Remarks on vacuum state and uniqueness of concentration process

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The zero-pressure gas dynamics system in the one-dimensional case was studied in [1]. This system has the form

$$\partial_t \rho + \partial_x(\rho u) = 0, \qquad \partial_t(\rho u) + \frac{\partial}{\partial x}(\rho u^2) = 0 \tag{1}$$

and, in the domains where the solution belongs to C^1, is, as is known, equivalent to the following system

$$\partial_t \rho + \partial_x(\rho u) = 0, \qquad \partial_t u + \frac{1}{2}\partial_x u^2 = 0. \tag{2}$$

In particular, the solution of the Cauchy problem is constructed in the case where the initial profile of velocity is an unstable step function. To construct the second component ρ of the solution, i.e., to solve the continuity equation, the authors [1] choose a class of functions invariant under the scaling transformation

$$x \to kx, \qquad t \to kt.$$

Indeed, the group of scaling transformations acts on the solution of the system considered. But these are particular solutions. For example, in [2], such solutions are considered in a quite different context. In [1], the assertion that a vacuum domain exists is derived from the assumption that such invariant solutions are unique. Such an assertion (in form, it pretends to be a description of some physically meaninful phenomenon) cannot be made based only on the consideration of *particular* solutions.

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The following natural question arises: Can solutions that are not contained in the class of solutions invariant under the action of the scaling group help to fill a vacuum?

More generally, the question can be formulated as follows: Do there exist any natural conditions ensuring the uniqueness of the Goursat problem solutions considered in [1]. In this small note, we give an affirmative answer to this question. Namely, for any initial distribution ρ with compact support (perhaps, with a first kind discontinuity), the solution of the Goursat problem is zero in the rarefaction domain.

In our considerations, we do not use the regularization procedure, which uniformly approximates the solution of the Cauchy problem for (1). This can be done using the simple formulas from [3], but the problem is very simple and does not require any special technical methods.

So the solution of the Cauchy problem for (1) with the initial conditions

$$u|_{t=0} = \begin{cases} u_l, & x < x_0, \\ u_r, & x > x_0, \end{cases} \qquad u_{l,r} = \text{const}, \quad u_l < u_r,$$

$$\rho|_{t=0} = \begin{cases} \rho_l, & x < x_0, \\ \rho_r, & x > x_0, \end{cases} \qquad \rho_{l,r} \geq 0,$$

for $t > 0$ has the form

$$u = \begin{cases} u_l, & x < x_0 + u_l t, \\ u_l + \frac{x - u_l t - x_0}{t}, & x \in [x_0 + u_l t, x_0 + u_r t], \\ u_r, & x > x_0 + u_r t, \end{cases} \tag{3}$$

and, respectively,

$$\rho = \begin{cases} \rho_l(x - u_l t), & x < x_0 + u_l t, \\ \rho_0(\frac{x - x_0}{t}) t^{-1}, & x \in (x_0 + u_l t, x_0 + u_r t), \\ \rho_r(x - u_l t), & x > x_0 + u_r t, \end{cases} \tag{4}$$

where $\rho_0 = \rho_0(z)$ is an arbitrary C^1-function.

Formulas (3), (4) can be verified by a direct substitution. We only note that since the function u in (3) is continuous, the Rankine–Hugoniot type conditions are identically satisfied on the lines $x = x_0 + u_j t$, $j = l, r$, because the equation for ρ is linear in ρ.

The following obvious assertion holds.

Lemma 1. *Let (u, ρ) be a δ-shock wave type solution in the sense of [4] to system (1), and let $\rho|_{t=0}$ have a compact support. Then for any finite t,*

$$\langle \rho(x,t), \zeta(x) \rangle = \langle \rho(x,0), \zeta(x) \rangle$$

for any test function $\zeta(x)$ equal to 1 on the support of $\rho(x,t)$; here $\langle , \rangle$ is the distribution action.

Proof. Paper [4] presents a method for constructing a weak asymptotic solution to system (1), i.e., a pair of functions u_ε, ρ_ε smooth for $\varepsilon > 0$ such that:

(1) The weak limits of u_ε, ρ_ε as $\varepsilon \to 0$ give a solution u, ρ to system (1) in the sense of the integral identity [4].

(2) The product $\rho_\varepsilon u_\varepsilon$ has a limit as $\varepsilon \to 0$ in the sense of distributions.

(3) The relation

$$\rho'_{\varepsilon t} + (\rho_\varepsilon u_\varepsilon)'_x = O_{D'}(\varepsilon)$$

holds, where $O_{D'}(\varepsilon)$ is a distribution such that

$$\langle O_{D'}(\varepsilon), \zeta(x) \rangle = O(\varepsilon)$$

for any test function $\zeta(x)$.

If $\rho|_{t=0}$ has a compact support, then ρ has a compact support for any finite t because of hyperbolicity, $\rho_\varepsilon = O(\varepsilon^N)$, $N \gg 1$, outside the support of ρ, and the function u is bounded [4].

Choosing a test function $\zeta(x)$ such that $\zeta(x) = 1$, $x \in \sup \rho$, and applying both sides of the equation for ρ to ζ, we obtain

$$\langle \rho'_{\varepsilon t}, \zeta(x) \rangle = O(\varepsilon).$$

Passing to the limit as $\varepsilon \to 0$, we obtain the statement of the lemma. $\square$

Corollary 1. *Let ρ have the form*

$$\rho = R(x,t) + \sum_{k=1}^{N} \rho R_k(t) \delta(x - \varphi_k),$$

where $R(x,t) \in C([0,T]; L^1(R^1))$ and $R_k(t), \varphi_k(t) \in C^1([0,T])$. Then

$$\frac{d}{dt}\left[\int_{R^1} R(x,t)\, dx + \sum_{k=1}^{N} R_k(t) \right] = 0.$$

Calculating the integral $\int_{R^1} \rho(x,t)\,dt$ for $t > 0$ ($\rho(x,t)$ is defined in (4)), we obtain

$$\int_{R^1} \rho(x,t)\,dx = \int_{x<x_0} \rho_l\,dx + \int_{x>x_0} \rho_r\,dx + \int_{u_l}^{u_r} \rho_0(z)\,dz.$$

Hence, since ρ is nonnegative and $\langle \rho, \zeta \rangle$ is preserved, we have

$$\rho_0(z) = 0.$$

We point out that we derived this relation without any assumptions on the properties of *particular* solutions to system (1). We also note that a (more general than that in [1]) assumption ensuring the uniqueness of the solution of the Goursat problem in the case under study could be the assumption that ρ is bounded. However, if simultaneously with the rarefaction wave we consider shock waves in the u-component, then δ-shock solutions arise, which is prohibited by the boundedness condition.

But if the initial condition for ρ is replaced by the condition

$$\rho|_{t=0} = \rho_l H(x_0 - x) + \rho_r H(x - x_0) + \hat{\rho}\delta(x - x_0),$$

then the choice of the function ρ_0 in (4) is restricted only by the condition

$$\int_{u_l}^{u_r} \rho_0(z)\,dz = \hat{\rho}$$

and the solution of this "singular" Goursat problem is not unique.

It is also interesting to note the following fact.

It is proved in [4] that for system (1) to have a solution of the form

$$\begin{aligned} u &= u_0(x,t) + u_1(x,t)H(-\varphi(t) - x), \\ v &= v_0(x,t) + v_1(x,t)H(\varphi(t) - x) + e(t)\delta(\varphi_x) \end{aligned}$$

in the sense of the integral identity introduced in [4], it is necessary that

$$\dot{e}_t(t) = \varphi_t([uv] - [v]\varphi_t)|_{x=\varphi}, \qquad \frac{d}{dt}(e\varphi_t) - ([u^2v] - [uv]\varphi_t)|_{x=\varphi}, \tag{5}$$

where $[f]|_{x=\varphi} = f(\varphi(t)+0) - f(\varphi(t)-0)$.

It is easy to see that these relations form a second-order system of equations for e, φ. The original system is a first-order system, hence the value

$\varphi_t(0)$ remains undetermined. In [4], it is shown that, in the case of constant u_i, v_i, $i = 1, 2$, system (5) has a unique solution if $e(0) = 0$. In this case, the solution is independent of $\varphi_t(0)$.

The formula for $\varphi(t)$ obtained in [4] (see Theorem 4.3) has the form

$$\varphi(t) = \begin{cases} \dfrac{e(0) + [uv] + \sqrt{e(0)^2 + 2e(0)\dot{e}(0)t + v_- v_+ (u_- - u_+)^2 t^2}}{[v]}, & [v] \neq 0, \\ \dfrac{e(0)\dot{\varphi}(0) + t v_0 [u]^2/2}{e(0) + t v_0 [u]}, \qquad [v] = 0. \end{cases}$$

It follows from the first equation in (5) that the values of the constants $\dot{e}(0)$ and $\dot{\varphi}(0)$ can be expressed linearly in terms of each other. From these formulas it is easily seen that, in the case $e(0) = 0$, the expression $\dot{\varphi}(0)$ ($\dot{e}(0)$) is not contained in the formula for $\varphi(t)$.

A similar statement also holds in the general case. Indeed, for $e(0) = 0$, relations (5) imply the following equation for $\dot{\varphi}(0)$:

$$\dot{\varphi}(0)^2 [v]^0 - 2\dot{\varphi}(0)[uv]^0 + [u^2 v]^0 = 0,$$

where $[\ ]^0 \stackrel{\text{def}}{=} [\ ]|_{t=0}$ and hence the quantities $[\ ]^0$ are functions of the argument $\varphi(0)$. Solving this equation under the additional condition $u^+|_{t=0} < \dot{\varphi}(0) < u^-|_{t=0}$, which is necessary for the existence of the desired δ-shock type solutions, we obtain

$$\dot{\varphi}(0) = \{([uv]^0)^2 - [v]^0 [u^2 v]^0\}^{1/2} ([v]^0)^{-1} \stackrel{\text{def}}{=} G(\varphi(0)).$$

Thus, in the case $e(0) = 0$, the missing constant is determined by the natural initial data of the problem.

It is also easy to verify that $\dot{e}(0) > 0$ in this case. Hence from the second equation in (5) we obtain $|\ddot{\varphi}(0)| < \infty$. Therefore, although the coefficient of the second derivative $\ddot{\varphi}$ vanishes for $t = 0$, system (5) has a smooth solution at least in the small in t. This can be proved as usual, by reducing the problem to a system of integral equations.

The case $v_1|_{t=0} = -[v]^0 = 0$ is considered similarly (see [4] for $u_i, v_i =$ const).

Thus, we see that the "singular" Cauchy problem for system (1) does not have the property that the solution is unique and the problem whose initial conditions do not contain the Dirac function has a unique solution.

References

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