1}{\varepsilon}\Big), \\ s_L - s_R = \kappa\rho_L^\varepsilon\Big(\sqrt{\dfrac{(\alpha-1)(\alpha^{2\varepsilon+1}-1)}{\alpha}} + \sqrt{2\varepsilon+1}\,\dfrac{\alpha^\varepsilon-1}{\varepsilon}\Big), \qquad \alpha>1; \end{cases}$$

$$S_2: \begin{cases} r_L - r_R = \kappa\rho_L^\varepsilon\Big(\sqrt{\dfrac{(\alpha-1)(\alpha^{2\varepsilon+1}-1)}{\alpha}} - \sqrt{2\varepsilon+1}\,\dfrac{\alpha^\varepsilon-1}{\varepsilon}\Big), \\ s_L - s_R = \kappa\rho_L^\varepsilon\Big(\sqrt{\dfrac{(\alpha-1)(\alpha^{2\varepsilon+1}-1)}{\alpha}} + \sqrt{2\varepsilon+1}\,\dfrac{\alpha^\varepsilon-1}{\varepsilon}\Big), \qquad \alpha<1. \end{cases}$$

*The rarefaction curves are*

$$R_1: r_L - r_R = 0, \quad s_L - s_R = 2\kappa\sqrt{2\varepsilon+1}\rho_L^\varepsilon\frac{\alpha^\varepsilon-1}{\varepsilon}, \qquad \alpha<1,$$

$$R_2: r_L - r_R = -2\kappa\sqrt{2\varepsilon+1}\rho_L^\varepsilon\frac{\alpha^\varepsilon-1}{\varepsilon}, \quad s_L - s = 0, \qquad \alpha>1.$$

The existence of entropy solutions to (2.1)–(2.2) was first established in Nishida-Smoller [24] by the Glimm scheme [13]. This was recently captured by the front-tracking method by Asakura [1].

**Lemma 2.2** (Asakura [1])**.** *Assume that there exist $\underline{\rho} < \bar\rho$ such that*

$$0 < \underline{\rho} \le \rho_0(x) \le \bar\rho < \infty. \tag{2.6}$$

*Then there exists a constant $N>0$ such that, if*

$$\varepsilon\, TV\{U_0\} \le N, \tag{2.7}$$

*for every $\delta>0$, the Cauchy problem* (2.1)–(2.2) *admits a $\delta$-approximate front-tracking solution $U^{\delta,\varepsilon}$, defined for all $t\ge 0$.*

By standard arguments (cf. [3]), we arrive at

**Lemma 2.3.** *Under assumptions* (2.6)–(2.7)*, the Cauchy problem* (2.1)–(2.2) *has an entropy solution $U^\varepsilon(t,x)$ defined for all $t\ge0$ obtained as a limit of the front-tracking approximations $U^{\delta,\varepsilon}$ in $L^1_{loc}$.*

Using our approach described in §1, we have the following theorem.

**Theorem 2.1.** *Assume that $U_0$ satisfies* (2.6)–(2.7)*. Let $\mathcal{S}$ be the Standard Riemann Semigroup to the isothermal Euler equations* (1.10) *and* (2.2)*. Let $U^\varepsilon$ be the entropy solution to* (2.1)–(2.2) *for $\varepsilon>0$ obtained by the front-tracking method. Then, for every $t>0$,*

$$\|\mathcal{S}_t U_0 - U^\varepsilon(t)\|_{L^1} = \mathcal{O}(1) TV\{U_0\}\, t\,\varepsilon. \tag{2.8}$$

**Proof.** As explained in §1, it suffices to consider the front-tracking approximations $U^{\delta,\varepsilon}$ to (2.1)–(2.2), where $\delta>0$ is the front-tracking parameter for $\varepsilon=(\gamma-1)/2\in[0,\varepsilon_0]$. By Lemma 2.2, $U^{\delta,\varepsilon}$ exist for all $t>0$ and converge to $U^\varepsilon$ as $\delta\downarrow 0$. Thus it suffices to show that, for each $t>0$,

$$\|\mathcal{S}_t U_0^\delta - U^{\delta,\varepsilon}(t)\|_{L^1} = \mathcal{O}(1)\,(TV\{U_0\}\,\varepsilon+\delta)\,t \qquad \text{for all } \delta>0. \tag{2.9}$$

In view of the approach developed in §1, establishing (2.9) is equivalent to studying the trajectory of the semigroup $\mathcal{S}$ to (1.10) with Riemann data front $(U_L,U_R)$ of the approximations $U^{\delta,\varepsilon}$ at the discontinuity point $(\tau,\bar x)$ of $U^{\delta,\varepsilon}$, but away from the interaction, over the time interval $(\tau,\tau+h)$. There are three cases to be investigated: shock fronts, rarefaction fronts, and non-physical fronts.

**Case** 1: **Shock Front** $(U_L,U_R)$**.** We first focus on a 1-shock front of (2.1), for which the speed of the front is given by

$$\sigma(\varepsilon) = \frac{\rho_L u_L - \rho_R u_R}{\rho_L-\rho_R}$$

and the strength is $\alpha=\rho_R/\rho_L$. We define the jump on the Riemann invariants along the shock curves to be

$$H^{(S)}(\alpha,\varepsilon) := \begin{pmatrix} r_L-r_R \\ s_L-s_R\end{pmatrix} = \begin{pmatrix} \kappa\rho_L^\varepsilon\big(\sqrt{\frac{(\alpha-1)(\alpha^{2\varepsilon+1}-1)}{\alpha}} - \sqrt{2\varepsilon+1}\,\frac{\alpha^\varepsilon-1}{\varepsilon}\big) \\ \kappa\rho_L^\varepsilon\big(\sqrt{\frac{(\alpha-1)(\alpha^{2\varepsilon+1}-1)}{\alpha}} + \sqrt{2\varepsilon+1}\,\frac{\alpha^\varepsilon-1}{\varepsilon}\big)\end{pmatrix}, \qquad \alpha\ge1 \text{ for 1-shock}; \tag{2.10}$$

$$H^{(S)}(\alpha,\varepsilon) := \begin{pmatrix} r_L - r_R \\ s_L - s_R \end{pmatrix} = \begin{pmatrix} \kappa\rho_L^{\varepsilon}\big(\sqrt{\frac{(\alpha-1)(\alpha^{2\varepsilon+1}-1)}{\alpha}} - \sqrt{2\varepsilon+1}\,\frac{\alpha^{\varepsilon}-1}{\varepsilon}\big) \\ \kappa\rho_L^{\varepsilon}\big(\sqrt{\frac{(\alpha-1)(\alpha^{2\varepsilon+1}-1)}{\alpha}} + \sqrt{2\varepsilon+1}\,\frac{\alpha^{\varepsilon}-1}{\varepsilon}\big) \end{pmatrix}, \qquad \alpha \le 1 \text{ for 2-shock.} \tag{2.11}$$

The Riemann solution to (1.10) (i.e., $\varepsilon = 0$) with data $(U_L, U_R)$ consists of 3 states, $U_L$, $U^*$, and $U_R$, joined by either 1- and 2- shocks, or by 1-shock and 2-rarefaction waves as shown in Figure 1. Without loss of generality, let $(\tau, \bar{x}) = (\tau, 0)$.

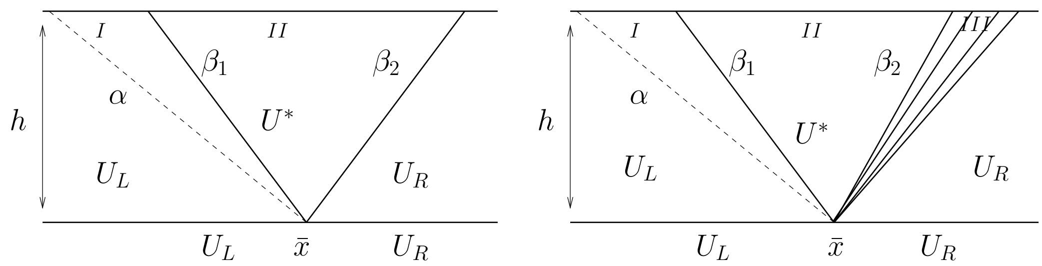


Figure 1. The Riemann solutions for $\varepsilon = 0$

Since $U_L$ and $U^*$ are joined by a 1-shock, we have

$$r_L - r^* = \kappa\big(\frac{\beta_1 - 1}{\sqrt{\beta_1}} - \ln\beta_1\big), \qquad s_L - s^* = \kappa\big(\frac{\beta_1 - 1}{\sqrt{\beta_1}} + \ln\beta_1\big) \tag{2.12}$$

with strength $\beta_1 = \rho^*/\rho_L$. Depending on whether the 2-wave is a shock or rarefaction wave of strength $\beta_2 = \rho_R/\rho^*$, we obtain either

$$r^* - r_R = \kappa\big(\frac{1-\beta_2}{\sqrt{\beta_2}} - \ln\beta_2\big), \quad s^* - s_R = \kappa\big(\frac{1-\beta_2}{\sqrt{\beta_2}} + \ln\beta_2\big), \tag{2.13}$$

or

$$r^* - r_R = -2\kappa\ln\beta_2, \quad s^* - s_R = 0, \tag{2.14}$$

respectively. We define $G(\beta_2)$ to be either

$$G(\beta_2) = \kappa(\frac{1-\beta_2}{\sqrt{\beta_2}} - \ln\beta_2, \frac{1-\beta_2}{\sqrt{\beta_2}} + \ln\beta_2)^{\top}, \qquad \beta_2 \le 1 \text{ for 2-shock,} \tag{2.15}$$

or

$$G(\beta_2) = (-2\kappa\ln\beta_2, 0)^{\top}, \qquad \beta_2 > 1 \text{ for 2-rarefaction wave.} \tag{2.16}$$

By (2.10) and (2.12)–(2.15), we have

$$H^{(S)}(\alpha,\varepsilon) = H^{(S)}(\beta_1, 0) + G(\beta_2).$$

It is easy to check that $H^{(S)}$ and $G$ are $C^2$–functions of their arguments. For example, we can view the term in $H^{(S)}$ as

$$\frac{\alpha^{\varepsilon}-1}{\varepsilon} = (\alpha-1)\int_0^1 (1+\mu(\alpha-1))^{\varepsilon-1}\,d\mu \in C^2. \tag{2.17}$$

It is a straightforward calculation to check that the rank of matrix

$$\Big(\frac{\partial H^{(S)}}{\partial\beta_1}, \frac{\partial G}{\partial\beta_2}\Big)\Big|_{\{\beta_1=\alpha,\beta_2=1\}} = \begin{pmatrix} \ldots & -2\kappa \\ \kappa(\frac{1}{2}\alpha^{1/2} + \frac{1}{2}\alpha^{-3/2} + \frac{1}{\alpha}) & 0 \end{pmatrix}$$

is two. The Implicit Function Theorem yields that $\beta_1$ and $\beta_2$ are $C^2$ functions of $(\alpha,\varepsilon)$:

$$\beta_1 = \beta_1(\alpha,\varepsilon), \qquad \beta_2 = \beta_2(\alpha,\varepsilon).$$

Moreover, $\beta_1(\alpha,0) = \alpha$, $\beta_2(\alpha,0) = 1$, $\beta_1(1,\varepsilon) = 1$, and $\beta_2(1,\varepsilon) = 1$. Thus,

$$\beta_1 = \alpha + \mathcal{O}(1)|\alpha-1|\,\varepsilon, \quad \beta_2 = 1 + \mathcal{O}(1)|\alpha-1|\,\varepsilon. \tag{2.18}$$

Let

$$\bar{U}(x) = \begin{cases} U_L & \text{for } x \in (\bar{x}-a, \bar{x}), \\ U_R & \text{for } x \in (\bar{x}, \bar{x}+a). \end{cases} \tag{2.19}$$

We now compare the Riemann solution $\mathcal{S}_h\bar{U}$ with the front states $(U_L, U_R)$ of $U^{\delta,\varepsilon}(\tau+h,x)$ over $x \in (\bar{x}-a, \bar{x}+a)$:

(i) Interval $I$: We first compare the speeds of the $\alpha$- and $\beta_1$-waves. Recall that the speed of the $\alpha$-wave is

$$\sigma(\varepsilon) = \frac{\rho_L u_L - \rho_R u_R}{\rho_L - \rho_R}. \tag{2.20}$$

The speed of the $\beta_1$-wave is

$$\sigma_1 = \frac{\rho_L u_L - \rho^* u^*}{\rho_L - \rho^*}. \tag{2.21}$$

The fact that $H^{(S)}$ is $C^2$ in $\varepsilon$ and $\rho_R, u_R \in C^2$ in $\varepsilon \in (0, \varepsilon_0)$ yields $\sigma(\varepsilon) \in C^2$ in $\varepsilon$ for $\alpha > 1$. As $\varepsilon \to 0+$, $\beta_1 \to \alpha$ and $(\rho^*, u^*) \to (\rho_R, u_R)$; hence $\sigma(0) = \sigma_1$. This implies that $\sigma(\varepsilon) = \sigma_1 + \mathcal{O}(1)\varepsilon$. Furthermore, the length of Interval I is $h|\sigma(\varepsilon) - \sigma_1| = \mathcal{O}(1)h\,\varepsilon$. Thus the total contribution in Interval I is

$$|U_L - U_R| \cdot [\text{length of interval}] = \mathcal{O}(1)h\,|U_L - U_R|\,\varepsilon. \tag{2.22}$$

(ii) Interval II. Assume that $U^*$ is connected to $U_R$ via a 2-shock of strength $\beta_2 = \rho_R/\rho^*$ and speed $\sigma_2$. Then, for $0 < \underline{\rho} \le \rho^* \le \overline{\rho} < \infty$,

$$\rho^* - \rho_R = \rho^*(1-\beta_2) = \mathcal{O}(1)|\alpha - 1|\varepsilon, \tag{2.23}$$
$$u^* - u_R = r^* - r_R + \kappa \ln \beta_2 = G_1(\beta_2) + \kappa \ln \beta_2 = \mathcal{O}(1)|\alpha-1|\varepsilon. \tag{2.24}$$

Furthermore, the length of Interval II is $h|\sigma_1 - \sigma_2| = \mathcal{O}(1)\,h$. Indeed,

$$\sigma_1 = \frac{\rho^* u^* - \rho_L u_L}{\rho^* - \rho_L} = u_L - 2\kappa\sqrt{\beta_1} < 0, \qquad \beta_1 > 1,$$

and

$$\sigma_2 = \frac{\rho_R u_R - \rho^* u^*}{\rho_R - \rho^*} = u_R + \frac{2\kappa}{\sqrt{\beta_2}} > 0, \qquad \beta_2 < 1.$$

Since, when $\varepsilon = 0$, there exists $\hat{\rho} > 0$ such that $0 < \hat{\rho} < \rho < \hat{\rho}^{-1}$, then $|\sigma_1| + |\sigma_2|$ is uniformly bounded, which implies $|\sigma_1 - \sigma_2| = |\sigma_1| + |\sigma_2| = \mathcal{O}(1)$.

Thus the total contribution on Interval II is

$$|U^* - U_R| \cdot [\text{length of interval}] = \mathcal{O}(1)\,h\,|\alpha - 1|\,\varepsilon. \tag{2.25}$$

Note that the total variation of the Riemann data is $|U_L - U_R| = \mathcal{O}(1)\,|\alpha - 1|$.

(iii) Interval III: If $U^*$ is connected with $U_R$ via a 2-rarefaction wave, then we also need to treat this interval. Denote the solution over this interval by $U(\xi)$, where $\xi = x/t \in [\xi_2^*, \xi_2] := [\lambda_2(U^*), \lambda_2(U_R)]$. Note that

$$|\xi_2^* - \xi_2| = \mathcal{O}(1)|U^* - U_R| = \mathcal{O}(1)|\alpha - 1|\,\varepsilon \tag{2.26}$$

by the estimates over Interval II. Thus, by (2.23) and (2.26), for $\xi \in [\xi_2^*, \xi_2]$,

$$\beta_2(\xi) := \frac{\rho(\xi)}{\rho^*} = \beta_2 + \mathcal{O}(1)|\xi_2^* - \xi_2| = \beta_2 + \mathcal{O}(1)|\alpha-1|\,\varepsilon. \tag{2.27}$$

Hence, we can estimate $|U(\xi) - U_R|$ as follows:

$$\rho(\xi) - \rho_R = \rho^* - \rho_R + \rho^*(\beta_2(\xi) - 1) = \mathcal{O}(1)|\alpha - 1|\,\varepsilon,$$
$$u(\xi) - u_R = r(\xi) - r_R + \kappa \ln \beta_2(\xi) = G_1(\beta_2(\xi)) + \kappa \ln \beta_2(\xi) = \mathcal{O}(1)|\alpha-1|\,\varepsilon.$$

Then the difference of the speeds is

$$|\sigma_2(\xi_2^*) - \sigma_2(\xi_2)| = |\lambda_2(U^*) - \lambda_2(U_R)| = \mathcal{O}(1)|U^* - U_R| = \mathcal{O}(1)|\alpha - 1|\,\varepsilon.$$

Thus, the total contribution over Interval III is

$$\int_{III} |\mathcal{S}_h\bar{U} - U_R|\,dx = \mathcal{O}(1)h\,|U_L - U_R||\alpha - 1|\,\varepsilon^2. \tag{2.28}$$

This completes the 1-shock front case. Similarly, we can treat the 2− shock front case for $\alpha < 1$. Therefore, we have

$$\begin{aligned}\frac{1}{h}\sum_{\substack{\text{shock}\\\text{fronts}}}\int_{\bar{x}-a}^{\bar{x}+a}|\mathcal{S}_h U^{\delta,\varepsilon}(\tau)-U^{\delta,\varepsilon}(\tau+h)|\,dx &= \mathcal{O}(1)\sum_{\substack{\text{shock}\\\text{fronts}}}(1+\varepsilon\,|\alpha-1|)|\alpha-1|\,\varepsilon\\ &= \mathcal{O}(1)\sum_{\substack{\text{shock}\\\text{fronts}}}(1+\varepsilon\,|\alpha-1|)|U^{\delta,\varepsilon}(\tau,\bar{x}-)-U^{\delta,\varepsilon}(\tau,\bar{x}+)|\,\varepsilon.\end{aligned}\tag{2.29}$$

**Case** 2: **Rarefaction Front** $(U_L, U_R)$**.** We first focus on the 1-rarefaction front case. Then the jump across this front is less than $\delta$ by construction. Denote the speed by

$$\sigma(\varepsilon) := \lambda_1^{\varepsilon}(U_R) = u_R - \kappa\sqrt{2\varepsilon+1}\rho_R^{\varepsilon+1}$$

and the strength by $\alpha = \rho_R/\rho_L$, whence $1-\alpha < \delta$.

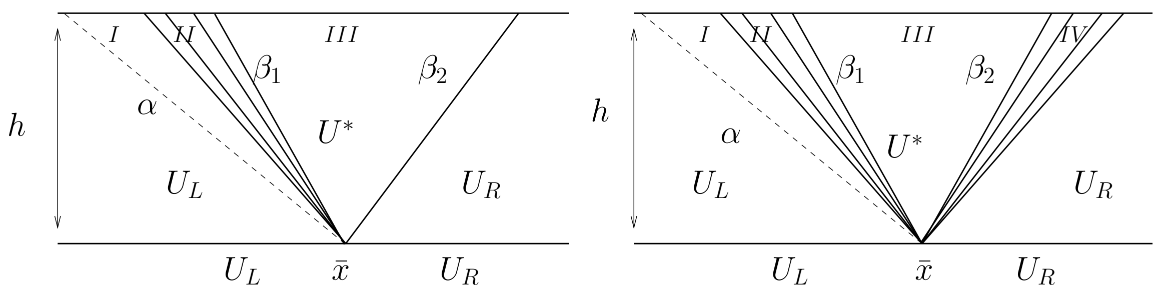


FIGURE 2. The Riemann solutions for $\varepsilon = 0$

To deal with the rarefaction fronts, we introduce

$$H^{(R)}(\alpha,\varepsilon) := \begin{pmatrix} r_L - r_R \\ s_L - s_R \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ -2\kappa\sqrt{2\varepsilon+1}\rho_L^{\varepsilon}\,\frac{\alpha^{\varepsilon}-1}{\varepsilon} \end{pmatrix}, & \alpha < 1 \quad \text{for 1-rarefaction wave,} \\ \begin{pmatrix} 2\kappa\sqrt{2\varepsilon+1}\rho_L^{\varepsilon}\,\frac{\alpha^{\varepsilon}-1}{\varepsilon} \\ 0 \end{pmatrix}, & \alpha > 1 \quad \text{for 2-rarefaction wave,} \end{cases}\tag{2.30}$$

corresponding to (2.10)–(2.11).

As before, the solution to (1.10) with Riemann data (2.19) consists of three states, $U_L$, $U^*$, and $U_R$ joined by either 1-rarefaction wave and a 2-shock or 1- and 2-rarefaction waves, as shown in Figure 2. Denote the strength of these waves by $\beta_1$ and $\beta_2$, respectively. Without loss of generality, let $\bar{x} = 0$. If $U^*$ and $U_R$ are joined by a 2-shock, then

$$U(t,x) = \mathcal{S}_t\bar{U}(x) = \begin{cases} U_L, & x/t < \lambda_1(U_L), \\ R_1(s;U_L), & x/t = \lambda_1(R_1(s;U_L)) \in [\lambda_1(U_L), \lambda_1(U^*)], \\ U^*, & x/t \in [\lambda_1(U^*), \sigma_2], \qquad \sigma_2 = \sigma_2(U^*, U_R), \\ U_R, & x/t > \sigma_2, \end{cases} \qquad t \in [0,h],\ x \in (-a,a);\tag{2.31}$$

otherwise

$$U(t,x) = \mathcal{S}_t\bar{U}(x) = \begin{cases} U_L, & x/t < \lambda_1(U_L), \\ R_1(s;U_L), & x/t = \lambda_1(R_1(s;U_L)) \in [\lambda_1(U_L), \lambda_1(U^*)], \\ U^*, & x/t \in [\lambda_1(U^*), \lambda_2(U^*)], \\ R_2(s;U^*), & x/t = \lambda_1(R_2(s;U_L)) \in [\lambda_1(U^*), \lambda_1(U_R)], \\ U_R, & x/t > \lambda_2(U_R), \end{cases} \qquad t \in [0,h],\ x \in (-a,a).\tag{2.32}$$

For $\alpha < 1$, we have the relation

$$H^{(R)}(\alpha,\varepsilon) = H^{(R)}(\beta_1, 0) + G(\beta_2),\tag{2.33}$$

where $G$ is given in (2.15)–(2.16), $\beta_1 = \rho^*/\rho_L$, and $\beta_2 = \rho_R/\rho^*$. Note that $H^{(R)}$ and $G$ are $C^2$ with respect to their arguments. It is easy to check that we can also apply the Implicit Function Theorem in this setting to obtain

$$\beta_1 = \beta_1(\alpha, \varepsilon), \quad \beta_2 = \beta_2(\alpha, \varepsilon),$$

and

$$\beta_1 = \alpha + \mathcal{O}(1)|\alpha - 1|\,\varepsilon, \quad \beta_2 = 1 + \mathcal{O}(1)|\alpha - 1|\,\varepsilon. \tag{2.34}$$

Then we proceed as before by estimating the contributions on the different intervals.

(i) Interval I: Denote the solution by $U(\xi)$, where $\xi = x/t \in [\xi_1, \xi_1^*] := [\lambda_1(U_L), \lambda_1(U_R)]$. Let

$$\beta_1(\xi) := \rho(\xi)/\rho_L, \tag{2.35}$$

hence $\beta_1(\xi_1) = 1$ and $\beta_1(\xi_1^*) = \rho^*/\rho_L$. We compare the speeds of the $\alpha$- and $\beta_1$-waves: The speed of $\beta_1$ is

$$\sigma_1(\xi) = \lambda_1(U(\xi)), \quad \lambda_1(U_L) \le \xi = x/t \le \lambda_1(U_R).$$

Since $\sigma(0) = \lambda_1(U_R)$, we obtain

$$|\sigma(\varepsilon) - \sigma_1| = \mathcal{O}(1)(|U_L - U_R| + \varepsilon) = \mathcal{O}(1)(\delta + \varepsilon)$$

for $\sigma_1 := \sigma_1(\xi_1) = \lambda_1(U_L)$. Moreover, the length of Interval I is $h\,|\sigma(\varepsilon) - \sigma_1| = \mathcal{O}(1)\,h\,(\delta + \varepsilon)$. Thus the total contribution over Interval I is

$$|U_L - U_R| \cdot [\text{length of interval}] = \mathcal{O}(1)h\,|U_L - U_R|\,(\delta + \varepsilon). \tag{2.36}$$

(ii) Interval II: Note that $|\xi_1^* - \xi_1| = \mathcal{O}(1)\delta$ so that

$$\beta_1(\xi) = \frac{\rho(\xi)}{\rho_L} = \beta_1 + \mathcal{O}(1)\,\delta. \tag{2.37}$$

We estimate $|U(\xi) - U_R|$ for $\xi \in [\xi_1, \xi_1^*]$ to obtain

$$\begin{aligned} \rho(\xi) - \rho_R &= \rho_L(1 - \alpha) + \rho_L(\beta_1(\xi) - 1) = \mathcal{O}(1)(\delta + |\alpha - 1|\varepsilon), \\ u(\xi) - u_R &= u_L - u_R - \kappa \ln \beta_1(\xi) = \mathcal{O}(1)(\delta + |\alpha - 1|\varepsilon). \end{aligned}$$

Now we compare the speeds to have

$$|\sigma_1(\xi_1) - \sigma_1(\xi_1^*)| = \mathcal{O}(1)|U_L - U^*| = \mathcal{O}(1)|U_L - U_R|\,\varepsilon.$$

Moreover, the length of Interval II is $h\,|\sigma_1(\xi_1) - \sigma_1(\xi_1^*)| = \mathcal{O}(1)\,h|U_L - U_R|$. Thus the total contribution is

$$\int_{II} |\mathcal{S}_h \bar{U} - U_R|\,dx = \mathcal{O}(1)h|U_L - U_R|\,(\delta + |\alpha - 1|\varepsilon)\,\varepsilon. \tag{2.38}$$

(iii) Interval III: Assume that $U^*$ is connected to $U_R$ via a 2-shock. We estimate $|U^* - U_R|$ to obtain

$$\begin{aligned} \rho^* - \rho_R &= \rho^*(\beta_2 - 1) = \mathcal{O}(1)|\alpha - 1|\,\varepsilon, \\ u^* - u_R &= r^* - r_R + \kappa \ln \beta_2 = G_1(\beta_2) + \kappa \ln \beta_2 = \mathcal{O}(1)|\alpha - 1|\,\varepsilon. \end{aligned}$$

As in Case 1, the length of Interval III is $h\,|\sigma_1(\xi_1^*) - \sigma_2| = \mathcal{O}(1)\,h$. Then the total contribution over Interval III is

$$|U^* - U_R| \cdot [\text{length of interval}] = \mathcal{O}(1)h\,|\alpha - 1|\,\varepsilon. \tag{2.39}$$

(iv) Interval IV: If $U^*$ is connected to $U_R$ via a 2-rarefaction wave, then we also need to treat this region. Denote the solution over this interval by $U(\xi)$, where $\xi = x/t \in [\xi_2^*, \xi_2] := [\lambda_2(U^*), \lambda_2(U_R)]$. Then we estimate $|U(\xi) - U_R|$ as before to have

$$\begin{aligned} \rho(\xi) - \rho_R &= \rho^*(1 - \beta_2) + \rho^*(\beta_2(\xi) - 1) = \mathcal{O}(1)(\delta + \varepsilon|\alpha - 1|), \\ u(\xi) - u_R &= r(\xi) - r_R + \kappa \ln \beta_2(\xi) = \mathcal{O}(1)(\delta + \varepsilon|\alpha - 1|). \end{aligned}$$

Moreover,

$$|\sigma_2(\xi_2^*) - \sigma_2(\xi_2)| = |\lambda_2(U^*) - \lambda_2(U_R)| = \mathcal{O}(1)|U^* - U_R| = \mathcal{O}(1)|U_L - U_R|\,\varepsilon,$$

hence the length of Interval IV is $h\,|\sigma_2(\xi_2^*)-\sigma_2(\xi_2)| = \mathcal{O}(1)\,h|U_L-U_R|\,\varepsilon$. Thus, the total contribution over Interval IV is

$$\int_{IV} |\mathcal{S}_h\bar{U} - U_R|\,dx = \mathcal{O}(1)h\,|U_L - U_R|\,(\delta + \varepsilon|\alpha - 1|)\,\varepsilon. \tag{2.40}$$

This completes the rarefaction front case with the following estimate:

$$\begin{aligned}
&\frac{1}{h}\sum_{\substack{\text{rarefaction}\\ \text{fronts}}} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau+h)|\,dx \\
&= \mathcal{O}(1) \sum_{\substack{\text{rarefaction}\\ \text{fronts}}} |\alpha - 1|\,(\delta + \varepsilon + \varepsilon|\alpha-1| + \varepsilon\delta + \varepsilon^2|\alpha - 1|) \\
&= \mathcal{O}(1) \sum_{\substack{\text{rarefaction}\\ \text{fronts}}} |U^{\delta,\varepsilon}(\tau,\bar{x}-) - U^{\delta,\varepsilon}(\tau,\bar{x}+)|(\delta + \varepsilon + |\alpha - 1|\,\varepsilon).
\end{aligned} \tag{2.41}$$

**Case 3: Non-Physical Front** $(U_L, U_R)$**.** By construction, the sum of strengths of these fronts is less than $\delta$. The speed of the front is $\hat{\lambda}$, strictly greater than the characteristic speed $\lambda_i$ for all $i = 1, 2$. We measure the strength of the front by $\alpha = \rho_R/\rho_L$ and know that $|U_L - U_R| = \mathcal{O}(1)|\alpha - 1|$.

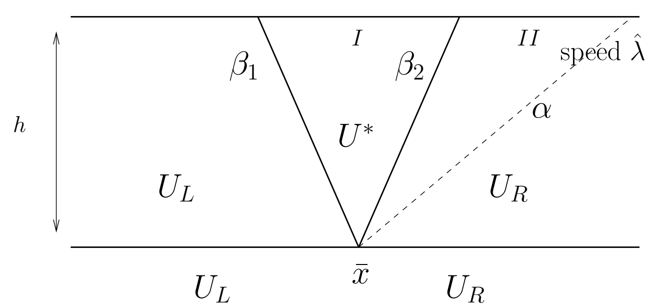


Figure 3. The Riemann solutions for $\varepsilon = 0$

Without loss of generality, we assume that the solution to the Riemann problem consists of two shocks with strengths $\beta_1$ and $\beta_2$. We treat the case of rarefaction waves as in Cases 1 and 2.

(i) Interval I: Because of the structure of system (1.10), the total variation is decreasing. Hence

$$|U^* - U_L| \leq |U_L - U_R| = \mathcal{O}(1)|\alpha - 1|.$$

Moreover, the length of Interval I is $h\,|\sigma_1 - \sigma_2| = \mathcal{O}(1)\,h$. Thus the total contribution over Interval I is

$$|U^* - U_L| \cdot [\text{length of interval}] = \mathcal{O}(1)\,h\,|\alpha - 1|. \tag{2.42}$$

(ii) Interval II: The length of the interval is $h\,|\sigma_2 - \hat{\lambda}| = \mathcal{O}(1)\,h$. Thus the total contribution is

$$|U_L - U_R| \cdot [\text{length of interval}] = \mathcal{O}(1)\,h\,|\alpha - 1|. \tag{2.43}$$

Thus,

$$\frac{1}{h}\sum_{\substack{\text{non-physical}\\ \text{fronts}}} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau+h)|\,dx = \mathcal{O}(1) \sum_{\substack{\text{non-physical}\\ \text{fronts}}} |\alpha - 1| < \delta. \tag{2.44}$$

By (2.29), (2.41), and (2.44), we conclude

$$
\begin{aligned}
\|\mathcal{S}_t U^{\delta,\varepsilon}(0) - U^{\delta,\varepsilon}(t)\|_{L^1} &\leq L \int_0^t \sum_{\text{fronts } x=\bar{x}(\tau)} \frac{1}{h} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\varepsilon}(\tau) - U^{\delta,\varepsilon}(\tau+h)|\, dx \\
&= \mathcal{O}(1) \left( (\varepsilon+\delta) \int_0^t TV U^{\delta,\varepsilon}(\tau)\, d\tau + \delta\, t \right) \\
&= \mathcal{O}(1)\, t\, (TV\{U_0\}\, \varepsilon + \delta). \qquad (2.45)
\end{aligned}
$$

The proof is complete. ■

**Remark 2.1.** *The $L^1$-estimate in $t$ in* (2.8) *of order one is in fact optimal, which can be easily seen when the initial data is Riemann data.*

## 3. Relativistic Euler equations for conservation of momentum

The relativistic Euler equations for conservation of momentum are

$$
\begin{cases}
\partial_t \left( \dfrac{(p+\rho\, c^2)}{c^2} \dfrac{u^2}{c^2-u^2} + \rho \right) + \partial_x \left( (p+\rho\, c^2) \dfrac{u}{c^2-u^2} \right) = 0, \\
\partial_t \left( (p+\rho\, c^2) \dfrac{u}{c^2-u^2} \right) + \partial_x \left( (p+\rho\, c^2) \dfrac{u^2}{c^2-u^2} + p \right) = 0,
\end{cases} \qquad (3.1)
$$

where $u$ is the classical velocity, $\rho$ the mass-energy density, and $p = p(\rho)$ is the pressure of the fluid, and $c$ is the speed of light. We consider a perfect polytropic fluid with the pressure-density relation: $p(\rho) = \kappa^2 \rho^\gamma$, $\gamma := 2\varepsilon + 1 > 1$. The corresponding physical region is

$$
\Omega = \{(\rho, u):\ 0 \leq \rho < \rho_{max},\ |u| < c\}, \qquad (3.2)
$$

where $\rho_{max} = \sup\{\rho : p'(\rho) \leq c^2\}$, which represents the fact that the fluid speed is always less than the light speed. System (3.1) can be written in the conservation form by setting $U = (\rho, \rho u)^\top$ and

$$
\begin{aligned}
W^\mu(U) = (W_1^\mu(U), W_2^\mu(U))^\top &= \left( \frac{(p+\rho\, c^2)}{c^2} \frac{u^2}{c^2-u^2} + \rho, (p+\rho\, c^2) \frac{u}{c^2-u^2} \right)^\top, \\
F^\mu(U) = (F_1^\mu(U), F_2^\mu(U))^\top &= \left( (p+\rho\, c^2) \frac{u}{c^2-u^2}, (p+\rho\, c^2) \frac{u^2}{c^2-u^2} + p \right)^\top,
\end{aligned}
$$

where $\mu = (\varepsilon, \frac{1}{c^2})$.

The system is strictly hyperbolic with the following two eigenvalues:

$$
\lambda_i(U) = \frac{u + (-1)^i \kappa \sqrt{2\varepsilon+1} \rho^\varepsilon}{1 + (-1)^i\, \kappa u \sqrt{2\varepsilon+1}\, \rho^\varepsilon / c^2}, \qquad (3.3)
$$

and two linearly independent eigenvectors

$$
r_i(U) = \left( \frac{(-1)^i}{c^2-u^2}, \frac{\kappa \sqrt{2\varepsilon+1}\, \rho^\varepsilon}{\kappa^2 + \rho^{2\varepsilon+1} + \rho\, c^2} \right)^\top, \qquad i = 1, 2. \qquad (3.4)
$$

Consider the Cauchy problem to (3.1) with initial data $U_0 = (\rho_0, \rho_0 u_0)$. Global BV solutions for $\varepsilon = 0$ were first constructed by Smoller-Temple [28] by the Glimm scheme [13] with large variation of the initial data. Later, Chen [8] established the result for $\varepsilon > 0$ under the usual condition on the initial data. In view of our approach in §1, to achieve our goal, we first need to obtain global BV solutions by the front-tracking method. As it is shown by Li-Ren [19], we expect that the total variation will be uniformly bounded, independent of $\mu$. We first state the following preliminary results.

**Lemma 3.1.** *The mapping $(\rho, u) \to (W_1^\mu, W_2^\mu)$ is uniformly 1-1, and the Jacobian of the mapping is continuous and non-zero in the region $\Omega \cap \{\rho > 0\}$ for $\varepsilon \in [0, \varepsilon_0]$ and $c \in [c_0, \infty)$. Moreover, the convergence*

$$
W^\mu \to (\rho, \rho u)^\top, \quad F^\mu(U) \to (\rho u, \rho u^2 + \kappa^2 \rho^{2\varepsilon+1})^\top \qquad (3.5)
$$

*as $c \to \infty$ is uniform in any bounded region $\{U : 0 < \rho < \bar{\rho},\ |u| < \bar{v} < c\}$, where $\bar{\rho}$ and $\bar{v}$ are positive constants.*

One of the important properties of (3.1) is the invariance under the Lorenz transformation:

$$\bar{t} = \frac{ct - \frac{\tau}{c}x}{\sqrt{1 - \frac{t^2}{c^2}}}, \quad \bar{x} = \frac{cx - c\tau\, t}{\sqrt{1 - \frac{t^2}{c^2}}}, \quad u = \frac{\tau + \bar{u}}{1 + \frac{\tau\bar{u}}{c^2}},$$

where the barred coordinates $(\bar{t}, \bar{x})$ move with velocity $\tau$ as measured in the unbarred coordinates $(t, x)$, $u$ denotes the velocity of a particle as measured in the unbarred frame, and $\bar{u}$ denotes the velocity in the barred frame.

Due to the Lorenz invariance, if there are two states $(\rho_L, u_L)$ and $(\rho_R, u_R)$ connected by a shock, we can assume the velocity state of the left-hand side of the shock in the barred coordinates is $\bar{u}_L = 0$. Hence the Lax entropy and Rankine-Hugoniot conditions imply (see [6, 8, 17, 19]):

**Lemma 3.2.** *The shock curves in the barred coordinates are given by*

$$\bar{S}_1\text{:}\ \bar{u}_R = -\kappa c^2\sqrt{\frac{(\alpha^{2\varepsilon+1}-1)(\alpha-1)}{(\kappa^2\alpha^{2\varepsilon+1}+\rho_L^{-2\varepsilon}c^2)(\kappa^2\rho_L^{2\varepsilon}+\alpha c^2)}}, \quad \alpha = \rho/\rho_L > 1, \qquad \text{for } \rho > \rho_L,\ \bar{u} < \bar{u}_L = 0;$$

$$\bar{S}_2\text{:}\ \bar{u}_R = -\kappa c^2\sqrt{\frac{(\alpha^{2\varepsilon+1}-1)(\alpha-1)}{(\kappa^2\alpha^{2\varepsilon+1}+\rho_L^{-2\varepsilon}c^2)(\kappa^2\rho_L^{2\varepsilon}+\alpha c^2)}}, \quad \alpha = \rho/\rho_L < 1, \qquad \text{for } 0 < \rho < \rho_L,\ \bar{u} < \bar{u}_L = 0.$$

*The rarefaction curves in the barred coordinates are given by*

$$\bar{R}_1\text{:}\ \frac{c}{2}\ln\big(\frac{c+u}{c-u}\big) + \sqrt{2\varepsilon+1}\frac{c}{\varepsilon}\left(\arctan\big(\frac{\kappa\rho^\varepsilon}{c}\big) - \arctan\big(\frac{\kappa}{c}\big)\right) = const. \qquad \text{for } 0 < \rho < \rho_L, \bar{u} > \bar{u}_L = 0;$$

$$\bar{R}_2\text{:}\ \frac{1}{2}c\ln\big(\frac{c+u}{c-u}\big) - c\frac{\sqrt{2\varepsilon+1}}{\varepsilon}\left(\arctan\big(\frac{\kappa\rho^\varepsilon}{c}\big) - \arctan\big(\frac{\kappa}{c}\big)\right) = const. \qquad \text{for } \rho > \rho_L, \bar{u} > \bar{u}_L = 0.$$

Since

$$c^2\sqrt{\frac{(\alpha^{2\varepsilon+1}-1)(\alpha-1)}{(\kappa^2\alpha^{2\varepsilon+1}+\rho_L^{-2\varepsilon}c^2)(\kappa^2\rho_L^{2\varepsilon}+\alpha c^2)}} = |\alpha-1|\sqrt{\frac{(2\varepsilon+1)\int_0^1(1+\lambda(\alpha-1))^{2\varepsilon}d\lambda}{(\kappa^2\alpha^{2\varepsilon+1}/c^2+\rho_L^{-2\varepsilon})(\kappa^2\rho_L^{2\varepsilon}/c^2+\alpha)}}$$

and

$$\begin{aligned}&\frac{1}{2}c\ln\big(\frac{c+u}{c-u}\big) \pm c\frac{\sqrt{2\varepsilon+1}}{\varepsilon}\left(\arctan\big(\frac{\kappa\rho^\varepsilon}{c}\big) - \arctan\big(\frac{\kappa}{c}\big)\right)\\ &\quad= \frac{1}{2}\int_{-1}^1\frac{ud\lambda}{1+\lambda u/c} \pm \int_0^1\frac{\sqrt{2\varepsilon+1}\,d\lambda}{1+\kappa^2(1+\lambda(\rho^\varepsilon-1))^2/c^2}\frac{\kappa(\rho^\varepsilon-1)}{\varepsilon},\end{aligned}$$

then these expressions can be viewed as $C^\infty$ functions of $(\alpha, \mu, \rho)$ near $\alpha = 1$ and for $\varepsilon \in [0, \varepsilon_0]$, $c \geq c_0$, and $\rho \in [\underline{\rho}, \overline{\rho}]$. The following proposition is crucial to obtain the wave interaction estimates.

**Proposition 3.3.** *Consider two shock curves of the first family, which start from the points $(r_0, s_1) = (r(U_0), s(U_1))$ and $(r_0, s_0) = (r(U_0), s(U_0))$ and are continued to the points $(r, s_2)$ and $(r, s)$, respectively. Then*

$$s_0 - s - (s_1 - s_2) = \mathcal{O}(1)\varepsilon(s_1 - s_0)(r_0 - r), \tag{3.6}$$

*where $\mathcal{O}(1)$ depends only on $\rho_0, \rho_1 \in [\underline{\rho}, \overline{\rho}]$ and $c_0$, independent of $\varepsilon > 0$ and $c \geq c_0 > 0$.*

**Proof.** Let $s_2 = s^*(\varepsilon, 1/c^2, \Delta r, \Delta s)$, where $\Delta r = r - r_0$ and $\Delta s = s_1 - s_0$. By Lemma 3.2 and following [24], $s^*$ is a smooth function of its arguments and satisfies

$$\begin{aligned}&s^*(0, 1/c^2, 0, 0) - s^*(0, 1/c^2, \Delta r, 0) - (s^*(0, 1/c^2, 0, \Delta s) - s^*(0, 1/c^2, \Delta r, \Delta s))\\ &= \rho_0 - \rho - (\rho_1 - \rho_2) = 0.\end{aligned}$$

Then

$$
\begin{aligned}
&(s_0 - s) - (s_1 - s_2)\\
&= s^*(\varepsilon, 1/c^2, 0, 0) - s^*(\varepsilon, 1/c^2, \Delta r, 0) - (s^*(\varepsilon, 1/c^2, 0, \Delta s) - s^*(\varepsilon, 1/c^2, \Delta r, \Delta s))\\
&= (s^*(\varepsilon, 1/c^2, 0, 0) - s^*(0, 1/c^2, 0, 0)) - (s^*(\varepsilon, 1/c^2, \Delta r, 0) - s^*(0, 1/c^2, \Delta r, 0))\\
&\quad - (s^*(\varepsilon, 1/c^2, 0, \Delta s) - s^*(0, 1/c^2, 0, \Delta s)) + (s^*(\varepsilon, 1/c^2, \Delta r, \Delta s) - s^*(0, 1/c^2, \Delta r, \Delta s))\\
&= \varepsilon \int_0^1 I_3 d\lambda,
\end{aligned}
$$

where

$$
I_3 = s^*_\varepsilon(\lambda\,\varepsilon, 1/c^2, 0, 0) - s^*_\varepsilon(\lambda\,\varepsilon, 1/c^2, \Delta r, 0) - s^*_\varepsilon(\lambda\,\varepsilon, 1/c^2, 0, \Delta s) + s^*_\varepsilon(\lambda\,\varepsilon, 1/c^2, \Delta r, \Delta s) = \mathcal{O}(1)\Delta r \Delta s. \quad (3.7)
$$

This completes the proof. ∎

We denote by $U_L$, $U_M$, and $U_R$ the three constant states from left to right, which are connected by two incoming waves, and denote by $\gamma + \beta \to \beta' + \gamma'$ the interaction of a 2−wave $\gamma$ with a 1−wave $\beta$ which produces an outgoing 1−wave $\beta'$ and a 2−wave $\gamma'$; the other case can be written in a similar way, while the rarefaction waves are denoted by 0. Using the above lemma, we can carry out the same steps as in [24] to derive the following interaction estimates.

**Lemma 3.4.** *Assume that $0 < \underline{\rho} < \overline{\rho} < \infty$. Then there exist $C > 0$ and $\delta \in (0,1)$ depending only on the system, $\underline{\rho}$, and $\overline{\rho}$ such that the following estimates hold for the corresponding interactions:*

(1) $\gamma + \beta \to \beta' + \gamma'$: *One of the following holds:*
   (a) $|\beta'| \le |\beta| + C\epsilon|\beta||\gamma|$, $|\gamma'| \le |\gamma| + C\varepsilon|\beta||\gamma|$;
   (b) $|\beta'| = |\beta| - \zeta$, $|\gamma'| \le |\gamma| + C\varepsilon|\beta||\gamma| + \eta$;
   (c) $|\gamma'| = |\gamma| - \zeta$, $|\beta'| \le |\beta| + C\varepsilon|\beta\gamma| + \eta$, $0 \le \eta \le \delta\zeta$;
(2) $\gamma + 0 \to 0 + \gamma'$: $|\gamma'| = |\gamma|$;
(3) $\gamma_1 + \gamma_2 \to 0' + \gamma'$: $|\gamma'| = |\gamma_1| + |\gamma_2|$;
(4) $\gamma + 0 \to \beta' + \gamma'$ *or* $\gamma + 0 \to \beta' + 0$: *There exist* 1−*shock* $\beta_0$ *and* 2−*shock* $\gamma_0$ *such that the interaction* $\gamma_0 + \beta_0 \to \beta' + \gamma'$ *is the same as in* (1) *and* $|\beta_0| + |\gamma_0| \le |\gamma| - C_0|\beta_0|$;
(5) $0 + \gamma \to \beta' + \gamma'$ *or* $0 + \gamma \to \beta' + 0$: $|\gamma'| + |\beta'| \le |\gamma| - C_0|\beta'|$;
(6) $0 + 0 \to 0' + 0'$,

*where $C$ and $C_0$ are constants independent of $\varepsilon, \beta, \gamma$, $\rho \in [\underline{\rho}, \overline{\rho}]$, and $c \ge c_0$.*

This lemma enables us to introduce the Glimm functional for the approximate solutions which are constructed by the front-tracking method with the same notations as in [1]. Let $J$ be the space-like curve and denote $S_j(J)$ as the set of $j$−shock waves crossing $J$, and $S(J) = S_1(J) \cap S_2(J)$. We define

$$
\begin{aligned}
L^-(J) &= \sum\{|\alpha| : \ \alpha \in S(J)\},\\
Q(J) &= \sum\{|\beta||\gamma| : \ \beta \in S_1(J),\ \gamma \in S_2(J) \text{ and } \beta \text{ and } \gamma \text{ are approaching}\},
\end{aligned}
$$

and set $F(J) := L^-(J) + 4C\varepsilon Q(J)$. Let $O$ stand for the initial $I$−curve as defined in [24]. Then we follow [24] to obtain

**Lemma 3.5.** *If $C\varepsilon F(O) \le C'$ for some constant $C'$, then $F(J_2) \le F(J_1)$ for $J_2 > J_1$. Therefore, $L^-(J) \le F(O)$.*

Following [1] (see also [24]), the $\delta$−approximate solutions $U^{\delta,\mu}$ are globally defined and converge to the entropy solution $U^\mu$ to (3.1) as $\delta \downarrow 0$. See Lemmas 2.2 and 2.3.

Now, we present the principal result of this section.

**Theorem 3.1.** *Suppose that $0 < \underline{\rho} \le \rho_0(x) \le \bar{\rho} < \infty$ and $\varepsilon\, TV\{U_0\} \le N$. Let $\mathcal{S}$ be the Standard Riemann Semigroup to the isothermal Euler equations* (1.10)–(1.11). *If $U^\mu$ is the entropy solution to* (3.1) *with $\varepsilon > 0$ and $c \ge c_0$ obtained by the front-tracking method, then, for every $t > 0$,*

$$
\|\mathcal{S}_t U_0 - U^\mu(t)\|_{L^1} = \mathcal{O}(1) TV\{U_0\}\, t\, \|\mu\|, \quad (3.8)
$$

*where $\|\mu\| = \varepsilon + 1/c^2$.*

**Proof.** We follow the proof of Theorem 2.1. Here, we emphasize only the estimates required to deal with system (3.1). As before, we study the three possible types of fronts.

**Case 1: Shock Front** $(U_L, U_R)$: Assuming that this is a 1-shock front of strength $\alpha = \rho_R/\rho_L > 1$ without loss of generality. By Lemma 3.2, we have

$$u_R = \frac{u_L + \bar{u}_R}{1 + \frac{u_L \bar{u}_R}{c^2}} \tag{3.9}$$

for $\tau = u_L$. The speed of the front is

$$\sigma(\mu) = \frac{1}{W_1^\mu(U_R) - W_1^\mu(U_L)} \left( \frac{(\kappa^2 \rho_R^{2\varepsilon+1} + \rho_R c^2) u_R}{c^2 - u_R^2} - \frac{(\kappa^2 \rho_L^{2\varepsilon+1} + \rho_L c^2) u_L}{c^2 - u_L^2} \right). \tag{3.10}$$

Note that $\bar{u}_R = \bar{u}_R(\alpha, \mu) \in C^2$ in $(\alpha, \varepsilon, \frac{1}{c^2})$ for $\alpha$ near 1, $\varepsilon \in [0, \varepsilon_0]$, and $c \geq c_0$, whence $u_R - u_L \in C^2(\alpha, \mu)$. Also, $\sigma(\mu) \in C^2(\mu)$ for $\alpha > 1$. Without loss of generality, we assume that the Riemann solution consists of a 1-shock and a 2-shock. We treat the 2 rarefaction wave as in the proof of Theorem 2.1. Then we have

$$\begin{aligned} u^* - u_L &= \kappa \frac{1 - \beta_1}{\sqrt{\beta_1}}, \quad \rho^* - \rho_L = \rho_L(\beta_1 - 1), \qquad \beta_1 \geq 1, \\ u_R - u^* &= \kappa \frac{\beta_2 - 1}{\sqrt{\beta_2}}, \quad \rho_R - \rho^* = \rho^*(\beta_2 - 1), \qquad \beta_2 \leq 1, \end{aligned}$$

where $\beta_1 = \rho^*/\rho_L$ and $\beta_2 = \rho_R/\rho^*$. Hence

$$\begin{aligned} \rho_R - \rho_L &= \rho_L(\beta_1 - 1) + \rho_R \frac{(\beta_2 - 1)}{\beta_2}, \\ u_R - u_L &= \kappa \frac{1 - \beta_1}{\sqrt{\beta_1}} + \kappa \frac{\beta_2 - 1}{\sqrt{\beta_2}}. \end{aligned}$$

We define

$$H^{(S)}(\alpha, \mu) := (\rho_R - \rho_L, u_R - u_L)^\top \in C^2 \quad \text{in } (\alpha, \mu)$$

for $\alpha$ near 1, $\varepsilon \in [0, \varepsilon_0]$, and $c \geq c_0$. Also,

$$G(\beta_2) := (\rho_R - \rho^*, u_R - u^*)^\top.$$

By the relation

$$H^{(S)}(\alpha, \mu) = H^{(S)}(\beta_1, 0, 0) + G(\beta_2)$$

and the Implicit Function Theorem, we obtain $C^2$ solutions $\beta_1 = \beta_1(\alpha, \mu)$ and $\beta_2 = \beta_2(\alpha, \mu)$ that satisfy

$$\beta_1(1, \mu) = \beta_2(1, \mu) = 1, \quad \beta_1(\alpha, 0, 0) = \alpha, \quad \beta_2(\alpha, 0, 0) = 1.$$

Therefore

$$\beta_1 = \beta_1(\alpha, \mu) = \alpha + \mathcal{O}(1)|\alpha - 1|\,\|\mu\|, \tag{3.11}$$

$$\beta_2 = \beta_2(\alpha, \mu) = 1 + \mathcal{O}(1)|\alpha - 1|\|\mu\| \tag{3.12}$$

for $\varepsilon \in [0, \varepsilon_0]$, $c \geq c_0$, and $\|\mu\| = \varepsilon + 1/c^2$. Now we compare the Riemann solution $\mathcal{S}_h \bar{U}$ with the front states $(U_L, U_R)$ of $U^{\delta,\mu}(\tau + h, x)$ for $x \in (\bar{x} - a, \bar{x} + a)$ (see Fig. 2).

(i) Interval I: Denote by $\sigma_i$ the speed of the $i$-shock. Then

$$\sigma_1 = \frac{\rho_L u_L - \rho^* u^*}{\rho_L - \rho^*} \tag{3.13}$$

and $\sigma(0, 0) = \sigma_1$. Since $\sigma(\mu) \in C^2$ for $\alpha > 1$, we have

$$|\sigma(\mu) - \sigma_1| = \mathcal{O}(1)\|\mu\|. \tag{3.14}$$

Moreover, the length of Interval I is $h\,|\sigma(\mu) - \sigma_1| = \mathcal{O}(1)\,h\,\|\mu\|$. Then the total contribution over Interval I is

$$|U_L - U_R| \cdot [\text{length of the interval}] = \mathcal{O}(1)\,h|U_L - U_R|\,\|\mu\|. \tag{3.15}$$

(ii) Interval II: We use (3.12) to estimate $|U^* - U_R|$:

$$\rho^* - \rho_R = \rho^*(1-\beta_2) = \mathcal{O}(1)|\alpha - 1|\,\|\mu\|,$$
$$u^* - u_R = G_1(\beta_2) = \mathcal{O}(1)|\alpha - 1|\,\|\mu\|.$$

Interval II has length $h\,|\sigma_1 - \sigma_2| = \mathcal{O}(1)\,h$. Thus the total contribution over Interval II is

$$|U^* - U_R| \cdot [\text{length of the interval}] = \mathcal{O}(1)\,h|U_L - U_R|\,\|\mu\|. \tag{3.16}$$

(iii) Interval III: We need to work on this interval only if there is a 2-rarefaction wave instead of a 2-shock. We treat this in the same way as Case 1 in Section 2.

Thus, for all shock fronts, we conclude

$$\frac{1}{h} \sum_{\substack{\text{shock}\\ \text{fronts}}} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\,dx$$
$$= \mathcal{O}(1) \sum_{\substack{\text{shock}\\ \text{fronts}}} (1 + \|\mu\|\,|\alpha-1|)|U^{\delta,\mu}(\tau,\bar{x}-) - U^{\delta,\mu}(\tau,\bar{x}+)|\,\|\mu\|. \tag{3.17}$$

**Case 2: Rarefaction Front** $(U_L, U_R)$. The strength of this front is less than $\delta$ by construction. If this is a 1-rarefaction front with speed

$$\sigma(\mu) := \lambda_1(\mu)(U_R) = \frac{u_R - \kappa\sqrt{2\varepsilon+1}\,\rho_R^{\varepsilon}}{1 - \kappa\, u_R\sqrt{2\varepsilon+1}\,\rho_R^{\varepsilon}/c^2} \tag{3.18}$$

and strength $\alpha = \rho_R/\rho_L$, then $1 - \alpha < \delta$. The solution $\mathcal{S}_h\bar{U}$ consists of three states as given in (2.31)–(2.32). Define $H^{(R)}$ similarly. Then we have the identity

$$H^{(R)}(\alpha,\mu) = H^{(R)}(\beta_1, 0, 0) + G(\beta_2) \tag{3.19}$$

and conclude (3.11)–(3.12). We study each interval in Figure 2 as proceed as in Case 2 of Theorem 2.1.

(i) Interval I: As before, if $\sigma_1$ is the speed of the $\beta_1$-wave, by (3.18), we have

$$|\sigma(\mu) - \sigma_1| = \mathcal{O}(1)(|U_L - U_R| + \|\mu\|) = \mathcal{O}(1)(\delta + \|\mu\|).$$

Thus the total contribution is

$$|U_L - U_R| \cdot [\text{length of interval}] = \mathcal{O}(1)h\,|U_L - U_R|(\delta + \|\mu\|).$$

(ii) Interval II: By (3.11),

$$|\mathcal{S}_h\bar{U}(x) - U_R| = \mathcal{O}(1)(\delta + |\alpha-1|\,\|\mu\|) \qquad \text{for}\, x \in (\bar{x}-a, \bar{x}+a), \tag{3.20}$$

where $\bar{U}$ is given in (2.19) and the length of the interval is $\mathcal{O}(1)h\,|U_L - U_R|\,\|\mu\|$.

(iii) Interval III and IV are treated similarly.

Thus, we have

$$\frac{1}{h} \sum_{\substack{\text{rarefaction}\\ \text{fronts}}} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\,dx$$
$$= \mathcal{O}(1) \sum_{\substack{\text{rarefaction}\\ \text{fronts}}} |U^{\delta,\mu}(\tau,\bar{x}-) - U^{\delta,\mu}(\tau,\bar{x}+)|\big(\delta + (1 + |\alpha-1| + \delta)\|\mu\|\big). \tag{3.21}$$

**Case 3: Non-Physical Front**. We follow the same argument as in Theorem 2.1 to obtain

$$\frac{1}{h} \sum_{\substack{\text{non-physical}\\ \text{shocks}}} \int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\,dx < \delta. \tag{3.22}$$

Thus, by (3.17) and (3.21)–(3.22), we conclude

$$\begin{aligned}\|\mathcal{S}_t U_0^\delta - U^{\delta,\mu}(t)\|_{L^1} &\le L \int_0^t \sum_{\text{fronts } x=\bar{x}(\tau)} \frac{1}{h}\int_{\bar{x}-a}^{\bar{x}+a} |\mathcal{S}_h U^{\delta,\mu}(\tau) - U^{\delta,\mu}(\tau+h)|\, dx \\ &= \mathcal{O}(1)\, t\, TV\{U_0\}(\|\mu\| + \delta), \end{aligned} \tag{3.23}$$

and the result follows. ■

Now consider the relativistic Euler equations (3.1) for $\varepsilon = 0$. In Colombo-Risebro [9], it was shown that (3.1) for $\varepsilon = 0$ generates a Standard Riemann Semigroup $\mathcal{S}^c$. Thus we have

**Theorem 3.2.** *Suppose that $0 < \underline{\rho} \le \rho_0(x) \le \bar{\rho} < \infty$ and $\varepsilon\, TV\{U_0\} \le N$. Let $\mathcal{S}^c$ be the Standard Riemann Semigroup to the relativistic Euler equations (3.1) for $\varepsilon = 0$. If $U^{\varepsilon,c}$ is the entropy solution to (3.1) for $\varepsilon > 0$ and $c_0 \ge c < \infty$ obtained by the front-tracking method, then, for every $t > 0$,*

$$\|\mathcal{S}_t^c U_0 - U^{\varepsilon,c}(t)\|_{L^1} = \mathcal{O}(1) TV\{U_0\}\, t\, \varepsilon, \tag{3.24}$$

*where $\mathcal{O}(1)$ depends only on $c_0 > 0$ and is independent of $\varepsilon, c \ge c_0, t,$ and $TV\{U_0\}$.*

The proof can be carried out in a similar way by combining the proofs of Theorems 2.1 and 3.1.

## 4. Isentropic Relativistic Euler Equations

The Euler system of conservation laws of baryon number and momentum in special relativity reads:

$$\begin{cases} \partial_t\left(\dfrac{n}{\sqrt{1-u^2/c^2}}\right) + \partial_x\left(\dfrac{nu}{\sqrt{1-u^2/c^2}}\right) = 0, \\ \partial_t\left(\dfrac{(\rho+p/c^2)u}{1-u^2/c^2}\right) + \partial_x\left(\dfrac{(\rho+p/c^2)u^2}{1-u^2/c^2} + p\right) = 0, \end{cases} \tag{4.1}$$

where $\rho$, $p$, $u$, and $n$ represent the proper energy density, the pressure, the particle speed, and the proper number density of baryons, respectively. The proper number density $n$ is determined by the first law of thermodynamics:

$$\theta dS = \frac{d\rho}{n} - \frac{p+\rho}{n^2} dn, \tag{4.2}$$

where $\theta$ is the temperature and $S$ the entropy per baryon. For isentropic fluids, $S$ is constant, hence

$$n = n(\rho) = n_0\, exp\Big(\int_1^\rho \frac{ds}{s + p(s)/c^2}\Big), \tag{4.3}$$

and for a perfect polytropic fluids, $p(\rho) = \kappa^2\rho^\gamma, \gamma \ge 1$. The corresponding physical region is

$$\Omega = \{U \,:\, 0 \le \rho < \rho_{max},\ |u| < c\} \tag{4.4}$$

as in the relativistic model discussed in the previous section. We rewrite (4.1) in conservation form by setting

$$W^\mu(U) = \left(\frac{n}{\sqrt{1-u^2/c^2}}, \frac{(\rho+p/c^2)u}{1-u^2/c^2}\right)^\top, \quad F^\mu(U) = \left(\frac{nu}{\sqrt{1-u^2/c^2}}, \frac{(\rho+p/c^2)u^2}{1-u^2/c^2} + p\right)^\top$$

for $U = (\rho, \rho u)^\top$, where $\mu = (\varepsilon, \frac{1}{c^2})$. The system is strictly hyperbolic in $\Omega$ and has two distinct eigenvalues when $\rho > 0$:

$$\lambda_1(U) = \frac{u - \sqrt{p'}}{1 - u\sqrt{p'}/c^2}, \qquad \lambda_2(U) = \frac{u + \sqrt{p'}}{1 - u\sqrt{p'}/c^2}. \tag{4.5}$$

The corresponding eigenvectors are

$$r_j(U) = \alpha_j(\rho, u)\left(\frac{n}{\rho + p/c^2}, \frac{u + (-1)^j\sqrt{p'}}{\sqrt{1-u^2/c^2}}\right)^\top, \qquad j = 1, 2, \tag{4.6}$$

where

$$\alpha_j(\rho, u) = \frac{(-1)^j 2\sqrt{p'}(\rho + p/c^2)}{\rho\, p'' + 2p' + (p\, p'' - 2(p')^2)/c^2}\,\frac{(1 + u\sqrt{p'}/c^2)^3}{1 - u^2/c^2)^{3/2}} \ne 0, \qquad j = 1, 2. \tag{4.7}$$

Moreover, the system is invariant under Lorenz transformation:

$$\bar{t} = \frac{ct - \frac{\tau}{c}x}{\sqrt{1 - \frac{t^2}{c^2}}}, \quad \bar{x} = \frac{cx - c\tau\, t}{\sqrt{1 - \frac{t^2}{c^2}}}, \quad u = \frac{\tau + \bar{u}}{1 + \frac{\tau\bar{u}}{c^2}},$$

as in the previous section. Then

**Lemma 4.1.** *The shock curves in the barred coordinates are given by*

$\bar{S}_1$: $\bar{u} = -c\dfrac{\sqrt{G^2(\rho) - H^2(\rho)}}{G(\rho)}$ *for* $\rho > \rho_L$, $\bar{u} < \bar{u}_L = 0$;

$\bar{S}_2$: $\bar{u} = -c\dfrac{\sqrt{G^2(\rho) - H^2(\rho)}}{G(\rho)}$ *for* $0 < \rho < \rho_L$, $\bar{u} < \bar{u}_L = 0$,

*where* $G(\rho) = 2n_L(\rho + p_L/c^2) > 0$ *and*

$$H(\rho) = -n(p - p_L)/c^2 + \sqrt{n^2(p - p_L)^2/c^4 + 4n_L^2(\rho + p_L/c^2)(\rho + p/c^2)}.$$

*The rarefaction curves in the barred coordinates are given by*

$\bar{R}_1$: $r := \dfrac{1}{2}c\ln\big(\dfrac{c+u}{c-u}\big) + \displaystyle\int_1^\rho \frac{\sqrt{p'(w)}}{w + p(w)/c^2}\,dw = const.$ *for* $0 < \rho < \rho_L, \bar{u} > \bar{u}_L = 0$;

$\bar{R}_2$: $s := \dfrac{1}{2}c\ln\big(\dfrac{c+u}{c-u}\big) - \displaystyle\int_1^\rho \frac{\sqrt{p'(w)}}{w + p(w)/c^2}\,dw = const.$ *for* $\rho > \rho_L, \bar{u} > \bar{u}_L = 0$.

Note that, as $c \to \infty$, system (4.1) and the curves in Lemma 4.1 tend to the corresponding expressions of the classical isentropic Euler equations (2.1) as presented in Section 2.

Global existence of entropy solutions to the Cauchy problem (4.1) was first established by Pant [25] for $\varepsilon = 0$ and later by Li-Shi [20] for $\varepsilon > 0$ with $c = 1$. See also Chen-Li [7] for the uniqueness and asymptotic stability of Riemann solutions. The recent result by Li-Geng [18] establishes the global existence of entropy solutions independently of $c$. However, all the results mentioned were obtained by the Glimm scheme. Now the goal is to establish the convergence of the front-tracking method and obtain the bounds independent of the adiabatic exponent $\gamma > 1$ and the speed of light $c \geq c_0$ for large $c_0$. The next lemma is useful to obtain the corresponding result to Proposition 3.3 in this setting.

**Lemma 4.2.** *There is a smooth function* $g(\mu, \rho, \rho_l)$ *such that*

$$g(\mu, \rho, \rho_L) = \frac{c\sqrt{G^2(\rho) - H^2(\rho)}}{G(\rho)} \tag{4.8}$$

*for* $c \geq c_0$, $\varepsilon \geq 0$, *and* $\rho \geq \rho_L$.

**Proof.** A direct computation yields

$$c^2(G^2(\rho) - H^2(\rho)) = I_1 + I_2 - \frac{2n^2(p - p_L)^2}{c^2}, \tag{4.9}$$

where

$$I_1 = \frac{2n_L\, n\sqrt{\rho + p_L/c^2}(\rho - \rho_L)(n - n_L)}{\sqrt{\rho + p_L/c^2} + \sqrt{\rho + p/c^2}} \tag{4.10}$$

and

$$I_2 = \frac{2n^3(p - p_L)^3}{c^4\big(\sqrt{n^2(p - p_L)^2/c^4 + 4n_L^2(\rho + p_L/c^2)(\rho + p/c^2)} + \sqrt{4n_L^2(\rho + p_L/c^2)(\rho + p/c^2)}\big)}. \tag{4.11}$$

On the other hand,

$$p - p_L = (2\varepsilon + 1)\int_0^1 (\rho_L + \lambda(\rho - \rho_L))^{2\varepsilon} d\lambda\,(\rho - \rho_L) \tag{4.12}$$

and

$$n - n_L = \int_0^1 \frac{\partial n}{\partial \rho}(\rho_L + \lambda(\rho - \rho_L))d\lambda\,(\rho - \rho_L). \tag{4.13}$$

Then there is a smooth function $g_1$ such that

$$c\sqrt{G^2(\rho) - H^2(\rho)} = (\rho - \rho_L) g_1(\mu, \rho, \rho_L) \tag{4.14}$$

for $c \geq c_0$, $\varepsilon \geq 0$, and $\rho \geq \rho_L$. Since $G > 0$, we arrive at the result. ■

Then, in a similar way, we have

**Proposition 4.3.** *Consider two shock curves of first family, which start from the points $(r_0, s_1) = (r(U_0), s(U_1))$ and $(r_0, s_0) = (r(U_0), s(U_0))$ and are continuous to the points $(r, s_2)$ and $(r, s)$, respectively. Then there exists a constant $c_0 > 0$ such that, for $c \geq c_0$,*

$$s_0 - s - (s_1 - s_2) = \mathcal{O}(1)\varepsilon(s_1 - s_0)(r_0 - r), \tag{4.15}$$

*where $\mathcal{O}(1)$ depends only $\rho_0, \rho_1 \in [\underline{\rho}, \overline{\rho}]$ and $c_0 > 0$.*

Again, following [1] (see also [24]), we obtain the wave interaction estimates corresponding to Lemma 3.4 and conclude the existence of a global entropy solution $U^\mu$ to (4.1) by the front-tracking method.

**Theorem 4.1.** *Suppose that $0 < \underline{\rho} \leq \rho_0(x) \leq \bar{\rho} < \infty$ and $\varepsilon TV\{U_0\} \leq N$. Let $\mathcal{S}$ be the Standard Riemann Semigroup to the isothermal Euler equations (1.10). If $U^\mu$ is the entropy solution to (4.1) for $\varepsilon > 0$ and $c \geq c_0$ constructed by the front tracking method, then, for every $t > 0$,*

$$\|\mathcal{S}_t U_0 - U^\mu(t)\|_{L^1} = \mathcal{O}(1) TV\{U_0\}\, t\, \|\mu\|,$$

*where $\|\mu\| = \varepsilon + 1/c^2$.*

**Proof.** The proof follows closely the proofs of Theorems 2.1 and 3.1. Given two front states $(U_L, U_R)$ of the $\delta$-front tracking approximate solution $U^{\delta,\mu}$ at a non-interaction point $(\bar{x}, \tau)$ with strength $\alpha$, consider the Standard Riemann Semigroup $\mathcal{S}_h \bar{U}$ to (1.10) that consists of three states of strength $\beta_1$ and $\beta_2$. In the case of a shock front, the speed of the $\alpha$-wave is

$$\sigma(\mu) = \frac{1}{W_1^\mu(U_R) - W_1^\mu(U_L)} \left( \frac{n_R u_R}{\sqrt{1 - u_R^2/c^2}} - \frac{n_L u_L}{\sqrt{1 - u_L^2/c^2}} \right),$$

and, as usual, one gets

$$|\sigma(\mu) - \sigma_1| = \mathcal{O}(1)\|\mu\|.$$

where $\sigma_1$ denotes the speed of the $\beta_1$-wave. Also, by employing the smoothness of the functions for $\alpha$ close to 1, $\varepsilon \in [0, \varepsilon_0]$, and $c \geq c_0$, we have

$$\beta_1 - \alpha = \beta_2 - 1 = \mathcal{O}(1)|\alpha - 1|\, \|\mu\|.$$

Using the standard techniques developed in the two previous sections, we prove the result. ■

**Acknowledgments**. Gui-Qiang Chen's research was supported in part by the National Science Foundation under Grants DMS-0505473, DMS-0426172, DMS-0244473, and an Alexandre von Humboldt Foundation Fellowship. Cleopatra Christoforou's research was supported in part by the National Science Foundation under Grants DMS-0244473 and DMS-0505473. Yongqian Zhang's research was supported in part by NSFC Project 10531020.

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