L^2 formulation of multidimensional scalar conservation laws

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Abstract

We show that Kruzhkov's theory of entropy solutions to multidimensional scalar conservation laws [Kr] can be entirely recast in L^2 and fits into the general theory of maximal monotone operators in Hilbert spaces. Our approach is based on a combination of level-set, kinetic and transport-collapse approximations, in the spirit of previous works by Giga, Miyakawa, Osher, Tsai and the author [Br1, Br2, Br3, Br4, GM, TGO].

1 A short review of Kruzhkov's theory

First order systems of conservation laws read:

$$\partial_t u + \sum_{i=1}^{d} \partial_{x_i}(Q_i(u)) = 0,$$

or, in short, using the nabla notation,

$$\partial_t u + \nabla_x \cdot (Q(u)) = 0, \tag{1}$$

where $u = u(t,x) \in \mathbb{R}^m$ depends on $t \geq 0$, $x \in \mathbb{R}^d$, and $\cdot$ denotes the inner product in $\mathbb{R}^d$. The Q_i (for $i = 1, \cdots, d$) are given smooth functions from

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$\mathbb{R}^m$ into itself. The system is called hyperbolic when, for each $\tau \in \mathbb{R}^d$ and each $U \in \mathbb{R}^m$, the $m \times m$ matrix $\sum_{i=1,d} \tau_i Q_i'(U)$ can be put in diagonal form with real eigenvalues. There is no general theory to solve globally in time the initial value problem for such systems of PDEs. (See [BDLL, Da, Ma, Se] for a general introduction to the field.) In general, smooth solutions are known to exist for short times but are expected to blow up in finite time. Therefore, it is usual to consider discontinuous weak solutions, satisfying additional 'entropy' conditions, to adress the initial value problem, but nothing is known, in general, about their existence . Some special situations are far better understood. First, for some special systems (enjoying 'linear degeneracy' or 'null conditions'), smooth solutions may be global (shock free), at least for 'small' initial data (see [Kl], for instance). Next, in one space dimension $d = 1$, for a large class of systems, existence and uniqueness of global weak entropy solutions have been (recently) proven for initial data of sufficiently small total variation [BB]. Still, in one space dimension, for a limited class of systems (typically for $m = 2$), existence of global weak entropy solutions have been obtained for large initial data by 'compensated compactness' arguments [Ta, Di, LPS]. Finally, there is a very comprehensive theory in the much simpler case of a *single* conservation laws, i.e. when $m = 1$. Then, equation (1) is called a 'scalar conservation law'. Kruzhkov [Kr] showed that such a scalar conservation law has a unique 'entropy solution' $u \in L^\infty$ for each given initial condition $u_0 \in L^\infty$. (If the derivative Q' is further assumed to be bounded, then we can substitute L^1_{loc} for L^∞ in this statement.) An entropy (or Kruzhkov) solution is an L^∞ function that satisfies the following distributional inequality

$$\partial_t C(u) + \nabla_x \cdot (Q^C(u)) \leq 0, \tag{2}$$

for all Lipschitz convex function $C : \mathbb{R} \to \mathbb{R}$, where the derivative of Q^C is defined by $(Q^C)' = C'Q'$. In addition, the initial condition u_0 is prescribed in L^1_{loc}, namely:

$$\lim_{t \to 0} \int_B |u(t,x) - u_0(x)| dx = 0, \tag{3}$$

for all compact subset B of $\mathbb{R}^d$. Beyond their existence and uniqueness, the Kruzhkov solutions enjoy many interesting properties. Each entropy solution $u(t,\cdot)$, with initial condition u_0, continuously depends on $t \geq 0$ in L^1_{loc} and can be written $T(t)u_0$, where $(T(t), t \geq 0)$ is a family of order preserving operators:

$$T(t)u_0 \ \geq \ T(t)\tilde{u}_0 \ , \ \ \forall t \geq 0, \tag{4}$$

whenever $u_0 \geq \tilde{u}_0$. Since constants are trivial entropy solutions to (1), it follows that if u_0 takes its values in some fixed compact interval, so does $u(t,\cdot)$ for all $t \geq 0$. Next, two solutions u and $\tilde{u}$, with $u_0 - \tilde{u}_0 \in L^1$, are L^1 stable with respect to their initial conditions:

$$\int |u(t,x) - \tilde{u}(t,x)|dx \leq \int |u_0(x) - \tilde{u}_0(x)|dx, \tag{5}$$

for all $t \geq 0$. As a consequence, the total variation $TV(u(t,\cdot))$ of a Kruzhkov solution u at time $t \geq 0$ cannot be larger than the total variation of its initial condition u_0. This easily comes from the translation invariance of (1) and from the following definition of the total variation of a function v:

$$TV(v) = \sup_{\eta \in \mathbb{R}^d,\ \eta \neq 0} \int \frac{|v(x+\eta) - v(x)|}{||\eta||} dx, \tag{6}$$

where $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^d$. The space L^1 plays a key role in Kruzhkov's theory. There is no L^p stability with respect to initial conditions in any $p > 1$. Typically, for $p > 1$, the Sobolev norm $||u(t,\cdot)||_{W^{1,p}}$ of a Kruzhkov solution blows up in finite time. This fact has induced a great amount of pessimism about the possibility of a unified theory of global solutions for general multidimensional systems of hyperbolic conservation laws. Indeed, simple linear systems, such as the wave equation (written as a first order system) or the Maxwell equations, are not well posed in any L^p but for $p = 2$ [Brn]. However, as shown in the present work, L^2 is a perfectly suitable space for entropy solutions to multidimensional scalar conservation laws, provided a different formulation is used, based on a combination of level-set, kinetic and transport-collapse approximations, in the spirit of previous works by Giga, Miyakawa, Osher, Tsai and the author [Br1, Br2, Br3, Br4, GM, TGO].

2 Kruzhkov solutions revisited

2.1 A maximal monotone operator in L^2

Subsequently, we restrict ourself, for simplicity, to initial conditions $u_0(x)$ valued in $[0,1]$ and spatially periodic of period 1 in each direction. In other words, the variable x will be valued in the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

Let us now introduce:

1) the space $L^2([0,1]\times\mathbb{T}^d)$ of all square integrable functions

$$(a,x)\in[0,1]\times\mathbb{T}^d\to Y(a,x)\in\mathbb{R}\ ,$$

2) the closed convex cone K of all $Y\in L^2$ such that $\partial_a Y\geq 0$ (in the sense of distributions),
3) the subdifferential of K defined at each point $Y\in K$ by:

$$\partial K(Y)=\{Z\in L^2,\quad \int(\tilde{Y}-Y)Z\,dadx\leq 0\ ,\quad \forall\tilde{Y}\in K\}\ , \tag{7}$$

4) the maximal monotone operator (MMO) (see [Brz]):

$$Y\to -q(a)\cdot\nabla_x Y+\partial K(Y), \tag{8}$$

where $q(a)=Q'(a)$, and the corresponding subdifferential equation [Brz]:

$$0\ \in\ \ \partial_t Y+q(a)\cdot\nabla_x Y+\partial K(Y). \tag{9}$$

From maximal monotone operator theory [Brz], we know that, for each initial condition $Y_0\in K$, there is a unique solution $Y(t,\cdot)\in K$ to (9), for all $t\geq 0$. More precisely, we will use the following definition (which includes the possibility of a left-hand side $q_0\in L^2([0,1])$):

Definition 2.1 *Y is a solution to*

$$q_0(a)\ \in\ \ \partial_t Y+q(a)\cdot\nabla_x Y+\partial K(Y), \tag{10}$$

with initial value $Y_0\in K$ and left-hand side $q_0\in L^2([0,1])$, if:
1) $t\to Y(t,\cdot)\in L^2$ is continuous and valued in K, with $Y(0,\cdot)=Y_0$,
2) Y satisfies, in the sense of distribution,

$$\frac{d}{dt}\int|Y-Z|^2dadx\leq 2\ \int(Y-Z)(q_0(a)-\partial_t Z-q(a)\cdot\nabla_x Z)dadx, \tag{11}$$

for each smooth function $Z(t,a,x)$ such that $\partial_a Z\geq 0$.

Proposition 2.2 *For each $Y_0\in K$, and $q_0\in L^2([0,1])$, there is a unique solution Y to (10) in the sense of Definition 2.1. If both Y_0 and q_0 belong to L^∞, then we have for all $t\geq 0$:*

$$-t\sup(-q_0)_+ +\inf Y_0\leq Y(t,\cdot)\leq \sup Y_0+t\sup(q_0)_+. \tag{12}$$

If $\nabla_x Y_0$ belongs to L^2, then so do $\partial_t Y(t,\cdot)$ and $\nabla_x Y(t,\cdot)$ for all $t \geq 0$. Two solutions Y and $\tilde{Y}$ to (10) (with different left-hand side q_0 and $\tilde{q}_0$) are L^2 stable with respect to their initial conditions Y_0 and $\tilde{Y}_0$ in K:

$$||Y(t,\cdot) - \tilde{Y}(t,\cdot)||_{L^2} \leq ||Y_0 - \tilde{Y}_0||_{L^2} + t||q_0 - \tilde{q}_0||_{L^2}. \tag{13}$$

for all $t \geq 0$. This is also true for all $p \geq 1$, when both $Y_0 - \tilde{Y}_0$ and $q_0 - \tilde{q}_0$ belong to L^p:

$$||Y(t,\cdot) - \tilde{Y}(t,\cdot)||_{L^p} \leq ||Y_0 - \tilde{Y}_0||_{L^p} + t||q_0 - \tilde{q}_0||_{L^p}. \tag{14}$$

For the sake of completeness, a brief proof of these (standard) results will be provided at the end of the paper.

2.2 The main result

Our main result is

Theorem 2.3 *Let $Y = Y(t,a,x)$ be a solution to the subdifferential equation (9) with initial condition $Y_0 \in L^\infty$, with $\partial_a Y_0 \geq 0$. Then,*

$$u(t,y,x) = \int_0^1 H(y - Y(t,a,x))da, \tag{15}$$

defines a one parameter family (parameterized by $y \in \mathbb{R}$) of Kruzhkov solution to (1), valued in $[0,1]$. In addition, all Kruzhkov solutions, with initial values in L^∞, can be recovered this way (up to a trivial rescaling).

Let us rapidly check the last statement of our main result. We must show that any Kruzhkov solution $U(t,x)$ with initial condition $U_0(x)$ valued in L^∞ can be recovered from a solution to (9). To do that, according to the first part of the theorem, it is enough to find an L^∞ function $Y_0(a,x)$ such that $\partial_a Y_0 \geq 0$ and

$$U_0(x) = \int_0^1 H(y - Y_0(a,x))da,$$

for some $y \in \mathbb{R}$, say $y = 1$. This is always possible, up to rescaling, by assuming:

$$r \leq U_0(x) \leq 1 - r$$

for some constant $r > 0$. Indeed, we set

$$u_0(y,x) = \max(0, \min(1, y\, U_0(x)))$$

so that $U_0(x) = u_0(1, x)$ and $\partial_y u_0 \geq 0$.
Then, for each fixed x, we solve $u_0(y, x) = a$ by $y = Y_0(a, x)$, setting:

$$Y_0(a, x) = \frac{a}{U_0(x)}, \quad \forall a \in [0, 1], \quad \forall x \in \mathbb{T}^d,$$

so that

$$u_0(y, x) = \int_0^1 H(y - Y_0(x, a))da.$$

(Notice that Y_0 is valued in $[0, r^{-1}]$.) Finally, according to the first part of the theorem, we get

$$U(t, x) = \int_0^1 H(1 - Y(t, x, a))da,$$

where Y is the solution to (9) with initial condition Y_0.

Remark

Notice that, for all $t \geq 0$, the level sets of Y and U are related by:

$$\{(a, x), \quad U(t, x) \geq a\} = \{(a, x), \quad Y(t, a, x) \leq 1\}.$$

Thus, the method of construction of Y_0 out of U_0 and the derivation of $U(t, x)$ from $Y(t, a, x)$ can be related to level-set methods in the spirit of [FSS, Gi1, OF, TGO]. This is why we may call 'level-set formulation' of scalar conservation law (1) the subdifferential equation given by (9)

Remark

The solutions $(t, x) \to u(t, y, x)$, parameterized by $y \in \mathbb{R}$, are automatically ordered in y. Indeed, $\partial_y u \geq 0$ immediately follows from representation formula (15). This is consistent with the order preserving property of Kruzhkov's theory (as explained in the first section).

2.3 A second result

The function $u(t, y, x)$, given by (15), can also be considered as a *single* Kruzhkov solution of a scalar conservation law in the enlarged $(1+d)$ dimensional space $\mathbb{R} \times \mathbb{T}^d$, namely

$$\partial_t u + \partial_y (Q_0(u)) + \nabla_x \cdot (Q(u)) = 0, \tag{16}$$

with $(y,x) \in \mathbb{R} \times \mathbb{T}^d$, provided:

1) Q_0 is zero,
2) the initial condition $u_0(y,x)$ is valued in $[0,1]$ and $\partial_y u_0 \geq 0$.
Furthermore, it turns out that, if we add the left-hand side $q_0(a) = Q_0'(a)$ to (9), so that we get (10):

$$q_0(a) \ \in \ \ \partial_t Y + q(a) \cdot \nabla_x Y + \partial K(Y),$$

and solve for Y, then the corresponding u given by (15) is a Kruzhkov solution to (16).
As a matter of fact, our proof will be done in this larger framework. We assume that q_0, q and Y_0 are given in L^∞, for simplicity. Without loss of generality, up to easy rescalings, we may assume that both q_0 and Y_0 are nonnegative, which simplifies some notations.

Theorem 2.4 *Assume that q_0 and q are given in L^∞, with $q_0 \geq 0$. Let $Y = Y(t,a,x)$ be a solution to the subdifferential equation (10), with initial condition $Y_0 \in L^\infty$, $Y_0 \geq 0$ and $\partial_a Y_0 \geq 0$. Then,*

$$u(t,y,x) = \int_0^1 H(y - Y(t,a,x))da, \tag{17}$$

is the unique Kruzhkov solution to (16) with initial condition:

$$u_0(y,x) = \int_0^1 H(y - Y_0(a,x))da. \tag{18}$$

In addition, Y is nonnegative and can be recovered from u as:

$$Y(t,a,x) = \int_0^\infty H(u(t,y,x) - a)dy. \tag{19}$$

Before proving the theorem, let us observe that the recovery of Y from u through (19) is just a consequence of the following elementary lemma which generalizes (in a standard way) the inversion of a strictly increasing function of one real variable:

Lemma 2.5 *Let: $a \in [0,1] \to Z(a) \in \mathbb{R}_+$ with $Z' \geq 0$. We define the generalized inverse of Z:*

$$v(y) = \int_0^1 H(y - Z(a))da, \quad \forall y \in \mathbb{R}.$$

Then $v' \geq 0$, $H(y - Z(a)) = H(v(y) - a)$ holds true a.e. in $(a, y) \in [0,1] \times \mathbb{R}$ and:

$$Z(a) = \int_0^\infty H(a - v(y))dy.$$

In addtion, for a pair (Z, v), $(\tilde{Z}, \tilde{v})$ of such functions, we have the co-area formula:

$$\int_0^1 |Z(a) - \tilde{Z}(a)|da = \int_0^1 \int_0^\infty |H(y - Z(a)) - H(y - \tilde{Z}(a))|dyda \qquad (20)$$

$$= \int_0^1 \int_0^\infty |H(v(y) - a) - H(\tilde{v}(y) - a)|dyda = \int_0^\infty |v(y) - \tilde{v}(y)|dy.$$

To recover (19), we notice first that $\partial_a Y \geq 0$ follows from the very definition 2.1 of a solution to (10). Next, $Y \geq 0$ follows from (12) and the assumptions $q_0 \geq 0$, $Y_0 \geq 0$. Then, we apply lemma 2.5, for each fixed $x \in \mathbb{T}^d$ and $t \geq 0$, by setting $Z(a) = Y(t, a, x)$ and $u(t, y, x) = v(y)$.

Remark

The function $f(t, a, y, x) = H(y - Y(t, a, x)) = H(u(t, y, x) - a)$ valued in $\{0, 1\}$ is nothing but the solution of the Lions-Perthame-Tadmor [LPT] 'kinetic formulation' of (16), which satisfies:

$$\partial_t f + q_0(a)\partial_y f + q(a) \cdot \nabla_x f = \partial_a \mu,$$

for some nonnegative measure $\mu(t, a, y, x)$.

Remark

As already mentioned, the solutions of (10) enjoys the L^p stability property with respect to initial conditions (14), not only for $p = 2$ but also for all $p \geq 1$. The case $p = 1$ is of particular interest. Let us consider two solutions Y and $\tilde{Y}$ of (10) and the corresponding Kruzhkov solutions u and $\tilde{u}$ given by Theorem 2.4. Using the co-area formula (20), we find, for all $t \geq 0$,

$$\int_{\mathbb{R}} \int_{\mathbb{T}^d} |u(t, y, x) - \tilde{u}(t, y, x)|dxdy =$$

$$= \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{T}^d} |H(u(t, y, x) - a) - H(\tilde{u}(t, y, x) - a)|dadxdy$$

$$= \int_0^1 \int_{\mathbb{R}} \int_{\mathbb{T}^d} |H(y - Y(t,a,x)) - H(y - \tilde{Y}(t,a,x))| dadxdy$$

$$= \int_0^1 \int_{\mathbb{T}^d} |Y(t,a,x) - \tilde{Y}(t,a,x)| dxda \leq \int_0^1 \int_{\mathbb{T}^d} |Y_0(a,x) - \tilde{Y}_0(a,x)| dxda$$

$$= \int_{\mathbb{R}} \int_{\mathbb{T}^d} |u_0(y,x) - \tilde{u}_0(y,x)| dxdy.$$

Thus, Kruzhkov's L^1 stability property is nothing but a *very* incomplete output of the much stronger L^p stability property provided by equation (10) for all $p \geq 1$.

Remark

As a matter of fact, in Theorem 2.4, it is possible to translate the L^p stability of the level set function Y in terms of the Kruzhkov solution u by using Monge-Kantorovich (MK) distances. Let us first recall that for two probability measures μ and ν compactly supported on $\mathbb{R}^D$, their p MK distance can be defined (see [Vi] for instance), for $p \geq 1$, by:

$$\delta_p^p(\mu, \nu) = \sup \int \phi(x) d\mu(x) + \int \psi(y) d\nu(y),$$

where the supremum is taken over all pair of continuous functions ϕ and ψ such that:

$$\phi(x) + \psi(y) \leq |x - y|^p, \quad \forall x, y \in \mathbb{R}^D.$$

In dimension $D = 1$, this definition reduces to:

$$\delta_p(\mu, \nu) = ||Y - Z||_{L^p},$$

where Y and Z are respectively the generalized inverse (in the sense of Lemma 2.5) of u and v defined on $\mathbb{R}$ by:

$$u(y) = \mu([-\infty, y]), \quad v(y) = \nu([-\infty, y]), \quad \forall y \in \mathbb{R}.$$

Next, observe that, for each $x \in \mathbb{T}^d$, the y derivative of the Kruzhkov solution $u(t,y,x)$, as described in Theorem 2.4, can be seen as a probability measure compactly supported on $\mathbb{R}$. (Indeed, $\partial_y u \geq 0$, $u = 0$ near $y = -\infty$ and $u = 1$ near $y = +\infty$.) Then, the L^p stability property simply reads:

$$\int_{\mathbb{T}^d} \delta_p^p(\partial_y u(t,\cdot,x), \partial_y \tilde{u}(t,\cdot,x)) dx \leq \int_{\mathbb{T}^d} \delta_p^p(\partial_y u_0(\cdot,x), \partial_y \tilde{u}_0(\cdot,x)) dx.$$

We refer to [BBL] and [CFL] for recent occurences of MK distances in the field of scalar conservation laws.

3 Proofs

Let us now prove Theorem 2.4 (which contains the first part of Theorem 2.3 as the special case $q_0 = 0$). The main idea is to provide, for both formulations (16) and (10), the same time-discrete approximation scheme, namely the 'transport-collapse' method [Br1, Br2, Br3, GM], and get the same limits.

3.1 A time-discrete approximation

We fix a time step $h > 0$ and approximate $Y(nh, a, x)$ by $Y_n(a, x)$, for each positive integer n. To get Y_n from Y_{n-1}, we perform two steps, making the following induction assumptions:

$$\partial_a Y_{n-1} \geq 0, \quad 0 \leq Y_{n-1} \leq \sup Y_0 + (n-1)h \sup q_0, \tag{21}$$

which are consistent with our assumptions on Y_0.

Predictor step

The first 'predictor' step amounts to solve the linear equation

$$\partial_t Y + q(a) \cdot \nabla_x Y = q_0(a) \tag{22}$$

for $nh - h < t < nh$, with Y_{n-1} as initial condition at $t = nh - h$. We exactly get at time $t = nh$ the predicted value:

$$Y_n^*(a, x) = Y_{n-1}(a, x - h\, q(a)) + h\, q_0(a). \tag{23}$$

Notice that, since q_0 is supposed to be nonnegative, the induction assumption (21) implies:

$$0 \leq Y_n^* \leq \sup Y_0 + nh \sup q_0. \tag{24}$$

However, although $\partial_a Y_{n-1}$ is nonnegative, the same may not be true for $\partial_a Y_n^* \geq 0$. This is why, we need a correction step.

Rearrangement step

In the second step, we 'rearrange' Y^* in increasing order with respect to $a \in [0, 1]$, for each fixed x, and get the corrected function Y_n. Let us recall some elementary facts about rearrangements:

Lemma 3.1 *Let: $a \in [0,1] \to X(a) \in \mathbb{R}_+$ an L^∞ function. Then, there is unique L^∞ function $Y : [0,1] \to \mathbb{R}_+$, such that $Y' \geq 0$ and:*

$$\int_0^1 H(y - Y(a))da = \int_0^1 H(y - X(a))da, \quad \forall y \in \mathbb{R}.$$

We say that Y is the rearrangement of X. In addition, for all $Z \in L^\infty$ such that $Z' \geq 0$, the following rearrangement inequality:

$$\int |Y(a) - Z(a)|^p da \leq \int |X(a) - Z(a)|^p da. \tag{25}$$

holds true for all $p \geq 1$.

So, we define $Y_n(a,x)$ to be, for each fixed x, the rearrangement of $Y_n^*(a,x)$ in $a \in [0,1]$:

$$\partial_a Y_n \geq 0, \quad \int_0^1 H(y - Y_n(a,x))da = \int_0^1 H(y - Y_n^*(a,x))da, \quad \forall y \in \mathbb{R}. \tag{26}$$

Equivalently, we may define the auxiliary function:

$$u_n(y,x) = \int_0^1 H(y - Y_n^*(a,x))da, \quad \forall y \in \mathbb{R}, \tag{27}$$

i.e.

$$u_n(y,x) = \int_0^1 H(y - h\ q_0(a) - Y_{n-1}(a, x - h\ q(a)))da, \tag{28}$$

and set:

$$Y_n(a,x) = \int_0^\infty H(a - u_n(y,x))dy. \tag{29}$$

At this point, Y_n is entirely determined by Y_{n-1} through formulae (23), (26), or, equivalently, through formulae (28), (29). Notice that, from the very definition (26) of the rearrangement step, u_n, defined by (27), can be equivalently written:

$$u_n(y,x) = \int_0^1 H(y - Y_n(a,x))da. \tag{30}$$

Also notice that, for all function $Z(a,x)$ such that $\partial_a Z \geq 0$, and all $p \geq 1$:

$$\int |Y_n(a,x) - Z(a,x)|^p dadx \leq \int |Y_n^*(a,x) - Z(a,x)|^p dadx \tag{31}$$

follows from the rearrangement inequality (25). Finlly, we see that $\partial_a Y_n \geq 0$ is automatically satisfied (this was the purpose of the rearrangement step) and

$$0 \leq Y_n \leq \sup Y_0 + nh \sup q_0.$$

follows form (24) (since the range of Y_n^* is preserved by the rearrangement step). So, the induction assumption (21) is enforced at step n and the scheme is well defined.

Remark

Observe that, for any fixed x, $u_n(y,x)$, as a function of y, is the (generalized) inverse of $Y_n(a,x)$, viewed as a function of a, in the sense of Lemma 2.5. Also notice that the level sets $\{(a,y);\ \ y \geq Y_n(a,x)\}$ and $\{(a,y);\ \ a \leq u_n(y,x)\}$ coincide.

The transport-collapse scheme revisited

The time-discrete scheme can be entirely recast in terms of u_n (defined by (30)). Indeed, introducing

$$ju_n(a,y,x) = H(u_n(y,x) - a), \tag{32}$$

we can rewrite (28), (29) in terms of u_n and ju_n only:

$$u_n(y,x) = \int_0^1 ju_{n-1}(y - h\ q_0(a), x - h\ q(a), a)da. \tag{33}$$

We observe that, formulae (32,33) exactly define the 'transport-collapse' (TC) approximation to (16), or, equivalently, its 'kinetic' approximation, according to [Br1, Br2, Br3, GM].

3.2 Convergence to the Kruzhkov solution

We are now going to prove that, on one hand, $Y_n(a,x)$ converges to $Y(t,a,x)$ as $nh \to t$, and, on the other hand, $u_n(y,x)$ converges to $u(t,y,x)$, where Y and u are respectively the unique solution to subdifferential equation (10) with initial condition $Y_0(a,x)$ and the unique Kruzhkov solution to (16) with initial condition

$$u_0(y,x) = \int_0^1 H(y - Y_0(a,x))da. \tag{34}$$

From the convergence analysis of the TC method [Br1, Br2, Br3, GM], we already know that, as $nh \to t$,

$$\int |u_n(y,x) - u(t,y,x)| dydx \to 0,$$

where u is the unique Kruzhkov solution with initial value u_0 given by (34). More precisely, if we extend the time discrete approximations $u_n(y,x)$ to all $t \in [0,T]$ by linear interpolation in time:

$$u^h(t,y,x) = u_{n+1}(y,x)\frac{t-nh}{h} + u_n(y,x)\frac{nh+h-t}{h}, \tag{35}$$

then $u^h - u$ converges to 0 in the space $C^0([0,T], L^1(\mathbb{R} \times \mathbb{T}^d))$ as $h \to 0$. Following (19), it is now natural to introduce the level-set function Y defined by (19) from the Kruzhkov solution:

$$Y(t,a,x) = \int_0^\infty H(a - u(t,y,x))dy.$$

(Notice that, at this point, we do not know that Y is a solution to the subdifferential formulation (10)!) Let us interpolate the Y_n by

$$Y^h(t,a,x) = Y_{n+1}(a,x)\frac{t-nh}{h} + Y_n(a,x)\frac{nh+h-t}{h}, \tag{36}$$

for all $t \in [nh, nh+h]$ and $n \geq 0$. By the co-area formula (20), we have

$$\int |Y(t,a,x) - Y_n(a,x)| dadx = \int |u(t,y,x) - u_n(y,x)| dydx.$$

Thus:

$$\sup_{t\in[0,T]} ||Y(t,\cdot) - Y^h(t,\cdot)||_{L^1} \leq \sup_{t\in[0,T]} ||u(t,\cdot) - u^h(t,\cdot)||_{L^1} \to 0,$$

and we conclude that the approximate solution Y^h must converge to Y in $C^0([0,T], L^1([0,1] \times \mathbb{T}^d))$ as $h \to 0$. Notice that, since the Y^h are uniformly bounded in L^∞, the convergence also holds true in $C^0([0,T], L^2([0,1] \times \mathbb{T}^d))$.

We finally have to prove that Y is the solution to the subdifferential formulation (10) with initial condition Y_0.

3.3 Consistency of the transport-collapse scheme

Let us check that the TC scheme is consistent with the subdifferential formulation (10) in its semi-integral formulation (11). For each smooth function $Z(t,a,x)$ with $\partial_a Z \geq 0$ and $p \geq 1$, we have

$$\int |Y_{n+1}(a,x) - Z(nh+h,a,x)|^p dadx$$

$$\leq \int |Y^*_{n+1}(a,x) - Z(nh+h,a,x)|^p dadx$$

(because of property (31) due to the rearrangement step (26))

$$= \int |Y_n(a, x - h\ q(a)) + h\ q_0(a) - Z(nh+h,a,x)|^p dadx$$

(by definition of the predictor step (23)

$$= \int |Y_n(a,x) + h\ q_0(a) - Z(nh+h,a,x+h\ q(a))|^p dadx$$

$$= \int |Y_n - Z(nh,\cdot)|^p dadx + h\ \Gamma + o(h)$$

where:

$$\Gamma = p \int (Y_n - Z(nh,\cdot))|Y_n - Z(nh,\cdot)|^{p-2}\{q_0 - \partial_t Z(nh,\cdot) - q \cdot \nabla_x Z(nh,\cdot)\} dadx$$

(by Taylor expanding Z about (nh,a,x)). Since the approximate solution provided by the TC scheme has a unique limit Y, as shown in the previous section, this limit must satisfy:

$$\frac{d}{dt}\int |Y-Z|^p dadx \leq p\ \int (Y-Z)|Y-Z|^{p-2}(q_0(a) - \partial_t Z - q(a)\cdot\nabla_x Z) dadx,$$

in the distributional sense in t. In particular, for $p = 2$, we exactly recover the semi-integral version (11) of (10). We conclude that the approximate solutions generated by the TCM scheme do converge to the solutions of (10) in the sense of Definition 2.1, which completes the proof of Theorem 2.4.

4 Viscous approximations

A natural regularization for subdifferential equation (10) amounts to substitute a barrier function for the convex cone K in $L^2([0,1] \times \mathbb{T}^d)$ of all functions Y such that $\partial_a Y \geq 0$. Typically, we introduce a convex function $\phi : \mathbb{R} \to]-\infty, +\infty]$ such that $\phi(\tau) = +\infty$ if $\tau < 0$, we define, for all $Y \in K$,

$$\Phi(Y) = \int \phi(\partial_a Y) da dx, \tag{37}$$

and set $\Phi(Y) = +\infty$ if Y does not belong to K. Typical examples are:

$$\phi(\tau) = -\log(\tau), \quad \phi(\tau) = \tau \log(\tau), \quad \phi(\tau) = \frac{1}{\tau}, \quad \forall \tau > 0.$$

Then, we considered the perturbed subdifferential equation

$$0 \in \partial_t Y + q(a) \cdot \nabla_x Y - q_0(a) + \varepsilon \partial \Phi(Y), \tag{38}$$

for $\varepsilon > 0$. The general theory of maximal monotone operators guarantees the convergence of the corresponding solutions to those of (10) as $\varepsilon \to 0$. It is not difficult (at least formally) to identify the corresponding perturbation to scalar conservation (16). Indeed, assuming $\phi(\tau)$ to be smooth for $\tau > 0$, we get, for each smooth function Y such that $\partial_a Y > 0$:

$$\partial \Phi(Y) = -\partial_a(\phi'(\partial_a Y)).$$

Thus, any smooth solution Y to (38), satisfying $\partial_a Y > 0$, solves the following parabolic equation:

$$\partial_t Y + q(a) \cdot \nabla_x Y - q_0(a) = \varepsilon \partial_a(\phi'(\partial_a Y)). \tag{39}$$

Introducing, the function $u(t, y, x)$ implicitely defined by

$$u(t, Y(t, a, x), x) = a,$$

we get (by differentiating with respect to a, t and x):

$$(\partial_y u)(t, Y(t, a, x), x) \partial_a Y(t, a, x) = 1,$$

$$(\partial_t u)(t, Y, x) + (\partial_y u)(t, Y, x) \partial_t Y = 0,$$

$$(\nabla_x u)(t, Y, x) + (\partial_y u)(t, Y, x) \nabla_x Y = 0.$$

Multiplying (39) by $(\partial_y u)(t, Y(t,a,x), x)$, we get:

$$-\partial_t u - q(u) \cdot \nabla_x u - q_0(u)\partial_y u = \varepsilon \partial_y (\phi'(\frac{1}{\partial_y u})). \tag{40}$$

In particular, in the case $\phi(\tau) = -\log \tau$, we recognize a linear viscous approximation to scalar conservation law (16):

$$\partial_t u + q(u) \cdot \nabla_x u + q_0(u)\partial_y u = \varepsilon \partial^2_{yy} u, \tag{41}$$

with viscosity only in the y variable.

Remark

Of course, these statements are not rigourous since the parabolic equations we have considered are degenerate and their solutions may not be smooth.

Remark

In the case of our main result, Theorem 2.3, we have $q_0 = 0$ and the variable y is just a dummy variable in (1). Thus, the corresponding regularized version

$$-\partial_t u - q(u) \cdot \nabla_x u = \varepsilon \partial_y (\phi'(\frac{1}{\partial_y u})). \tag{42}$$

includes viscous effects not on the space variable x but rather on the 'parameter' $y \in \mathbb{R}$. This unusual type of regularization has already been used and analyzed in the level-set framework developped by Giga for Hamilton-Jacobi equations [Gi2], and by Giga, Giga, Osher, Tsai for scalar conservation laws [GG, TGO].

5 Related equations

A similar method can be applied to some special systems of conservation laws. A typical example (which was crucial for our understanding) is the 'Born-Infeld-Chaplygin' system considered in [Br4], and the related concept of 'order-preserving strings'. This system reads:

$$\partial_t(hv) + \partial_y(hv^2 - hb^2) - \partial_x(hb) = 0, \tag{43}$$

$$\partial_t h + \partial_y(hv) = 0, \quad \partial_t(hb) - \partial_x(hv) = 0,$$

where h, b, v are real valued functions of time t and two space variables x, y. In [Br4], this system is related to the following subdifferential system:

$$0 \in \partial_t Y - \partial_x W + \partial K(Y), \quad \partial_t W = \partial_x Y, \tag{44}$$

where (Y, W) are real valued functions of (t, a, x) and K is the convex cone of all Y such that $\partial_a Y \geq 0$. The (formal) correspondence between (43) and (44) is obtained by setting:

$$h(t, x, Y(t, x, a))\partial_a Y(t, x, a) = 1,$$

$$v(t, x, Y(t, x, a)) = \partial_t Y(t, x, a), \quad b(t, x, Y(t, x, a)) = \partial_x Y(t, x, a).$$

Unfortunately, this system is very special (its smooth solutions are easily integrable). In our opinion, it is very unlikely that L^2 formulations can be found for general hyperbolic conservation laws as easily as in the multidimensional scalar case.

6 Appendix: proof of Proposition 2.2

In the case when q_0 and Y_0 belong to L^∞ and are nonnegative, we already know, from the convergence of the TC scheme, that there is a solution Y to (10), with initial value Y_0, in the sense of definition 2.1. From (21), we also get for such solutions, when $q_0 \geq 0$ and $Y_0 \geq 0$,

$$0 \leq Y(t, \cdot) \leq \sup Y_0 + t \sup q_0, \quad \forall t \geq 0.$$

By elementary rescalings, we can remove the assumptions that both Y_0 and q_0 are nonnegative and get estimate (12).

Let us now examine some additional properties of the solutions to (10) obtained from the TC approximations. First, we observe that, in the TC scheme,

1) the predictor step (a translation in the x variable by $h\, q(a)$ plus an addition of $h\, q_0(a)$) is isometric in all L^p spaces,

2) the corrector step (an increasing rearrangement in the a variable) is non-expansive in all L^p.

Thus the scheme is non-expansive in all $L^p([0, 1] \times \mathbb{T}^d)$. More precisely, for two different initial conditions Y_0 and $\tilde{Y}_0$, and two different data q_0 and $\tilde{q}_0$, all in L^∞, we get for the corresponding approximate solutions Y_n and $\tilde{Y}_n$:

$$||Y_n - \tilde{Y}_n||_{L^p} \leq ||Y_{n-1} - \tilde{Y}_{n-1}||_{L^p} + h||q_0 - \tilde{q}_0||_{L^p} \;. \tag{45}$$

This shows that (14) holds true for all solutions of (10) generated by the TC scheme.
Since the scheme is also invariant under translations in the x variable, we get the following a priori estimate:

$$||\nabla_x Y_n||_{L^p} \leq ||\nabla_x Y_0||_{L^p}. \tag{46}$$

Finally, let us compare two solutions of the scheme Y_n and $\tilde{Y}_n = Y_{n+1}$ obtained with initial condition $\tilde{Y}_0 = Y_1$. Using (45), we deduce:

$$\int |Y_{n+1}(a,x) - Y_n(a,x)|^p dadx \leq \int |Y_1(a,x) - Y_0(a,x)|^p dadx$$

$$\leq \int |Y_1^*(a,x) - Y_0(a,x)|^p dadx = \int |Y_0(a, x-h\; q(a)) + h\; q_0(a) - Y_0(a,x)|^p dadx.$$

So we get a second a priori estimate:

$$||Y_{n+1} - Y_n||_{L^p} \leq (||q_0||_{L^p} + ||q||_{L^\infty} ||\nabla_x Y_0||_{L^p})h. \tag{47}$$

Thus the solutions Y to (10) obtained from the TC scheme satisfy the a priori bounds:

$$||\nabla_x Y(t,\cdot)||_{L^p} \leq ||\nabla_x Y_0||_{L^p}, \tag{48}$$

$$||\partial_t Y(t,\cdot)||_{L^p} \leq ||q_0||_{L^p} + ||q||_{L^\infty} ||\nabla_x Y_0||_{L^p}. \tag{49}$$

Notice that, at this level, we still do not know if solutions, in the sense of Definition 2.1 exist when $Y_0 \in K$ and $q_0 \in L^2([0,1])$ are not in L^∞ and we know nothing about their uniqueness. This can be easily addressed by standard functional analysis arguments.

Existence for general data

Let $Y_0 \in K$ and $q_0 \in L^2([0,1])$. We can find two Cauchy sequences in L^2, labelled by $k \in \mathbb{N}$, namely $Y_0^k \in K$ and $q_0^k \in L^2([0,1])$, made of smooth functions, with limits Y_0 and q_0 respectively. Let us denote by Y^k the corresponding solutions, generated by the TC scheme. Because of their L^2 stability, they satisfy:

$$\sup_{t\in[0,T]} ||Y^k(t,\cdot) - Y^{k'}(t,\cdot)||_{L^2} \leq ||Y_0^k - Y_0^{k'}||_{L^p} + T||q_0^k - q_0^{k'}||_{L^2}.$$

So, Y^k is a Cauchy sequence in $C^0([0,T], L^2)$ of solutions of (10) in the sense of Definition 2.1, with a definite limit Y. Definition 2.1 is clearly stable under this convergence process. So, we conclude that Y satisfies the requirements of Definition 2.1 and is a solution with initial condition Y_0 and left-hand side q_0. Notice that, through our approximation process, we keep the a priori estimates (48),(49), for general data $q_0 \in L^2([0,1])$.

Uniqueness

Let us consider a solution Y to (10), with initial condition $Y_0 \in K$ and left-hand side $q_0 \in L^2([0,1])$, in the sense of Definition 2.1. By definition $Y(t,\cdot) \in K$ depends continuously of $t \in [0,T]$ in L^2. From definition (11), using $Z = 0$ as a test function, we see that:

$$\frac{d}{dt}||Y(t,\cdot)||_{L^2}^2 \leq 2\int Y(t,a,x)q_0(a)\,dadx \leq ||Y(t,\cdot)||_{L^2}^2 + ||q||_{L^2}^2,$$

which implies that the L^2 norm $Y(t,\cdot)$ stays uniformly bounded on any finite interval $[0,T]$. Thus, $T > 0$ being fixed, we can mollify Y and get, for each $\epsilon \in]0,1]$ a smooth function Y_ϵ, valued in K, so that:

$$\sup_{t\in[0,T]} ||Y(t,\cdot) - Y_\epsilon(t,\cdot)||_{L^2} \leq \epsilon. \tag{50}$$

Let us now consider an initial condition Z_0 such that $\nabla_x Z_0$ belongs to L^2. We know that there exist a solution Z to (10), still in the sense of Definition 2.1, obtained by TC approximation, for which both $\partial_t Z(t,\cdot)$ and $\nabla_x Z(t,\cdot)$ stay uniformly bounded in L^2 for all $t \in [0,T]$. This function Z has enough regularity to be used as a test function in (11) when expressing that Y is a solution in the sense of Definition 2.1. So, for each smooth nonnegative function $\theta(t)$, compactly supported in $]0,T[$, we get from (11):

$$\int\{\theta'(t)|Y - Z|^2 + 2\theta(t)(Y - Z)(q_0(a) - \partial_t Z - q(a)\cdot\nabla_x Z)\}dadxdt \geq 0.$$

Substituting Y_ϵ for Y, we have, thanks to estimate (50),

$$\int\{\theta'(t)|Y_\epsilon - Z|^2 + 2\theta(t)(Y_\epsilon - Z)(q_0(a) - \partial_t Z - q(a)\cdot\nabla_x Z)\}dadxdt \geq -C\epsilon,$$

where C is a constant depending on θ, Z, q_0 and q only. Since Z is also a solution, using Y_ϵ as a test function, we get from formulation (11):

$$\int\{\theta'(t)|Z - Y_\epsilon|^2 + 2\theta(t)(Z - Y_\epsilon)(q_0(a) - \partial_t Y_\epsilon - q(a)\cdot\nabla_x Y_\epsilon)\}dadxdt \geq 0.$$

Adding up these two inequalities, we deduce:

$$\int \{2\theta'(t)|Y_\epsilon - Z|^2 + 2\theta(t)(Y_\epsilon - Z)(\partial_t(Y_\epsilon - Z) + q(a)\cdot\nabla_x(Y_\epsilon - Z))\} da dx dt \geq -C\epsilon.$$

Integrating by part in $t \in [0,T]$ and $x \in \mathbb{T}^d$, we simply get:

$$\int \theta'(t)|Y_\epsilon - Z|^2 da dx dt \geq -C\epsilon.$$

Letting $\epsilon \to 0$, we deduce:

$$\frac{d}{dt}\int |Y - Z|^2 da dx \leq 0.$$

We conclude, at this point, that:

$$||Y(t,\cdot) - Z(t,\cdot)||_{L^2} \leq ||Y_0 - Z_0||_{L^2}, \quad \forall t \in [0,T]$$

This immediately implies the uniqueness of Y. Indeed, any other solution $\tilde{Y}$ with initial condition Y_0 must also satisfy:

$$||\tilde{Y}(t,\cdot) - Z(t,\cdot)||_{L^2} \leq ||Y_0 - Z_0||_{L^2}.$$

Thus, by the triangle inequality:

$$||\tilde{Y}(t,\cdot) - Y(t,\cdot)||_{L^2} \leq 2||Y_0 - Z_0||_{L^2}.$$

Since $Z_0 \in K$ is any function such that $\nabla_x Z_0$ belongs to L^2, we can make $||Y_0 - Z_0||_{L^2}$ arbitrarily small and conclude that $\tilde{Y} = Y$, which completes the proof of uniqueness.

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