On the piecewise smoothness of entropy solutions to scalar conservation laws for a large class of initial data

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Abstract

In this paper, we prove that if the initial data do not belong to a certain subset of C^k, which is hitherto smallest in the sense of inclusion relation of sets, then the solutions of scalar conservation laws are piecewise smooth. In particular, our initial data allow centered compression waves, which is the case not covered by Dafermos (1974) and Schaeffer (1973). More precisely, we are concerned with the structure of the solutions in the neighborhoods of the points at which only a C^{k+1} shock generates, while there can be infinite number of intervals, the characteristics from each of these intervals will meet at a point in any small neighborhood. We give sufficient and almost necessary conditions of the initial data for a degenerate point at which a C^{k+1} shock generates. It is also shown that there are finitely many shocks for smooth initial data (in the Schwartz space) except a certain subset of $\mathscr{S}(\mathbb{R})$ of the first category. It should be pointed out that this subset is smaller than those used in previous works. We point out that Thom's theory of catastrophes [10], which plays a key role in Schaeffer [9], can not be used to analyze the larger class of initial data considered in this work.

1 Introduction

Consider the Cauchy problem for the scalar hyperbolic conservation law:

$$(1.1) \qquad \begin{cases} u_t + f(u)_x = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = \phi & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

where f is C^{k+1} smooth and uniformly convex, i.e., $\partial^2 f/\partial u^2 \geq \epsilon > 0$, the initial data are C^k smooth and bounded, with $3 \leq k \leq \infty$. In general, the problem (1.1) does not admit a global smooth solution even if the initial data are smooth, but for arbitrary bounded measurable initial data a unique global weak solution does exist. The structure of entropy solution has been studied by many authors, e.g., Chen-Zhang [1], Dafermos [2], Lax [5], Li-Wang [6, 7], Oleinik [8], Schaeffer [9], and Tadmor & Tassa [11].

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The main results of this work will be obtained by using the minimizing process of $F(x,t,u)$ introduced by Lax [5]: For each initial function ϕ we define a function $H \times \mathbb{R}$:

$$(1.2) \qquad F(x,t,u) = tg(u) + \Phi(x - ta(u))$$

where $H = \mathbb{R} \times (0, \infty)$, $a(u) = f'(u)$,

$$g(u) = ua(u) - f(u), \quad \Phi(y) = \int_0^y \phi(x)dx.$$

Lax has proved that for almost all (x,t) there exists a unique value of u which minimizes $F(x,t,\bullet)$, and $u(x,t)$, the function defined(almost everywhere) to equal the value of u which minimizes $F(x,t,\bullet)$, is in fact the solution of (1.1). It follows from the convexity hypothesis and the fact that the initial function is bounded that $F(x,t,u) \to +\infty$ as $u \to \pm\infty$. Therefore, $F(x,t,\bullet)$ always has a minimum, and a minimizing value u must be a critical point of F, a solution of the equation

$$(1.3) \qquad (\partial F/\partial u)(x,t,u) = 0.$$

Note that

$$(1.4) \qquad (\partial F/\partial u)(x,t,u) = ta'(u)\{u - \phi(x - ta(u))\},$$

and neither of the factors outside the brackets vanishes. It can be verified that the differential of $\partial F/\partial u$ never vanishes when $\partial F/\partial u = 0$, so (1.3) defines a smooth surface S in $H \times \mathbb{R}$. we record here the following relations that will be needed below:

$$(1.5) \qquad\qquad (\partial F/\partial x)(x,t,u) = u \qquad \text{on } S,$$
$$(1.6) \qquad\qquad (\partial F/\partial t)(x,t,u) = -f(u) \quad \text{on } S.$$

Let us introduce some notations and definitions. First let

$$L_1 = \{x \in \mathbb{R} \mid (a(\phi(x)))' < 0, (a(\phi(x)))'' = 0, (a(\phi(x)))''' = 0\}$$
$$L_2 = \{x \in \mathbb{R} \mid (a(\phi(x)))' < 0, (a(\phi(x)))'' = 0, \cdots, (a(\phi(x)))^{(k)} = 0, k \geq 3\}$$

and

$$\begin{aligned}L_3 \;=\; &\{x \in \mathbb{R} \mid (a(\phi(x)))' < 0, (a(\phi(x)))'' = 0, \cdots, (a(\phi(x)))^{(k)} = 0, (k \geq 3) \;\&\\ &\text{there exists a point } \xi \in (x, x + \delta) \text{ such that } (a(\phi(\xi)))'' < 0 \text{ or}\\ &\text{a point } \eta \in (x - \delta, x) \text{ such that } (a(\phi(\eta)))'' > 0 \text{ for any } \delta > 0\}.\end{aligned}$$

Remark 1.1 It can be verified that L_2 is a proper subset of L_1 when $k > 3$ and L_3 is a proper subset of L_2 when $k \geq 3$.

Definition 1.1 Let u_0 be a minimizing value for $F(x_0, t_0, \bullet)$, u_0 is called non-degenerate (degenerate) if $F_{uu}(x_0, t_0, u_0) \neq 0 (= 0)$.

Definition 1.2 The solution $u(x,t)$ is said to be C^k piecewise smooth if every bounded subset of $(-\infty, \infty) \times [0, \infty)$ intersects at most a finite number of shocks, every shock is piecewise C^{k+1} smooth curve, and $u(x,t)$ is C^k smooth on the complement of the shock set.

To summarize some previous result, we will first give some definitions. Let

$$U = \{(x,t) : \exists \text{ unique minimizing value for } F(x,t,\bullet), \text{ on which } F_{uu} \neq 0\},$$
$$\Gamma_1 = \{(x,t) : \exists \text{ two minimizing values for } F(x,t,\bullet), \text{ on which } F_{uu} \neq 0\},$$
$$\Gamma_0^{(c)} = \{(x,t) : \exists \text{ three minimizing values for } F(x,t,\bullet), \text{ on which } F_{uu} \neq 0\},$$
$$\Gamma_0^{(f)} = \{(x,t) : \exists \text{ unique minimizing value for } F(x,t,\bullet), \text{ on which } F_{uu} = 0, F_u^{(4)} \neq 0\}.$$

Moreover, let $\mathscr{M}_{(x,t)}$ be the set consisting of all the minimizing values for $F(x,t,\bullet)$, and

$$\overline{\Gamma}_1 = \{(x,t) : \exists \text{ two connected components of } \mathscr{M}_{(x,t)}\},$$
$$\overline{\Gamma}_0^{(f)} = \{(x,t) : \exists \text{ unique connected component } [\alpha, \beta] \text{ of } \mathscr{M}_{(x,t)}, \text{ and } F_{uu}(x,t,\alpha) = 0\},$$
$$\overline{\Gamma}_0^{(c)} = \{(x,t) : \exists n \text{ connected components of } \mathscr{M}_{(x,t)}, \text{ where } n \geq 3\}.$$

It is shown by Schaeffer [9] that U is an open subset of $H = \mathbb{R} \times (0, \infty)$, on which the solution $u(x,t)$ is smooth; Γ_1 is a union of smooth curves across which the minimizing function has a jump discontinuity and that $\Gamma_0^{(c)}$ and $\Gamma_0^{(f)}$ consist of isolated points at which the curves in Γ_1 begin to collide. It is obvious that $\Gamma_0^{(f)}$ is a proper subset of $\overline{\Gamma}_0^{(f)}$, Γ_1 is a proper subset of $\overline{\Gamma}_1$ and $\Gamma_0^{(c)}$ is a proper subset of $\overline{\Gamma}_0^{(c)}$. It has been proved by Schaeffer [9] that the solutions of (1.1) are piecewise smooth and the total number of shocks is finite for smooth initial data (in the Schwartz space) except in a certain subset of $\mathscr{S}(\mathbb{R})$ of the first category. In other words, there is a set $\Omega \subset \mathscr{S}(\mathbb{R})$ of first category such that for $\phi \in \mathscr{S}(\mathbb{R}) \sim \Omega$

$$(1.7) \qquad\qquad\qquad H = U \cup \Gamma_1 \cup \Gamma_0^{(c)} \cup \Gamma_0^{(f)}.$$

Lemma 1.4 in [9] is very important, which shows that any point $(x_0, t_0) \in \Gamma_0^{(f)}$ has a neighborhood Θ such that $\Gamma_1 \cap \Theta$ consists of a half-curve originating at (x_0, t_0). The minimizing function is smooth on $\Theta' \sim \Gamma_1$, where $\Theta' = \Theta \sim \{(x_0, t_0)\}$. The proof is an adaption of standard techniques from the theory of singularities of differentiable mappings, especially Thom's theory of catastrophes [10]. Schaeffer made a serious attempt to make this material accessible to analysts, including reproving the so called universal unfolding of the Riemann-Hugoniot catastrophe, one of the seven elementary catastrophes of Thom [10]. Section 2 in [9] is devoted to prove the unfolding theorem for the Riemann–Hugoniot catastrophe.

It is naturally to ask if the conclusions of Lemma 1.4 in Schaeffer [9] are true when

$$(1.8) \quad F_u(x_0, t_0, u_0) = 0, \quad F_{uu}(x_0, t_0, u_0) = 0, \cdots, F_u^{(2n-1)}(x_0, t_0, u_0) = 0, F_u^{(2n)}(x_0, t_0, u_0) > 0,$$

where $n \geq 3$ is some integer; or even $F_u^{(m)}(x_0, t_0, u_0) = 0, (m = 1, 2, \cdots)$. Unfortunately, the unfolding theorem for the Riemann–Hugoniot catastrophe is not applicable in this case since $F(x,t,u)$ is unstable as an unfolding of codimension two of $F(x_0, t_0, u)$ when

$$(1.9) \qquad F_u(x_0, t_0, u_0) = 0, \quad F_{uu}(x_0, t_0, u_0) = 0, \quad F_u^{(3)}(x_0, t_0, u_0), \quad F_u^{(4)}(x_0, t_0, u_0) = 0.$$

One of the main purposes of this work is to show that the conclusions of Lemma 1.4 in [9] are true under the conditions $L_3 = \emptyset$, which is more general than the condition (1.9). The method used in our proof is different from the one used by Schaeffer, which is elementary but technical.

It is shown by Li and Wang [6] that there is a set $\Omega_1 \subset \Omega \subset \mathscr{S}(\mathbb{R})$ of first category such that for $\phi \in \mathscr{S}(\mathbb{R}) \sim \Omega_1$ the solutions are piecewise smooth. The authors also give an explicit conditions on Ω_1:

$$(1.10) \qquad \Omega_1 = \{\phi \in \mathbb{R} \mid (a(\phi(x)))' < 0, (a(\phi(x)))'' = 0, (a(\phi(x)))''' = 0, x \in \mathbb{R}\}.$$

Dafermos [2] introduced the concept of generalized characteristic to study the solution structures of hyperbolic conservation laws. He proved that generically the solutions generated by initial data in C^k are piecewise smooth and do not contain centered compression waves. In other words, there is a set $\Omega_2 \subset C^k$ of the first category such that for $\phi \in C^k \sim \Omega_2$ the solutions are piecewise smooth, where

$$(1.11) \qquad \Omega_2 = \{\phi \in C^k \mid (a(\phi(x)))' < 0, (a(\phi(x)))'' = 0, \cdots, (a(\phi(x)))^{(k)} = 0, x \in \mathbb{R}\}.$$

In this paper, we first generalize Lemma 1.4 in Schaeffer [9], i.e., the case that $(x_0, t_0) \in \Gamma_0^{(f)}$ to the case that $(x_0, t_0) \in \overline{\Gamma}_0^{(f)}$. We will show that $\overline{\Gamma}_1$ is a union of C^{k+1} curves, except the points at which at least one connected component of $\mathscr{M}$ is not an isolated point, across which the minimizing function has a jump discontinuity, $\overline{\Gamma}_0^{(f)}$ consists of isolated points at which the curves in Γ_1 begin. $\overline{\Gamma}_0^{(c)}$ consists of isolated points at which the curves in Γ_1 collide. As a result, we will prove that there is some neighborhood of (x_0, t_0) such that a unique C^{k+1} smooth shock originating at (x_0, t_0), while there may be infinite number of intervals on which $(a(\phi(x)))' < 0$, $(a(\phi(x)))'' = 0$ (i.e., $a(\phi(x))$ is linearly decreasing) the characteristics from each of these intervals will meet at a point in any small neighborhood of (x_0, t_0). We also show that for any point $(x_0, t_0) \in \overline{\Gamma}_0^{(c)}$ i.e. there exist finitely many, say n, connected components of $\mathscr{M}_{(x_0,t_0)}$, there exists a neighborhood Θ of (x_0, t_0) such that $\Gamma_1 \cap \Theta$ is the union of n half shocks, $n-1$ terminating and one originating at (x_0, t_0). The shocks are piecewise C^{k+1} smooth and shocks are not differentiable only at points of intersection with other shocks and points which are centers of centered compression waves. The minimizing function is C^k smooth on each n components of $\Theta' \sim \Gamma_1$. We prove that there is a set $\Omega_3 \subset C^k(\mathbb{R})$ of the first category such that for $\phi \in C^k(\mathbb{R}) \sim \Omega_3$, $H = U \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$ and the minimizing process leads to piecewise smooth solution of initial value problem (1.1) pointwise in H, where

$$(1.12) \quad \Omega_3 \;=\; \{\phi \in C^k \mid (a(\phi(x)))' < 0, (a(\phi(x))'' = 0, \cdots, (a(\phi(x)))^{(k)} = 0, (k \geq 3) \;\&$$
$$\text{there exists a point } \xi \in (x, x + \delta)\text{such that } (a(\phi(\xi)))'' < 0 \text{ or}$$
$$\text{a point } \eta \in (x - \delta, x) \text{ such that } (a(\phi(\eta)))'' > 0 \text{ for each } \delta > 0, x \in \mathbb{R}\}.$$

Here Ω_3 is a proper subset of Ω_2. To our knowledge, Ω_3 has been the smallest in the sense of inclusion relation of sets. We also prove that there is a set $\Omega_4 \subset \mathscr{S}(\mathbb{R})$ of the first category such that for $\phi \in \mathscr{S}(\mathbb{R}) \sim \Omega_4$, there are only finite number of shocks. It is pointed out that piecewise smooth solutions with the underlying initial conditions may contain centered compression waves. This indicates that the class of piecewise smooth solutions with C^k initial data excluding Ω_3 obtained in this paper is wider than the class of the piecewise solutions with C^k initial data excluding Ω_2 considered by Dafermos.

This paper is arranged as follows: we study the local structure of the solutions in Section 2 and we prove the solutions are piecewise smooth based on the results in Section 2 in Section 3.

2   Local solution structure

In this section, we will study the local structure of the solutions of (1.1). Some main results in this section are listed below:

- In Theorem 2.1, we study the structure of the solutions in the neighborhoods of the points at which a shock generates. We show that there is a neighborhood such that a unique C^{k+1} smooth shock originates, even there are infinite number of intervals, on which $(a(\phi(x)))' < 0$, $(a(\phi(x)))'' = 0$, i.e., $a(\phi(x))$ is linearly decreasing, the characteristics from each of these intervals will meet at a point in any small neighborhood of (x_0, t_0), while all of these points do not generate shocks.

- In Theorem 2.2, we study the points at which finitely many shocks collide to form a new shock. All of the shocks are C^{k+1} smooth except at points of interaction with other shocks and points belonging to $\overline{\Gamma}_1 \sim \Gamma_1$, and the shocks are not differentiable at points of interaction with other shocks and points which are centers of centered compression waves. In other words, all the shocks are piecewise C^{k+1} curves.

Let $y(x, t, u) = x - ta(u)$. For each $(x, t, u) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$, it is easy to see that $y(x, t, u)$ is just the intersection point of a straight line passing through (x, t) with slope $a(u)$ with the line $t = 0$. Note that

$$(2.1) \qquad\qquad F_u(x, t, u) = ta'(u)\{u - \phi(x - a(u)t)\}.$$

If $F_u(x, t, u) = 0$, then

$$(2.2) \qquad\qquad F_{uu}(x, t, u) = ta'(u)[1 + (a(\phi(y(x, t, u))))'t].$$

If $F_u(x, t, u) = 0$ and $F_{uu}(x, t, u) = 0$, then

$$\begin{aligned}
F_{uuu}(x, t, u) &= ta'(u)[(a(\phi(y(x, t, u))))'t]_u = t^2 a'(u)[(a(\phi(y(x, t, u))))']_u \\
&= t^2 a'(u)[(a(\phi(y(x, t, u))))']_{y(x,t,u)}(y(x, t, u))_u \\
(2.3) \qquad\qquad &= (-t)^3 [a'(u)]^2 (a(\phi(y(x, t, u))))''.
\end{aligned}$$

By induction, if $F_u^{(m)}(x, t, u) = 0, (m = 1, \cdots, n - 1)$, then we have

$$(2.4) \qquad F_u^{(n)}(x, t, u) = (-t)^n [a'(u)]^{n-1} (a(\phi(y(x, t, u))))^{(n-1)}, \qquad (n \geq 3).$$

In view of (2.1)-(2.4), it can be verified that if (x, t, u) satisfies

$$(2.5) \qquad F_u^{(m)}(x, t, u) = 0 \ (m = 1, \cdots, 2n - 1), \qquad F_u^{(2n)}(x, t, u) > 0, \ (n \geq 2),$$

then we have, for $y(x, t, u) = x - ta(u)$,

$$(2.6) \qquad \begin{cases} (a(\phi(y(x, t, u))))' < 0, \ (a(\phi(y(x, t, u))))^{(m)} = 0, \quad (m = 2, 3, \cdots, 2n - 2) \\ (a(\phi(y(x, t, u))))^{(2n-1)} > 0, \quad (n \geq 2). \end{cases}$$

On the other hand, if x_0 satisfies (2.6) then for $u = \phi(x_0)$,

$$t = -\frac{1}{(a(\phi(x_0)))'}, \quad x = x_0 - \frac{a(\phi(x_0))}{(a(\phi(x_0)))'}.$$

Consequently, (2.5) is satisfied.

2.1 Some useful lemmas

Lemma 2.1 ([9]) *Let the set U be set defined in the last section and let $u(x,t)$ be the minimizing function of F defined by (1.2). Then we have*

- *U is an open subset of H, and $u(x,t)$ is smooth on U.*

- *Any point $(x_0, t_0) \in \Gamma_1$ has a neighborhood Θ such that $\Gamma_1 \cap \Theta$ is a smooth curve $x = \gamma(t)$ passing through (x_0, t_0). The minimizing function $u(x,t)$ is smooth on both components of $\Theta \sim \Gamma_1$.*

The above lemma is from Lemmas 1.1 and 1.2 of [9]. Let $u_1(x,t)$ and $u_2(x,t)$ be the corresponding solutions of (1.3) defined in some neighborhood Θ of (x_0, t_0). We see that for nearby (x,t) the minimum of $F(x,t,\bullet)$ is assumed at either $u_1(x,t)$ or $u_2(x,t)$, or both. Hence every point of Θ belongs to U or Γ_1. Every point in the two components Θ_1 and Θ_2 of $\Theta \sim \Gamma_1$ belongs to U. Let $u_i(x,t)$ be the unique minimizing value for $F(x,t,\bullet)$ for $(x,t) \in \Theta_i$ $(i = 1,2)$. Any curve in Γ_1 which separates two components of U is defined by an equation in the following form

$$(2.7) \qquad\qquad F(x,t,u_2(x,t)) - F(x,t,u_1(x,t)) = 0$$

and it follows from (1.5),(1.6) that the jump relation

$$(2.8) \qquad\qquad \dot{\gamma}(t) = [f(u)]/[u],$$

where $[u] = u_1 - u_2, [f(u)] = f(u_1) - f(u_2)$ is satisfied along the curve. Thus γ is C^{k+1}.

Now we turn to discuss the connection between the critical point and the characteristic. Suppose u_0 is a critical point of $F(x_0, t_0, \bullet)$, i.e. $F_u(x_0, t_0, u_0) = 0$, then $u_0 = \phi(x_0 - t_0 a(u_0)) = \phi(y(x_0, t_0, u_0))$ due to (2.1) and so there exists a characteristic

$$(2.9) \qquad C_0: \quad x = y(x_0, t_0, u_0) + t a(\phi(y(x_0, t_0, u_0))) = x_0 + (t - t_0) a(u_0)(t > 0),$$

passing through (x_0, t_0) and $F_u(x,t,u_0) = 0$ for each $(x,t) \in C_0$. On the other hand, consider a characteristic $C_1 : x = \overline{x} + t a(\phi(\overline{x})), t > 0$ then $F_u(x,t,\phi(\overline{x})) = 0$, for $(x,t) \in C_1$ due to the fact that $\phi(\overline{x}) = \phi(x - t a(\phi(\overline{x})))$ and (2.1). This implies that $\phi(\overline{x})$ is a critical point of $F(x,t,\bullet)$.

Naturally, it may be asked if $\phi(\overline{x})$ is a minimizing value for $F(x,t,\bullet)$ for $(x,t) \in C_1$. The following lemma gives an answer.

Lemma 2.2 *Assume $\phi(x)$ is bounded and C^k smooth and let*

$$C = \{(x,t) : x = x_0 + t a(\phi(x_0)), t > 0\}.$$

Then precisely one of the following statements must hold:

- $C \subset U$ and $\phi(x_0)$ is the unique minimizing value for $F(x, t, \bullet)$; or

- there exists a point $(x_1, t_1) \in C$ such that $\phi(x_0)$ is either the unique minimizing value for $F(x_1, t_1, \bullet)$ which is degenerate or one of at least two minimizing values for $F(x_1, t_1, \bullet)$. Thus, $\phi(x_0)$ is the minimizing value for $F(x, t, \bullet)$ for $(x, t) \in C^- := C \cap \{(x, t) : t_1 > t > 0\} \subset U$ while $\phi(x_0)$ is no longer the minimizing value for $F(x, t, \bullet)$ for $(x, t) \in C^+ := C \cap \{(x, t) : t > t_1\}$.

Proof: First we claim that $\phi(x_0)$ will be no longer a minimizing value for $F(x, t, \bullet)$ for $(x, t) \in C^+$ if there exist at least two minimizing values for $F(x_1, t_1, \bullet)$ and $\phi(x_0)$ is one of them. Let $\phi(\tilde{x}_0) \neq \phi(x_0)$ be another minimizing value for $F(x_1, t_1, \bullet)$. Computing the directional derivative along $x = x_0 + a(\phi(x_0))t$:

$$\frac{d}{d\lambda} = \frac{\partial}{\partial t} + \frac{x_1 - x_0}{t_1} \frac{\partial}{\partial x};$$

$$\frac{d}{d\lambda} F(x, t, \phi(x_0)) \mid_{(x_1, t_1)} = g(\phi(x_0)) = h(\frac{x_1 - x_0}{t_1}),$$

where

$$h(u) = ua^{-1}(u) - f(a^{-1}(u)) = g(a^{-1}(u)),$$

$a^{-1}(u)$ is the inverse function of $a(u)$. It can be verified that $h(u)$ is convex since $h''(u) = 1/a'(u) > 0$. In fact, it follows from $x_1 - x_0/t_1 = a(\phi(x_0))$ that $\phi(x_0) = a^{-1}((x_1 - x_0)/t_1)$. This gives $g(\phi(x_0)) = h(\frac{x_1 - x_0}{t_1})$. Hence

$$\frac{d}{d\lambda} F(x, t, \phi(\tilde{x}_0))$$

$$= g(\phi(\tilde{x}_0)) - a(\phi(\tilde{x}_0))\phi(x_1 - t_1 a(\phi(\tilde{x}_0))) + \frac{x_1 - x_0}{t_1} \phi(x_1 - t_1 a(\phi(\tilde{x}_0)))$$

$$= h(\frac{x_1 - \tilde{x}_0}{t_1}) + \frac{\tilde{x}_0 - \tilde{x}_0}{t_1} h'(\frac{x_1 - \tilde{x}_0}{t_1}).$$

Using the convexity of h, we get

$$\frac{d}{d\lambda}(F(x, t, \phi(x_0)) - F(x, t, \phi(\tilde{x}_0))) \mid_{(x_1, t_1)}$$

$$= \frac{1}{2!} h''(\frac{x_1 - \theta x_0 - (1 - \theta)\tilde{x}_0}{t_1})(\frac{\tilde{x}_0 - x_0}{t_1})^2 > 0,$$

where $0 < \theta < 1$. Thus $\phi(x_0)$ is no longer a minimizing value for $F(x, t, \bullet)$ for $(x, t) \in C^+$.

If $\phi(x_0)$ is a degenerate minimizing value for $F(x_1, t_1, \bullet)$, namely

$$F_{uu}(x_1, t_1, \phi(x_0)) = t_1(a(\phi(x_0)))'[1 + t_1(a(\phi(x_0)))'] = 0.$$

Consequently, $F_u(x, t, \phi(x_0)) = 0$ and $F_{uu}(x, t, \phi(x_0)) < 0$ for $(x, t) \in C^+$, which implies that $\phi(x_0)$ is a local maximizing value for $F(x, t, u)$ for $(x, t) \in C^+$. Obviously, $\phi(x_0)$ is not a minimizing value for $F(x, t, u)$. By noting that $(x, t) \in C$ (t is sufficiently small) $F_u(x, t, u) = 0$ has a unique minimizing value $\phi(x_0)$, and $F_{uu}(x, t, \phi(x_0)) > 0$, the proof is completed. $\square$

Lemma 2.2 is Lemma 6 in Li-Wang [6]. For self-contained and convenience, its proof will be given below.

Definition 2.1 *A characteristic C_{t_c}: $x = x_0 + a(\phi(x_0))t$ starting from point $(x_0, 0)$, $0 < t < t_c(\leq \infty)$ is called valid if $\phi(x_0)$ is a unique minimizing value for $F(x, t, \bullet)$ for $(x, t) \in C_{t_c}$.*

Suppose u_0 is a minimizing value for $F(x_0, t_0, \bullet)$ (not necessarily unique). Consider a characteristic $C : x = y(x_0, t_0, u_0) + ta(\phi(y(x_0, t_0, u_0))) = x_0 + (t - t_0)a(u_0)(t \geq 0)$, which passes through (x_0, t_0) and $(y(x_0, t_0, u_0), 0)$. Consider the segment $C_{t_0} : x = x_0 + (t - t_0)a(u_0)(0 < t < t_0)$. It is valid according to Lemma 2.2. Therefore, a minimizing value for $F(x_0, t_0, \bullet)$ defines a part of valid characteristic passing through (x_0, t_0), on the other hand a valid characteristic provides a minimizing value for $F(x, t, \bullet)$ for each given point (x, t) belonging to the valid part of the characteristic. Such kind of characteristic will be often used hereafter.

Lemma 2.3 *Suppose that $L_3 = \emptyset$. Then for each point $(x, t) \in H$, there are finitely many connected components of $\mathcal{M}_{(x,t)}$.*

Proof: We see each connected component of $\mathcal{M}$ is either an isolated point or a closed interval. Let $\mathcal{M}_1 = [u_i^-, u_i^+]$, $\mathcal{M}_2 = [u_{i+1}^-, u_{i+1}^+]$ $(u_i^+ < u_{i+1}^-)$ be two neighboring connected components of $\mathcal{M}$. Thus there exists a point $u_i \in (u_{i+1}^-, u_i^+)$ such that

$$(2.10) \qquad\qquad F_u(x, t, u_i) = 0, \quad F_{uu}(x, t, u_i) \leq 0,$$

where u_i is a local maximizing value for $F(x, t, \bullet)$. Set $y_i^\pm = y(x, t, u_i^\pm)$, $y_{i+1}^\pm = y(x, t, u_{i+1}^\pm)$ and $y_i = y(x, t, u_i)$. Now we claim

$$(2.11) \qquad\qquad \text{there exists a point } x_i' \in [y_{i+1}^-, y_i] \text{ such that } (a(\phi(x_i')))'' < 0.$$

$$(2.12) \qquad\qquad \text{Otherwise, } (a(\phi(y)))'' \geq 0 \text{ for each } y \in [y_{i+1}^-, y_i].$$

Two cases need to be considered.

Case 1: $F_{uu}(x, t, u_{i+1}^-) \neq 0$. If this is the case, on one hand we have $(a(\phi(y_{i+1}^-)))' > -1/t \geq (a(\phi(y_i)))'$ in view of (2.10); and on the other hand $(a(\phi(y_{i+1}^-)))' \leq (a(\phi(y_i)))'$ due to (2.12). This gives a contradiction; hence Case 1 does not hold.

Case 2: $F_{uu}(x, t, u_{i+1}^-) = 0$. In this case, since $L_3 = \emptyset$, we have

$$(2.13) \qquad \begin{array}{l} \text{There exists a point } \delta_0 > 0 \text{ such that } (a(\phi(y)))'' \geq 0 \text{ for each} \\ y \in [y_{i+1}^-, y_{i+1}^- + \delta_0] \text{ and } (a(\phi(z)))'' \leq 0 \text{ for each } z \in [y_{i+1}^+ - \delta_0, y_{i+1}^+]. \end{array}$$

Next we claim that

$$(2.14) \qquad \text{there exists a point } \xi \in (y_{i+1}^-, y_{i+1}^- + \delta) \text{ such that } (a(\phi(\xi)))'' > 0 \text{ for each } \delta > 0.$$

Otherwise according to (2.13), there exists a constant $\delta_1 > 0$ such that $(a(\phi(\xi)))'' \equiv 0$ for each $\xi \in [y_{i+1}^-, y_{i+1}^- + \delta_1]$. Set $y(x, t, \alpha') = y_{i+1}^- + \delta_1$, where $\alpha' < u_{i+1}^-$. Then $(a(\phi(\xi)))' = -1/t$ for each $\xi \in [y_{i+1}^-, y(x, t, \alpha')]$, which implies $a(\phi(\xi)) = -\xi/t + c$, where c is a constant. Let $\xi = y_{i+1}^-$. We have

$$c = \frac{1}{t}[y_{i+1}^- + ta(\phi(y_{i+1}^-))] = \frac{x}{t},$$

and so

$$(2.15) \qquad\qquad a(\phi(\xi)) = -\frac{1}{t}\xi + \frac{x}{t}.$$

Consequently, $a(u) = a(\phi(y(x, t, u)))$, if we set $\xi = y(x, t, u)$ in (2.15) and notice $y(x, t, u) = x - ta(u)$ for $u \in [\alpha', u_{i+1}^-]$, which implies that $F_u(x, t, u) \equiv 0$, $F(x, t, u)$ is a constant for $u \in [\alpha', u_{i+1}^-]$. This leads to a contradiction since $[u_{i+1}^-, u_{i+1}^+]$ is a connected component, hence Case 2 does not hold. Thus we complete the proof of (2.14). Similarly, we have the following conclusion

(2.16)   there exists a point $\eta \in (y_{i+1}^+ - \delta, y_{i+1}^+]$ such that $(a(\phi(\eta)))'' < 0$ for each $\delta > 0$.

We see $(a(\phi(y_i)))' \leq (a(\phi(y_{i+1}^-)))' = -1/t$ since $F_{uu}(x, t, u_{i+1}^-) = 0$. On the other hand, $(a(\phi(y_i)))' > (a(\phi(y_{i+1}^-)))'$ according to (2.12) and (2.14). This is a contradiction since $(a(\phi(y_i)))' \leq (a(\phi(y_{i+1}^-)))'$. Thus the assertion (2.11) is true. By a similar argument, there exists a point $x_i'' \in [y_i, y_i^+]$ such that $(a(\phi(x_i'')))'' > 0$. The above arguments yield

(2.17)   there exists a point $x_i' \in [y_{i+1}^-, y_i]$ and a point $x_i'' \in [y_i, y_i^+]$
             such that $(a(\phi(x_i')))'' < 0$ and $(a(\phi(x_i'')))'' > 0$.

Next we claim there are only finitely many connected components of $\mathscr{M}$. If this is not true, then there exists a monotone increasing sequence $\{\mathscr{M}_i\} = ([u_i^-, u_i^+])$. The sequence $\{[y_i^+, y_i^-]\}$ is monotone decreasing and bounded, then it must be convergent to a point x_0. It is easy to know that

$$\lim_{i \to \infty} x_i' = \lim_{i \to \infty} x_i'' = \lim_{i \to \infty} y_i = x_0.$$

It follows from $(a(\phi(y_i)))' \leq -1/t$ that $(a(\phi(x_0)))' \leq -\frac{1}{t} < 0$. On the other hand, $(a(\phi(x_0)))'' = 0, \cdots, (a(\phi(x_0)))^{(k)} = 0$ in light of (2.17). For each $\delta > 0$, there exist $i_0 > 0$ such that $[y_i^+, y_i^-] \subset (x_0 - \delta, x_0 + \delta)$ for each $i > i_0$. We see $x_0 < x_i' < x_i''$ for each $i > i_0$ since the sequence $\{[y_i^+, y_i^-]\}$ is monotone decreasing. Observing $(a(\phi(x_i')))'' < 0$, $x_i' \in (x_0, x_0 + \delta)$, which implies $x_0 \in L_3$. This is contradicts $L_3 = \emptyset$. The proof is completed. $\square$

Same result by Li and Wang [7] to Lemma 2.3 has been obtained under the hypothesis that ϕ is locally finite to f.

Below we give some lemmas which will be used in the proof of one of our main results, Theorem 2.1.

Lemma 2.4 *The locus of the degenerate points on all the characteristics originating from some neighborhood of $[y(x_0, t_0, \beta), y(x_0, t_0, \alpha)]$ form two half curves in the neighborhood of (x_0, t_0), for $t > t_0$ with a unique common point (x_0, t_0). Each of them is continuous and consists of union of countable open arcs, on which the curve is C^k and points as the end point of the arcs on which the curve is not differentiable but the left derivative continuous on the left and right derivative continuous on the right. The graphs of them are demonstrated in Figure 1.*

Proof: Note that by (2.1) the critical set of $F(x, t, \bullet)$ is contained in the compact interval $J = \{u : |u| \leq M\}$, where $M = \sup_y |\phi(y)|$. First we claim that

(2.18)   for each open interval $(\alpha - \varepsilon, \beta + \varepsilon)$, there exists an open
             neighborhood Θ of (x_0, t_0) such that all the minimizing values
             for $F(x, t, \bullet)$ belong to $(\alpha - \varepsilon, \beta + \varepsilon)$ for (x, t) belonging to Θ.

Otherwise there exists a sequence (x_n, t_n) convergent to (x_0, t_0) and a sequence u_n belonging to $J \sim (\alpha - \varepsilon, \beta + \varepsilon)$ such that $F(x_n, t_n, u_n) = \min_{u \in \mathbb{R}} F(x_n, t_n, u)$ $(n = 1, 2, \ldots)$. Since the set
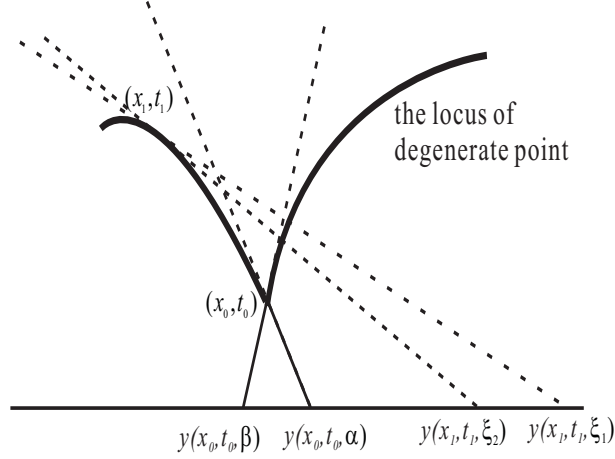
Figure 1: The locus of the degenerate points on all the characteristics. The interval $(y(x_1, t_1, \xi_2), y(x_1, t_1, \xi_1))$ is the linearly decreasing interval, on which $(a(\phi(\bullet)))'' = 0$ and $(a(\phi(\bullet)))' < 0$. There may be infinitely many such intervals in the neighborhood of $(y(x_1, t_1, \xi_2), y(x_1, t_1, \xi_1))$.

$J \sim (\alpha - \varepsilon, \beta + \varepsilon)$ is compact and we can select a subsequence of u_n written again as u_n for convenience convergent to u_1 belonging to the set $J \sim (\alpha - \varepsilon, \beta + \varepsilon)$. Then $F(x_0, t_0, u_0) = \lim_{n \to \infty} F(x_n, t_n, u_n) = F(x_0, t_0, u_1)$, since the function $m(x, t) = \min_{u \in \mathbb{R}} F(x, t, u)$ is continuous of x and t (see Lemma A.1 in the Appendix). This implies that there are at least two connected components of $\mathcal{M}_{(x_0, t_0)}$. It is a contradiction since there is a unique connected component of $\mathcal{M}_{(x_0, t_0)}$. Thus the assertion (2.18) is true. According to (2.18), we can find a sufficiently small constant $\varepsilon_0 > 0$ and a neighborhood Θ of (x_0, t_0) such that for each $(x, t) \in \Theta$, $u \in (\alpha - \varepsilon_0, \beta + \varepsilon_0)$, $x - ta(u)$ belongs to $(y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \alpha) + \delta_0)$.

Two half curves mentioned above can be written in the following form

$$(2.19) \qquad \begin{cases} x(\xi) = \xi - \dfrac{a(\phi(\xi))}{(a(\phi(\xi)))'} \\[2mm] t(\xi) = -\dfrac{1}{(a(\phi(\xi)))'} \end{cases} \qquad \xi \in [y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0]$$

$$(2.20) \qquad \begin{cases} x(\xi) = \xi - \dfrac{a(\phi(\xi))}{(a(\phi(\xi)))'} \\[2mm] t(\xi) = -\dfrac{1}{(a(\phi(\xi)))'} \end{cases} \qquad \xi \in [y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta)].$$

We see they have a common point (x_0, t_0). Moreover, the slope of the line tangent to the curve (2.19) at (x_0, t_0) is $a(\alpha)$ and the slope of the line tangent to the curve (2.20) at (x_0, t_0) is $a(\beta)$. By a direct computation and in light of (2.2), we have

$$F_u(x(\xi), t(\xi), \phi(\xi)) = 0, \quad F_{uu}(x(\xi), t(\xi), \phi(\xi)) = 0.$$

Thus, the curve (2.19) is the locus of the degenerate points on all the characteristics that start from the interval $[y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0]$ and the curve (2.20) is the locus of the degenerate

points on all the characteristics that start from the interval $[y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta)]$. Let $(x_l, t_1), (x_r, t_1) \in \Theta$ $(t_1 > t_0)$ be the intersection points of the line $t = t_1$ with the curves (2.19) and (2.20). Now we claim $x_r - x_l > 0$. The proof can be found in Lemma A.2 of the Appendix. Thus these two curves have a unique common point (x_0, t_0). Each of them is continuous and consists of union of countable open arcs, on which the curve is C^k and points as the end point of the arcs on which the curve is not differentiable but the left derivative continuous on the left and right derivative continuous on the right; the proof this statement will be postponed to Lemma A.4 in Appendix. $\square$

Lemma 2.5 *All the characteristics that originate from the interval* $(y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0)$ $((y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta)))$ *can only intersect with each other after the time* t_0 *and for any two characteristics that originate from the points on the right (left) of* $(y(x_0, t_0, \alpha), 0)$ $((y(x_0, t_0, \beta), 0))$, *the time when the one closer to* $(y(x_0, t_0, \alpha), 0)$ $((y(x_0, t_0, \beta), 0))$ *touches the curve* (2.19) $((2.20))$ *is not later than the time when they intersect with each other, where* $\delta_0 > 0$ *is a constant.*

Proof: Consider the following two characteristics:

$$\text{(2.21)} \qquad x = \overline{x}_1 + ta(\phi(\overline{x}_1)),$$
$$\text{(2.22)} \qquad x = \overline{x}_2 + ta(\phi(\overline{x}_2)),$$

where $y(x_0, t_0, \alpha) < \overline{x}_1 < \overline{x}_2 < y(x_0, t_0, \alpha) + \delta_0$. Suppose (x_{12}, t_{12}) is the intersection point of them; i.e.,

$$t_{12} = -\frac{\overline{x}_1 - \overline{x}_2}{a(\phi(\overline{x}_1)) - a(\phi(\overline{x}_2))} = -\frac{1}{(a(\phi(\overline{x}_{12})))'},$$

where $\overline{x}_1 < \overline{x}_{12} < \overline{x}_2$. Let $t_1 = -1/(a(\phi(\overline{x}_1)))'$ and $t_2 = -1/(a(\phi(\overline{x}_2)))'$. Thus $F_u(x_i, t_i, \phi(\overline{x}_i)) = 0$ and $F_{uu}(x_i, t_i, \phi(\overline{x}_i)) = 0$, where $x_i = \overline{x}_i + t_i a(\phi(\overline{x}_i))$ $(i = 1, 2)$. It can be verified that

$$\text{(2.23)} \qquad t_0 < t_1 \le t_{12} \le t_2$$

according to (2.13) and (2.14). A similar conclusion can also be obtained if $\overline{x}_1, \overline{x}_2 \in (y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta))$. This implies that all the characteristics that originate from the intervals $(y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0)$ and $((y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta)))$ can only intersect with each other after $t = t_0$; and for any two characteristics that originate from the points on the right (left) of $(y(x_0, t_0, \alpha), 0)$ and $((y(x_0, t_0, \beta), 0))$, the time when the one closer to $(y(x_0, t_0, \alpha), 0)$ and $((y(x_0, t_0, \beta), 0))$ touches the curve (2.19) $((2.20))$ is not later than the time when they intersect with each other. Thus we complete the proof. $\square$

Lemma 2.6 *All the minimizing values for* $F(x, t, \bullet)$ *are non-degenerate for* $(x, t) \in \Theta'$ *and there exists a unique minimizing value for* $F(x, t, \bullet)$ *for* $(x, t) \in \Theta'$ *lying under and on the curves* (2.19) *and* (2.20).

Proof: We consider two possible cases.
Case 1: under the curves. There is a unique minimizing value for $F(x, t, \bullet)$ for $(x, t) \in \Theta' \cap \{t \le t_0\}$ which is non-degenerate due to Lemma 2.5. Consider a point $(x_1, t_1) \in \Theta$, if $x_1 < x_l$ $(x_1 > x_r)$. Then if follows from Lemma 2.5 that there exists a unique minimizing value

for $F(x_1, t_1, \bullet)$ which is non-degenerate.

Case 2: on the curves. If $x_1 = x_l (x_1 = x_r)$, we claim that

(2.24)   there exists a unique minimizing value for $F(x_1, t_1, \bullet)$ and it is non-degenerate.

In fact, there exists a unique minimizing value u_1 for $F(x_1, t_1, \bullet)$ such that $y(x_1, t_1, u_1)$ belongs to $(y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta))$ according to Lemma 2.5. If there exists another minimizing value u_2 for $F(x_1, t_1, \bullet)$ such that $y(x_1, t_1, u_2)$ belongs to $(y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0)$. Then there exists a characteristic passing through (x_1, t_1):

(2.25) $$x = y(x_1, t_1, u_2) + ta(\phi(y(x_1, t_1, u_2))), \quad 0 < t \le t_1.$$

Let

(2.26) $$x = \xi + ta(\phi(\xi)), \quad 0 < t \le t_1$$

be the line tangent to the curve (2.19) from below at the point (x_1, t_1). Since

$$t_1 = -\frac{1}{(a(\phi(\xi)))'} = -\frac{1}{(a(\phi(y(x_1, t_1, u_2))))'},$$

according to the argument of assertion (2.14), we know if $(a(\phi(\xi)))' < 0$, $(a(\phi(\xi)))'' = 0$, for each $\xi \in [\xi_1, \xi_2]$, then all the characteristics that start from the interval $[\xi_1, \xi_2]$ meet at a point, say (x, t) and so $F(x, t, u)$ is a constant for each $u \in [\phi(\xi_2), \phi(\xi_1)]$, thus we have $F(x_1, t_1, \phi(\xi)) = F(x_1, t_1, u_2)$, which implies $\phi(\xi)$ is a minimizing value for $F(x_1, t_1, \bullet)$. Thus the characteristic (2.26) is valid.

Let u_1 and $[u_3, \phi(\xi)]$ be the connected components of $\mathscr{M}_{(x_1, t_1)}$. In light of Lemma A.3 in Appendix, there exists an open neighborhood $\overline{\Theta}$ of (x_1, t_1) such that $\overline{\Theta} \cap \Gamma_1$ contains a half curve $x = \gamma^-(t)$ terminating at (x_1, t_1). Furthermore, we have

(2.27) $$\frac{d\gamma^-(t)}{dt} = \frac{f(u_1(x, t)) - f(u_2(x, t))}{u_1(x, t) - u_2(x, t)},$$

$y(x, t, u_1(x, t)) \in (y(x_1, t_1, u_1), y(x_1, t_1, u_1) + \delta)$, $y(x, t, u_2(x, t)) \in (\xi - \delta, \xi)$, $(t < t_1)$. Let $t \to t_1-$, we have

$$\lim_{t \to t_1 - 0} \frac{d\gamma^-(t)}{dt} = \frac{f(u_1) - f(\phi(\xi))}{u_1 - \phi(\xi)} \in (a(\phi(\xi)), a(u_1)).$$

On the other hand, the slope of the line tangent to the curve (2.19) at (x, t) tending to (x_1, t_1) is convergent to $a(\phi(\xi))$. Consequently the curve $x = \gamma^-(t)$ lies between the characteristic (2.26) and the curve (2.19) for t close to t_1. Consider a point $(\tilde{x}, \tilde{t}) = (\gamma^-(\tilde{t}), \tilde{t})$, there exists a minimizing value $u(\tilde{x}, \tilde{t})$ for $F(\tilde{x}, \tilde{t}, \bullet)$ such that $y(\tilde{x}, \tilde{t}, u(\tilde{x}, \tilde{t})) \in (\xi - \delta, \xi)$, thus there exists a valid characteristic passing through $(\tilde{x}, \tilde{t})$. On the other hand, it touches the curve (2.19) and becomes invalid according to Lemma 2.2, which is a contradiction. Hence the assertion (2.24) is true. $\square$

It is easy to see that the shock will not touch the curves (2.19) and (2.20), which implies the shock lies between these two curves, so it must be smooth.
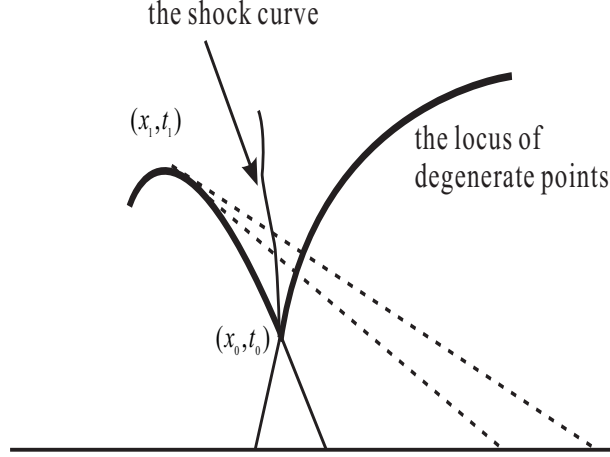
Figure 2: The local structure of the solution.

2.2 Shock generation results

Theorem 2.1 *Assume $L_3 = \emptyset$ and suppose that there is a unique connected component $[\alpha, \beta]$ of $\mathcal{M}_{(x_0,t_0)}$, and $F_{uu}(x_0, t_0, \alpha) = 0$ then (x_0, t_0) has a neighborhood Θ such that $\Gamma_1 \cap \Theta$ consists of a half-curve originating at (x_0, t_0). The minimizing function is smooth on $\Theta' \sim \Gamma_1$. The local structure of the solution is given in Figure 2.*

Proof. We will prove that there exists a C^{k+1} smooth shock originating at (x_0, t_0). For each given $t_1 > t_0$, let $A_{t_1} = \{x \mid (x, t_1) \in \Theta$, there exists a valid characteristic which originates from a point on the left of the point $(y(x_0, t_0, \beta), 0)$ and passes through $(x, t_1)\}$. $B_{t_1} = \{x \mid (x, t_1) \in \Theta$, there exists a valid characteristic which originates from a point on the right of the point $(y(x_0, t_0, \alpha), 0)$ and passes through $(x, t_1)\}$. According to the assertion (2.24), there exists a unique valid characteristic passing through (x_l, t_1), then $(x_l, t_1) \in A_{t_1}$, which implies that $A_{t_1} \neq \emptyset$. Similar, $B_{t_1} \neq \emptyset$. Let x_1 be the supremum of the set A_{t_1} and x_2 be the infimum of the set B_{t_1}.
Next we will show:

(2.28)   There are two valid characteristics passing through (x_1, t_1). One originates from a point on the left of the point $(y(x_0, t_0, \beta), 0)$ the other originates from a point on the right of the point $(y(x_0, t_0, \alpha), 0)$.

First we claim that

(2.29)   there are at least two minimizing values for $F(x_1, t_1, \bullet)$.

To see this, if there is a unique minimizing value u_1 of u for $F(x_1, t_1, \bullet)$ then there exists a unique characteristic passing through (x_1, t_1), which satisfies one of the following cases:
case 1: it originates from a point on the left of the point $(y(x_0, t_0, \beta), 0)$ and u_1 is non-degenerate. In this case, according to the Lemma 2.1, we can find a neighborhood $U(x_1)$ of x_1 such that for each $x \in U(x_1), (x, t_1)$, there exists only one characteristic that originates from a point on the left of the point $(y(x_0, t_0, \beta), 0)$. Since x_1 is the supremum of the set A_{t_1}, there exists a point $\tilde{x} \in U(x_1)$ such that there exists a characteristic passing through $(\tilde{x}, t_1)$ that originates from a

point on the left of the point $(y(x_0, t_0, \beta), 0)$. This is contradictory to the property of $U(x_1)$.
case 2: it originates from a point on the right of the point $(y(x_0, t_0, \alpha), 0)$. Similar to the case 1 above, a contradiction can also be obtained.

Next, we claim that

$$(2.30) \qquad\qquad \text{there are at most two minimizing values for } F(x_1, t_1, \bullet)$$

Otherwise, there exist two minimizing values u_1 and u_2 for $F(x_1, t_1, \bullet)$ such that $y(x_1, t_1, u_1) = x_1 - t_1 a(u_1)$ and $y(x_1, t_1, u_2) = x_1 - t_1 a(u_2)$ belong to $(y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta))$ or $(y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0)$. Suppose $y(x_1, t_1, u_1), y(x_1, t_1, u_2)$ belong to $(y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0)$. Then there exist two characteristics

$$(2.31) \qquad\qquad\qquad\qquad x = \overline{x}_1 + ta(\phi(\overline{x}_1))$$

$$(2.32) \qquad\qquad\qquad\qquad x = \overline{x}_2 + ta(\phi(\overline{x}_2))$$

and they are valid, where $\overline{x}_1 < \overline{x}_2$. Suppose that $(\overline{x}, \overline{t})$ is the intersection point of them. Then we have

$$(2.33) \qquad\qquad \overline{x} = \frac{\overline{x}_1 a(\phi(\overline{x}_2)) - \overline{x}_2 a(\phi(\overline{x}_1))}{a(\phi(\overline{x}_2)) - a(\phi(\overline{x}_1))}, \quad \overline{t} = \frac{\overline{x}_1 - \overline{x}_2}{a(\phi(\overline{x}_2)) - a(\phi(\overline{x}_1))}.$$

Let

$$(2.34) \qquad\qquad \overline{\overline{t}} = -\frac{1}{(a(\phi(\overline{x}_1)))'}, \quad \overline{\overline{x}} = \overline{x}_1 - \frac{a(\phi(\overline{x}_1))}{(a(\phi(\overline{x}_1)))'}.$$

Then $(\overline{\overline{x}}, \overline{\overline{t}})$ lies on the characteristic (2.31), $\overline{t} \geq \overline{\overline{t}}$ and $(\overline{\overline{x}}, \overline{\overline{t}})$ satisfies

$$F_u(\overline{\overline{x}}, \overline{\overline{t}}, \phi(x_1)) = 0, \qquad F_{uu}(\overline{\overline{x}}, \overline{\overline{t}}, \phi(x_1)) = 0.$$

Hence, $\phi(x_1)$ is a degenerate point of $F(\overline{\overline{x}}, \overline{\overline{t}}, \bullet)$. It follows from Lemma 2.2 that $\phi(x_1)$ is no longer the minimum point of $F(x, t, \bullet)$ for (x, t) belonging to the characteristics $x = x_1 + ta(\phi(x_1))$ ($t > \overline{\overline{t}}$). If $\overline{t} > \overline{\overline{t}}$, then the characteristic (2.31) becomes invalid when $t > \overline{\overline{t}}$. This is contradictory to the fact that the characteristic (2.31) is valid. If $\overline{t} = \overline{\overline{t}}$, then $\phi(\overline{x}_1)$ is a degenerate minimizing value for $F(x_1, t_1, \bullet)$ which is a contradiction according to the assertion (2.24).

Combining (2.29) and (2.30), we have the following conclusion

$$(2.35) \quad
\begin{array}{l}
\text{there are only two minimizing values for } F(x_1, t_1, \bullet) \text{ and they are} \\
\text{non-degenerate, which implies there are two valid characteristics passing through} \\
(x_1, t_1). \text{ One originates from a point on the left of the point } (y(x_0, t_0, \beta), 0) \\
\text{the other originates from a point on the right of the point } (y(x_0, t_0, \alpha), 0).
\end{array}$$

By a similar argument to the assertion (2.28), we have

$$(2.36) \quad
\begin{array}{l}
\text{There are two valid characteristics passing through } x_2. \text{ One originates} \\
\text{from a point on the left of the point } (y(x_0, t_0, \beta), 0) \text{ the other originates} \\
\text{from a point on the right of the point } (y(x_0, t_0, \alpha), 0).
\end{array}$$

Now we claim

$$(2.37) \qquad\qquad\qquad\qquad x_1 = x_2.$$

If not, i.e., $x_1 \neq x_2$, then it follows from the assertions (2.28) and (2.36) that there is a valid characteristic passing through (x_1, t_1) and a valid characteristic passing through (x_2, t_1) such that they intersect with each other at a time earlier than t_1. This is a contradiction since they will become invalid after the time when they intersect according to Lemma 2.2. Therefore $x_1 = x_2$. In this way a unique curve $x = \gamma(t)$ is defined for $t \geq t_0$. Therefor there are only two minimizing values for $F(\gamma(t), t, \bullet)$.

Consider any point $(x_1, t_1) \in \Theta \cap \{t > t_0\}$, if $x_1 < \gamma(t_1)(x_1 > \gamma(t_1))$, We see that there is a unique value of u which minimizes $F(x_1, t_1, \bullet)$ since it lies on the left of the point $(\gamma(t_1), t_1)$, thus the minimizing function is smooth at the point (x, t) in view of Lemma 2.1. If $x_1 = \gamma(t_1)$, then there are only two values of u which minimizes $F(x_1, t_1, \bullet)$ and they are non-degenerate according to (2.35). Combining Lemma 2.2 and the above arguments, we complete the proof. $\square$

Remark 2.1 Lemma 1.4 in Schaeffer [9] is a special case of Theorem 2.1, even if we set $\alpha = \beta$ in Theorem 2.1. Theorem 2.1 shows that even around a point or an interval $[y(x_0, t_0, \beta), y(x_0, t_0, \alpha)]$ on $t = 0$, there can be infinite number of intervals, on which $(a(\phi(x)))' < 0$, $(a(\phi(x)))'' = 0$, i.e., $a(\phi(x))$ is linearly decreasing, the characteristics from each of these intervals will meet at a point in any small neighborhood of (x_0, t_0), no shocks generate at all of these points, while there is only a C^{k+1} smooth shock in this neighborhood. On the other hand, this theorem implies our solutions contain centered compression waves.

Theorem 2.2 *Assume* $L_3 = \emptyset$. *If* $[u_1^-, u_1^+], [u_2^-, u_2^+], \cdots, [u_n^-, u_n^+]$ *are the n connected components of* $\mathscr{M}_{(x_0, t_0)}$ *(suppose* $u_1^- \leq u_1^+ < u_2^- \leq u_2^+ \cdots < u_n^- \leq u_n^+$*), where* $n \geq 2$, *then* (x_0, t_0) *has a neighborhood* Θ *such that* $\Gamma_1 \cap \Theta$ *consists of n half-curves, one originating at* (x_0, t_0) *and the other* $(n - 1)$ *terminating at* (x_0, t_0). *Moreover, the minimizing function is smooth on* $\Theta' \sim \Gamma_1$.

Proof: Set

(2.38)
$$l_i^+ : x = y(x_0, t_0, u_i^+) + ta(\phi(y(x_0, t_0, u_i^+)))$$

(2.39)
$$l_i^- : x = y(x_0, t_0, u_i^-) + ta(\phi(y(x_0, t_0, u_i^-)))$$

where $i = 1, \cdots, n$, $0 < t < t_0$. By an argument similar to the proof of assertion (2.18), there exists a neighborhood Θ of (x_0, t_0) and n intervals $(y(x_0, t_0, u_i^+) - \delta, y(x_0, t_0, u_i^-) + \delta)$ $(i = 1, \cdots, n)$, $\delta < \frac{1}{2} \min_{1 \leq i \leq n-1} \{a(u_{i+1}^-) - a(u_i^+)\}$ such that for each given $(x, t) \in \Theta$ and each minimizing value $u(x, t)$ for $F(x, t, \bullet)$, $y(x, t, u(x, t))$ belongs to $(y(x_0, t_0, u_i^+) - \delta, y(x_0, t_0, u_i^-) + \delta)$ for some $i \in \{1, \cdots, n\}$. In light of Lemma A.3 in Appendix, we can deduce that all the minimizing values for $F(x, t, \bullet)$ are non-degenerate for $(x, t) \in \Theta' \cap \{t \leq t_0\}$ here $\Theta' = \Theta \sim \{(x_0, t_0)\}$. Furthermore, there exists a unique half-curve terminating at (x_0, t_0) denoted as $x = \gamma_i^-(t)$ is defined for $t \leq t_0$, which lies in the triangle domain G_{i+1}^i formed by the line $t = 0$, the characteristic l_i^+ and l_{i+1}^-, moreover there exist two minimizing values for $F(x, t, \bullet)$ and they are non-degenerate ,where $x = \gamma_i^-(t)(i = 1, \cdots, n - 1)$, according to Lemma A.3 in Appendix. By the same way used in Theorem 2.1, there exists a half-curve originating at (x_0, t_0) denoted as $x = \gamma^+(t)$, $(t \geq t_0)$. Thus, we complete the proof. $\square$

Corollary 2.1 *Any point* $(x_0, t_0) \in \overline{\Gamma}_1$ *has a neighborhood* Θ *such that* $\overline{\Gamma}_1 \cap \Theta$ *is a curve* $x = \gamma(t)$ *passing through* (x_0, t_0) *and* $x = \gamma(t)$ *is* C^{k+1} *smooth at each point except* $t = t_0$. *The minimizing function* $u(x, t)$ *is smooth on both components of* $\Theta \sim \overline{\Gamma}_1$.

The smoothness of the curve $x = \gamma(t)$ can be decided by the following cases:

Case 1: $\mathscr{M} = \{u_1, u_2\}$ for $F(x_0, t_0, \bullet)$. In this case,

- If $F_{uu}(x_0, t_0, u_1) = 0$ or $F_{uu}(x_0, t_0, u_2) = 0$, then the curve $x = \gamma(t)$ is C^1 smooth. In fact, $x = \gamma(t)$ is only C^1 at the point $t = t_0$ since $u_1(x, t)$ and $u_2(x, t)$ are continuous, but $u_{1x}(x, t_0) \to \infty$ or $u_{2x}(x, t_0) \to \infty$ as $x \to x_0 - o$ or $x \to x_0 + o$ although it is C^{k+1} at each point except $t = t_0$.

- If $F_{uu}(x_0, t_0, u_1) \neq 0$ and $F_{uu}(x_0, t_0, u_2) \neq 0$, then the curve $x = \gamma(t)$ is C^{k+1} smooth.

Case 2: $\mathscr{M} = [u_1^-, u_1^+] \cup [u_2^-, u_2^+]$, where $u_1^- < u_1^+$ or $u_2^- < u_2^+$. In this case, the curve $x = \gamma(t)$ is continuous at the point $t = t_0$.

Throughout this section, we can see the shocks are piecewise C^{k+1} smooth except at points of interaction with other shocks and points belonging to $\overline{\Gamma}_1 \sim \Gamma_1$, and the shocks are not differentiable at points of interaction with other shocks and points which are centers of centered compression waves. In other words, all the shocks curves are piecewise C^{k+1} curves.

3   Piecewise Smoothness

In Section 2, we have studied the local structure of the solutions under the condition that $L_3 = \emptyset$. We can deduce that the solutions are piecewise smooth based on the local structure of the solutions under the same condition. On the other hand we can obtain the asymptotical curves of the shock curves using the technique similar to the Lemma 4.1 of Schaeffer in [9], thus we can show there can be finitely many shock curves under certain conditions. Now we will show all of the results mentioned above in detail. There are only finitely many connected components of $\mathscr{M}$ for $F(x, t, \bullet)$ for each given point $(x, t) \in H$ since $L_3 = \emptyset$, and so we can know easily $(x, t) \in U \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$ for each $(x, t) \in \mathbb{R} \times (0, \infty)$ i.e. $H = U \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$ provided that $L_3 = \emptyset$. By Lemma 2.1, $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$ is a closed subset of H, and Γ is covered by neighborhoods of the type described in Theorem 2.1 and Theorem 2.2. For any compact set $K \subset H$, by choosing a finite subcover of $K \cap \Gamma$ we see that $K \cap \Gamma$ consists of the union of a finite number of shock and each shock is piecewise C^{k+1} curve. Therefore the minimizing function $u(x, t)$ is piecewise C^k smooth if $L_3 = \emptyset$. It is worth pointing out that our piecewise smooth solution contain centered compression waves, which implies the class of piecewise smooth solutions with C^k initial data except a first category subset Ω_3 in this paper is wider than the class of the piecewise solutions with C^k initial data, except a first category subset Ω_2 considered by Dafermos in [2].

Dafermos [2] has proved that the initial data belonging to Ω_2 is of first category. Thus Ω_3 as a proper subset of Ω_2 is of first category. Consequently we get the following theorem.

Theorem 3.1 *There is a set $\Omega_3 \subset C^k(\mathbb{R})$ of the first category such that for $\phi \in \mathscr{S}(\mathbb{R}) \sim \Omega_3$, $H = U \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$, where Ω_3 is a proper subset of Ω_2 . the solutions are piecewise smooth.*

Next we show there are finite number of shocks under certain condition.
Li-Wang [6] in Theorem 4 in Section 3 and Theorem 5 in Section 4 have proved that there is an open dense set $\triangle \subset \mathscr{S}(\mathbb{R})$ such that for $\phi \in \triangle$, the associated function Φ attains its minimum

over $\mathbb{R}$ only at points a_i ($|a_i| < \infty$) and $\phi^{(l_i)}(a_i) \neq 0$, $\phi^{(l_i-1)}(a_i) = \cdots = \phi(a_i) = 0$, where l_i ($i = 1, \cdots, m$) is some integer, then for sufficiently large t there are precisely $m+1$ smooth shocks. The proofs of them are similar to Lemma 4.1 and Proposition 4.2 in Section 4 in Schaeffer [9], so we omit them. Consequently we see there is an open dense set $\triangle \subset \mathscr{S}(\mathbb{R})$ such that for $\phi \in \triangle$, there are finitely many shock curves for sufficiently large t. Choose T such that for $t > T$ there are only finitely many smooth shock curves. It is a simple to show that there is a constant C such that no shocks form in the region $\{(x,t) : |x| \geq C, 0 \leq t \leq T\}$. Of course only finitely many shocks form in the compact region $\{(x,t) : |x| \leq C, 0 \leq t \leq T\}$, so we can get the following theorem:

Theorem 3.2 *There is a set $\Omega_4 = \Omega_3 \cup \triangle^c \subset \mathscr{S}(\mathbb{R})$ of the first category such that for $\phi \in \mathscr{S}(\mathbb{R}) \sim \Omega_4$, $H = U \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_0^{(f)} \cup \overline{\Gamma}_0^{(c)}$, the solutions of (1.1) are piecewise smooth and there are finite number of shocks, where $\triangle^c = \mathscr{S}(\mathbb{R}) \sim \triangle$.*

4 Concluding remarks

In this work we proved that if the initial data do not belong to a very small subset of C^k then the solutions of scalar conservation laws are piecewise smooth. It is important to understand the conditions under which the solution of the conservation law (1.1) is *piecewise smooth*, since most practical cases deal with the piecewise smooth solutions. For this reason, there have many studies on approximation methods for conservation laws whose solutions are piecewise smooth. For example, for systems of conservation laws, Goodman and Xin [3] proved that the viscosity methods approximating piecewise smooth solutions with finitely many *noninteracting* shocks have a local first-order rate of convergence away from the shocks; on the other hand, for scalar conservation laws, the global rate of convergence for the viscosity methods can be obtained [14, 15], and the point-wise rate of convergence for the viscosity methods has been obtained [12, 13].

In this work, we have introduced a new approach for studying the solution structures for the conservation laws, which is particularly suitable for handling the larger class of initial data considered in this works. We point out that Thom's theory of catastrophes [10], which plays a key role in Schaeffer [9], can not be used to analyze the larger class of initial data. The main motivation of this study is to develop a new analysis approach which can be extended to study the solution structures for the Hamilton-Jacobi equations. The study along this direction is under investigation, and some relevant results will be reported elsewhere.

A Appendix

Lemma A.1 *Set $m(x,t) = \min_{u \in \mathbb{R}} F(x,t,u)$. Then $m(x,t)$ is a continuous function of x, t in the semi-plane $t > 0$.*

Proof: For each fixed point (x_0, t_0), let $m(x_0, t_0) = F(x_0, t_0, u_0)$ and (x_n, t_n) with $n \geq 1$ be any sequence convergent to (x_0, t_0). Now we prove $\lim_{n \to \infty} m(x_n, t_n) = m(x_0, t_0)$. In fact, letting

$m(x_n, t_n) = F(x_n, t_n, u_n)$ gives

(A.1)
$$m(x_0, t_0) - m(x_n, t_n) \leq F(x_0, t_0, u_n) - F(x_n, t_n, u_n)$$
$$= t_0 g(u_n) + \Phi(x_0 - t_0 a(u_n)) - t_n g(u_n) - \Phi(x_n - t_n a(u_n))$$
$$= (t_0 - t_n)g(u_n) + \Phi(x_0 - t_0 a(u_n)) - \Phi(x_n - t_n a(u_n))$$
$$= (t_0 - t_n)g(u_n) + \phi(\xi)[x_0 - x_n + (t_n - t_0)a(u_n)]$$
$$\leq M(|x_n - x_0| + |t_n - t_0|) \to 0, \quad \text{as } n \to \infty,$$

where ξ is between $x_0 - t_0 a(u_n)$ and $x_n - t_n a(u_n)$, M is a positive constant. On the other hand, since $F(x, t, u_0)$ is continuous at (x_0, t_0) as u_0 is fixed,

(A.2) $$m(x_0, t_0) - m(x_n, t_n) \geq F(x_0, t_0, u_0) - F(x_n, t_n, u_0) \to 0, \quad \text{as } n \to \infty.$$

Combining (A.1) and (A.2), we have $m(x_0, t_0) - m(x_n, t_n) \to 0$, as $n \to \infty$. Thus $m(x, t)$ is continuous at (x_0, t_0). $\square$

Lemma A.2 *If $(x_l, t_1), (x_r, t_1) \in \Theta$ $(t_1 > t_0)$ are the intersection points of the line $t = t_1$ with the curves (2.19) and (2.20), then $x_r - x_l > 0$.*

Proof: It follows from (2.19) and (2.20) that there exist points $\overline{x}_1 \in (y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0)$ and $\overline{x}_2 \in (y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta))$ such that $x_l = \overline{x}_1 + t_1 a(\phi(\overline{x}_1))$ and $x_r = \bar{x}_2 + t_1 a(\phi(\bar{x}_2))$. Consequently,

(A.3) $$x_r - x_l = x_r - x_2 + x_2 - x_1 + x_1 - x_l$$
$$= \overline{x}_2 + t_1 a(\phi(\overline{x}_2)) - (y(x_0, t_0, \beta) + t_1 a(\phi(y(x_0, t_0, \beta))))$$
$$+ y(x_0, t_0, \beta) + t_1 a(\phi(y(x_0, t_0, \beta))) - (y(x_0, t_0, \alpha) + t_1 a(\phi(y(x_0, t_0, \alpha))))$$
$$+ y(x_0, t_0, \alpha) + t_1 a(\phi(y(x_0, t_0, \alpha))) - (\overline{x}_1 + t_1 a(\phi(\overline{x}_1))),$$

where $x_2 = y(x_0, t_0, \beta)) + t_1 a(\phi(y(x_0, t_0, \beta)))$ and $x_1 = y(x_0, t_0, \beta)) + t_1 a(\phi(y(x_0, t_0, \beta)))$. Observe

(A.4) $$x_2 - x_1$$
$$= y(x_0, t_0, \beta) + t_1 a(\phi(y(x_0, t_0, \beta))) - (y(x_0, t_0, \alpha) + t_1 a(\phi(y(x_0, t_0, \alpha))))$$
$$= y(x_0, t_0, \beta) - y(x_0, t_0, \alpha) + t_1(a(\phi(y(x_0, t_0, \beta))) - a(\phi(y(x_0, t_0, \alpha))))$$
$$= y(x_0, t_0, \beta) - y(x_0, t_0, \alpha) + t_1(a(\phi(\xi)))'(y(x_0, t_0, \beta) - y(x_0, t_0, \alpha))$$
$$= (y(x_0, t_0, \beta) - y(x_0, t_0, \alpha))[1 + t_1(-\frac{1}{t_0})] \geq 0,$$

where $y(x_0, t_0, \beta) < \xi < y(x_0, t_0, \alpha)$. Moreover,

(A.5) $$x_r - x_2$$
$$= \overline{x}_2 + t_1 a(\phi(\overline{x}_2)) - (y(x_0, t_0, \beta) + t_1 a(\phi(y(x_0, t_0, \beta))))$$
$$= \overline{x}_2 - y(x_0, t_0, \beta) - \frac{1}{(a(\phi(\overline{x}_2)))'}[a(\phi(\overline{x}_2)) - a(\phi(y(x_0, t_0, \beta)))]$$
$$= \overline{x}_2 - y(x_0, t_0, \beta) + \frac{1}{(a(\phi(\overline{x}_2)))'}[(a(\phi(\overline{x}_2)))'(y(x_0, t_0, \beta) - \overline{x}_2)$$
$$+ (y(x_0, t_0, \beta) - \overline{x}_2)^2 \int_0^1 (1-t)(a(\phi))''(\overline{x}_2 + t(y(x_0, t_0, \beta) - \overline{x}_2))dt]$$
$$= \frac{1}{(a(\phi(\overline{x}_2)))'}(y(x_0, t_0, \beta) - \overline{x}_2)^2 \int_0^1 (1-t)(a(\phi))''(\overline{x}_2 + t(y(x_0, t_0, \beta) - \overline{x}_2))dt] > 0,$$

for each $\overline{x}_2 \in (y(x_0, t_0, \beta) - \delta_0, y(x_0, t_0, \beta))$, where we have used the fact $(a(\phi(\overline{x}_2)))' < 0$, (2.13) and (2.16). Similarly, we have $x_1 - x_l > 0$. It follows from (A.3)-(A.5) that $x_r - x_l = x_r - x_2 + x_2 - x_1 + x_1 - x_l > 0$. $\quad\square$

Lemma A.3 *Suppose $L_3 = \emptyset$. If $[u_1^-, u_1^+]$ and $[u_2^-, u_2^+]$ are two neighboring connected components of $\mathscr{M}_{(x_1, t_1)}$, where $u_1^- \le u_1^+ < u_2^- \le u_2^+]$, then there exists an open neighborhood $\overline{\Theta}$ of (x_1, t_1) such that $\overline{\Theta} \cap G \cap \Gamma_1$ contains a half curve $x = \gamma^-(t)$ terminating at (x_1, t_1), where G is the triangle domain formed by $t = 0$, the characteristics $x = y(x_1, t_1, u_2^-) + ta(u_2^-)$ and $x = y(x_1, t_1, u_1^+) + ta(u_1^+)$.*

Proof: By an argument similar to the proof of assertion (2.18), there exists a neighborhood $\overline{\Theta}$ of (x_1, t_1) and two intervals $(y(x_1, t_1, u_1^+) - \delta, y(x_1, t_1, u_1^+))$ and $(y(x_1, t_1, u_2^-), y(x_1, t_1, u_2^-) + \delta)$ such that for each given $(x, t) \in \overline{\Theta}' \cap G$ and any minimizing value, say u, for $F(x, t, \bullet)$, $y(x, t, u)$ belongs to $(y(x_1, t_1, u_1^+) - \delta, y(x_1, t_1, u_1^+))$ or $(y(x_1, t_1, u_2^-), y(x_1, t_1, u_2^-) + \delta)$. Now we claim that: for each $(x, t) \in \overline{\Theta}' \cap G$ there exists a unique minimizing value $u(x, t)$ for $F(x, t, \bullet)$ such that $y(x, t, u(x, t)) \in (y(x_1, t_1, u_1^+) - \delta, y(x_1, t_1, u_1^+))$ and it is non-degenerate, where $\overline{\Theta}' = \overline{\Theta} \sim \{(x_1, t_1)\}$. In fact, there are only two cases to be considered

Case 1: $F_{uu}(x_1, t_1, u_1^+) = 0$. In this case, if there exist a point $(\tilde{x}, \tilde{t}) \in \overline{\Theta}' \cap G$ such that there exist at least two minimizing values u^* and u^{**} for $F(\tilde{x}, \tilde{t}, \bullet)$ and $y(\tilde{x}, \tilde{t}, u^*), y(\tilde{x}, \tilde{t}, u^{**}) \in (y(x_1, t_1, u_1^+) - \delta, y(x_1, t_1, u_1^+))$. Thus there exist at least two characteristics passing through $(\tilde{x}, \tilde{t})$:

(A.6) $$x = y(\tilde{x}, \tilde{t}, u^*) + ta(u^*),$$

(A.7) $$x = y(\tilde{x}, \tilde{t}, u^{**}) + ta(u^{**}).$$

On the other hand, by using the fact that $(a(\phi(x)))'$ is monotone decreasing on the interval $(y(x_1, t_1, u_1^+) - \delta, y(x_1, t_1, u_1^+))$ and the assertion (2.16), we have

$$\tilde{t} = -\frac{y(\tilde{x}, \tilde{t}, u^*) - y(\tilde{x}, \tilde{t}, u^{**})}{a(u^*) - a(u^{**})} = -\frac{1}{(a(\phi(\eta)))'} > t_1,$$

where η is between $y(\tilde{x}, \tilde{t}, u^*)$ and $y(\tilde{x}, \tilde{t}, u^{**})$. This result is contradictory to the fact that $\tilde{t} < t_1$. The claim is therefore proved.

Case 2: $F_{uu}(x_1, t_1, u_1^+) \ne 0$. In this case, since $F_{uu}(x_1, t_1, u_1^+) \ne 0$, it follows from the Implicit Function Theorem that there is an open neighborhood $U(u_1^+)$ of u_1 such that for (x, t) close to (x_1, t_1) the equation $(\partial F / \partial u)(x, t, u) = 0$ has a unique solution $u(x, t) \in U(u_1^+)$. Consequently, there is a unique minimizing value $u(x, t)$ such that $y(x, t, u(x, t)) \in (y(x_1, t_1, u_1^+) - \delta, y(x_1, t_1, u_1^+))$.

Similarly we can deduce that There exists a unique minimizing value $u(x, t)$ for $F(x, t, \bullet)$ for each $(x, t) \in \overline{\Theta}' \cap G$, such that $y(x, t, u(x, t)) \in (y(x_1, t_1, u_2^-), y(x_1, t_1, u_2^-) + \delta)$ and it is non-degenerate.

Let $A_{t_2} = \{x \mid (x, t_2) \in \overline{\Theta}' \cap G\}$. There exists a characteristic which originates from a point on the left of the point $(y(x_1, t_1, u_1^+), 0)$ and passes through $(x_1, t_2)\}$. Let $B_{t_2} = \{x \mid (x, t_2) \in \overline{\Theta}' \cap G\}$. There exists a characteristic which originates from a point on the right of the point $(y(x_1, t_1, u_2^-), 0)$ and passes through $(x_1, t_2)\}$. Let $\overline{x}_1$ be the infimum of the set A_{t_2} and $\overline{x}_2$ be the supremum of the set B_{t_2}. By an argument similar to the proofs of the assertion (2.35) and (2.36), it can be shown that there exist two minimizing values for $F(\overline{x}_1, t_2, \bullet)$ which are non-degenerate.

This implies there are two valid characteristics passing through $(\overline{x}_1, t_2)$, one originated from a point on the left of the point $(y(x_1, t_1, u_1^+), 0)$ and the other originated from a point on the right of the point $(y(x_1, t_1, u_2^-), 0)$. Similarly, it can be shown that there exist two minimizing values for $F(\overline{x}_2, t_2, \bullet)$ which are non-degenerate. This implies that there are two valid characteristics passing through $(\overline{x}_2, t_2)$, one originated from a point on the left of the point $(y(x_1, t_1, u_1^+), 0)$ and the other originated from a point on the right of the point $(y(x_1, t_1, u_2^-), 0)$. By the same argument for obtaining the assertion (2.37) we have

$$(A.8) \qquad\qquad\qquad\qquad \overline{x}_1 = \overline{x}_2.$$

Consequently, a unique half-curve terminating at (x_1, t_1) denoted by $x = \gamma^-(t)$ is defined for $t \leq t_1$. Thus we complete the proof. $\square$

Lemma A.4 *The curve (2.19) (respective (2.20)) is continuous and is the closure of union of countable open arcs, on which the curve is C^k. The points at which the curve (2.19) (respective (2.20))is not differentiable belong to the end points of the open arcs but the left derivative continuous on the left and right derivative continuous on the right.*

Proof: In fact, it is easy to compute that

$$(A.9) \qquad\qquad x'(\xi) = \frac{(a(\phi(\xi)))'(a(\phi(\xi)))''}{(a(\phi(\xi)))'^2}, \quad t'(\xi) = \frac{(a(\phi(\xi)))''}{(a(\phi(\xi)))'^2},$$

for $\xi \in [y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0]$. Let

$$(A.10) \qquad\qquad A_t = \{\xi \mid t(\xi) = t\}, \quad A_x = \{\xi \mid x(\xi) = x\},$$

where x and t are given. Obviously, A_t is closed. Since $L_3 = \emptyset$, we have $(a(\phi(\xi)))'' \geq 0$ for each $\xi \in [y(x_0, t_0, \alpha), y(x_0, t_0, \alpha) + \delta_0]$. We first prove

$$(A.11) \qquad\qquad\qquad x(\xi) \text{ is a function of } t(\xi).$$

For any fixed $t_1 = t(\xi)$ with $\xi \in A_{t_1}$, there exists a unique $x_1(\xi)$ to t_1. In fact, if $A_{t_1} = \{\xi_1\}$, we have $x_1 = x(\xi_1)$ which is unique. If $A_{t_1} = [\alpha_1, \beta_1], \alpha_1 \neq \beta_1$, we have $t'(\xi) = 0$, which implies $x'(\xi) = 0$ in light of (A.9), for each $\xi \in [\alpha_1, \beta_1]$. Consequently, $t(\xi_1) = t(\xi_2)$ implies $x(\xi_1) = t(\xi_2)$, here $\xi_1, \xi_2 \in [\alpha_1, \beta_1]$. Then $x_1(\xi)$ is unique. This completes the proof of assertion (A.11). The curve (2.19) can be denoted by $x = \beta(t)$. Next we claim

$$(A.12) \qquad\qquad\qquad x = \beta(t) \text{ is a continuous function of } t.$$

For each fixed point t_1, we only need to prove

$$\lim_{t \to t_1+} \beta(t) = \lim_{t \to t_1-} \beta(t) = \beta(t_1).$$

First we prove $\lim_{t \to t_1-} \beta(t) = \beta(t_1)$. Let $t_n < t_1$, $n \geq 1$, be any sequence convergent to t_1, $t_n = t(\xi_n)$ and $t_1 = t(\xi_1)$, where $\xi_n = \sup A_{t_n}$ and $\xi_1 = \inf A_{t_1}$. We will show $\lim_{n \to \infty} \beta(t_n) = \beta(t_1)$. Notice the function t is an increasing function of ξ, we have $\lim_{n \to \infty} \xi_n = \xi_1$. Consequently,

$$\lim_{n \to \infty} \beta(t_n) = \lim_{n \to \infty} \beta(t(\xi_n)) = \lim_{n \to \infty} x(\xi_n) = x(\xi_1) = \beta(t_1).$$

Then $\lim_{t\to t_1-} \beta(t) = \beta(t_1)$. Similarly, we have $\lim_{t\to t_1+} \beta(t) = \beta(t_1)$. So the assertion (A.12) is true. Finally we prove

(A.13) $x = \beta(t)$ is differentiable at point t_1 $\iff$ the set A_{t_1} contains only one point.

Let $A_{t_1} = [\xi_1, \xi_2]$ and $t_n < t_1$, $n \geq 1$, be any sequence convergent to t_1, $t_n = t(\xi_n)$ and $t_1 = t(\xi_1)$, with $\xi_n = \sup A_{t_n}$ and $\xi_1 = \inf A_{t_1}$. Observe

$$
\begin{aligned}
\text{(A.14)} \qquad \frac{\beta(t_n) - \beta(t_1)}{t_n - t_1} &= \frac{x(\xi_n) - x(\xi_1)}{t(\xi_n) - t(\xi_1)} \\
&= \frac{\xi_n - a(\phi(\xi_n))/(a(\phi(\xi_n)))' - \xi_1 + a(\phi(\xi_1))/(a(\phi(\xi_1)))'}{1/(a(\phi(\xi_1)))' - 1/(a(\phi(\xi_n)))'} \\
&= a(\phi(\xi_1)) + T_1,
\end{aligned}
$$

where the last term is non-negative:

$$
\begin{aligned}
\text{(A.15)} \qquad T_1 &:= \frac{\xi_n - \xi_1 + \Big(a(\phi(\xi_1)) - a(\phi(\xi_n))\Big)/(a(\phi(\xi_n)))'}{1/(a(\phi(\xi_1)))' - 1/(a(\phi(\xi_n)))'} \\
&= (a(\phi(\xi_n)))'(a(\phi(\xi_1)))' \frac{\xi_n - \xi_1 + \Big(a(\phi(\xi_1)) - a(\phi(\xi_n))\Big)/(a(\phi(\xi_n)))'}{(a(\phi(\xi_n)))' - (a(\phi(\xi_1)))'} \\
&= (a(\phi(\xi_n)))'(a(\phi(\xi_1)))' \frac{(a(\phi(\tilde{\xi})))''(\xi_n - \xi_1)^2}{2(a(\phi(\xi_n)))'[(a(\phi(\xi_n)))' - (a(\phi(\xi_1)))']} \geq 0,
\end{aligned}
$$

where $\tilde{\xi}$ lies between ξ_n and ξ_1 and satisfies

$$
a(\phi(\xi_1)) - a(\phi(\xi_n)) = (a(\phi(\xi_n)))'(\xi_1 - \xi_n) + \frac{1}{2!}(a(\phi(\tilde{\xi})))''(\xi_1 - \xi_n)^2.
$$

In the last step of (A.15), we have used the facts that $(a(\phi(\tilde{\xi})))'' \geq 0$ and $(a(\phi(\xi_n)))' - (a(\phi(\xi_1)))' < 0$, which are due to the monotonicity of $(a(\phi(\xi)))'$, the assertion (2.17) and $\xi_n < \xi_1$. Similarly,

(A.16) $$\frac{\beta(t_n) - \beta(t_1)}{t_n - t_1} = a(\phi(\xi_n)) + T_2,$$

where

$$
\begin{aligned}
\text{(A.17)} \qquad T_2 &:= \frac{\xi_n - \xi_1 + \Big(a(\phi(\xi_1)) - a(\phi(\xi_n))\Big)/(a(\phi(\xi_1)))'}{1/(a(\phi(\xi_1)))' - 1/(a(\phi(\xi_n)))'} \\
&= -(a(\phi(\xi_n)))'(a(\phi(\xi_1)))' \frac{(a(\phi(\hat{\xi})))''(\xi_n - \xi_1)^2}{2(a(\phi(\xi_1)))'[(a(\phi(\xi_n)))' - (a(\phi(\xi_1)))']} \leq 0,
\end{aligned}
$$

where $\hat{\xi}$ is between ξ_n and ξ_1 and satisfies

$$
a(\phi(\xi_n)) - a(\phi(\xi_1)) = (a(\phi(\xi_1)))'(\xi_n - \xi_1) + \frac{1}{2!}(a(\phi(\hat{\xi})))''(\xi_n - \xi_1)^2.
$$

Combining (A.14)-(A.17) gives

(A.18) $$a(\phi(\xi_1)) \leq \frac{\beta(t_n) - \beta(t_1)}{t_n - t_1} \leq a(\phi(\xi_n)).$$

Consequently,

$$(A.19) \qquad \lim_{t_n \to t_1-} \frac{\beta(t_n) - \beta(t_1)}{t_n - t_1} = a(\phi(\xi_1)),$$

which implies that

$$(A.20) \qquad \lim_{t \to t_1-} \frac{\beta(t) - \beta(t_1)}{t - t_1} = a(\phi(\xi_1)).$$

Similarly,

$$(A.21) \qquad \lim_{t \to t_1+} \frac{\beta(t) - \beta(t_1)}{t - t_1} = a(\phi(\xi_2)).$$

The above two results yield $\beta'_-(t_1) = a(\phi(\xi_1))$ and $\beta'_+(t_1) = a(\phi(\xi_2))$. Thus the function $\beta(t)$ is differentiable at t_1 if and only if $\xi_1 = \xi_2$. This verifies the assertion (A.13). It is easy to show that there are countable points $(\beta(t), t)$ such that the set A_t contains more than one point. Moreover, it can be shown that the set

$$(A.22) \qquad \{(\beta(t), t) \mid A_t = \{\xi\}, (a(\phi(\xi)))'' \neq 0\}$$

is an open set, and is therefore a union of disjoint open arcs. On the other hand, the set

$$(A.23) \qquad \{(\beta(t), t) \mid A_t = [\xi_1, \xi_2], (a(\phi(\xi_1)))'' = 0\}$$

is a closed set and contains at most countable points. Each point belonging to the set (A.23) is an end point of some open arcs of the set (A.22). This implies that the curve (2.19) (respective (2.20)) is continuous and is the closure of union of countable open arcs, on which the curve is C^k. The points at which the curve (2.19) (respective (2.20)) is not differentiable belong to the set (A.23); and the left derivative is continuous on the left and the right derivative is continuous on the right. $\square$

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