PERIODIC CONSERVATIVE SOLUTIONS OF THE CAMASSA–HOLM EQUATION

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Abstract. We show that the periodic Camassa–Holm equation $u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0$ possesses a global continuous semigroup of weak conservative solutions for initial data $u|_{t=0}$ in H^1_{per}. The result is obtained by introducing a coordinate transformation into Lagrangian coordinates. To characterize conservative solutions it is necessary to include the energy density given by the positive Radon measure μ with $\mu_{\mathrm{ac}} = (u^2 + u_x^2)\,dx$. The total energy is preserved by the solution.

1. Introduction

The Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0 \tag{1.1}$$

was first studied extensively in 1993 [6, 7]. It can be derived as a model for shallow water waves. Furthermore, the equation can be derived in the context of geodesic flows of a certain invariant metric on the Bott–Virasoro group [25, 3].

The equation possesses many fascinating properties that has made it a popular equation. In particular, it is bi-Hamiltonian, completely integrable, has infinitely many conserved quantities, and has solitary waves, called (multi)peakons, that interact like KdV-solitons. Another interesting aspect is that it enjoys wave-breaking in finite time in the sense that the spatial derivative u_x of the solution blows up while the solution u itself as well as its energy, the H^1-norm, both remain finite. Continuation of the solution beyond wave breaking has been a challenge. Several entropy conditions that single out the proper continuation have been analyzed.

The Cauchy problem for (1.1) has been studied in two different settings; on the full line $\mathbb{R}$ and the periodic case on $[0,1]$. We here address the latter case, and for reasons of space we restrict the general references mainly to the periodic case. Constantin and Moulinet have proved [14, p. 60] that for initial data $u|_{t=0} = \bar{u} \in H^1([0,1])$ such that $\bar{m} = \bar{u} - \bar{u}_{xx}$ is a non-negative Radon-measure, the equation (1.1) possesses a unique solution $u \in C^1((0,\infty), L^2([0,1])) \cap C((0,\infty), H^1([0,1]))$. Furthermore, the quantities $\int_{[0,1]} u\,dx$, $\int_{[0,1]}(u^2 + u_x^2)\,dx$, and $\int_{[0,1]}(u^3 + uu_x^2)\,dx$ are all conserved quantities. Regarding blow-up, Constantin and Escher [13] have derived the following result. Let $u_0 \in H^3([0,1])$. Then there exists a maximal $T > 0$ such that (1.1) has a unique solution $u \in C([0,T), H^3([0,1])) \cap C^1([0,T), H^2([0,1]))$. If $\bar{u}$ is non-zero and $\int_{[0,1]}(\bar{u}^3 + \bar{u}\bar{u}_x^2)\,dx = 0$, then T is finite. See also [10, 26].

The question about how to continue the solution beyond wave-breaking can be nicely studied in the case of multipeakons (we here give the description on the full

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line). Multipeakons are given by (see, e.g., [21] and references therein)

$$u(t,x)=\sum_{i=1}^{n}p_i(t)e^{-|x-q_i(t)|}, \tag{1.2}$$

where the $(p_i(t),q_i(t))$ satisfy the explicit system of ordinary differential equations

$$\dot q_i=\sum_{j=1}^{n}p_je^{-|q_i-q_j|},\quad \dot p_i=\sum_{j=1}^{n}p_ip_j\operatorname{sgn}(q_i-q_j)e^{-|q_i-q_j|}.$$

Observe that the solution (1.2) is not smooth even with continuous functions $(p_i(t),q_i(t))$; one possible way to interpret (1.2) as a weak solution of (1.1) is to rewrite the equation (1.1) as

$$u_t+\big(\frac{1}{2}u^2+(1-\partial_x^2)^{-1}(u^2+\frac{1}{2}u_x^2)\big)_x=0.$$

Wave breaking may appear when at least two of the q_i's coincide. If all the $p_i(0)$ have the same sign, the peakons move in the same direction, the solution experiences no wave breaking, and one has a global solution. Higher peakons move faster than the smaller ones, and when a higher peakon overtakes a smaller, there is an exchange of mass, but no wave breaking takes place. Furthermore, the $q_i(t)$ remain distinct. However, if some of $p_i(0)$ have opposite sign, wave breaking may incur. For simplicity, consider the case with $n=2$ and one peakon $p_1(0)>0$ (moving to the right) and one antipeakon $p_2(0)<0$ (moving to the left). In the symmetric case ($p_1(0)=-p_2(0)$ and $q_1(0)=-q_2(0)<0$) the solution will vanish pointwise at the collision time t^* when $q_1(t^*)=q_2(t^*)$, that is, $u(t^*,x)=0$ for all $x\in\mathbb{R}$. Clearly, at least two scenarios are possible; one is to let $u(t,x)$ vanish identically for $t>t^*$, and the other possibility is to let the peakon and antipeakon "pass through" each other in a way that is consistent with the Camassa–Holm equation. In the first case the energy $\int(u^2+u_x^2)\,dx$ decreases to zero at t^*, while in the second case, the energy remains constant except at t^*. Clearly, the well-posedness of the equation is a delicate matter in this case. The first solution could be denoted a dissipative solution, while the second one could be called conservative. Other solutions are also possible. Global dissipative solutions of a more general class of equations were recently derived by Coclite, Holden, and Karlsen [8, 9]. In their approach the solution was obtained by first regularizing the equation by adding a small diffusion term ϵu_{xx} to the equation, and subsequently analyzing the vanishing viscosity limit $\epsilon\to 0$.

Recently, a rather different approach to the Camassa–Holm equation was taken by Bressan and Constantin [4]. The method has been further studied and extended to the hyperelastic-rod wave equation, see [22, 24]. As a first step one reformulates the Camassa–Holm equation (1.1) as the following system

$$u_t+uu_x+P_x=0, \tag{1.3a}$$

$$P-P_{xx}=u^2+\frac{1}{2}u_x^2. \tag{1.3b}$$

The equations are further reformulated as a semilinear system of ordinary differential equations taking values in a Banach space. This formulation allows one to continue the solution beyond collision time, giving a global conservative solution where the energy is conserved for almost all times. Thus in the context of peakon-antipeakon collisions one considers the solution where the peakons and antipeakons "pass through" each other. Local existence of the semilinear system is obtained by a contraction argument. Furthermore, the reformulation allows for a global solution where all singularities disappear. Going back to the original function u, one obtains a global solution of the Camassa–Holm equation. The well-posedness, i.e., the uniqueness and stability of the solution, is resolved as follows. In addition to the solution u, one includes a family of non-negative Radon measures μ_t with density $u_x^2\,dx$ with respect to the Lebesgue measure. The pair (u,μ_t) constitutes a continuous semigroup, in particular, one has uniqueness and stability. See also [5, 17].

In this paper we follow [22, 24] rather than [4], and reformulate the equation using a transformation that corresponds to the transformation between Eulerian and Lagrangian coordinates. Let $u = u(t,x)$ denote the solution, and $y(t,\xi)$ the corresponding characteristics, thus $y_t(t,\xi) = u(t,y(t,\xi))$. Our new variables are $y(t,\xi)$,

$$U(t,\xi) = u(t,y(t,\xi)), \quad H(t,\xi) = \int_{-\infty}^{y(t,\xi)} (u^2+u_x^2)\,dx \tag{1.4}$$

where U corresponds to the Lagrangian velocity while H could be interpreted as the Lagrangian cumulative energy distribution. In the periodic case one defines

$$\begin{aligned} Q &= \frac{1}{2(e-1)}\int_0^1 \sinh(y(\xi)-y(\eta))(U^2y_\xi+H_\xi)(\eta)\,d\eta \\ &\quad -\frac14\int_0^1 \operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)(U^2y_\xi+H_\xi)\,d\eta, \end{aligned} \tag{1.5}$$

$$\begin{aligned} P &= \frac{1}{2(e-1)}\int_0^1 \cosh(y(\xi)-y(\eta))(U^2y_\xi+H_\xi)(\eta)\,d\eta \\ &\quad +\frac14\int_0^1 \exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)(U^2y_\xi+H_\xi)\,d\eta. \end{aligned} \tag{1.6}$$

Then one can show that

$$\begin{cases} y_t = U, \\ U_t = -Q, \\ H_t = U^3 - 2PU, \end{cases} \tag{1.7}$$

is equivalent to the Camassa–Holm equation. Global existence of solutions of (1.7) is obtained starting from a contraction argument, see Theorem 2.7. The uniqueness issue is resolved by considering the set $\mathcal{D}$ (see Definition 3.1) which consists of pairs (u,μ) such that $(u,\mu)\in\mathcal{D}$ if $u\in H^1_{\rm per}$ and μ is a positive Radon measure with period one, and whose absolutely continuous part satisfies $\mu_{\rm ac} = (u^2+u_x^2)\,dx$. With three Lagrangian variables (y,U,H) versus two Eulerian variables (u,μ), it is clear that there can be no bijection between the two coordinates systems. However, we define a group of transformations which acts on the Lagrangian variables and lets the system of equations (1.7) invariant. We are able to establish a bijection between the space of Eulerian variables and the space of Lagrangian variables when we identify variables that are invariant under the action of the group. This bijection allows us to transform the results obtained in the Lagrangian framework (in which the equation is well-posed) into the Eulerian framework (in which the situation is much more subtle). In particular, and this constitutes the main result of this paper, we obtain a metric $d_{\mathcal{D}}$ on $\mathcal{D}$ and a continuous semi-group of solutions on $(\mathcal{D},d_{\mathcal{D}})$. The distance $d_{\mathcal{D}}$ gives $\mathcal{D}$ the structure of a complete metric space. This metric is compared with some more standard topologies, and we obtain that convergence in $H^1_{\rm per}$ implies convergence in $(\mathcal{D},d_{\mathcal{D}})$ which itself implies convergence in L^∞, see Propositions 5.1 and 5.2. The properties of the spaces as well as the various mappings between them are described in great detail, see Section 3. Our main result, Theorem 4.2, states that there exists a continuous semigroup $T:\mathcal{D}\times\mathbb{R}\to\mathcal{D}$ such that, for any $(\bar u,\bar\mu)\in\mathcal{D}$, if we denote $(u(t),\mu(t)) = T_t(\bar u,\bar\mu)$, then $u(t)$ is a weak solution of the Camassa–Holm equation. The topology on $\mathcal{D}$ is of course given by the metric $d_{\mathcal{D}}$

and, by continuity of the semigroup, we mean that if $(\bar{u}_n, \bar{\mu}_n) \to (\bar{u}, \bar{\mu})$ in $\mathcal{D}$, then $(u_n(t), \mu_n(t) \to (u(t), \mu(t))$ in $\mathcal{D}$, i.e., we use the same topology on the set of initial data as on the set of solutions, which shows that the complete metric $d_{\mathcal{D}}$ is the appropriate metric for conservative solutions of the Camassa–Holm equation. The associated measure $\mu(t)$ has constant total mass, i.e., $\mu(t)([0,1)) = \mu(0)([0,1))$ for all t, which corresponds to the total energy of the system. This is the reason why our solutions are called conservative.

Many of the ideas used in this article originate from [21] where the case of the full line is treated and some of the proofs are indeed adaptations to the periodic case. There is, however, one significant and important difference. It concerns the group of transformations here denoted $\tilde{G}$ acting on the Lagrangian variables. By introducing Lagrangian variables, one introduces a degree of arbitrariness which is captured by the group of transformation acting on the new variables and which is removed when one takes the quotient space. The determination of the correct group is crucial as it enables us to return to the Eulerian coordinates via the quotient space and to construct the continuous semigroup of solutions in Eulerian coordinates. On the full line, this group consists of the group of diffeomorphism with some regularity condition denoted G, which is also a natural choice taking into account the geometric interpretation of the equation, see [3, 15]. In the periodic case, the same group G is needed but we also have to take into account that we introduce an additional degree of freedom by considering the cumulative energy. Indeed, the energy is like a potential and defined up to a constant. In the case of the full line, we normalize this constant to zero at $-\infty$. We cannot do that in the periodic case and instead we expand the group G to $\tilde{G} = G \times \mathbb{R}$. The action of the group $(\mathbb{R}, +)$ corresponds to the degree of freedom resulting from the fact that the energy is defined up to a constant.

The method described here can be studied in detail for multipeakons, see [21] for details on the full line. The results can be extended, as in [24], to show global existence of conservative periodic solutions for the generalized hyperelastic-rod wave equation

$$\begin{cases} u_t + f(u)_x + P_x = 0, \\ P - P_{xx} = g(u) + \dfrac{1}{2} f''(u) u_x^2, \end{cases} \tag{1.8}$$

where $f, g \in C^\infty(\mathbb{R})$ and f is strictly convex. Observe that if $g(u) = u^2$ and $f(u) = \frac{u^2}{2}$, then (1.8) is the classical Camassa–Holm equation (1.1).

Furthermore, the methods presented in this paper can be used to derive numerical methods that converge to conservative solutions rather than dissipative solutions. This contrasts finite difference methods that normally converge to dissipative solutions, see [23] for a proof of convergence of a upwind scheme in the periodic case, and [19] for the related Hunter–Saxton equation. See also [20].

2. Global solutions in Lagrangian coordinates

2.1. **Equivalent system.** We consider periodic functions. For the sake of simplicity, we will only consider functions of unit period, that is, $g(\xi + 1) = g(\xi)$. The results are of course valid for any period after making the necessary adjustments. We introduce the space V_1 defined as

$$V_1 = \{f \in H^1_{\mathrm{loc}}(\mathbb{R}) \mid f(\xi + 1) = f(\xi) + 1 \text{ for all } \xi \in \mathbb{R}\}.$$

Functions in V_1 map the unit interval into itself in the sense that if u is periodic with period 1, then $u \circ f$ is also periodic with period 1. We define the characteristics $y\colon \mathbb{R} \to V_1$, $t \mapsto y(t, \cdot)$ as the solutions of

$$y_t(t, \xi) = u(t, y(t, \xi)). \tag{2.1}$$

Assuming that u is smooth, it is not hard to check that (1.3) yields

$$(u^2+u_x^2)_t+(u(u^2+u_x^2))_x=(u^3-2Pu)_x. \tag{2.2}$$

We define the Lagrangian energy cumulative distribution as

$$H(t,\xi)=\int_{y(t,0)}^{y(t,\xi)}(u^2+u_x^2)(t,x)\,dx. \tag{2.3}$$

Using (2.2) and (2.1), we obtain

$$\frac{dH}{dt}=\left[(u^3-2Pu)\circ y\right]_0^{\xi}. \tag{2.4}$$

From (2.3), the periodicity of u and the fact that $y\in V_1$, we can check that, for all $\xi\in\mathbb{R}$,

$$H(t,\xi+1)-H(t,\xi)=H(t,1)-H(t,0).$$

Moreover, from (2.4), we can verify that $H(t,1)-H(t,0)$ is constant in time so that $H(t,1)-H(t,0)=H(0,1)-H(0,0)$. For all t, H belongs to the vector space V defined as follows

$$\begin{aligned}V=\{f\in H^1_{\mathrm{loc}}(\mathbb{R})\mid{}&\text{there exists }\alpha\in\mathbb{R}\\&\text{such that }f(\xi+1)=f(\xi)+\alpha,\text{ for all }\xi\in\mathbb{R}\}.\end{aligned}$$

We equip V with the norm $\|f\|_V=\|f\|_{H^1([0,1])}$. Later we will prove that V is a Banach space. To simplify the notation, we will denote $H^1([0,1])$ by H^1 and follow the same convention for the other norms we will consider.

We now derive formally a system equivalent to (1.3). From the definition of the characteristics, it follows that

$$U_t(t,\xi)=u_t(t,y)+y_t(t,\xi)u_x(t,y)=-P_x\circ y\,(t,\xi). \tag{2.5}$$

This last term can be expressed uniquely in term of U, y, and H. From (1.3b), we obtain the following explicit expression for P,

$$P(t,x)=\frac{1}{2}\int_{\mathbb{R}}e^{-|x-z|}(u^2(t,z)+\frac{1}{2}u_x^2(t,z))\,dz. \tag{2.6}$$

Thus we have

$$P_x\circ y\,(t,\xi)=-\frac{1}{2}\int_{\mathbb{R}}\operatorname{sgn}(y(t,\xi)-z)e^{-|y(t,\xi)-z|}(u^2(t,z)+\frac{1}{2}u_x^2(t,z))\,dz$$

and, after the change of variables $z=y(t,\eta)$,

$$\begin{aligned}P_x\circ y(t,\xi)=-\frac{1}{2}\int_{\mathbb{R}}\Big[&\operatorname{sgn}(y(t,\xi)-y(t,\eta))e^{-|y(t,\xi)-y(t,\eta)|}\\&\times\Big(u^2(t,y(t,\eta))+\frac{1}{2}u_x^2(t,y(t,\eta))\Big)\,y_\xi(t,\eta)\Big]\,d\eta.\end{aligned} \tag{2.7}$$

We have

$$H_\xi=(u^2+u_x^2)\circ y\,y_\xi. \tag{2.8}$$

Therefore, (2.7) can be rewritten as

$$P_x\circ y\,(\xi)=-\frac{1}{4}\int_{\mathbb{R}}\operatorname{sgn}(y(\xi)-y(\eta))\exp(-|y(\xi)-y(\eta)|)\big(U^2y_\xi+H_\xi\big)(\eta)\,d\eta \tag{2.9}$$

where the t variable has been dropped to simplify the notation. Later we will prove that y is an increasing function for any fixed time t. If, for the moment, we take this for granted, then $P_x\circ y$ is equivalent to Q where

$$Q(t,\xi)=-\frac{1}{4}\int_{\mathbb{R}}\operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+H_\xi\big)(\eta)\,d\eta, \tag{2.10}$$

and, slightly abusing the notation, we write

$$P(t,\xi)=\frac{1}{4}\int_{\mathbb{R}}\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+H_\xi\big)(\eta)\,d\eta. \tag{2.11}$$

The derivatives of Q and P are given by

$$Q_\xi=-\frac{1}{2}H_\xi-\left(\frac{1}{2}U^2-P\right)y_\xi \quad\text{and}\quad P_\xi=Qy_\xi. \tag{2.12}$$

For $\xi\in[0,1]$, using the fact that $y(\xi+1)=y(\xi)+1$ and the periodicity of H_ξ and U, these expressions can be rewritten as

$$\begin{aligned}Q=&\frac{1}{2(e-1)}\int_0^1\sinh(y(\xi)-y(\eta))(U^2y_\xi+H_\xi)(\eta)\,d\eta\\&-\frac{1}{4}\int_0^1\operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)(U^2y_\xi+H_\xi)\,d\eta\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}P=&\frac{1}{2(e-1)}\int_0^1\cosh(y(\xi)-y(\eta))(U^2y_\xi+H_\xi)(\eta)\,d\eta\\&+\frac{1}{4}\int_0^1\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)(U^2y_\xi+H_\xi)\,d\eta.\end{aligned} \tag{2.14}$$

Thus $P_x\circ y$ and $P\circ y$ can be replaced by equivalent expressions given by (2.10) and (2.11) which only depend on our new variables U, H, and y. We now derive a new system of equations, formally equivalent to the Camassa–Holm equation. Equations (2.5), (2.4) and (2.1) give us

$$\begin{cases}y_t=U,\\U_t=-Q,\\H_t=\big[U^3-2PU\big]_0^\xi.\end{cases} \tag{2.15}$$

Differentiating (2.15) yields

$$\begin{cases}y_{\xi t}=U_\xi,\\U_{\xi t}=\frac{1}{2}H_\xi+\left(\frac{1}{2}U^2-P\right)y_\xi,\\H_{\xi t}=-2Q\,Uy_\xi+\big(3U^2-2P\big)\,U_\xi.\end{cases} \tag{2.16}$$

The system (2.16) is semilinear with respect to the variables y_ξ, U_ξ, and H_ξ.

2.2. **Existence and uniqueness of solutions of the equivalent system.** In this section, we focus our attention on the system of equations (2.15) and prove, by a contraction argument, that it admits a unique solution. Let Id denote the identity, i.e., $\mathrm{Id}(\xi)=\xi$. We claim that the linear map $\Lambda\colon(\sigma,h)\mapsto f=\sigma+h\,\mathrm{Id}$ is an homeomorphism from $H^1_{\mathrm{per}}\times\mathbb{R}$ to V where H^1_{per} denotes the Banach space

$$H^1_{\mathrm{per}}=\{\sigma\in H^1_{\mathrm{loc}}(\mathbb{R})\mid\sigma(\xi+1)=\sigma(\xi)\text{ for all }\xi\in\mathbb{R}\}$$

with the norm $\|\sigma\|_{H^1_{\mathrm{per}}}=\|\sigma\|_{H^1}$. It is clear that Λ is invertible and, for any $f\in V$, its inverse $(\sigma,h)=\Lambda^{-1}f$ is given by $h=f(1)-f(0)$ and $\sigma=f-h\,\mathrm{Id}$. Let $f=\Lambda(\sigma,h)$, we have

$$\|f\|_{H^1}\le\|\sigma\|_{H^1}+|h|\,\|\mathrm{Id}\|_{H^1}=\|\sigma\|_{H^1}+\sqrt{\frac{2}{3}}\,|h|$$

and therefore Λ is continuous. Conversely,

$$|h|=|f(1)-f(0)|\le2\,\|f\|_{L^\infty}\le2C\,\|f\|_{H^1}\,,$$

and

$$\|\sigma\|_{H^1} \leq \|f\|_{H^1} + 2\|f\|_{L^\infty}\|\mathrm{Id}\|_{H^1} \leq (1 + 2\sqrt{\frac{2}{3}}C)\|f\|_{H^1}$$

where the constant C denotes the constant of the Sobolev embedding $H^1 \subset L^\infty$. Hence, Λ^{-1} is continuous. Since $H^1_{\mathrm{per}} \times \mathbb{R}$ is a Banach space, V is also a Banach space. We introduce $\zeta = y - \mathrm{Id}$ and $(\sigma, h) = \Lambda^{-1}(H)$, i.e., $h = H(t,1) - H(t,0)$ and $\sigma = H - h\,\mathrm{Id}$. The system (2.15) is then equivalent to

$$\begin{cases} \zeta_t = U, \\ U_t = -Q, \\ \sigma_t = \left[U^3 - 2PU\right]_0^\xi, \\ h_t = 0. \end{cases} \tag{2.17}$$

We will prove that the system (2.17) is a well-posed system of ordinary differential equations in the Banach space E where

$$E = H^1_{\mathrm{per}} \times H^1_{\mathrm{per}} \times H^1_{\mathrm{per}} \times \mathbb{R}.$$

There is a bijection $(\zeta, U, \sigma, h) \mapsto (y, U, H)$ between E and $V_1 \times H^1_{\mathrm{per}} \times V$ given by $y = \zeta + \mathrm{Id}$, $H = \sigma + h\,\mathrm{Id}$ and U is unchanged, so that in the remaining we will use both set of variables. However, for the contraction argument it is important to have a Banach space and we use E and the variables (ζ, U, σ, h) (note that V_1 and a fortiori $V_1 \times H^1_{\mathrm{per}} \times V$ are not Banach spaces). The following lemma gives the Lipschitz bounds we need on Q and P.

Lemma 2.1. *For any $X = (\zeta, U, \sigma, h)$ in E, we define the maps $\mathcal{Q}$ and $\mathcal{P}$ as $\mathcal{Q}(X) = Q$ and $\mathcal{P}(X) = P$ where Q and P are given by (2.10) and (2.11), respectively. Then, $\mathcal{P}$ and $\mathcal{Q}$ are Lipschitz maps on bounded sets from E to H^1_{per}. Moreover, we have*

$$Q_\xi = -\frac{1}{2}(\sigma_\xi + h) - \left(\frac{1}{2}U^2 - P\right)(1 + \zeta_\xi), \tag{2.18}$$

$$P_\xi = Q(1 + \zeta_\xi). \tag{2.19}$$

Proof. Let $B_M = \{X = (\zeta, U, \sigma, h) \in E \mid \|X\|_E \leq M\}$. Let us first prove that $\mathcal{P}$ and $\mathcal{Q}$ are Lipschitz maps from B_M to L^∞_{per}. Let $X = (\zeta, U, \sigma, h)$ and $\tilde{X} = (\tilde{\zeta}, \tilde{U}, \tilde{\sigma}, \tilde{h})$ be two elements of B_M. We have

$$\|y\|_{L^\infty} = \|\mathrm{Id} + \zeta\|_{L^\infty} \leq 1 + C\|\zeta\|_{H^1} \leq (1 + CM).$$

and $\|\tilde{y}\|_{L^\infty} \leq (1 + CM)$. Since the map $x \mapsto \cosh x$ is locally Lipschitz, it is Lipschitz on $\{x \in \mathbb{R} \mid |x| \leq 2(1 + CM)\}$. Hence, if we denote by C a generic constant that only depends on M, we have

$$\begin{aligned} |\cosh(y(\xi) - y(\eta)) - \cosh(\tilde{y}(\xi) - \tilde{y}(\eta))| &\leq C\,|y(\xi) - \tilde{y}(\xi) - y(\eta) + \tilde{y}(\eta)| \\ &\leq C\left\|\zeta - \tilde{\zeta}\right\|_{L^\infty} \end{aligned}$$

for all ξ, η in $[0,1]$. It follows that, for all $\xi \in [0,1]$,

$$\begin{aligned} &\left\|\cosh(y(\xi) - y(\,\cdot\,))U^2 y_\xi(\,\cdot\,) - \cosh(\tilde{y}(\xi) - \tilde{y}(\,\cdot\,))\tilde{U}^2\tilde{y}_\xi(\,\cdot\,)\right\|_{L^2} \\ &\qquad \leq C\big(\left\|\zeta - \tilde{\zeta}\right\|_{L^\infty} + \left\|U - \tilde{U}\right\|_{L^\infty} + \left\|\zeta_\xi - \tilde{\zeta}_\xi\right\|_{L^2}\big) \end{aligned}$$

and the map $X = (\zeta, U, \sigma, h) \mapsto \frac{1}{2(e-1)}\int_0^1 \cosh(y(\xi) - y(\eta))(U^2 y_\xi)(\eta)\,d\eta$ which corresponds to the first term in (2.14) is Lipschitz from B_M to L^∞_{per}. We handle the other terms in (2.14) in the same way and we prove that $\mathcal{P}$ is Lipschitz from B_M to L^∞_{per}. Similarly, one proves that $\mathcal{Q}\colon B_M \to L^\infty_{\mathrm{per}}$ is Lipschitz. Direct differentiation gives us the expressions (2.12) for the derivatives P_ξ and Q_ξ of P and Q. Since, as we have just proved, $\mathcal{P}$ and $\mathcal{Q}$ are Lipschitz from B_M to L^∞_{per}, we have

$$\begin{aligned}
&\left\|\mathcal{Q}(X)_\xi - \mathcal{Q}(\tilde{X})_\xi\right\|_{L^2}\\
&\quad= \left\| y_\xi \mathcal{P}(X) - \tilde{y}_\xi \mathcal{P}(\tilde{X}) - \frac{1}{2}(U^2 y_\xi - \tilde{U}^2 \tilde{y}_\xi + \sigma_\xi - \tilde{\sigma}_\xi + h - \tilde{h})\right\|_{L^2}\\
&\quad\leq C\big(\left\|\mathcal{P}(X) - \mathcal{P}(\tilde{X})\right\|_{L^\infty} + \left\|U - \tilde{U}\right\|_{L^\infty} + \left\|\zeta_\xi - \tilde{\zeta}_\xi\right\|_{L^2} + \left\|\sigma_\xi - \tilde{\sigma}_\xi\right\|_{L^2} + \left|h - \tilde{h}\right| \big)\\
&\quad\leq C\left\|X - \tilde{X}\right\|_E
\end{aligned}$$

where we have used the fact that U and $\tilde{U}$ are bounded in B_M so that $\left\|U^2 - \tilde{U}^2\right\|_{L^\infty} \leq C\left\|U - \tilde{U}\right\|_{L^\infty}$. Hence, we have proved that $\mathcal{Q}\colon B_M \to H^1_{\text{per}}$ is Lipschitz. We prove that $\mathcal{P}\colon B_M \to H^1_{\text{per}}$ in the same way, and this concludes the proof of the lemma. $\square$

In the next theorem, by using a contraction argument, we prove the short-time existence of solutions to (2.15).

Theorem 2.2. *Given $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{\sigma}, h)$ in E, there exists a time T depending only on $\left\|\bar{X}\right\|_E$ such that the system (2.15) admits a unique solution in $C^1([0,T], E)$ with initial data $\bar{X}$.*

Proof. Solutions of (2.15) can be rewritten as

$$X(t) = \bar{X} + \int_0^t F(X(\tau))\,d\tau \tag{2.20}$$

where $F\colon E \to E$ is given by $F(X) = (U, -\mathcal{Q}(X), [U^3 - 2\mathcal{P}(X)U]_0^\xi, 0)$ where $X = (\zeta, U, \sigma, h)$. The integrals are defined as Riemann integrals of continuous functions on the Banach space E. To prove that $X \mapsto [U^3 - 2\mathcal{P}(U)U]_0^\xi$ is Lipschitz from bounded set of E to H^1_{per}, we proceed as in the proof of Lemma 2.1. Hence, F is Lipschitz on bounded set, and the theorem follows from the standard theory of ordinary differential equations, see, for example, [1]. $\square$

We now turn to the proof of existence of global solutions of (2.15). We are interested in a particular class of initial data that we are going to make precise later, see Definition 2.5. In particular, we will only consider initial data that belong to $\left[W^{1,\infty}_{\text{per}}\right]^3 \times \mathbb{R}$ where $W^{1,\infty}_{\text{per}} = \{f \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) \mid f(\xi+1) = f(\xi) \text{ for all } \xi \in \mathbb{R}\}$, which is a Banach space for the norm $\|f\|_{W^{1,\infty}_{\text{per}}} = \|f\|_{W^{1,\infty}}$. Of course, $\left[W^{1,\infty}_{\text{per}}\right]^3 \times \mathbb{R}$ is a subset of E. Given $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{\sigma}, \bar{h}) \in [W^{1,\infty}]^3 \times \mathbb{R}$, we consider the short-time solution $X = (\zeta, U, \sigma, h) \in C([0,T], E)$ of (2.15) given by Theorem 2.2. Using the fact that $\mathcal{Q}$ and $\mathcal{P}$ are locally Lipschitz (Lemma 2.1) and, since $X \in C([0,T], E)$, we can prove that P and Q belongs to $C([0,T], H^1_{\text{per}})$. We now consider U, P, and Q as given functions in $C([0,T], H^1_{\text{per}})$. Then, for any *fixed* $\xi \in \mathbb{R}$, we can solve the system of ordinary differential equations in $\mathbb{R}^3$ given by

$$\begin{cases}
\dfrac{d}{dt}\alpha(t,\xi) = \beta(t,\xi),\\[2mm]
\dfrac{d}{dt}\beta(t,\xi) = \dfrac{1}{2}(\gamma(t,\xi) + \bar{h}) + \left[(\dfrac{1}{2}U^2 - P)(t,\xi)\right](1 + \alpha(t,\xi)),\\[2mm]
\dfrac{d}{dt}\gamma(t,\xi) = -\left[2(Q\,U)(t,\xi)\right](1 + \alpha(t,\xi)) + \left[(3U^2 - 2P)(t,\xi)\right]\beta(t,\xi),
\end{cases} \tag{2.21}$$

and which is obtained by substituting ζ_ξ, U_ξ and σ_ξ in (2.16) by the unknowns α, β and γ, respectively. We also replaced $h(t)$ by $\bar{h}$ since $h(t) = \bar{h}$ for all t. We have to specify the initial conditions for (2.21). Let $\mathcal{A}$ be the following set

$$\mathcal{A} = \{\xi \in \mathbb{R} \mid \left|\bar{U}_\xi(\xi)\right| \leq \left\|\bar{U}_\xi\right\|_{L^\infty},\ \left|\bar{\sigma}_\xi(\xi)\right| \leq \left\|\bar{\sigma}_\xi\right\|_{L^\infty},\ \left|\bar{\zeta}_\xi(\xi)\right| \leq \left\|\bar{\zeta}_\xi\right\|_{L^\infty}\}.$$

Since we assumed $\bar{X} \in [W^{1,\infty}]^3 \times \mathbb{R}$, we have that $\mathcal{A}$ has full measure, that is, $\operatorname{meas}(\mathcal{A}^c) = 0$. For $\xi \in \mathcal{A}$ we define $(\alpha(0,\xi), \beta(0,\xi), \gamma(0,\xi)) = (\bar{\zeta}_\xi(\xi), \bar{U}_\xi(\xi), \bar{\sigma}_\xi(\xi))$. However, for $\xi \in \mathcal{A}^c$ we take $(\alpha(0,\xi), \beta(0,\xi), \gamma(0,\xi)) = (0,0,0)$.

Lemma 2.3. *Given initial condition $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{\sigma}, \bar{h}) \in [W^{1,\infty}]^3 \times \mathbb{R}$, we consider the solution $X = (\zeta, U, \sigma, h) \in C^1([0,T], E)$ of (2.21) given by Theorem 2.2. Then, $X \in C^1([0,T], [W^{1,\infty}]^3 \times \mathbb{R})$. The functions $\alpha(t,\xi)$, $\beta(t,\xi)$ and $\gamma(t,\xi)$ which are obtained by solving (2.21) for any fixed given ξ with the initial condition specified above, coincide for almost every ξ and for all time t with ζ_ξ, U_ξ and σ_ξ, respectively, that is, for all $t \in [0,T]$, we have*

$$(\alpha(t,\xi), \beta(t,\xi), \gamma(t,\xi)) = (\zeta_\xi(t,\xi), U_\xi(t,\xi), \sigma_\xi(t,\xi)) \tag{2.22}$$

for almost every $\xi \in \mathbb{R}$.

Thus, this lemma allows us to pick up a special representative for $(\zeta_\xi, U_\xi, \sigma_\xi)$ given by (α, β, γ), which is defined for all $\xi \in \mathbb{R}$ and which, for any given ξ, satisfies the ordinary differential equation (2.21) in $\mathbb{R}^3$. In the remaining we will of course identify the two and set $(\zeta_\xi, U_\xi, \sigma_\xi)$ equal to (α, β, γ). To prove this lemma, we will need the following proposition which is adapted from [28, p. 134, Corollary 2].

Proposition 2.4. *Let R be a bounded linear operator on a Banach space X into a Banach space Y. Let f be in $C([0,T], X)$. Then, Rf belongs to $C([0,T], Y)$ and therefore is Riemann integrable, and $\int_{[0,T]} Rf(t)\,dt = R\int_{[0,T]} f(t)\,dt$.*

Proof of Lemma 2.3. We introduce the Banach space of *everywhere* bounded periodic function B^∞_{per} whose norm is naturally given by $\|f\|_{B^\infty_{\mathrm{per}}} = \sup_{\xi\in[0,1]} |f(\xi)|$. Obviously, the periodic continuous functions belong to B^∞_{per}. We define (α, β, γ) as the solution of (2.21) in $\left[B^\infty_{\mathrm{per}}\right]^3$ with initial data as given above. Thus, strictly speaking, this is a different definition than the one given in the lemma but we will see that they are in fact equivalent. We note that the system (2.21) is affine (it consists of a sum of a linear transformation and a constant) and therefore it is not hard to prove, by using a contraction argument in $\left[B^\infty_{\mathrm{per}}\right]^3$, the short-time existence of solutions. Moreover, due to the affine structure, a direct application of Gronwall's lemma shows that the solution exists on $[0,T]$, the interval on which (ζ, U, σ, h) is defined. For any given ξ, the map $f \mapsto f(\xi)$ from B^∞_{per} to $\mathbb{R}$ is linear and continuous (the space B^∞_{per} was precisely introduced in order to make this map continuous). Hence, after applying this map to each term in (2.21) written in integral form and using Proposition 2.4, we recover the original definition of α, β and γ as solutions, for any given $\xi \in \mathbb{R}$, of the system (2.21) of ordinary differential equations in $\mathbb{R}^3$. The derivation map $\frac{d}{d\xi}$ is continuous from H^1_{per} into L^2_{per}. We can apply it to each term in (2.15) written in integral from and, by Proposition 2.4, this map commutes with the integral. We end up with, after using (2.18) and (2.19),

$$\begin{cases} \zeta_\xi(t) = \bar{\zeta}_\xi + \displaystyle\int_0^t U_\xi(\tau)\,d\tau, \\ U_\xi(t) = \bar{U}_\xi + \displaystyle\int_0^t \left(\frac{1}{2}(\sigma_\xi + \bar{h}) + (\frac{1}{2}U^2 - P)(1+\zeta_\xi)\right)(\tau)\,d\tau, \\ \sigma_\xi(t) = \bar{\sigma}_\xi + \displaystyle\int_0^t \left(-2Q\,U(1+\zeta_\xi) + (3U^2 - 2P)U_\xi\right)(\tau)\,d\tau. \end{cases} \tag{2.23}$$

The map from B^∞_{per} to L^2_{per} is also continuous, we can apply it to (2.21) written in integral form, and again use Proposition 2.4. Then, we subtract each equation in (2.23) from the corresponding one in (2.21), take the norm and add them. After introducing $Z(t) = \|\alpha(t,\cdot) - \zeta_\xi(t,\cdot)\|_{L^2} + \|\beta(t,\cdot) - U_\xi(t,\cdot)\|_{L^2} + \|\gamma(t,\cdot) - \sigma_\xi(t,\cdot)\|_{L^2}$, we end up with the following equation

$$Z(t) \le Z(0) + C\int_0^t Z(\tau)\,d\tau$$

where C is a constant which, again, only depends on the $C([0,T],H^1)$-norms, of U, P, and Q. By assumption on the initial conditions, we have $Z(0) = 0$ because $\alpha(0) = \bar\zeta_\xi$, $\beta(0) = \bar U_\xi$, $\gamma(0) = \bar\sigma_\xi$ almost everywhere and therefore, by Gronwall's lemma, we get $Z(t) = 0$ for all $t \in [0,T]$. This is just a reformulation of (2.22), and this concludes the proof of the lemma. □

It is possible to carry out the contraction argument of Theorem 2.2 in the Banach space $[W^{1,\infty}_{\rm per}]^3 \times \mathbb{R}$ but the topology on this space turns out to be too strong for our purpose and that is why we prefer E whose topology is weaker. Our goal is to find solutions of (1.3) with initial data $\bar u$ in $H^1_{\rm per}$. Theorem 2.2 gives us the existence of solutions to (2.15) for initial data in E. Therefore we have to find initial conditions that match the initial data $\bar u$ and belong to E. A natural choice would be to use $\bar y(\xi) = y(0,\xi) = \xi$ and $\bar U(\xi) = u(\xi)$. Then $y(t,\xi)$ gives the position of the particle which is at ξ at time $t = 0$. But, if we make this choice, then $\bar H_\xi = \bar u^2 + \bar u_x^2$ and H_ξ does not belong to $L^2_{\rm per}$ in general. We consider instead $(\bar y, \bar U, \bar H)$ given by the relations

$$\begin{aligned}
&\int_0^{\bar y(\xi)} (\bar u^2 + \bar u_x^2)\,dx + \bar y(\xi) = (1+\bar h)\xi,\\
&\bar U(\xi) = \bar u\circ\bar y(\xi) \;\text{ and }\; \bar H(\xi) = \int_0^{\bar y(\xi)} \left(\bar u^2 + \bar u_x^2\right)\,dx,
\end{aligned}\tag{2.24}$$

where $\bar h = \int_0^1 (\bar u^2 + \bar u_x^2)\,dx = \|\bar u\|^2_{H^1_{\rm per}}$. The definition of $\bar y$ is implicit, it is well-defined as the function $y \mapsto \int_0^y (\bar u^2 + \bar u_x^2)\,dx + y$ is continous, strictly increasing and therefore invertible. Later (see Remark 3.10), we will prove that $(\bar y - \mathrm{Id}, \bar U, \bar H - \bar h\,\mathrm{Id}, \bar h)$ belongs to $\mathcal{G}$ where $\mathcal{G}$ is defined as follows.

Definition 2.5. The set $\mathcal{G}$ is composed of all $(\zeta, U, \sigma, h) \in E$ such that

$$(\zeta, U, \sigma, h) \in \left[W^{1,\infty}\right]^3 \times \mathbb{R}, \tag{2.25a}$$

$$y_\xi \ge 0, H_\xi \ge 0, y_\xi + H_\xi > 0 \text{ almost everywhere}, \tag{2.25b}$$

$$y_\xi H_\xi = y_\xi^2 U^2 + U_\xi^2 \text{ almost everywhere}, \tag{2.25c}$$

where we denote $y(\xi) = \zeta(\xi) + \xi$ and $H = \sigma + h\,\mathrm{Id}$.

If $(\zeta, U, \sigma, h) \in \mathcal{G}$, then $h \ge 0$. Indeed, since $H_\xi \ge 0$, H is an increasing function and $h = H(1) - H(0) \ge 0$. Note that if all functions are smooth and $y_\xi > 0$, we have $u_x \circ y = \frac{U_\xi}{y_\xi}$ and condition (2.25c) is equivalent to (2.8). For initial data in $\mathcal{G}$, the solution of (2.15) exists globally.

Lemma 2.6. *Given initial data $\bar X = (\bar\zeta, \bar U, \bar\sigma, \bar h)$ in $\mathcal{G}$, let $X(t) = (\zeta(t), U(t), \sigma(t), h(t))$ be the short-time solution of (2.15) in $C([0,T],E)$ for some $T > 0$ with initial data $(\bar\zeta, \bar U, \bar\sigma, \bar h)$. Then,*

(i) $X(t)$ belongs to $\mathcal{G}$ for all $t \in [0,T]$,

(ii) for almost every $t \in [0,T]$, $y_\xi(t,\xi) > 0$ for almost every $\xi \in \mathbb{R}$.

We denote by $\mathcal{A}$ the set where the absolute values of $\bar\zeta_\xi(\xi)$, $\bar\sigma_\xi(\xi)$, and $\bar U_\xi(\xi)$ all are smaller than $\|\bar X\|_{[W^{1,\infty}]^3\times\mathbb{R}}$ and where the inequalities in (2.25b) and (2.25c) are satisfied for $\bar y_\xi$, $\bar U_\xi$ and $\bar H_\xi$. By assumption, we have $\mathrm{meas}(\mathcal{A}^c) = 0$ and we set $(\bar\zeta_\xi, \bar U_\xi, \bar\sigma_\xi)$ equal to zero on $\mathcal{A}^c$. Thus, as allowed by Lemma 2.3, we choose a special representative for $(\zeta(t,\xi), U(t,\xi), \sigma(t,\xi))$ whose derivative satisfies (2.16) as an ordinary differential equation, for every $\xi \in \mathbb{R}$. The proof of this lemma is almost the same as in [22]. We repeat it here for completeness.

Proof. (i) We already proved in Lemma 2.3 that the space $[W^{1,\infty}]^3 \times \mathbb{R}$ is preserved and $X(t)$ satisfies (2.25a) for all $t \in [0,T]$. Let us prove that (2.25c) and the inequalities in (2.25b) hold for any $\xi \in \mathcal{A}$ and therefore almost everywhere. We consider a fixed ξ in $\mathcal{A}$ and drop it in the notation when there is no ambiguity. From (2.16), we have, on the one hand,

$$(y_\xi H_\xi)_t = y_{\xi t} H_\xi + H_{\xi t} y_\xi = U_\xi H_\xi + (3U^2 U_\xi - 2y_\xi QU - 2PU_\xi) y_\xi,$$

and, on the other hand,

$$\begin{aligned}(y_\xi^2 U^2 + U_\xi^2)_t &= 2y_{\xi t} y_\xi U^2 + 2y_\xi^2 U_t U + 2U_{\xi t} U_\xi \\ &= 3U_\xi U^2 y_\xi - 2PU_\xi y_\xi + H_\xi U_\xi - 2y_\xi^2 QU.\end{aligned}$$

Thus, $(y_\xi H_\xi - y_\xi^2 U^2 - U_\xi^2)_t = 0$, and since $y_\xi H_\xi(0) = (y_\xi^2 U^2 + U_\xi^2)(0)$, we have $y_\xi H_\xi(t) = (y_\xi^2 U^2 + U_\xi^2)(t)$ for all $t \in [0,T]$. We have proved (2.25c). Let us introduce t^* given by

$$t^* = \sup\{t \in [0,T] \mid y_\xi(t') \geq 0 \text{ for all } t' \in [0,t]\}.$$

Here we recall that we consider a fixed $\xi \in \mathcal{A}$ and dropped it in the notation. Assume that $t^* < T$. Since $y_\xi(t)$ is continuous with respect to time, we have

$$y_\xi(t^*) = 0. \tag{2.26}$$

Hence, from (2.25c) that we just proved, $U_\xi(t^*) = 0$ and, by (2.16),

$$y_{\xi t}(t^*) = U_\xi(t^*) = 0. \tag{2.27}$$

From (2.16), since $y_\xi(t^*) = U_\xi(t^*) = 0$, we get

$$y_{\xi tt}(t^*) = U_{\xi t}(t^*) = \frac{1}{2} H_\xi(t^*). \tag{2.28}$$

If $H_\xi(t^*) = 0$, then $(y_\xi, U_\xi, H_\xi)(t^*) = (0,0,0)$ and, by the uniqueness of the solution of (2.16), seen as a system of ordinary differential equations, we must have $(y_\xi, U_\xi, H_\xi)(t) = 0$ for all $t \in [0,T]$. This contradicts the fact that $y_\xi(0)$ and $H_\xi(0)$ cannot vanish at the same time ($\bar{y}_\xi + \bar{H}_\xi > 0$ for all $\xi \in \mathcal{A}$). If $H_\xi(t^*) < 0$, then $y_{\xi tt}(t^*) < 0$ and, because of (2.26) and (2.27), there exists a neighborhood $\mathcal{U}$ of t^* such that $y(t) < 0$ for all $t \in \mathcal{U} \setminus \{t^*\}$. This contradicts the definition of t^*. Hence, $H_\xi(t^*) > 0$ and, since we now have $y_\xi(t^*) = y_{\xi t}(t^*) = 0$ and $y_{\xi tt}(t^*) > 0$, there exists a neighborhood of t^* that we again denote $\mathcal{U}$ such that $y_\xi(t) > 0$ for all $t \in \mathcal{U} \setminus \{t^*\}$. This contradicts the fact that $t^* < T$, and we have proved the first inequality in (2.25b), namely that $y_\xi(t) \geq 0$ for all $t \in [0,T]$. Let us prove that $H_\xi(t) \geq 0$ for all $t \in [0,T]$. This follows from (2.25c) when $y_\xi(t) > 0$. Now, if $y_\xi(t) = 0$, then $U_\xi(t) = 0$ from (2.25c) and we have seen that $H_\xi(t) < 0$ would imply that $y_\xi(t') < 0$ for some t' in a punctured neighborhood of t, which is impossible. Hence, $H_\xi(t) \geq 0$ and we have proved the second inequality in (2.25b). Assume that the third inequality in (2.25c) does not hold. Then, by continuity, there exists a time $t \in [0,T]$ such that $(y_\xi + H_\xi)(t) = 0$. Since y_ξ and H_ξ are positive, we must have $y_\xi(t) = H_\xi(t) = 0$ and, by (2.25c), $U_\xi(t) = 0$. Since zero is a solution of (2.16), this implies that $y_\xi(0) = U_\xi(0) = H_\xi(0)$, which contradicts $(y_\xi + H_\xi)(0) > 0$.

(ii) We define the set

$$\mathcal{N} = \{(t,\xi) \in [0,T] \times \mathbb{R} \mid y_\xi(t,\xi) = 0\}.$$

Fubini's theorem gives us

$$\operatorname{meas}(\mathcal{N}) = \int_{\mathbb{R}} \operatorname{meas}(\mathcal{N}_\xi)\, d\xi = \int_{[0,T]} \operatorname{meas}(\mathcal{N}_t)\, dt \tag{2.29}$$

where $\mathcal{N}_\xi$ and $\mathcal{N}_t$ are the ξ-section and t-section of $\mathcal{N}$, respectively, that is,

$$\mathcal{N}_\xi = \{t \in [0,T] \mid y_\xi(t,\xi) = 0\} \text{ and } \mathcal{N}_t = \{\xi \in \mathbb{R} \mid y_\xi(t,\xi) = 0\}.$$

Let us prove that, for all $\xi \in \mathcal{A}$, $\operatorname{meas}(\mathcal{N}_\xi) = 0$. If we consider the sets $\mathcal{N}_\xi^n$ defined as

$$\mathcal{N}_\xi^n = \{t \in [0,T] \mid y_\xi(t,\xi) = 0 \text{ and } y_\xi(t',\xi) > 0 \text{ for all } t' \in [t - 1/n, t + 1/n] \setminus \{t\}\},$$

then

$$\mathcal{N}_\xi = \bigcup_{n\in\mathbb{N}} \mathcal{N}_\xi^n. \tag{2.30}$$

Indeed, for all $t \in \mathcal{N}_\xi$, we have $y_\xi(t,\xi) = 0$, $y_{\xi t}(t,\xi) = 0$ from (2.25c) and (2.16) and $y_{\xi tt}(t,\xi) = \frac{1}{2}H_\xi(t,\xi) > 0$ from (2.16) and (2.25b) (y_ξ and H_ξ cannot vanish at the same time for $\xi \in \mathcal{A}$). This implies that, on a small punctured neighborhood of t, y_ξ is strictly positive. Hence, t belongs to some $\mathcal{N}_\xi^n$ for n large enough. This proves (2.30). The set $\mathcal{N}_\xi^n$ consists of isolated points that are countable since, by definition, they are separated by a distance larger than $1/n$ from one another. This means that $\operatorname{meas}(\mathcal{N}_\xi^n) = 0$ and, by the subadditivity of the measure, $\operatorname{meas}(\mathcal{N}_\xi) = 0$. It follows from (2.29) and since $\operatorname{meas}(\mathcal{A}^c) = 0$ that

$$\operatorname{meas}(\mathcal{N}_t) = 0 \text{ for almost every } t \in [0,T]. \tag{2.31}$$

We denote by $\mathcal{K}$ the set of times such that $\operatorname{meas}(\mathcal{N}_t) > 0$, i.e.,

$$\mathcal{K} = \{t \in \mathbb{R}_+ \mid \operatorname{meas}(\mathcal{N}_t) > 0\}. \tag{2.32}$$

By (2.31), $\operatorname{meas}(\mathcal{K}) = 0$. For all $t \in \mathcal{K}^c$, $y_\xi > 0$ almost everywhere and, therefore, $y(t,\xi)$ is strictly increasing and invertible (with respect to ξ). $\square$

We are now ready to prove global existence of solutions to (2.15).

Theorem 2.7. *For any $\bar{X} = (\bar{y}, \bar{U}, \bar{H}) \in \mathcal{G}$, the system (2.15) admits a unique global solution $X(t) = (y(t), U(t), H(t))$ in $C^1(\mathbb{R}_+, E)$ with initial data $\bar{X} = (\bar{y}, \bar{U}, \bar{H})$. We have $X(t) \in \mathcal{G}$ for all times. If we equip $\mathcal{G}$ with the topology inducted by the E-norm, then the map $S\colon \mathcal{G} \times \mathbb{R}_+ \to \mathcal{G}$ defined as*

$$S_t(\bar{X}) = X(t)$$

is a continuous semigroup.

In the formulation of Theorem 2.7, we write (y, U, H) where we really should have written (ζ, U, σ, h) with $y = \zeta + \mathrm{Id}$ and $H = \sigma + h\,\mathrm{Id}$. In the remaining we will continue abusing the notation in the same way because the relevant variables are really (y, U, H) which correspond to Lagrangian variables.

Proof. The solution has a finite time of existence T only if $\|(\zeta, U, \sigma, h)(t,\,\cdot\,)\|_E$ blows up when t tends to T because, otherwise, by Theorem 2.2, the solution can be prolongated by a small time interval beyond T. Let (ζ, U, σ, h) be a solution of (2.15) in $C([0,T), E)$ with initial data $(\bar{\zeta}, \bar{U}, \bar{\sigma}, \bar{h})$. We want to prove that

$$\sup_{t\in[0,T)} \|(\zeta(t,\,\cdot\,), U(t,\,\cdot\,), \sigma(t,\,\cdot\,), h(t)\|_E < \infty. \tag{2.33}$$

It is clear from (2.17) that $h(t) = \bar{h}$ for all time. We now consider a fixed time $t \in [0,T)$ and to simplify the notation we omit it in the notation. From (2.15), we infer that $H(0) = 0$. Since $H_\xi \geq 0$, H is an increasing function and $\|H\|_{L^\infty} \leq H(1) = H(1) - H(0) = h$. Hence, as $\sigma = H - h\,\mathrm{Id}$, $\|\sigma\|_{L^\infty} \leq 2h$ and $\sup_{t\in[0,T)} \|\sigma(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}$

is bounded by $2\bar{h}$. For ξ and η in $[0,1]$, we have $|y(\xi)-y(\eta)|\leq 1$ because y is increasing and $y(1)-y(0)=1$. From (2.25c), we infer $U^2y_\xi\leq H_\xi$ and, from (2.13), we obtain

$$|Q|\leq\frac{1}{e-1}\int_0^1\sinh(y(\xi)-y(\eta))H_\xi(\eta)\,d\eta+\int_0^1 e^{-|y(\xi)-y(\eta)|}H_\xi(\eta)\,d\eta.$$

Hence, $|Q|\leq C(H(1)-H(0))=Ch=C\bar{h}$ for some constant C and $\sup_{t\in[0,T)}\|Q(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}$ is finite. Similarly, one prove that $\sup_{t\in[0,T)}\|P(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}<\infty$. Since $U_t=-Q$, it follows that $\sup_{t\in[0,T)}\|U(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}<\infty$ and, since $\zeta_t=U$, $\sup_{t\in[0,T)}\|\zeta(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})}$ is also finite. We have proved that

$$C_1=\sup_{t\in[0,T)}\{\|U(t,\,\cdot\,)\|_{L^\infty}+\|P(t,\,\cdot\,)\|_{L^\infty}+\|Q(t,\,\cdot\,)\|_{L^\infty}\}$$

is finite. Let $Z(t)=\|y_\xi(t,\,\cdot\,)\|_{L^2}+\|U_\xi(t,\,\cdot\,)\|_{L^2}+\|H_\xi(t,\,\cdot\,)\|_{L^2}$. Using the semi-linearity of (2.16), we obtain

$$Z(t)\leq Z(0)+C\int_0^t Z(\tau)\,d\tau$$

where C is a constant depending only on C_1. It follows from Gronwall's lemma that $\sup_{t\in[0,T)}Z(t)$ is finite and, as $\zeta_\xi=y_\xi-\mathrm{Id}$ and $\sigma=H-\bar{h}\,\mathrm{Id}$, it proves that (2.33) holds. From standard theory for ordinary differential equations we infer that S_t is a continuous semi-group. $\square$

3. From Eulerian to Lagrangian coordinates and vice versa

Even if the H^1-norm is conserved by the equation and therefore H^1_{per} could be seen as the natural space for the equation, the conservative solutions are not well-posed in this space. There are cases, see [4, 20, 21] for the non periodic case, where the energy density $(u^2+u_x^2)\,dx$ becomes a singular measure. The appropriate space which makes the conservative solution into a semigroup is the $\mathcal{D}$ defined as:

Definition 3.1. The set $\mathcal{D}$ is composed of all pairs (u,μ) such that u belongs to H^1_{per} and μ is a positive periodic Radon measure whose absolute continuous part, μ_{ac}, satisfies

$$\mu_{\mathrm{ac}}=(u^2+u_x^2)\,dx. \tag{3.1}$$

A Radon measure μ is said to be 1-periodic if $\mu(1+B)=\mu(B)$ for all Borel sets B. The equivalent system (2.15) was derived by using the characteristics. Since y satisfies (2.1), y, for a given ξ, can also be seen as the position of a particle evolving in the velocity field u, where u is the solution of the Camassa–Holm equation. We are then working in *Lagrangian coordinates*. In [15], the Camassa–Holm equation is derived as a geodesic equation on the group of diffeomorphism equipped with a right-invariant metric. In the present paper, the geodesic curves correspond to $y(t,\,\cdot\,)$. Note that y does not remain a diffeomorphism since it can become non invertible, which agrees with the fact that the solutions of the geodesic equation may break down, see [11]. The right-invariance of the metric can be interpreted as an invariance with respect to relabeling as noted in [3]. This is a property that we also observe in our setting. We denote by G the subgroup of the group of homeomorphisms on the unit circle defined as follows: $f\in G$ if f is invertible,

$$f\in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}),\ f(\xi+1)=f(\xi)+1\text{ for all }\xi\in\mathbb{R},\text{ and} \tag{3.2}$$

$$f-\mathrm{Id}\ \text{ and }\ f^{-1}-\mathrm{Id}\ \text{ both belong to }W^{1,\infty}_{\mathrm{per}}. \tag{3.3}$$

The set G can be interpreted as the set of relabeling functions. For any $\alpha>1$, we introduce the subsets G_α of G defined by

$$G_\alpha=\{f\in G\mid\ \|f-\mathrm{Id}\|_{W^{1,\infty}}+\left\|f^{-1}-\mathrm{Id}\right\|_{W^{1,\infty}}\leq\alpha\}.$$

The subsets G_α do not possess the group structure of G. The next lemma provides a useful characterization of G_α.

Lemma 3.2. *Let $\alpha \geq 0$. If f belongs to G_α, then $1/(1+\alpha) \leq f_\xi \leq 1+\alpha$ almost everywhere. Conversely, if f satisfies (3.2) and there exists $c \geq 1$ such that $1/c \leq f_\xi \leq c$ almost everywhere, then $f \in G_\alpha$ for some α depending only on c.*

Proof. Given $f \in G_\alpha$, let B be the set of points where f^{-1} is differentiable. Rademacher's theorem says that $\operatorname{meas}(B^c) = 0$. For any $\xi \in f^{-1}(B)$, we have

$$\lim_{\xi' \to \xi} \frac{f^{-1}(f(\xi')) - f^{-1}(f(\xi))}{f(\xi') - f(\xi)} = (f^{-1})_\xi(f(\xi))$$

because f is continuous and f^{-1} is differentiable at $f(\xi)$. On the other hand, we have

$$\frac{f^{-1}(f(\xi')) - f^{-1}(f(\xi))}{f(\xi') - f(\xi)} = \frac{\xi' - \xi}{f(\xi') - f(\xi)}.$$

Hence, f is differentiable for any $\xi \in f^{-1}(B)$ and

$$f_\xi(\xi) \geq \frac{1}{\|(f^{-1})_\xi\|_{L^\infty}} \geq \frac{1}{1+\alpha}. \tag{3.4}$$

The estimate (3.4) holds only on $f^{-1}(B)$ but, since $\operatorname{meas}(B^c) = 0$ and f^{-1} is Lipschitz and one-to-one, $\operatorname{meas}(f^{-1}(B^c)) = 0$ (see, e.g., [2, Remark 2.72]), and therefore (3.4) holds almost everywhere. We have $f_\xi \leq 1 + \|f_\xi - 1\|_{L^\infty} \leq 1+\alpha$.

Let us now consider a function f that satisfies (3.2) and such that $1/c \leq f_\xi \leq c$ almost everywhere for some $c \geq 1$. Since $f_\xi \geq 1/c$ almost everywhere, f is strictly increasing and, since it is also continuous, it is invertible. As f is Lipschitz, we can make the following change of variables (see, for example, [2]) and get that, for all ξ_1, ξ_2 in $\mathbb{R}$ such that $\xi_1 < \xi_2$,

$$f^{-1}(\xi_2) - f^{-1}(\xi_1) = \int_{[f^{-1}(\xi_1), f^{-1}(\xi_2)]} \frac{f_\xi}{f_\xi}\, d\xi \leq c(\xi_2 - \xi_1).$$

Hence, f^{-1} is Lipschitz and $(f^{-1})_\xi \leq c$. We have $f^{-1}(\xi') - \xi' = \xi - f(\xi)$ for $\xi' = f(\xi)$ and therefore $\|f - \mathrm{Id}\|_{L^\infty} = \left\|f^{-1} - \mathrm{Id}\right\|_{L^\infty}$. Finally, we get

$$\begin{aligned}
\|f - \mathrm{Id}\|_{W^{1,\infty}} + \left\|f^{-1} - \mathrm{Id}\right\|_{W^{1,\infty}} &\leq 2\|f - \mathrm{Id}\|_{L^\infty} + 2 \\
&\quad + \|f_\xi\|_{L^\infty} + \left\|(f^{-1})_\xi\right\|_{L^\infty} \\
&\leq 2\|f - \mathrm{Id}\|_{L^\infty} + 2 + 2c.
\end{aligned}$$

Since $\|f - \mathrm{Id}\|_{L^\infty} \leq \int_0^1 (|f_\xi| + 1)\, d\xi \leq c + 1$, the lemma is proved. ∎

We define the subsets $\mathcal{F}_\alpha$ and $\mathcal{F}$ of $\mathcal{G}$ as follows

$$\mathcal{F}_\alpha = \{X = (y, U, H) \in \mathcal{G} \mid \frac{1}{1+h}(y + H) \in G_\alpha\},$$

and

$$\mathcal{F} = \{X = (y, U, H) \in \mathcal{G} \mid \frac{1}{1+h}(y + H) \in G\}.$$

We recall that $h = H(\xi + 1) - H(\xi) = H(1) - H(0)$. For $\alpha = 0$, $G_0 = \{\mathrm{Id}\}$. As we will see, the space $\mathcal{F}_0$ will play a special role. These sets are relevant only because they are in some sense preserved by the governing equation (2.15) as the next lemma shows.

Lemma 3.3. *The space $\mathcal{F}$ is preserved by the governing equation (2.15). More precisely, given $\alpha, T \geq 0$ and $\bar{X} \in \mathcal{F}_\alpha$, we have*

$$S_t(\bar{X}) \in \mathcal{F}_{\alpha'}$$

for all $t \in [0, T]$ where α' only depends on T, α and $\left\|\bar{X}\right\|_E$.

Proof. Let $\bar{X} = (\bar{y}, \bar{U}, \bar{H}) \in \mathcal{F}_\alpha$, we denote $X(t) = (y(t), U(t), H(t))$ the solution of (2.15) with initial data $\bar{X}$ and set $v(t,\xi) = \frac{1}{1+h}(y(t,\xi)+H(t,\xi))$, $\bar{v}(\xi) = \frac{1}{1+h}(\bar{y}(\xi)+\bar{H}(\xi))$. By definition, we have $\bar{v} \in G_\alpha$ and, from Lemma 3.2, $1/c \leq \bar{v}_\xi \leq c$ almost everywhere, for some constant $c > 1$ depending only α. We consider a fixed ξ and drop it in the notation. Applying Gronwall's inequality backward in time to (2.16), we obtain

$$|y_\xi(0)| + |H_\xi(0)| + |U_\xi(0)| \leq e^{CT}\left(|y_\xi(t)| + |H_\xi(t)| + |U_\xi(t)|\right) \tag{3.5}$$

for some constant C which depends on $\|X(t)\|_{C([0,T],E)}$, which itself depends only on $\left\|\bar{X}\right\|_E$ and T. From (2.25c), we have

$$|U_\xi(t)| \leq \sqrt{y_\xi(t)H_\xi(t)} \leq \frac{1}{2}(y_\xi(t) + H_\xi(t)).$$

Hence, since y_ξ and H_ξ are positive, (3.5) gives us

$$\frac{1+h}{c} \leq \bar{y}_\xi + \bar{H}_\xi \leq \frac{3}{2}e^{CT}(y_\xi(t) + H_\xi(t)),$$

and $v_\xi(t) = \frac{1}{1+h}(y_\xi(t) + H_\xi(t)) \geq \frac{2}{3c}e^{-CT}$. Similarly, by applying Gronwall's lemma forward in time, we obtain $v_\xi = \frac{1}{1+h}(y_\xi(t) + H_\xi(t)) \leq \frac{3}{2}ce^{CT}$. We have $\frac{1}{1+h}(y+H)(t,\xi+1) = \frac{1}{1+h}(y+H)(t,\xi)+1$. Hence, applying Lemma 3.2, we obtain that $\frac{1}{1+h}(y(t,\cdot\,) + H(t,\cdot\,)) \in G_{\alpha'}$ and therefore $X(t) \in \mathcal{F}_{\alpha'}$ for some α' depending only on α, T and $\left\|\bar{X}\right\|_E$. $\square$

For the sake of simplicity, for any $X = (y, U, H) \in \mathcal{F}$ and any function $f \in G$, we denote $(y \circ f, U \circ f, H \circ f)$ by $X \circ f$. We denote by $\tilde{G}$ the product group $G \times \mathbb{R}$. The group operation on $\tilde{G}$ is given by $(f_1, \gamma_1) \cdot (f_2, \gamma_2) = (f_2 \circ f_1, \gamma_1 + \gamma_2)$ where (f_1, γ_1) and (f_2, γ_2) are two elements of $\tilde{G}$. We define the map $\Phi\colon \tilde{G} \times \mathcal{F} \to \mathcal{F}$ as follows

$$\begin{cases} \bar{y} = y \circ f, \\ \bar{H} = H \circ f + \gamma, \\ \bar{U} = U \circ f, \end{cases}$$

where $(\bar{y}, \bar{U}, \bar{H}) = \Phi\{(f, \gamma), (y, U, H)\}$. We denote $(\bar{y}, \bar{U}, \bar{H}) = (f, \gamma) \bullet (y, U, H)$.

Proposition 3.4. *The map Φ defines a group action of $\tilde{G}$ on $\mathcal{F}$.*

Proof. It is clear that Φ satisfies the fundamental property of a group action, that is, $(f_2, \gamma_2) \bullet ((f_1, \gamma_1) \bullet X) = (f_1 \circ f_2, \gamma_1 + \gamma_2) \bullet X$ for all $X \in \mathcal{F}$ and all (f_1, γ_1) and (f_2, γ_2) in $\tilde{G}$. It remains to prove that $(f, \gamma) \bullet X$ indeed belongs to $\mathcal{F}$. It is not hard to check that $(0, \gamma) \bullet X$ belongs to $\mathcal{F}$. Thus, by the group action property, we only have to show that $(f, 0) \bullet X = X \circ f$ belongs to $\mathcal{F}$. We denote $\bar{X} = (\bar{y}, \bar{U}, \bar{H}) = X \circ f$. As compositions of two Lipschitz maps, $\bar{y}$, $\bar{U}$ and $\bar{H}$ are Lipschitz. It is not hard to check that $\bar{y}(\xi+1) = \bar{y}(\xi)+1$, $\bar{U}(\xi+1) = \bar{U}(\xi)$ and $\bar{H}(\xi+1) = \bar{H}(\xi)+H(1)-H(0)$, for all $\xi \in \mathbb{R}$. Let us prove that

$$\bar{y}_\xi = y_\xi \circ f\, f_\xi,\ \bar{U}_\xi = U_\xi \circ f\, f_\xi \text{ and } \bar{H}_\xi = H_\xi \circ f\, f_\xi \tag{3.6}$$

almost everywhere. Let B_1 be the set where y is differentiable and B_2 the set where $\bar{y}$ and f are differentiable. Using Radamacher's theorem, we get that $\mathrm{meas}(B_1^c) = \mathrm{meas}(B_2^c) = 0$. For $\xi \in B_3 = B_2 \cap f^{-1}(B_1)$, we consider a sequence ξ_i converging to ξ $(\xi_i \neq \xi)$. We have

$$\frac{y(f(\xi_i)) - y(f(\xi))}{f(\xi_i) - f(\xi)}\,\frac{f(\xi_i) - f(\xi)}{\xi_i - \xi} = \frac{\bar{y}(\xi_i) - \bar{y}(\xi)}{\xi_i - \xi}. \tag{3.7}$$

Since f is continuous, $f(\xi_i)$ converges to $f(\xi)$ and, as y is differentiable at $f(\xi)$, the left-hand side of (3.7) tends to $y_\xi \circ f(\xi)\, f_\xi(\xi)$. The right-hand side of (3.7) tends to $\bar{y}_\xi(\xi)$, and we get that

$$y_\xi(f(\xi))f_\xi(\xi) = \bar{y}_\xi(\xi) \tag{3.8}$$

for all $\xi \in B_3$. Since f^{-1} is Lipschitz, one-to-one and $\operatorname{meas}(B_1^c) = 0$, we have $\operatorname{meas}(f^{-1}(B_1)^c) = 0$ and therefore (3.8) holds everywhere. One proves the two other identities in (3.6) similarly. From Lemma 3.2, we have that $f_\xi > 0$ almost everywhere. Then, using (3.6) we easily check that (2.25b) and (2.25c) are fulfilled. Thus, we have proved that $(\bar{y}, \bar{U}, \bar{H})$ fulfills (2.25). We have $\bar{h} = \bar{H}(1) - \bar{H}(0) = H(1) - H(0) = h$. Hence, $\frac{1}{1+\bar{h}}(\bar{y} + \bar{H}) = \frac{1}{1+h}(y + H) \circ f$ which implies, since $\frac{1}{1+h}(y + H)$ and f belongs to G and G is a group, that $\frac{1}{1+\bar{h}}(\bar{y} + \bar{H}) \in G$. Therefore $\bar{X} \in \mathcal{F}$ and the proposition is proved. $\square$

Since $\tilde{G}$ is acting on $\mathcal{F}$, we can consider the quotient space $\mathcal{F}/\tilde{G}$ of $\mathcal{F}$ with respect to the group action. We denote by $\Pi(X) = [X]$ the projection of $\mathcal{F}$ into the quotient space $\mathcal{F}/\tilde{G}$. Let us introduce the subset $\mathcal{H}$ of $\mathcal{F}_0$ defined as follows

$$\mathcal{H} = \{(y, U, H) \in \mathcal{F}_0 \mid \int_0^1 y(\xi)\, d\xi = 0\}.$$

It turns out that $\mathcal{H}$ contains one and only one representative in $\mathcal{F}$ of each element of $\mathcal{F}/\tilde{G}$, that is, there exists a bijection between $\mathcal{H}$ and $\mathcal{F}/\tilde{G}$. In order to prove this we introduce two maps $\Gamma_1 \colon \mathcal{F} \to \mathcal{F}_0$ and $\Gamma_2 \colon \mathcal{F}_0 \to \mathcal{H}$ defined as follows

$$\Gamma_1(X) = X \circ f^{-1}$$

with $f = \frac{1}{1+h}(y + H) \in \mathcal{F}$ and $X = (y, U, H)$, and

$$\begin{cases} \bar{y} = y(\xi - a) \\ \bar{H} = H(\xi - a) + (1 + h)a \\ \bar{U} = U(\xi - a) \end{cases}$$

with $a = \int_0^1 y(\xi)\, d\xi$ and $(\bar{y}, \bar{U}, \bar{H}) = \Gamma_2(y, U, H)$. In fact, $\Gamma_1(X) = (f^{-1}, 0) \bullet X$ and $\Gamma_2(X) = (\tau_a, (1 + h)a) \bullet X$ where τ_a denotes the translation by a. After noting that the group action let invariant the quantity $h = H(1) - H(0)$, it is not hard to check that $\Gamma_1(X)$ indeed belongs to $\mathcal{F}_0$, that is, $\frac{1}{1+\bar{h}}(\bar{y} + \bar{H}) = \mathrm{Id}$. Let us prove that $(\bar{y}, \bar{U}, \bar{H}) = \Gamma_2(y, U, H)$ belongs to $\mathcal{H}$ for any $(y, U, H) \in \mathcal{F}_0$. On the one hand, we have

$$\frac{1}{1+\bar{h}}(\bar{y} + \bar{H})(\xi) = \frac{1}{1+h}\left[(y + H) \circ (\xi - a) + (1 + h)a\right] = \xi$$

because $\bar{h} = h$ and $\frac{1}{1+h}(y + H) = \mathrm{Id}$ as $(y, U, H) \in \mathcal{F}_0$. On the other hand,

$$\int_0^1 \bar{y}(\xi)\, d\xi = \int_{-a}^{1-a} y(\xi)\, d\xi = \int_0^1 y(\xi)\, d\xi + \int_{-a}^0 y(\xi)\, d\xi + \int_1^{1-a} y(\xi)\, d\xi$$

and, since $y(\xi + 1) = y(\xi) + 1$, we obtain

$$\int_0^1 \bar{y}(\xi)\, d\xi = \int_0^1 y(\xi)\, d\xi + \int_{-a}^0 y(\xi)\, d\xi + \int_0^{-a} y(\xi)\, d\xi - a = \int_0^1 y(\xi)\, dx - a = 0.$$

Thus $\Gamma_2(X) \in \mathcal{H}$. We denote the composition map $\Gamma_2 \circ \Gamma_1$ from $\mathcal{F}$ to $\mathcal{H}$ by Γ. We have

$$\Gamma(X) = (\tau_a, (1 + h)a) \bullet ((f^{-1}, 0) \bullet X) = (f^{-1} \circ \tau_a, (1 + h)a) \bullet X$$

where f and a has been defined above. Thus, $\Gamma(X)$ belongs to the same equivalence class as X, and we can define the map $\tilde{\Gamma} \colon \mathcal{F}/\tilde{G} \to \mathcal{H}$ on the quotient space as $\tilde{\Gamma}([X]) = \Gamma(X)$ for any representantive X of $[X]$. It is easily checked that Γ_1 and

Γ_2 let invariant $\mathcal{H}$ so that $\Gamma_{|\mathcal{H}} = \mathrm{Id}_{|\mathcal{H}}$. Hence, $\tilde{\Gamma} \circ \Pi_{|\mathcal{H}} = \mathrm{Id}_{|\mathcal{H}}$ and it follows that $\tilde{\Gamma}$ is a bijection from $\mathcal{F}/\tilde{G}$ to $\mathcal{H}$.

Any topology defined on $\mathcal{H}$ is naturally transported into $\mathcal{F}/\tilde{G}$ by the bijection $\tilde{\Gamma}$. We equip $\mathcal{H}$ with the metric induced by the E-norm, i.e., $d_{\mathcal{H}}(X, X') = \|X - X'\|_E$ for all $X, X' \in \mathcal{H}$. Since $\mathcal{H}$ is closed in E, this metric is complete. We define the metric on $\mathcal{F}/\tilde{G}$ as

$$d_{\mathcal{F}/\tilde{G}}([X], [X']) = \left\| \tilde{\Gamma}([X]) - \tilde{\Gamma}([X']) \right\|_E,$$

for any $[X], [X'] \in \mathcal{F}/\tilde{G}$. Then, $\mathcal{F}/\tilde{G}$ is isometrically isomorphic with $\mathcal{H}$ and the metric $d_{\mathcal{F}/\tilde{G}}$ is complete.

Lemma 3.5. *Given $\alpha \geq 0$. The restriction of Γ to $\mathcal{F}_\alpha$ is a continuous map from $\mathcal{F}_\alpha$ to $\mathcal{H}$.*

Proof. We prove first that Γ_1 is continuous from $\mathcal{F}_\alpha$ to $\mathcal{F}_0$ and then, that Γ_2 is continuous from $\mathcal{F}_0$ to $\mathcal{H}$. We equip $\mathcal{F}_\alpha$ with the topology induced by the E-norm. Let $X_n = (y_n, U_n, H_n) \in \mathcal{F}_\alpha$ be a sequence that converges to $X = (y, U, H)$ in $\mathcal{F}_\alpha$. We denote $\bar{X}_n = (\bar{y}_n, \bar{U}_n, \bar{H}_n) = \Gamma_1(X_n)$ and $\bar{X} = (\bar{y}, \bar{U}, \bar{H}) = \Gamma_1(X)$. By definition of $\mathcal{F}_0$, we have $\bar{H}_n = -\bar{\zeta}_n + \bar{h}_n\xi$ (recall that $\zeta_n = y_n - \mathrm{Id}$). Let us prove first that $\bar{H}_n$ tends to $\bar{H}$ in L^∞_{per}. We denote $f_n = \frac{1}{1+h_n}(y_n + H_n)$, $f = \frac{1}{1+h}(y + H)$, and we have $f_n, f \in G_\alpha$. Thus $\bar{H}_n - \bar{H} = (H_n - H) \circ f_n{}^{-1} + \bar{H} \circ f \circ f_n{}^{-1} - \bar{H}$ and we have

$$\left\|\bar{H}_n - \bar{H}\right\|_{L^\infty} \leq \|H_n - H\|_{L^\infty} + \left\|\bar{H} \circ f - \bar{H} \circ f_n\right\|_{L^\infty}. \tag{3.9}$$

From the definition of $\mathcal{F}_0$, we know that $\bar{H}$ is Lipschitz with Lipschitz constant smaller than $1 + \bar{h}_n$. Hence,

$$\left\|\bar{H} \circ f - \bar{H} \circ f_n\right\|_{L^\infty} \leq (1 + \bar{h}_n) \|f_n - f\|_{L^\infty}. \tag{3.10}$$

Since H_n and f_n converges to H and f, respectively, in L^∞_{per} and $\bar{h}_n = h_n$ converges to h, from (3.9) and (3.10), we get that $\bar{H}_n$ converges to $\bar{H}$ in L^∞_{per}. Let us prove now that $\bar{H}_{n,\xi}$ tend to $\bar{H}_\xi$ in L^2_{per}. We have $\bar{H}_{n,\xi} - \bar{H}_\xi = \frac{H_{n,\xi}}{f_{n,\xi}} \circ f_n{}^{-1} - \frac{H_\xi}{f_\xi} \circ f^{-1}$ which can be decomposed into

$$\bar{H}_{n,\xi} - \bar{H}_\xi = \left(\frac{H_{n,\xi} - H_\xi}{f_{n,\xi}}\right) \circ f_n{}^{-1} + \frac{H_\xi}{f_{n,\xi}} \circ f_n{}^{-1} - \frac{H_\xi}{f_\xi} \circ f^{-1}. \tag{3.11}$$

Since $f_n \in G_\alpha$, there exists a constant $c > 0$ independent of n such that $1/c \geq f_{n,\xi} \geq c$ almost everywhere, see Lemma 3.2. We have

$$\left\|\left(\frac{H_{n,\xi} - H_\xi}{f_{n,\xi}}\right) \circ f_n{}^{-1}\right\|^2_{L^2} = \int_0^1 (H_{n,\xi} - H_\xi)^2 \frac{1}{f_{n,\xi}}\, d\xi \leq c \|H_{n,\xi} - H_\xi\|^2_{L^2}, \tag{3.12}$$

where we have made the change of variables $\xi' = f_n{}^{-1}(\xi)$. Hence, the left-hand side of (3.12) converges to zero. If we can prove that $\frac{H_\xi}{f_{n,\xi}} \circ f_n{}^{-1} \to \frac{H_\xi}{f_\xi} \circ f^{-1}$ in L^2_{per}, then, using (3.11), we get that $\bar{H}_{n,\xi} \to \bar{H}_\xi$ in L^2_{per}, which is the desired result. We recall that, since the space V and $H^1_{\mathrm{per}} \times \mathbb{R}$ are homeomorphic, $H_n \to H$ in V is equivalent to $(\sigma_n, h_n) \to (\sigma, h)$ in $H^1_{\mathrm{per}} \times \mathbb{R}$. We have

$$\frac{H_\xi}{f_{n,\xi}} \circ f_n{}^{-1} = \frac{(\bar{H}_\xi \circ f) f_\xi}{f_{n,\xi}} \circ f_n{}^{-1} = (\bar{H}_\xi \circ g_n) g_{n,\xi}$$

where $g_n = f \circ f_n{}^{-1}$. Let us prove that $\lim_{n\to\infty} \|g_{n,\xi} - 1\|_{L^2} = 0$. We have, after using a change of variables,

$$\|g_{n,\xi} - 1\|^2_{L^2} = \int_0^1 \left(\frac{f_\xi}{f_{n,\xi}} \circ f_n{}^{-1} - 1\right)^2 d\xi = c \|f_\xi - f_{n,\xi}\|^2_{L^2}. \tag{3.13}$$

Hence, since $f_{n,\xi} \to f_\xi$ in $L^2_{\rm per}$, $\lim_{n\to\infty} \|g_{n,\xi} - 1\|_{L^2} = 0$. We have

$$\left\|\bar H_\xi \circ g_n g_{n,\xi} - \bar H_\xi\right\|_{L^2} \le \left\|\bar H_\xi \circ g_n\right\|_{L^\infty} \|g_{n,\xi} - 1\|_{L^2} + \left\|\bar H_\xi \circ g_n - \bar H_\xi\right\|_{L^2}. \quad (3.14)$$

We have $\left\|\bar H_\xi \circ g_n\right\|_{L^\infty} \le 1 + h_n$ since, as we already noted, $\bar H$ is Lipschitz with Lipschitz constant smaller than $1 + \bar h_n = 1 + h_n$. Hence, the first term in the sum in (3.14) converges to zero. As far as the second term is concerned, one can always approximate $\bar H_\xi$ in $L^2_{\rm per}$ by a periodic continuous function v. After observing that $1/c^2 \le g_{n,\xi} \le c^2$ almost everywhere, we can prove, as we have done several times now, that $\|H_\xi \circ g_n - v \circ g_n\|^2_{L^2} \le c^2 \|H_\xi - v\|^2_{L^2}$ and $v \circ g_n$ can be chosen arbitrarily close to $H_\xi \circ g_n$ in L^2 independently of n, that is, for all $\varepsilon > 0$, there exists v such that

$$\|H_\xi \circ g_n - v \circ g_n\|_{L^2} \le \frac{\varepsilon}{3} \text{ and } \|H_\xi - v\|_{L^2} \le \frac{\varepsilon}{3} \quad (3.15)$$

for all n. By Lebesgue's dominated convergence theorem, we have $v \circ g_n \to v$ in $L^2_{\rm per}$. Hence, for n large enough, we have $\|v \circ g_n - v\|_{L^2} \le \frac{\varepsilon}{3}$ which, together with (3.15), implies $\left\|\bar H_\xi \circ g_n - \bar H_\xi\right\|_{L^2} \le \varepsilon$, and $\bar H_\xi \circ g_n \to \bar H_\xi$ in $L^2_{\rm per}$. It remains to prove that Γ_2 is continuous from $\mathcal{F}_0$ to $\mathcal{H}$. We consider a sequence $X_n = (y_n, U_n, H_n)$ in $\mathcal{F}_0$ which converges to $X = (y, U, H)$ and denote $a_n = \int_0^1 y_n(\xi)\,d\xi$ and $a = \int_0^1 y(\xi)\,d\xi$. We set $\bar X_n = \Gamma_2(X_n)$ and $\bar X = \Gamma_2(X)$. Since $y_n \to y$ in L^∞, $a_n \to a$. We have $\bar y_n = y_n \circ \tau_{a_n}$, $\bar H = H_n \circ \tau_{a_n} + (1 + h_n)a_n$, $\bar U_n = U \circ \tau_{a_n}$ where $h_n = H_n(1) - H_n(0)$. Since, $\tau_{a_n} \to \tau_a$ in H^1 and $\tau_{a_n,\xi} = 1$ so that the $\tau_{a_n,\xi}$ are clearly uniformly bounded away from zero and infinity, we can repeat the proof of continuity of Γ_1 and prove that $y_n \circ \tau_{a_n} \to y \circ \tau_a$, $H_n \circ \tau_{a_n} \to H \circ \tau_a$ and $U_n \circ \tau_{a_n} \to U \circ \tau_a$ in H^1. Then, as $a_n \to a$, it follows that $\bar X_n \to \bar X$ and the continuity of Γ_2 is proved. ☐

Remark 3.6. The map Γ_1 is not continuous from $\mathcal{F}$ to $\mathcal{F}_0$ and therefore neither is the map Γ from $\mathcal{F}$ to $\mathcal{H}$. The spaces $\mathcal{F}_\alpha$ were precisely introduced in order to make the map Γ continuous.

3.1. **Continuous semigroup of solutions in $\mathcal{F}/\tilde G$.** We denote by $S\colon \mathcal{F} \times \mathbb{R}_+ \to \mathcal{F}$ the continuous semigroup which to any initial data $\bar X \in \mathcal{F}$ associates the solution $X(t)$ of the system of differential equation (2.15) at time t. As we indicated earlier, the Camassa–Holm equation is invariant with respect to relabeling, more precisely, using our terminology, we have the following result.

Theorem 3.7. *For any $t > 0$, the map $S_t\colon \mathcal{F} \to \mathcal{F}$ is $\tilde G$-equivariant, that is,*

$$S_t\left((f,\gamma) \bullet X\right) = (f,\gamma) \bullet S_t(X) \quad (3.16)$$

for any $X \in \mathcal{F}$ and $(f,\gamma) \in \tilde G$. Hence, the map $\tilde S_t$ from $\mathcal{F}/\tilde G$ to $\mathcal{F}/\tilde G$ given by

$$\tilde S_t([X]) = [S_t X]$$

is well-defined. It generates a continuous semigroup.

Proof. For any $X_0 = (y_0, U_0, H_0) \in \mathcal{F}$ and $(f,\gamma) \in \tilde G$, we denote $\bar X_0 = (\bar y_0, \bar U_0, \bar H_0) = (f,\gamma) \bullet X_0$, $X(t) = S_t(X_0)$ and $\bar X(t) = S_t(\bar X_0)$. We claim that $(f,\gamma) \bullet X(t)$ satisfies (2.15) and therefore, since $(f,\gamma) \bullet X(t)$ and $\bar X(t)$ satisfy the same system of differential equation with the same initial data, they are equal. We denote $\hat X(t) = (\hat y(t), \hat U(t), \hat H(t)) = (f,\gamma) \bullet X(t)$. We have

$$\hat U_t = \frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi - \eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(\hat y(\xi) - y(\eta))\big)\left[U(\eta)^2 y_\xi(\eta) + H_\xi(\eta)\right] d\eta. \quad (3.17)$$

We have $\hat y_\xi(\xi) = y_\xi(f(\xi)) f_\xi(\xi)$ and $\hat H_\xi(\xi) = H_\xi(f(\xi)) f_\xi(\xi)$ for almost every $\xi \in \mathbb{R}$. Hence, after the change of variable $\eta = f(\eta')$, we get from (3.17) that

$$\hat U_t = \frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi - \eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(\hat y(\xi) - \hat y(\eta))\big)\left[\hat U(\eta)^2 \hat y_\xi(\eta) + \hat H_\xi(\eta)\right] d\eta.$$

We treat similarly the other terms in (2.15), and it follows that $(\hat{y}, \hat{U}, \hat{H})$ is a solution of (2.15). Since $(\hat{y}, \hat{U}, \hat{H})$ and $(\bar{y}, \bar{U}, \bar{H})$ satisfy the same system of ordinary differential equations with the same initial data, they are equal, i.e., $\bar{X}(t) = (f, \gamma) \bullet X(t)$ and (3.16) is proved. We have the following diagram:

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\Pi"] & \mathcal{F}/\tilde{G} \\ \mathcal{F}_\alpha \arrow[u, "\Gamma"] & \\ \mathcal{H} \arrow[u, "S_t"] \arrow[r, "\Pi"] & \mathcal{F}/\tilde{G} \arrow[uu, "\tilde{S}_t"'] \end{tikzcd} \tag{3.18}$$

on a bounded domain of $\mathcal{H}$ whose diameter together with t determines the constant α, see Lemma 3.3. By the definition of the metric on $\mathcal{F}/\tilde{G}$, the map Π is an isometry from $\mathcal{H}$ to $\mathcal{F}/\tilde{G}$. Hence, from the diagram (3.18), we see that $\tilde{S}_t \colon \mathcal{F}/\tilde{G} \to \mathcal{F}/\tilde{G}$ is continuous if and only if $\Gamma \circ S_t \colon \mathcal{H} \to \mathcal{H}$ is continuous. Let us prove that $\Gamma \circ S_t \colon \mathcal{H} \to \mathcal{H}$ is sequentially continuous. We consider a sequence $X_n \in \mathcal{H}$ that converges to $X \in \mathcal{H}$ in $\mathcal{H}$, that is, $\lim_{n\to\infty} \|X_n - X\|_E = 0$. From Theorem 2.7, we get that $\lim_{n\to\infty} \|S_t(X_n) - S_t(X)\|_E = 0$. Since $X_n \to X$ in E, there exists a constant $C \geq 0$ such that $\|X_n\| \leq C$ for all n. Lemma 3.3 gives us that $S_t(X_n) \in \mathcal{F}_\alpha$ for some α which depends on C and t. Hence, $S_t(X_n) \to S_t(X)$ in $\mathcal{F}_\alpha$. Then, by Lemma 3.5, we obtain that $\Gamma \circ S_t(X_n) \to \Gamma \circ S_t(X)$ in $\mathcal{H}$. □

3.2. **Maps between the two coordinate systems.** Our next task is to derive the correspondence between Eulerian coordinates (functions in $\mathcal{D}$) and Lagrangian coordinates (functions in $\mathcal{F}/\tilde{G}$). Earlier we considered initial data in $\mathcal{D}$ with a special structure: The energy density μ was given by $(u^2 + u_x^2)\,dx$ and therefore μ did not have any singular part. The set $\mathcal{D}$ however allows the energy density to have a singular part and a positive amount of energy can concentrate on a set of Lebesgue measure zero. We constructed corresponding initial data in $\mathcal{F}_0$ by the means of (2.24). This construction can be generalized to the set $\mathcal{D}$. To any positive periodic Radon measure μ, we associate the function F_μ defined as

$$F_\mu(x) = \begin{cases} \mu([0,x)) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\mu([x,0)) & \text{if } x < 0. \end{cases}$$

The function F_μ is lower-semicontinuous, increasing and

$$F_\mu(b) - F_\mu(a) = \mu([a,b)) \tag{3.19}$$

for all $a < b$ in $\mathbb{R}$, see for example [16]. We denote by $L \colon \mathcal{D} \to \mathcal{F}/\tilde{G}$ the map transforming Eulerian coordinates into Lagrangian coordinates whose definition is contained in the following theorem.

Theorem 3.8. *For any (u, μ) in $\mathcal{D}$, let*

$$h = \mu([0,1)), \tag{3.20a}$$

$$y(\xi) = \sup\left\{y \mid F_\mu(y) + y < (1+h)\xi\right\}, \tag{3.20b}$$

$$H(\xi) = (1+h)\xi - y(\xi), \tag{3.20c}$$

$$U(\xi) = u \circ y(\xi)\,. \tag{3.20d}$$

Then $(y, U, H) \in \mathcal{F}_0$. We define $L(u, \mu) \in \mathcal{F}/\tilde{G}$ to be the equivalence class of (y, U, H).

Proof. From (3.19), since μ is periodic, we obtain that

$$F_\mu(z+1) = F_\mu(z) + h. \tag{3.21}$$

Hence, for all $z \in \mathbb{R}$, we have $F_\mu(z+1) + z + 1 < (1+h)(\xi+1)$ if and only if $F_\mu(z) + z < (1+h)\xi$, and it follows that $y(\xi+1) = y(\xi)+1$ and $H(\xi+1) = H(\xi)+h$ for all $\xi \in \mathbb{R}$. The function F_μ is increasing. Hence, the function $z \mapsto F_\mu(z) + z$ and therefore y are also increasing. Let us prove that y is Lipschitz with Lipschitz constant at most $1+h$. We consider ξ, ξ' in $\mathbb{R}$ such that $\xi < \xi'$ and $y(\xi) < y(\xi')$ (the case $y(\xi) = y(\xi')$ is straightforward). It follows from the definition that there exists an increasing sequence, x_i', and a decreasing one, x_i such that $\lim_{i\to\infty} x_i = y(\xi)$, $\lim_{i\to\infty} x_i' = y(\xi')$ with $F_\mu(x_i') + x_i' < (1+h)\xi'$ and $F_\mu(x_i) + x_i \geq (1+h)\xi$. Subtracting the second inequality from the first, we obtain

$$F_\mu(x_i') - F_\mu(x_i) + x_i' - x_i < (1+h)(\xi' - \xi). \tag{3.22}$$

For i large enough, since by assumption $y(\xi) < y(\xi')$, we have $x_i < x_i'$ and therefore $F_\mu(x_i') - F_\mu(x_i) = \mu([x_i, x_i')) \geq 0$. Hence, $x_i' - x_i < (1+h)(\xi' - \xi)$. Letting i tend to infinity, we get $y(\xi') - y(\xi) \leq (1+h)(\xi' - \xi)$. Hence, y is Lipschitz with Lipschitz constant bounded by $1+h$ and, by Rademacher's theorem, it is differentiable almost everywhere. Following [16], we decompose μ into its absolute continuous, singular continuous and singular part, denoted μ_{ac}, μ_{sc} and μ_{s}, respectively. Here, since $(u, \mu) \in \mathcal{D}$, we have $\mu_{\text{ac}} = (u^2 + u_x^2)\,dx$. The support of μ_{s} consists of a countable set of points. The points of discontinuity of F_μ exactly coincide with the support of μ_{s} (see [16]). Let A denote the complement of $y^{-1}(\text{supp}(\mu_{\text{s}}))$. We claim that for any $\xi \in A$, we have

$$F_\mu(y(\xi)) + y(\xi) = (1+h)\xi. \tag{3.23}$$

From the definition of $y(\xi)$ follows the existence of an increasing sequence x_i which converges to $y(\xi)$ and such that $F_\mu(x_i) + x_i < (1+h)\xi$. Since F_μ is lower semi-continuous, $\lim_{i\to\infty} F_\mu(x_i) = F_\mu(y(\xi))$ and therefore

$$F_\mu(y(\xi)) + y(\xi) \leq (1+h)\xi. \tag{3.24}$$

Let us assume that $F_\mu(y(\xi)) + y(\xi) < (1+h)\xi$. Since $y(\xi)$ is a point of continuity of F_μ, we can then find an x such that $x > y(\xi)$ and $F_\mu(x) + x < (1+h)\xi$. This contradicts the definition of $y(\xi)$ and proves our claim (3.23). In order to check that (2.25c) is satisfied, we have to compute y_ξ and U_ξ. We define the set B_1 as

$$B_1 = \left\{ x \in \mathbb{R} \mid \lim_{\rho \downarrow 0} \frac{1}{2\rho} \mu((x-\rho, x+\rho)) = (u^2 + u_x^2)(x) \right\}.$$

Since $(u^2 + u_x^2)\,dx$ is the absolutely continuous part of μ, we have, from Besicovitch's derivation theorem (see [2]), that $\text{meas}(B_1^c) = 0$. Given $\xi \in y^{-1}(B_1)$, we denote $x = y(\xi)$. We claim that for all $i \in \mathbb{N}$, there exists $0 < \rho < \frac{1}{i}$ such that $x - \rho$ and $x + \rho$ both belong to $\text{supp}(\mu_{\text{s}})^c$. Assume namely the opposite. Then for any $z \in (x - \frac{1}{i}, x + \frac{1}{i}) \setminus \text{supp}(\mu_{\text{s}})$, we have that $z' = 2x - z$ belongs to $\text{supp}(\mu_{\text{s}})$. Thus we can construct an injection between the uncountable set $(x - \frac{1}{i}, x + \frac{1}{i}) \setminus \text{supp}(\mu_{\text{s}})$ and the countable set $\text{supp}(\mu_{\text{s}})$. This is impossible, and our claim is proved. Hence, since y is surjective, we can find two sequences ξ_i and ξ_i' in A such that $\frac{1}{2}(y(\xi_i) + y(\xi_i')) = y(\xi)$ and $y(\xi_i') - y(\xi_i) < \frac{1}{i}$. We have, by (3.19) and (3.23), since $y(\xi_i)$ and $y(\xi_i')$ belong to A,

$$\mu([y(\xi_i), y(\xi_i'))) + y(\xi_i') - y(\xi_i) = (1+h)(\xi_i' - \xi_i). \tag{3.25}$$

Since $y(\xi_i) \notin \text{supp}(\mu_{\text{s}})$, $\mu(\{y(\xi_i)\}) = 0$ and $\mu([y(\xi_i), y(\xi_i'))) = \mu((y(\xi_i), y(\xi_i')))$. Dividing (3.25) by $\xi_i' - \xi_i$ and letting i tend to ∞, we obtain

$$y_\xi(\xi)(u^2 + u_x^2)(y(\xi)) + y_\xi(\xi) = 1 + h \tag{3.26}$$

where y is differentiable in $y^{-1}(B_1)$, that is, almost everywhere in $y^{-1}(B_1)$. We now derive a short lemma which will be useful several times in this proof.

Lemma 3.9. *Given an increasing Lipschitz function $f\colon \mathbb{R} \to \mathbb{R}$, for any set B of measure zero, we have $f_\xi = 0$ almost everywhere in $f^{-1}(B)$.*

Proof of Lemma 3.9. The lemma follows directly from the area formula:

$$\int_{f^{-1}(B)} f_\xi(\xi)\,d\xi = \int_{\mathbb{R}} \mathcal{H}^0\left(f^{-1}(B) \cap f^{-1}(\{x\})\right)\,dx \tag{3.27}$$

where $\mathcal{H}^0$ is the multiplicity function, see [2] for the formula and the precise definition of $\mathcal{H}^0$. The function $\mathcal{H}^0\left(f^{-1}(B) \cap f^{-1}(\{x\})\right)$ is Lebesgue measurable (see [2]) and it vanishes on B^c. Hence, $\int_{f^{-1}(B)} f_\xi\,d\xi = 0$ and therefore, since $f_\xi \geq 0$, $f_\xi = 0$ almost everywhere in $f^{-1}(B)$. ☐

We apply Lemma 3.9 to B_1^c and get, since $\operatorname{meas}(B_1^c) = 0$, that $y_\xi = 0$ almost everywhere on $y^{-1}(B_1^c)$. On $y^{-1}(B_1)$, we proved that y_ξ satisfies (3.26). It follows that $0 \leq y_\xi \leq 1 + h$ almost everywhere, which implies, since $H_\xi = 1 + h - y_\xi$, that $H_\xi \geq 0$. In the same way as we proved that y was Lipschitz with Lipschitz constant at most $1 + h$, we can prove that the function $\xi \mapsto \int_{-\infty}^{y(\xi)} u_x^2\,dx$ is also Lipschitz. Indeed, from (3.22), for i large enough, we have

$$\int_{x_i}^{x_i'} u_x^2\,dx \leq \mu_{\mathrm{ac}}([x_i, x_i')) \leq \mu([x_i, x_i')) = F_\mu(x_i') - F_\mu(x_i) < (1+h)(\xi' - \xi).$$

Since $\lim_{i\to\infty} x_i' = y(\xi')$ and $\lim_{i\to\infty} x_i = y(\xi)$, letting i tend to infinity, we obtain $\int_{y(\xi)}^{y(\xi')} u_x^2\,dx < (1+h)(\xi' - \xi)$ and the function $\xi \mapsto \int_0^{y(\xi)} u_x^2\,dx$ is Lipchitz with Lipschitz coefficient at most $1 + h$. For all $(\xi, \xi') \in \mathbb{R}^2$, we have, after using the Cauchy–Schwarz inequality,

$$\begin{aligned}
|U(\xi') - U(\xi)| &= \int_{y(\xi)}^{y(\xi')} u_x\,dx \\
&\leq \sqrt{y(\xi') - y(\xi)}\sqrt{\int_{y(\xi)}^{y(\xi')} u_x^2\,dx} \\
&\leq (1+h)\,|\xi' - \xi|
\end{aligned} \tag{3.28}$$

because y and $\xi \mapsto \int_0^{y(\xi)} u_x^2\,dx$ are Lipschitz with Lipschitz constant at most $1 + h$. Hence, U is also Lipschitz and therefore differentiable almost everywhere. We denote by B_2 the set of Lebesgue points of u_x in B_1, i.e.,

$$B_2 = \{x \in B_1 \mid \lim_{\rho\to 0} \frac{1}{\rho}\int_{x-\rho}^{x+\rho} u_x(t)\,dt = u_x(x)\}.$$

We have $\operatorname{meas}(B_2^c) = 0$. We choose a sequence ξ_i and ξ_i' such that $\frac{1}{2}(y(\xi_i) + y(\xi_i')) = x$ and $y(\xi_i') - y(\xi_i) \leq \frac{1}{i}$. Thus

$$\frac{U(\xi_i') - U(\xi_i)}{\xi_i' - \xi_i} = \frac{\int_{y(\xi_i)}^{y(\xi_i')} u_x(t)\,dt}{y(\xi_i') - y(\xi_i)}\,\frac{y(\xi_i') - y(\xi_i)}{\xi_i' - \xi_i}.$$

Hence, letting i tend to infinity, we get that for every ξ in $y^{-1}(B_2)$ where U and y are differentiable, that is, almost everywhere on $y^{-1}(B_2)$,

$$U_\xi(\xi) = y_\xi(\xi) u_x(y(\xi)). \tag{3.29}$$

From (3.28) and using the fact that $\xi \mapsto \int_0^{y(\xi)} u_x^2\,dx$ is Lipschitz with Lipschitz constant at most $1+h$, we get

$$\left|\frac{U(\xi')-U(\xi)}{\xi'-\xi}\right| \leq \sqrt{1+h}\sqrt{\frac{y(\xi')-y(\xi)}{\xi'-\xi}}.$$

Hence,

$$|U_\xi(\xi)| \leq \sqrt{y_\xi(\xi)}. \tag{3.30}$$

Since $\operatorname{meas}(B_2^c)=0$, we have by Lemma 3.9, that $y_\xi=0$ almost everywhere on $y^{-1}(B_2^c)$. Hence, $U_\xi=0$ almost everywhere on $y^{-1}(B_2^c)$. Thus, we have computed U_ξ almost everywhere. It remains to verify (2.25c). We have, after using (3.26) and (3.29), that $y_\xi H_\xi = y_\xi(1+h-y_\xi) = y_\xi^2(u^2+u_x^2)\circ y$ and, finally, $y_\xi H_\xi = y_\xi^2 U^2 + U_\xi^2$ almost everywhere on $y^{-1}(B_2)$. On $y^{-1}(B_2^c)$, we have $y_\xi = U_\xi = 0$ almost everywhere. Therefore (2.25c) is satisfied almost everywhere. By definition, we have $\frac{1}{1+h}(y+H)=\mathrm{Id}$, which concludes the proof of the theorem. □

Remark 3.10. If μ is absolutely continuous, then $\mu=(u^2+u_x^2)dx$ and, from (3.23), we get

$$\int_0^{y(\xi)} (u^2+u_x^2)\,dx + y(\xi) = (1+h)\xi$$

for all $\xi\in\mathbb{R}$.

At the very beginning, $H(t,\xi)$ was introduced as the energy contained in a strip between $y(0,\xi)$ and $y(t,\xi)$, see (2.3). This interpretation still holds. We obtain μ, the energy density in Eulerian coordinates, by pushing forward by y the energy density in Lagrangian coordinates, $H_\xi\,d\xi$. We recall that the push-forward of a measure ν by a measurable function f is the measure $f_\#\nu$ defined as

$$f_\#\nu(B) = \nu(f^{-1}(B))$$

for all Borel sets B. We are led to the map M which transforms Lagrangian coordinates into Eulerian coordinates and whose definition is contained in the following theorem.

Theorem 3.11. *Given any element $[X]$ in $\mathcal{F}/\tilde{G}$. Then, (u,μ) defined as follows*

$$u(x) = U(\xi) \text{ for any } \xi \text{ such that } x = y(\xi), \tag{3.31a}$$

$$\mu = y_\#(H_\xi\,d\xi) \tag{3.31b}$$

belongs to $\mathcal{D}$ and is independent of the representative $X=(y,U,H)\in\mathcal{F}$ we choose for $[X]$. We denote by $M\colon \mathcal{F}/\tilde{G}\to\mathcal{D}$ the map which to any $[X]$ in $\mathcal{F}/\tilde{G}$ associates (u,μ) as given by (3.31).

Proof. First we have to prove that the definition of u makes sense. Since y is surjective, there exists ξ, which may not be unique, such that $x=y(\xi)$. It remains to prove that, given ξ_1 and ξ_2 such that $x=y(\xi_1)=y(\xi_2)$, we have

$$U(\xi_1)=U(\xi_2). \tag{3.32}$$

Since $y(\xi)$ is an increasing function in ξ, we must have $y(\xi)=x$ for all $\xi\in[\xi_1,\xi_2]$ and therefore $y_\xi(\xi)=0$ in $[\xi_1,\xi_2]$. From (2.25c), we get that $U_\xi(\xi)=0$ for all $\xi\in[\xi_1,\xi_2]$ and (3.32) follows. It is not hard to check that $u(x+1)=u(x)$.

Since y is proper and $H_\xi\,d\xi$ is a Radon measure, we have, see [2, Remark 1.71], that μ is also a Radon measure. The fact that $y(\xi+1)=y(\xi)+1$ implies that $y^{-1}(B+1)=1+y^{-1}(B)$ and, since H_ξ is periodic, it follows that μ is also periodic. For any $\bar{X}=(\bar{y},\bar{U},\bar{H})\in\mathcal{F}$ which is equivalent to X, we denote $(\bar{u},\bar{\mu})$ the pair given by (3.31) when we replace X by $\bar{X}$. There exists $(f,\gamma)\in\tilde{G}$ such that $X=(f,\gamma)\bullet\bar{X}$.

For any x, there exists ξ' such that $x = \bar{y}(\xi')$ and $\bar{u}(x) = \bar{U}(\xi')$. Let $\xi = f^{-1}(\xi')$. As $x = \bar{y}(\xi') = y(\xi)$, by (3.31a), we get $u(x) = U(\xi)$ and, since $U(\xi) = \bar{U}(\xi')$, we finally obtain $\bar{u}(x) = u(x)$. For any continuous function with compact support ϕ, we have

$$\int_{\mathbb{R}} \phi \, d\bar{\mu} = \int_{\mathbb{R}} \phi \circ \bar{y}(\xi') \bar{H}_\xi(\xi') \, d\xi',$$

see [2]. Hence, after making the change of variables $\xi' = f(\xi)$, we obtain

$$\int_{\mathbb{R}} \phi \, d\bar{\mu} = \int_{\mathbb{R}} \phi \circ \bar{y} \circ f(\xi) \, \bar{H}_\xi \circ f(\xi) \, f_\xi(\xi) \, d\xi$$

and, since $H_\xi = \bar{H}_\xi \circ f f_\xi$ almost everywhere,

$$\int_{\mathbb{R}} \phi \, d\bar{\mu} = \int_{\mathbb{R}} \phi \circ y(\xi) H_\xi(\xi) \, d\xi = \int_{\mathbb{R}} \phi \, d\mu.$$

Since ϕ was arbitrary in $C_c(\mathbb{R})$, we get $\bar{\mu} = \mu$. This proves that X and $\bar{X}$ give raise to the same pair (u, μ), which therefore does not depend on the representative of $[X]$ we choose.

Let us prove that $u \in H^1$. We have $u \in L^\infty$, u periodic and it remains to prove that $u_x \in L^2_{\mathrm{loc}}(\mathbb{R})$. Given a bounded open set $\mathcal{U}$ of $\mathbb{R}$, for any smooth function ϕ with compact support in $\mathcal{U}$, we have, using the change of variables $x = y(\xi)$,

$$\int_{\mathcal{U}} u(x)\phi_x(x) \, dx = \int_{y^{-1}(\mathcal{U})} U(\xi)\phi_x(y(\xi))y_\xi(\xi) \, d\xi = -\int_{y^{-1}(\mathcal{U})} U_\xi(\xi)(\phi \circ y)(\xi) \, d\xi \tag{3.33}$$

after integrating by parts. Let $B_1 = \{\xi \in y^{-1}(\mathcal{U}) \mid y_\xi(\xi) > 0\}$. Because of (2.25c), and since $y_\xi \geq 0$ almost everywhere, we have $U_\xi = 0$ almost everywhere on B_1^c. Hence, we can restrict the integration domain in (3.33) to B_1. We divide and multiply by $\sqrt{y_\xi}$ the integrand in (3.33) and obtain, after using the Cauchy–Schwarz inequality,

$$\left| \int_{\mathcal{U}} u\phi_x \, dx \right| = \left| \int_{B_1} \frac{U_\xi}{\sqrt{y_\xi}} (\phi \circ y) \sqrt{y_\xi} \, d\xi \right| \leq \sqrt{\int_{B_1} \frac{U_\xi^2}{y_\xi} \, d\xi} \sqrt{\int_{B_1} (\phi \circ y)^2 y_\xi \, d\xi}.$$

By (2.25c), we have $\frac{U_\xi^2}{y_\xi} \leq H_\xi$. Hence, after another change of variables, we get

$$\left| \int_{\mathcal{U}} u\phi_x \, dx \right| \leq \sqrt{\int_{y^{-1}(\mathcal{U})} H_\xi \, d\xi} \, \|\phi\|_{L^2(\mathcal{U})} . \tag{3.34}$$

Since $\lim_{\xi \to \pm\infty} y(\xi) = \pm\infty$, $y^{-1}(\mathcal{U})$ is bounded and (3.34) implies that $u_x \in L^2(\mathcal{U})$. As $\mathcal{U}$ was an arbitrary bounded open set, it follows that $u_x \in L^2_{\mathrm{loc}}$.

Let us prove that the absolutely continuous part of μ is equal to $(u^2 + u_x^2) \, dx$. We introduce the sets Z and B defined as follows

$$Z = \Big\{ \xi \in \mathbb{R} \mid \ y \text{ is differentiable at } \xi \text{ and } y_\xi(\xi) = 0 \\ \text{or } y \text{ or } U \text{ are not differentiable at } \xi \Big\}$$

and

$$B = \{x \in y(Z)^c \mid u \text{ is differentiable at } x\}.$$

Since u belongs to H^1, it is differentiable almost everywhere. We have, since y is Lipschitz and by the definition of Z, that $\mathrm{meas}(y(Z)) = \int_Z y_\xi(\xi) \, d\xi = 0$. Hence, $\mathrm{meas}(B^c) = 0$. For any $\xi \in y^{-1}(B)$, we denote $x = y(\xi)$. By necessity, we have $\xi \in Z^c$. Let ξ_i be a sequence converging to ξ such that $\xi_i \neq \xi$ for all i. We write

$x_i = y(\xi_i)$. Since $y_\xi(\xi) > 0$, for i large enough, $x_i \neq x$. The following quantity is well-defined

$$\frac{U(\xi_i) - U(\xi)}{\xi_i - \xi} = \frac{u(x_i) - u(x)}{x_i - x}\,\frac{x_i - x}{\xi_i - \xi}.$$

Since u is differentiable at x and ξ belongs to Z^c, we obtain, after letting i tend to infinity, that

$$U_\xi(\xi) = u_x(y(\xi))y_\xi(\xi). \tag{3.35}$$

For all subsets B' of B, we have

$$\mu(B') = \int_{y^{-1}(B')} H_\xi \, d\xi = \int_{y^{-1}(B')} \left(U^2 + \frac{U_\xi^2}{y_\xi^2}\right) y_\xi \, d\xi.$$

We can divide by y_ξ in the integrand above because y_ξ does not vanish on $y^{-1}(B)$. After a change of variables and using (3.35), we obtain

$$\mu(B') = \int_{B'} (u^2 + u_x^2)\, dx. \tag{3.36}$$

Since (3.36) holds for any set $B' \subset B$ and $\mathrm{meas}(B^c) = 0$, we have $\mu_{\mathrm{ac}} = (u^2 + u_x^2)\, dx$. ☐

The next theorem shows that the transformation from Eulerian to Lagrangian coordinates is a bijection.

Theorem 3.12. *The maps M and L are invertible. We have*

$$L \circ M = \mathrm{Id}_{\mathcal{F}/\tilde{G}} \text{ and } M \circ L = \mathrm{Id}_{\mathcal{D}}\,.$$

Proof. Given $[X]$ in $\mathcal{F}/\tilde{G}$, we choose $X = (y, U, H) = \tilde{\Gamma}([X])$ as a representative of $[X]$ and consider (u, μ) given by (3.31) for this particular X. Note that, from the definition of $\tilde{\Gamma}$, we have $X \in \mathcal{H}$. Let $\bar{X} = (\bar{y}, \bar{U}, \bar{H})$ be the representative of $L(u, \mu)$ in $\mathcal{F}_0$ given by the formulas (3.20). We claim that $(\bar{y}, \bar{U}, \bar{H})$ and (y, U, H) are equivalent and therefore $L \circ M = \mathrm{Id}_{\mathcal{F}/\tilde{G}}$. Let

$$g(x) = \sup\{\xi \in \mathbb{R} \mid y(\xi) < x\}. \tag{3.37}$$

It is not hard to prove, using the fact that y is increasing and continuous, that

$$y(g(x)) = x \tag{3.38}$$

and $y^{-1}([a, b)) = [g(a), g(b))$ for any $a < b$ in $\mathbb{R}$. Hence, by (3.31b), we have

$$\mu([a, b)) = \int_{y^{-1}([a,b))} H_\xi \, d\xi = \int_{g(a)}^{g(b)} H_\xi \, d\xi = H(g(b)) - H(g(a)). \tag{3.39}$$

Since $X \in \mathcal{F}_0$, $y + H = (1 + h)\,\mathrm{Id}$ and from (3.38) we obtain $H(g(b)) = (1+h)g(b) - b$ and a similar expression for $H(g(a))$. From (3.39) and the definition of F_μ, it follows then that

$$F_\mu(x) + x = (1 + h)(g(x) - g(0)). \tag{3.40}$$

Inserting this result into the definition of $\bar{y}$, we obtain that

$$\bar{y}(\xi) = \sup\{x \in \mathbb{R} \mid g(x) < \xi + g(0)\}. \tag{3.41}$$

For any given $\xi \in \mathbb{R}$, let us consider an increasing sequence x_i tending to $\bar{y}(\xi)$ such that $g(x_i) < \xi + g(0)$; such sequence exists by (3.41). Since y is increasing and using (3.38), it follows that $x_i \leq y(\xi + g(0))$. Letting i tend to ∞, we obtain $\bar{y}(\xi) \leq y(\xi + g(0))$. Assume that $\bar{y}(\xi) < y(\xi + g(0))$. Then, there exists x such that $\bar{y}(\xi) < x < y(\xi + g(0))$ and equation (3.41) then implies that $g(x) \geq \xi + g(0)$. Hence, $x \geq y(\xi + g(0))$, as y is increasing, and contradicts the fact that $x < y(\xi + g(0))$. Thus we have

$$\bar{y}(\xi) = y(\xi + g(0)). \tag{3.42}$$

Since $\bar{X} \in \mathcal{F}_0$, $\bar{H}(\xi) = (1+h)\xi - \bar{y}(\xi)$. Hence, $\bar{H}(\xi) = (1+h)\xi - y(\xi + g(0))$ and, since $y(\xi + g(0)) + H(\xi + g(0)) = (1+h)(\xi + g(0))$,

$$\bar{H}(\xi) = H(\xi + g(0)) - (1+h)g(0). \tag{3.43}$$

It is not hard to prove that

$$\bar{U}(\xi) = U(\xi + g(0)) \tag{3.44}$$

From (3.42), (3.43) and (3.44), it follows, as claimed, that X and $\bar{X}$ are equivalent, as $\bar{X} = (\tau_{-g(0)}, -(1+h)g(0)) \bullet X$. In fact, we have $X = \Gamma_2(\bar{X})$. Thus we have proved that $L \circ M = \mathrm{Id}_{\mathcal{F}/\tilde{G}}$.

Given (u,μ) in $\mathcal{D}$, we denote by (y,U,H) the representative of $L(u,\mu)$ in $\mathcal{F}_0$ given by (3.20). Then, let $(\bar{u},\bar{\mu}) = M \circ L(u,\mu)$. We claim that $(\bar{u},\bar{\mu}) = (u,\mu)$. Let g be the function defined as before by (3.37). The same computation that leads to (3.40) now gives

$$F_{\bar{\mu}}(x) + x = (1+h)(g(x) - g(0)). \tag{3.45}$$

Given $\xi \in \mathbb{R}$, we consider an increasing sequence x_i which converges to $y(\xi)$ and such that $F_\mu(x_i) + x_i < (1+h)\xi$. The existence of such sequence is guaranteed by (3.20b). Passing to the limit and since F_μ is lower semi-continuous, we obtain $F_\mu(y(\xi)) + y(\xi) \leq (1+h)\xi$. We take $\xi = g(x)$ and get

$$F_\mu(x) + x \leq (1+h)g(x). \tag{3.46}$$

From the definition of g, there exists an increasing sequence ξ_i which converges to $g(x)$ such that $y(\xi_i) < x$. The definition (3.20b) of y tells us that $F_\mu(x) + x \geq (1+h)\xi_i$. Letting i tend to infinity, we obtain $F_\mu(x) + x \geq (1+h)g(x)$ which, together with (3.46), yields

$$F_\mu(x) + x = (1+h)g(x). \tag{3.47}$$

Comparing (3.47) and (3.45) we get that $F_\mu = F_{\bar{\mu}} + (1+h)g(0)$. Hence $\bar{\mu} = \mu$. It is clear from the definitions that $\bar{u} = u$. Hence, $(\bar{u},\bar{\mu}) = (u,\mu)$ and $M \circ L = \mathrm{Id}_{\mathcal{D}}$. $\square$

4. Continuous semigroup of solutions on $\mathcal{D}$

The fact that we have been able to establish a bijection between the two coordinate systems, $\mathcal{F}/\tilde{G}$ and $\mathcal{D}$, enables us now to transport the topology defined in $\mathcal{F}/\tilde{G}$ into $\mathcal{D}$. On $\mathcal{D}$ we define the distance $d_{\mathcal{D}}$ which makes the bijection L between $\mathcal{D}$ and $\mathcal{F}/\tilde{G}$ into an isometry:

$$d_{\mathcal{D}}((u,\mu),(\bar{u},\bar{\mu})) = d_{\mathcal{F}/\tilde{G}}(L(u,\mu), L(\bar{u},\bar{\mu})).$$

Since $\mathcal{F}/\tilde{G}$ equipped with $d_{\mathcal{F}/\tilde{G}}$ is a complete metric space, we have the following theorem.

Theorem 4.1. *$\mathcal{D}$ equipped with the metric d_D is a complete metric space.*

For each $t \in \mathbb{R}$, we define the map T_t from $\mathcal{D}$ to $\mathcal{D}$ by

$$T_t = M\tilde{S}_t L.$$

We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{D} & \mathcal{F}/\tilde{G} \arrow[l, "M"'] \\ \mathcal{D} \arrow[u, "T_t"] \arrow[r, "L"] & \mathcal{F}/\tilde{G} \arrow[u, "\tilde{S}_t"'] \end{tikzcd} \tag{4.1}$$

Our main theorem reads as follows.

Theorem 4.2. *$T\colon \mathcal{D}\times\mathbb{R}_+\to\mathcal{D}$ (where $\mathcal{D}$ is defined by Definition 3.1) defines a continuous semigroup of solutions of the Camassa–Holm equation, that is, given $(\bar u,\bar\mu)\in\mathcal{D}$, if we denote $t\mapsto(u(t),\mu(t))=T_t(\bar u,\bar\mu)$ the corresponding trajectory, then u is a weak solution of the Camassa–Holm equation (1.3). Moreover μ is a weak solution of the following transport equation for the energy density*

$$\mu_t+(u\mu)_x=(u^3-2Pu)_x. \tag{4.2}$$

Furthermore, we have that

$$\mu(t)([0,1))=\mu(0)([0,1)) \text{ for all } t, \tag{4.3}$$

and

$$\mu(t)([0,1))=\mu_{\mathrm{ac}}(t)([0,1))=\|u(t)\|^2_{H^1_{\mathrm{per}}}=\mu(0)([0,1)) \text{ for almost all } t. \tag{4.4}$$

Remark 4.3. We denote the unique solution described in the theorem as a *conservative* weak solution of the Camassa–Holm equation.

Proof. The proof is similar to the non periodic case. We want to prove that, for all $\phi\in C^\infty(\mathbb{R}_+\times\mathbb{R})$ with compact support,

$$\int_{\mathbb{R}_+\times\mathbb{R}}\left[-u(t,x)\phi_t(t,x)+u(t,x)u_x(t,x)\phi(t,x)\right]dxdt=\int_{\mathbb{R}_+\times\mathbb{R}}-P_x(t,x)\phi(t,x)\,dxdt \tag{4.5}$$

where P is given by (2.11) or equivalently (2.6). Let $(y(t),U(t),H(t))$ be a representative of $L(u(t),\mu(t))$ which is solution of (2.15). Since y is Lipschitz in ξ and invertible for $t\in\mathcal{K}^c$ (see (2.32) for the definition of $\mathcal{K}$, in particular, we have $\operatorname{meas}(\mathcal{K})=0$), we can use the change of variables $x=y(t,\xi)$ and, using (3.29), we get

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}}&\left[-u(t,x)\phi_t(t,x)+u(t,x)u_x(t,x)\phi(t,x)\right]dxdt\\&=\int_{\mathbb{R}_+\times\mathbb{R}}\left[-U(t,\xi)y_\xi(t,\xi)\phi_t(t,y(t,\xi))+U(t,\xi)U_\xi(t,\xi)\phi(t,y(t,\xi))\right]d\xi dt.\end{aligned} \tag{4.6}$$

Using the fact that $y_t=U$ and $y_{\xi t}=U_\xi$, one easily checks that

$$(Uy_\xi\phi\circ y)_t-(U^2\phi)_\xi=Uy_\xi\phi_t\circ y-UU_\xi\phi\circ y+U_ty_\xi\phi\circ y. \tag{4.7}$$

After integrating (4.7) over $\mathbb{R}_+\times\mathbb{R}$, the left-hand side of (4.7) vanishes and we obtain

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}}&\left[-Uy_\xi\,\phi_t\circ y+UU_\xi\,\phi\circ y\right]d\xi dt\\&=\frac14\int_{\mathbb{R}_+\times\mathbb{R}^2}\Big[\operatorname{sgn}(\xi-\eta)e^{-\{\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta)\}}\times\big(U^2y_\xi+H_\xi\big)(\eta)y_\xi(\xi)\phi\circ y(\xi)\Big]\,d\eta d\xi dt\end{aligned} \tag{4.8}$$

by (2.15). Again, to simplify the notation, we deliberately omitted the t variable. On the other hand, by using the change of variables $x=y(t,\xi)$ and $z=y(t,\eta)$ when $t\in\mathcal{K}^c$, we have

$$\begin{aligned}-\int_{\mathbb{R}_+\times\mathbb{R}}P_x(t,x)\phi(t,x)\,dxdt=\frac12\int_{\mathbb{R}_+\times\mathbb{R}^2}&\Big[\operatorname{sgn}(y(\xi)-y(\eta))e^{-|y(\xi)-y(\eta)|}\\&\times\big(u^2(t,y(\eta))+\frac12u_x^2(t,y(\eta))\big)\phi(t,y(\xi))y_\xi(\eta)y_\xi(\xi)\Big]\,d\eta d\xi dt.\end{aligned}$$

Since, from Lemma 2.6, y_ξ is strictly positive for $t \in \mathcal{K}^c$ and almost every ξ, we can replace $u_x(t, y(t,\eta))$ by $U_\xi(t,\eta)/y_\xi(t,\eta)$, see (3.29), in the equation above and, using the fact that y is an increasing function and the identity (2.25c), we obtain

$$-\int_{\mathbb{R}_+\times\mathbb{R}} P_x(t,x)\phi(t,x)\,dxdt = \frac{1}{4}\int_{\mathbb{R}_+\times\mathbb{R}^2}\Big[\operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta)\big) \\ \times\big(U^2y_\xi+H_\xi\big)(\eta)y_\xi(\xi)\phi(t,y(\xi))\Big]\,d\eta d\xi dt. \quad (4.9)$$

Thus, comparing (4.8) and (4.9), we get

$$\int_{\mathbb{R}_+\times\mathbb{R}}[-Uy_\xi\,\phi_t(t,y)+UU_\xi\,\phi]\,d\xi dt = -\int_{\mathbb{R}_+\times\mathbb{R}} P_x(t,x)\phi(t,x)\,dxdt$$

and (4.5) follows from (4.6). Similarly, one proves that $\mu(t)$ is solution of (4.2). We have $y^{-1}([0,1)) = [g(0), g(1))$ where g is given by (3.37). From (3.37) and the fact that $y(\xi+1) = y(\xi)+1$ for all ξ, we infer that $g(x+1) = g(x)+1$. Hence, it follows from (3.31a), since H_ξ is periodic, that

$$\mu(t)([0,1)) = \int_{[g(0),g(0)+1)} H_\xi\,d\xi = \int_{[0,1)} H_\xi\,d\xi = H(t,1)-H(t,0)$$

which is constant in time, from the governing equation (2.15). Hence, (4.3) is proved. We know from Lemma 2.6 (*ii*) that, for $t\in\mathcal{K}^c$, $y_\xi(t,\xi) > 0$ for almost every $\xi\in\mathbb{R}$. Given $t\in\mathcal{K}^c$ (the time variable is suppressed in the notation when there is no ambiguity), we have, for any Borel set B,

$$\mu(t)(B) = \int_{y^{-1}(B)} H_\xi\,d\xi = \int_{y^{-1}(B)}\left(U^2+\frac{U_\xi^2}{y_\xi^2}\right)y_\xi\,d\xi \quad (4.10)$$

from (2.25c) and because $y_\xi(t,\xi) > 0$ almost everywhere for $t\in\mathcal{K}^c$. Since y is one-to-one when $t\in\mathcal{K}^c$ and $u_x\circ y y_\xi = U_\xi$ almost everywhere, we obtain from (4.10) that

$$\mu(t)(B) = \int_B (u^2+u_x^2)(t,x)\,dx.$$

Hence, as $\operatorname{meas}(\mathcal{K}) = 0$, (4.4) is proved. □

5. The topology on $\mathcal{D}$

The metric $d_\mathcal{D}$ gives to $\mathcal{D}$ the structure of a complete metric space while it makes continuous the semigroup T_t of conservative solutions for the Camassa–Holm equation as defined in Theorem 4.2. In that respect, it is a suitable metric for the Camassa–Holm equation. However, as the definition of $d_\mathcal{D}$ is not straightforward, this metric is not so easy to manipulate and in this section we compare it with more standard topologies. More precisely, we establish that convergence in $H^1_{\rm per}$ implies convergence in $(\mathcal{D}, d_\mathcal{D})$, which itself implies convergence in $L^\infty_{\rm per}$.

Proposition 5.1. *The map*

$$u\mapsto (u,(u^2+u_x^2)dx)$$

is continuous from $H^1_{\rm per}$ into $\mathcal{D}$. In other words, given a sequence $u_n\in H^1_{\rm per}$ converging to u in $H^1_{\rm per}$, then $(u_n,(u_n^2+u_{nx}^2)dx)$ converges to $(u,(u^2+u_x^2)dx)$ in $\mathcal{D}$.

Proof. Let $X_n = (y_n, U_n, H_n)$ and $X = (y, U, H)$ be the representatives in $\mathcal{F}_0$ given by (3.20) of $L(u_n,(u_n^2+u_{nx}^2)dx)$ and $L(u,(u^2+u_x^2)dx)$, respectively. By definition of the topology of $\mathcal{D}$ and $\mathcal{F}/\tilde{G}$, we have to prove that $\Gamma(X_n)\to\Gamma(X)$ in $\mathcal{H}$. Since Γ is continuous from $\mathcal{F}_0$ into $\mathcal{H}$, see Lemma 3.5, it is enough to prove that $X_n\to X$

in E. We write $g_n = u_n^2 + u_{n,x}^2$ and $g = u^2 + u_x^2$; g_n and g are periodic. Following Remark 3.10, we have

$$\int_0^{y(\xi)} g(x)\,dx + y(\xi) = (1+h)\xi\ , \quad \int_0^{y_n(\xi)} g_n(x)\,dx + y_n(\xi) = (1+h_n)\xi \tag{5.1}$$

and, after taking the difference between the two equations, we obtain

$$\int_0^{y(\xi)} (g-g_n)(x)\,dx + \int_{y_n(\xi)}^{y(\xi)} g_n(x)\,dx + y(\xi) - y_n(\xi) = (h_n - h)\xi. \tag{5.2}$$

Since g_n is positive, $\left|y - y_n + \int_{y_n}^{y} g_n(x)\,d\xi)\right| = |y-y_n| + \left|\int_{y_n}^{y} g_n(x)\,d\xi)\right|$ and (5.2) implies

$$|y(\xi) - y_n(\xi)| \le \int_0^{y(\xi)} |g-g_n|\ dx + |h_n - h|\,|\xi| \le \|g-g_n\|_{L^1} + |h_n - h|$$

because $y(0) = 0$ and therefore $[0, y(\xi)] \subset [0,1]$ for $\xi \in [0,1]$. Since $u_n \to u$ in H^1, $g_n \to g$ in L^1 and it follows that $\zeta_n \to \zeta$ and $H_n \to H$ in L^∞. We recall that $\zeta(\xi) = y(\xi) - \xi$ and $H = h\xi - \zeta$ (as $X, X_n \in \mathcal{F}_0$). We have

$$U_n - U = u_n \circ y_n - u \circ y_n + u \circ y_n - u \circ y. \tag{5.3}$$

Since $u_n \to u$ in L^∞, $u_n \circ y_n \to u \circ y_n$ in L^∞. Moreover, since u is uniformly continuous $[0,1]$ and $y_n \to y$ in L^∞, $u \circ y_n - u \circ y$ in L^∞. Hence, it follows from (5.3) that $U_n \to U$ in L^∞. The measures $(u^2 + u_x^2)dx$ and $(u_n^2 + u_{n,x}^2)dx$ have, by definition, no singular part and in that case (3.26) holds almost everywhere, that is,

$$y_\xi = \frac{1}{g \circ y + 1} \quad \text{and} \quad y_{n,\xi} = \frac{1}{g_n \circ y_n + 1} \tag{5.4}$$

almost everywhere. Hence,

$$\begin{aligned} \zeta_{n,\xi} - \zeta_\xi &= (g \circ y - g_n \circ y_n) y_{n,\xi} y_\xi \\ &= (g \circ y - g \circ y_n) y_{n,\xi} y_\xi + (g \circ y_n - g_n \circ y_n) y_{n,\xi} y_\xi. \end{aligned} \tag{5.5}$$

Since $0 \le y_\xi \le 1 + h$, we have

$$\begin{aligned} \int_{[0,1]} |g \circ y_n - g_n \circ y_n|\, y_{n,\xi} y_\xi\, d\xi &\le (1+h) \int_{[0,1]} |g \circ y_n - g_n \circ y_n|\, y_{n,\xi}\, d\xi \\ &= (1+h) \|g - g_n\|_{L^1}. \end{aligned} \tag{5.6}$$

Let $C = \sup_n (1 + h_n)$. For any $\varepsilon > 0$, there exists a continuous function v with compact support such that $\|g - v\|_{L^1} \le \varepsilon/3C$. We can decompose the first term in the right-hand side of (5.5) into

$$\begin{aligned} (g \circ y - g \circ y_n) y_{n,\xi} y_\xi &= (g \circ y - v \circ y) y_{n,\xi} y_\xi \\ &\quad + (v \circ y - v \circ y_n) y_{n,\xi} y_\xi + (v \circ y_n - g \circ y_n) y_{n,\xi} y_\xi. \end{aligned} \tag{5.7}$$

Then, we have

$$\int_{[0,1]} |g \circ y - v \circ y|\, y_{n,\xi} y_\xi\, d\xi \le (1+h_n) \int |g \circ y - v \circ y|\, y_\xi\, d\xi \le C \|g - v\|_{L^1} \le \varepsilon/3$$

and, similarly, we obtain $\int_{[0,1]} |g \circ y_n - v \circ y_n|\, y_{n,\xi} y_\xi\, d\xi \le \varepsilon/3$. Since $y_n \to y$ in L^∞ and v is continuous, by applying the Lebesgue dominated convergence theorem, we obtain $v \circ y_n \to v \circ y$ in L^1 and we can choose n big enough so that

$$\int_{[0,1]} |v \circ y - v \circ y_n|\, y_{n,\xi} y_\xi\, d\xi \le C^2 \|v \circ y - v \circ y_n\|_{L^1} \le \varepsilon/3.$$

Hence, from (5.7), we get that $\int_{[0,1]}|g\circ y - g\circ y_n|\,y_{n,\xi}y_\xi\,d\xi \leq \varepsilon$ so that

$$\lim_{n\to\infty}\int_{[0,1]}|g\circ y - g\circ y_n|\,y_{n,\xi}y_\xi\,d\xi = 0,$$

and, from (5.5) and (5.6), it follows that $\zeta_{n,\xi}\to\zeta_\xi$ in L^1. Since $X_n\in\mathcal{F}_0$, $\zeta_{n,\xi}$ is bounded in L^∞ and we finally get that $\zeta_{n,\xi}\to\zeta_\xi$ in L^2 and, by (3.20c), $H_{n,\xi}\to H_\xi$ in L^2. It remains to prove that $U_{n,\xi}\to U_\xi$ in L^2. Since $y_{n,\xi}$, $H_{n,\xi}$ and U tend to y_ξ, H_ξ and U in L^2, respectively and $\|U_n\|_{L^\infty}$, $\|y_{n,\xi}\|_{L^\infty}$ are uniformly bounded, it follows from (2.25c) that

$$\lim_{n\to\infty}\|U_{n,\xi}\|_{L^2} = \|U_\xi\|_{L^2}\,. \tag{5.8}$$

Once we have proved that $U_{n,\xi}$ converges weakly to U_ξ, then (**??**) will imply that $U_{n,\xi}\to U_\xi$ strongly in L^2, see, for example, [28, section V.1]. For any continuous function ϕ with compact support, we have

$$\int_{\mathbb{R}} U_{n,\xi}\phi\,d\xi = \int_{\mathbb{R}} u_{n,x}\circ y_n y_{n,\xi}\phi\,d\xi = \int_{\mathbb{R}} u_{n,x}\,\phi\circ {y_n}^{-1}\,d\xi. \tag{5.9}$$

By assumption, we have $u_{n,x}\to u_x$ in L^2. Since $y_n\to y$ in L^∞, the support of $\phi\circ {y_n}^{-1}$ is contained in some compact that can be chosen to be independent of n. Thus, after using Lebesgue's dominated convergence theorem, we obtain that $\phi\circ {y_n}^{-1}\to\phi\circ y^{-1}$ in L^2 and therefore

$$\lim_{n\to\infty}\int_{\mathbb{R}} U_{n,\xi}\phi\,d\xi = \int_{\mathbb{R}} u_x\,\phi\circ y^{-1}\,d\xi = \int_{\mathbb{R}} U_\xi\phi\,d\xi. \tag{5.10}$$

From (5.8), we have that $U_{n,\xi}$ is bounded and therefore, by a density argument, (5.10) holds for any function ϕ in L^2 and $U_{n,\xi}\rightharpoonup U_\xi$ weakly in L^2. ☐

Proposition 5.2. *Let (u_n,μ_n) be a sequence in $\mathcal{D}$ that converges to (u,μ) in $\mathcal{D}$. Then*

$$u_n\to u \text{ in } L^\infty_{\text{per}} \quad\text{and}\quad \mu_n \overset{*}{\rightharpoonup} \mu.$$

Proof. We denote by $X_n=(y_n,U_n,H_n)$ and $X=(y,U,H)$ the representative in $\mathcal{H}$ of $L(u_n,\mu_n)$ and $L(u,\mu)$. Let $C=\sup_n(1+h_n)$. For any $x\in\mathbb{R}$, there exists ξ_n and ξ, which may not be unique, such that $x=y_n(\xi_n)$ and $x=y(\xi)$. We set $x_n=y_n(\xi)$. We have

$$u_n(x)-u(x) = u_n(x)-u_n(x_n)+U_n(\xi)-U(\xi), \tag{5.11}$$

and

$$\begin{aligned}
|u_n(x)-u_n(x_n)| &= \left|\int_\xi^{\xi_n} U_{n,\xi}(\eta)\,d\eta\right| \\
&\leq \sqrt{|\xi_n-\xi|}\left(\int_\xi^{\xi_n} U_{n,\xi}^2\,d\eta\right)^{1/2} && \text{(Cauchy–Schwarz)} \\
&\leq \sqrt{|\xi_n-\xi|}\left(\int_\xi^{\xi_n} y_{n,\xi}H_{n,\xi}\,d\eta\right)^{1/2} && \text{(from (2.25c))} \\
&\leq C\sqrt{|\xi_n-\xi|}\sqrt{|y_n(\xi_n)-y_n(\xi)|} && \text{(since } H_{n,\xi}\leq C) \\
&= C\sqrt{|\xi_n-\xi|}\sqrt{|y(\xi)-y_n(\xi)|} \\
&\leq C\sqrt{|\xi_n-\xi|}\,\|y-y_n\|_{L^\infty}^{1/2}\,.
\end{aligned} \tag{5.12}$$

We have

$$|y_n(\xi_n)-y_n(\xi)| = |y(\xi)-y_n(\xi)| \leq \|y_n-y\|_{L^\infty}\,. \tag{5.13}$$

Without loss of generality, we can assume that $\|y_n - y\|_{L^\infty} < 1$ so that (5.13) implies $|\xi_n - \xi| < 1$ as y_n is increasing and $y_n(\bar{\xi}+1) = y_n(\bar{\xi}) + 1$ for all $\bar{\xi}$. Hence, (5.12) implies

$$|u_n(x) - u_n(x_n)| \leq C \|y - y_n\|_{L^\infty}^{1/2}. \tag{5.14}$$

Since $y_n \to y$ and $U_n \to U$ in L^∞, it follows from (5.11) and (5.14) that $u_n \to u$ in L^∞. By weak-star convergence, we mean that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi \, d\mu_n = \int_{\mathbb{R}} \phi \, d\mu \tag{5.15}$$

for all continuous functions with compact support. It follows from (3.31b) that

$$\int_{\mathbb{R}} \phi \, d\mu_n = \int_{\mathbb{R}} \phi \circ y_n H_{n,\xi} \, d\xi \quad \text{and} \quad \int_{\mathbb{R}} \phi \, d\mu = \int_{\mathbb{R}} \phi \circ y H_\xi \, d\xi \tag{5.16}$$

see [2, Definition 1.70]. Since $y_n \to y$ in L^∞, the support of $\phi \circ y_n$ is contained in some compact which can be chosen independently of n and, from Lebesgue's dominated convergence theorem, we have that $\phi \circ y_n \to \phi \circ y$ in L^2. Hence, since $H_{n,\xi} \to H_\xi$ in L^2,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi \circ y_n H_{n,\xi} \, d\xi = \int_{\mathbb{R}} \phi \circ y H_\xi \, d\xi,$$

and (5.15) follows from (5.16).

□

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