VANISHING VISCOSITY METHOD FOR TRANSONIC FLOW

GUI-QIANG CHEN, MARSHALL SLEMROD, AND DEHUA WANG

Dedicated to Constantine M. Dafermos on the Occasion of His 65th Birthday

Abstract. A vanishing viscosity method is formulated for two-dimensional transonic steady irrotational compressible fluid flows with adiabatic constant $\gamma \in [1, 3)$. This formulation allows a family of invariant regions in the phase plane for the corresponding viscous problem, which implies an upper bound uniformly away from cavitation for the viscous approximate velocity fields. Mathematical entropy pairs are constructed through the Loewner-Morawetz relation by entropy generators governed by a generalized Tricomi equation of mixed elliptic-hyperbolic type, and the corresponding entropy dissipation measures are analyzed so that the viscous approximate solutions satisfy the compensated compactness framework. Then the method of compensated compactness is applied to show that a sequence of solutions to the artificial viscous problem, staying uniformly away from stagnation, converges to an entropy solution of the inviscid transonic flow problem.

1. Introduction

In two significant papers written a decade apart, Morawetz [27, 28] presented a program for proving the existence of weak solutions to the equations governing two-dimensional steady irrotational inviscid compressible flow in a channel or exterior to an airfoil. As is well known, the classical results of Shiffman [35] and Bers [4] apply when the upstream speed is sufficiently small, for which the flow remains subsonic and the governing equations are elliptic (also see [12, 14, 15, 16, 17, 18, 19, 20]). However, beyond a certain speed at infinity (determined by the flow geometry), the flow becomes transonic which, coupled with nonlinearity, yields shock formation (cf. [26]). Morawetz's program in [27, 28] was to imbed the problem within an assumed viscous framework for which the compensated compactness framework (see Section 6 of this paper) would be satisfied. Under this assumption, Morawetz proved that solutions of the as yet unidentified viscous problem have a convergent subsequence whose limit is a solution of the transonic flow problem.

The purpose of this paper is to present such a viscous formulation, hence completing part but not all of Morawetz's program. Specifically, a vanishing viscosity method is formulated for two-dimensional transonic steady irrotational compressible fluid flows with adiabatic constant $\gamma \in [1, 3)$ to ensure a family of invariant regions for the corresponding

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viscous problem, which implies an upper bound uniformly away from cavitation for the viscous approximate velocity fields. Mathematical entropy pairs are constructed through the Loewner-Morawetz relation by entropy generators governed by a generalized Tricomi equation of mixed elliptic-hyperbolic type, and the corresponding entropy dissipation measures are analyzed so that the viscous approximate solutions satisfy the compensated compactness framework. Then the method of compensated compactness is applied to show that a sequence of solutions to the viscous problem, staying uniformly away from stagnation, converges to an entropy solution of the inviscid transonic flow problem.

On the other hand, Morawetz's assumption of no stagnation points for the flow has not been removed; and the slip boundary condition $(u,v)\cdot\mathbf{n}=0$ on the obstacle is only satisfied as the inequality $(u,v)\cdot\mathbf{n}\geq 0$ for the general case (the desired condition $(u,v)\cdot\mathbf{n}=0$ may be achieved when a solution has no jump with certain regularity along the obstacle), where (u,v) is the fluid velocity field and $\mathbf{n}$ is the unit normal on the obstacle pointing into the flow. Both issues may reflect rather complicated boundary layer behavior for transonic flow [33] and require further investigation, which are the topics for subsequent research. In addition, we note that, when $\gamma\geq 3$, cavitation is indeed possible and will be the topic of a sequel to this paper.

This paper is divided into nine sections after the introduction. In Section 2, the classical fluid equations are presented for isentropic and isothermal irrotational planar flow. In Section 3, the fluid equations are rewritten in the polar variables $(u,v)=(q\cos\theta, q\sin\theta)$. In Section 4, we analyzes the behavior of the Riemann invariants in the supersonic region, which guides the design of artificial viscous terms in our vanishing viscosity method. In Section 5, we continue our analysis and set boundary conditions for the viscous system. This yields a family of invariant regions in the (u,v) fluid phase plane for $\gamma\in[1,3)$, where γ is the ratio of specific heats if $\gamma>1$ and $\gamma=1$ denotes the case of constant temperature. In particular, any invariant region is uniformly bounded away from the cavitation circle in the phase plane, which yields a upper bound uniformly away from cavitation in the viscous approximate solutions. In Section 6, we formulate a compensated compactness framework for steady flow. In Section 7, we first construct all mathematical entropy pairs for the potential flow system through the Loewner-Morawetz relation by entropy generators governed by the generalized Tricomi equation of mixed elliptic-hyperbolic type. Then we introduce the notion of entropy solutions through an entropy pair by a convex entropy generator suggested by the work of Osher-Hafez-Whitlow [32]. In Section 8, we use the previously derived uniform L^∞ bounds on (u^ϵ, v^ϵ), plus the convex entropy generator introduced in Section 7 to guarantee our problem is within the compensated compactness framework. In Section 9, we develop Morawetz's argument in [28] for proving convergence of a subsequence of solutions of our viscous problem to the irrotational fluid problem. Finally, in Section 10, we prove the existence of smooth solutions to our viscous problem.

2. Mathematical Equations

Consider the artificial viscous system in a domain $\Omega\subset\mathbb{R}^2$:

$$\begin{cases} v_x-u_y=R_1, \\ (\rho u)_x+(\rho v)_y=R_2, \end{cases} \tag{2.1}$$

where R_1 and R_2 are the artificial viscosity terms to be determined, and (u, v) is the flow velocity field. For a polytropic gas with the adiabatic exponent $\gamma > 1$, $p = p(\rho) = \rho^\gamma/\gamma$ is the normalized pressure, the renormalized density ρ is given by Bernoulli's law:

$$\rho = \rho(q) = \big(1 - \frac{\gamma - 1}{2} q^2\big)^{\frac{1}{\gamma-1}}, \tag{2.2}$$

where q is the flow speed defined by $q^2 = u^2 + v^2$. The sound speed c is defined as

$$c^2 = p'(\rho) = 1 - \frac{\gamma - 1}{2} q^2. \tag{2.3}$$

At the cavitation point $\rho = 0$,

$$q = q_{cav} := \sqrt{\frac{2}{\gamma - 1}}.$$

At the stagnation point $q = 0$, the density reaches its maximum $\rho = 1$. Bernoulli's law (2.2) is valid for $0 \le q \le q_{cav}$. At the sonic point $q = c$, (2.3) implies $q^2 = \frac{2}{\gamma+1}$. Define the critical speed q_{cr} as

$$q_{cr} := \sqrt{\frac{2}{\gamma + 1}}.$$

We rewrite Bernoulli's law (2.2) in the form

$$q^2 - q_{cr}^2 = \frac{2}{\gamma + 1}\left(q^2 - c^2\right). \tag{2.4}$$

Thus the flow is subsonic when $q < q_{cr}$, sonic when $q = q_{cr}$, and supersonic when $q > q_{cr}$.

For the isothermal flow ($\gamma = 1$), $p = c^2\rho$ where $c > 0$ is the constant sound speed, and the density ρ is given by Bernoulli's law:

$$\rho = \rho_0 \exp\big(-\frac{u^2 + v^2}{2c^2}\big) \tag{2.5}$$

for some constant $\rho_0 > 0$. In this case, $q_{cr} = c$. After scaling, we can take $c = \rho_0 = 1$.

3. Formulation in Polar Coordinate Phase Plane

We now use the polar coordinates in the phase plane:

$$u = q\cos\theta, \qquad v = q\sin\theta,$$

and rewrite the viscous conservation laws (2.1) in terms of (q, θ). Write the second equation of (2.1) as

$$\rho_x u + \rho u_x + \rho_y v + \rho v_y = R_2$$

or

$$\rho'(q) q_x u + \rho u_x + \rho'(q) q_y v + \rho v_y = R_2,$$

and use

$$q = \sqrt{u^2 + v^2}, \quad q_x = \frac{1}{q}(u u_x + v v_x), \quad q_y = \frac{1}{q}(u u_y + v v_y), \quad \rho'(q) = -\frac{\rho q}{c^2}$$

to find

$$(c^2 - u^2)u_x - uv(v_x + u_y) + (c^2 - v^2)v_y = \frac{c^2}{\rho} R_2.$$

Then (2.1) becomes

$$A \begin{bmatrix} u \\ v \end{bmatrix}_x + B \begin{bmatrix} u \\ v \end{bmatrix}_y = \begin{bmatrix} -R_1 \\ \frac{c^2}{\rho} R_2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & -1 \\ c^2 - u^2 & -uv \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ -uv & c^2 - v^2 \end{bmatrix}.$$

Thus, in terms of (q, θ), we obtain

$$A_1 \begin{bmatrix} q \\ \theta \end{bmatrix}_x + B_1 \begin{bmatrix} q \\ \theta \end{bmatrix}_y = \begin{bmatrix} -R_1 \\ \frac{1}{\rho q} R_2 \end{bmatrix}, \tag{3.1}$$

where

$$A_1 = \begin{bmatrix} -\sin\theta & -q\cos\theta \\ \frac{c^2-q^2}{c^2 q}\cos\theta & -\sin\theta \end{bmatrix}, \qquad B_1 = \begin{bmatrix} \cos\theta & -q\sin\theta \\ \frac{c^2-q^2}{c^2 q}\sin\theta & \cos\theta \end{bmatrix}.$$

In terms of (ρ, θ), using

$$\rho_x = \rho_q q_x = -\frac{\rho q}{c^2} q_x, \quad \rho_y = -\frac{\rho q}{c^2} q_y,$$

and thus

$$\begin{bmatrix} q \\ \theta \end{bmatrix}_x = \begin{bmatrix} -\frac{c^2}{\rho q}\rho_x \\ \theta_x \end{bmatrix}, \qquad \begin{bmatrix} q \\ \theta \end{bmatrix}_y = \begin{bmatrix} -\frac{c^2}{\rho q}\rho_y \\ \theta_y \end{bmatrix},$$

we find

$$A_2 \begin{bmatrix} \rho \\ \theta \end{bmatrix}_x + B_2 \begin{bmatrix} \rho \\ \theta \end{bmatrix}_y = \begin{bmatrix} -R_1 \\ -R_2 \end{bmatrix}, \tag{3.2}$$

where

$$A_2 = \begin{bmatrix} \frac{c^2}{\rho q}\sin\theta & -q\cos\theta \\ \frac{1}{q}(c^2 - q^2)\cos\theta & \rho q\sin\theta \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -\frac{c^2}{\rho q}\cos\theta & -q\sin\theta \\ \frac{1}{q}(c^2 - q^2)\sin\theta & -\rho q\cos\theta \end{bmatrix}.$$

We symmetrize (3.2) to obtain

$$A_3 \begin{bmatrix} \rho \\ \theta \end{bmatrix}_x + B_3 \begin{bmatrix} \rho \\ \theta \end{bmatrix}_y = \begin{bmatrix} -R_1 \\ \frac{q^2}{c^2-q^2} R_2 \end{bmatrix}, \tag{3.3}$$

where

$$A_3 = \begin{bmatrix} \frac{c^2}{\rho q}\sin\theta & -q\cos\theta \\ -q\cos\theta & \frac{-\rho q^3 \sin\theta}{c^2-q^2} \end{bmatrix}, \qquad B_3 = \begin{bmatrix} -\frac{c^2}{\rho q}\cos\theta & -q\sin\theta \\ -q\sin\theta & \frac{\rho q^3 \cos\theta}{c^2-q^2} \end{bmatrix}.$$

4. Choice of Artificial Viscosity from an Analysis of Riemann Invariants

The two matrices A_1, B_1 in (3.1) commute, thus their transposes commute and they have common eigenvectors. The eigenvalues of A_1 and B_1 are

$$\lambda_\pm = -\sin\theta \pm \frac{\sqrt{q^2-c^2}}{c}\cos\theta, \quad \mu_\pm = \cos\theta \pm \frac{\sqrt{q^2-c^2}}{c}\sin\theta.$$

The left eigenvectors of A_1 are

$$(\mp\frac{\sqrt{q^2-c^2}}{qc}, 1),$$

and thus the Riemann invariants $W_\pm$ satisfy

$$\frac{\partial W_\pm}{\partial\theta} = 1, \quad \frac{\partial W_\pm}{\partial q} = \mp\frac{\sqrt{q^2-c^2}}{qc} \qquad \text{for } q \geq c. \tag{4.1}$$

Multiply (3.1) by $(\frac{\partial W_\pm}{\partial q}, \frac{\partial W_\pm}{\partial\theta})$ to obtain

$$\lambda_\pm\frac{\partial W_\pm}{\partial x} + \mu_\pm\frac{\partial W_\pm}{\partial y} = -\frac{\partial W_\pm}{\partial q}R_1 + \frac{1}{\rho q}\frac{\partial W_\pm}{\partial\theta}R_2. \tag{4.2}$$

We now consider the viscosity terms of the form:

$$R_1 = \varepsilon\nabla\cdot(\sigma_1(\rho,\theta)\nabla\theta), \quad R_2 = \varepsilon\nabla\cdot(\sigma_2(\rho,\theta)\nabla\rho). \tag{4.3}$$

In particular, for a special choice of σ_1 and σ_2, we have the desired relations for the Riemann invariants.

Proposition 4.1. *If*

$$\sigma_1 = 1, \quad \sigma_2 = 1 - \frac{c^2}{q^2} \qquad \textit{for } q > c, \tag{4.4}$$

then the Riemann invariants $W_\pm$ satisfy the following equations:

$$\frac{qc}{\sqrt{q^2-c^2}}\left(\lambda_\pm\frac{\partial W_\pm}{\partial x} + \mu_\pm\frac{\partial W_\pm}{\partial y}\right) \mp \varepsilon\Delta W_\pm = -\varepsilon c\frac{(\gamma-3)q^2+4c^2}{2\rho^2q^2\sqrt{q^2-c^2}}|\nabla\rho|^2 \tag{4.5}$$

for $\gamma \geq 1$.

Proof. We first focus on the "+" part, since the "-" part can be done analogously. The proof is divided into three steps.

Step 1: Substitute (4.1) into (4.2) to obtain

$$\lambda_+\frac{\partial W_+}{\partial x} + \mu_+\frac{\partial W_+}{\partial y} = \frac{\sqrt{q^2-c^2}}{qc}R_1 + \frac{1}{\rho q}R_2. \tag{4.6}$$

Multiplication of (4.6) by $\frac{qc}{\sqrt{q^2-c^2}}$ gives

$$\frac{qc}{\sqrt{q^2-c^2}}\left(\lambda_+\frac{\partial W_+}{\partial x} + \mu_+\frac{\partial W_+}{\partial y}\right) = \frac{\partial W_+}{\partial\theta}R_1 + \frac{c}{\rho\sqrt{q^2-c^2}}R_2.$$

Using $\frac{d\rho}{dq} = -\rho q/c^2$, we have

$$\frac{\partial W_+}{\partial\rho} = \frac{\partial W_+}{\partial q}\frac{dq}{d\rho} = \frac{c\sqrt{q^2-c^2}}{\rho q^2},$$

and then

$$\frac{c}{\rho\sqrt{q^2-c^2}} = \frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial \rho}.$$

Therefore,

$$\frac{qc}{\sqrt{q^2-c^2}}\left(\lambda_+\frac{\partial W_+}{\partial x} + \mu_+\frac{\partial W_+}{\partial y}\right) = \frac{\partial W_+}{\partial \theta}R_1 + \frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial \rho}R_2. \tag{4.7}$$

For the choice of viscosity terms in (4.3), we obtain

$$\begin{aligned}
&\frac{qc}{\sqrt{q^2-c^2}}\left(\lambda_+\frac{\partial W_+}{\partial x} + \mu_+\frac{\partial W_+}{\partial y}\right)\\
&= \varepsilon\frac{\partial W_+}{\partial \theta}\nabla\cdot(\sigma_1(\rho,\theta)\nabla\theta) + \varepsilon\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial \rho}\nabla\cdot(\sigma_2(\rho,\theta)\nabla\rho)\\
&= \varepsilon\frac{\partial U_+}{\partial \theta}\left(\frac{\partial \sigma_1}{\partial \rho}\nabla\rho\cdot\nabla\theta + \frac{\partial \sigma_1}{\partial \theta}|\nabla\theta|^2\right) + \varepsilon\sigma_1(\rho,\theta)\frac{\partial W_+}{\partial \theta}\Delta\theta\\
&\quad + \varepsilon\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial \rho}\left(\frac{\partial \sigma_2}{\partial \rho}|\nabla\rho|^2 + \frac{\partial \sigma_2}{\partial \theta}\nabla\rho\cdot\nabla\theta\right) + \varepsilon\sigma_2(\rho,\theta)\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial \rho}\Delta\rho.
\end{aligned} \tag{4.8}$$

Step 2: From a direct calculation, we know

$$\Delta W_\pm = \frac{\partial^2 W_\pm}{\partial \rho^2}|\nabla\rho|^2 + \frac{\partial W_\pm}{\partial \rho}\Delta\rho + \Delta\theta,$$

then substitution for $\Delta\theta$ in (4.8) gives

$$\begin{aligned}
&\frac{qc}{\sqrt{q^2-c^2}}\left(\lambda_+\frac{\partial W_+}{\partial x} + \mu_+\frac{\partial W_+}{\partial y}\right)\\
&= \varepsilon\frac{\partial W_+}{\partial \theta}\left(\frac{\partial \sigma_1}{\partial \rho}\nabla\rho\cdot\nabla\theta + \frac{\partial \sigma_1}{\partial \theta}|\nabla\theta|^2\right) + \varepsilon\sigma_1(\rho,\theta)\frac{\partial W_+}{\partial \theta}\left(\Delta W_+ - \frac{\partial^2 W_+}{\partial \rho^2}|\nabla\rho|^2 - \frac{\partial W_+}{\partial \rho}\Delta\rho\right)\\
&\quad + \varepsilon\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial \rho}\left(\frac{\partial \sigma_2}{\partial \rho}|\nabla\rho|^2 + \frac{\partial \sigma_2}{\partial \theta}\nabla\rho\cdot\nabla\theta\right) + \varepsilon\sigma_2(\rho,\theta)\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial \rho}\Delta\rho.
\end{aligned}$$

To eliminate $\Delta\rho$, we choose

$$\sigma_2(\rho) = \sigma_1(\rho)\frac{q^2-c^2}{q^2},$$

where σ_1 and σ_2 are independent of θ. Then we have

$$\begin{aligned}
&\frac{qc}{\sqrt{q^2-c^2}}\left(\lambda_+\frac{\partial W_+}{\partial x} + \mu_+\frac{\partial W_+}{\partial y}\right)\\
&= \varepsilon\frac{\partial \sigma_1}{\partial \rho}\frac{\partial W_+}{\partial \theta}\nabla\rho\cdot\nabla\theta + \varepsilon\sigma_1(\rho)\frac{\partial W_+}{\partial \theta}\left(\Delta W_+ - \frac{\partial^2 W_+}{\partial \rho^2}|\nabla\rho|^2\right) + \varepsilon\frac{\partial \sigma_2}{\partial \rho}\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial \rho}|\nabla\rho|^2.
\end{aligned}$$

Finally, write $\nabla\theta = \nabla W_+ - \frac{\partial W_+}{\partial\rho}\nabla\rho$ so that

$$\begin{aligned}&\frac{qc}{\sqrt{q^2-c^2}}\left(\lambda_+\frac{\partial W_+}{\partial x}+\mu_+\frac{\partial W_+}{\partial y}\right)-\varepsilon\frac{\partial\sigma_1}{\partial\rho}\frac{\partial W_+}{\partial\theta}\nabla\rho\cdot\nabla W_+-\varepsilon\sigma_1\frac{\partial W_+}{\partial\theta}\Delta W_+\\&=-\varepsilon\frac{\partial\sigma_1}{\partial\rho}\frac{\partial W_+}{\partial\rho}\frac{\partial W_+}{\partial\theta}|\nabla\rho|^2-\varepsilon\sigma_1\frac{\partial^2 W_+}{\partial\rho^2}\frac{\partial W_+}{\partial\theta}|\nabla\rho|^2+\varepsilon\frac{\partial\sigma_2}{\partial\rho}\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial\rho}|\nabla\rho|^2.\end{aligned}\tag{4.9}$$

A convenient choice of σ_1 and σ_2 is as in (4.4):

$$\sigma_1 = 1, \qquad \sigma_2 = \frac{q^2-c^2}{q^2}.$$

Thus, using $\frac{\partial W_+}{\partial\theta} = 1$, we obtain

$$\frac{qc}{\sqrt{q^2-c^2}}\left(\lambda_+\frac{\partial W_+}{\partial x}+\mu_+\frac{\partial W_+}{\partial y}\right)-\varepsilon\Delta W_+=-\varepsilon\left(\frac{\partial^2 W_+}{\partial\rho^2}-\frac{d\sigma_2}{d\rho}\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial\rho}\right)|\nabla\rho|^2.\tag{4.10}$$

Step 3: We now compute the term

$$\frac{\partial^2 W_+}{\partial\rho^2}-\frac{d\sigma_2}{d\rho}\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial\rho}$$

in (4.10).

First we check the simple isothermal case ($\gamma = 1$) for which

$$\rho = e^{-\frac{q^2}{2}}, \quad c = 1; \qquad \frac{d\rho}{dq} = -\rho q.$$

Then

$$\begin{aligned}\frac{\partial W_+}{\partial q} &= -\frac{\sqrt{q^2-1}}{q},\\ \frac{\partial W_+}{\partial\rho} &= \frac{\partial W_+}{\partial q}\frac{dq}{d\rho} = \frac{\sqrt{q^2-1}}{\rho q^2},\\ \frac{\partial^2 W_+}{\partial\rho^2} &= \frac{-q^4+2q^2-2}{\rho^2q^4\sqrt{q^2-1}},\end{aligned}$$

and

$$\sigma_2 = \frac{q^2-1}{q^2} = 1-\frac{1}{q^2}, \qquad \frac{d\sigma_2}{d\rho} = \frac{d\sigma_2}{dq}\frac{dq}{d\rho} = -\frac{2}{\rho q^4}.$$

Thus,

$$\frac{\partial^2 W_+}{\partial\rho^2}-\frac{d\sigma_2}{d\rho}\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial\rho} = -\frac{q^2-2}{\rho^2q^2\sqrt{q^2-1}},$$

and (4.10) becomes, in the case $\gamma = 1$, $c = 1$,

$$\frac{q}{\sqrt{q^2-1}}\left(\lambda_+\frac{\partial W_+}{\partial x}+\mu_+\frac{\partial W_+}{\partial y}\right)-\varepsilon\Delta W_+ = \varepsilon\frac{q^2-2}{\rho^2q^2\sqrt{q^2-1}}|\nabla\rho|^2.\tag{4.11}$$

We now consider the case $\gamma > 1$ for which

$$c^2 = 1-\frac{\gamma-1}{2}q^2, \qquad \sigma_2 = \frac{q^2-c^2}{q^2} = \frac{\gamma+1}{2}-\frac{1}{q^2},$$

$$\frac{\partial W_+}{\partial q} = -\frac{\sqrt{q^2-c^2}}{qc}, \qquad \frac{d\rho}{dq} = -\frac{\rho q}{c^2}.$$

Then

$$\begin{aligned}
\frac{\partial W_+}{\partial \rho} &= \frac{\partial W_+}{\partial q}\frac{dq}{d\rho} = -\frac{\sqrt{q^2-c^2}}{qc}\left(-\frac{c^2}{\rho q}\right) = \frac{c\sqrt{q^2-c^2}}{\rho q^2},\\
\frac{d\sigma_2}{d\rho} &= \frac{d\sigma_2}{dq}\frac{dq}{d\rho} = -\frac{2c^2}{\rho q^4},\\
\frac{\partial^2 W_+}{\partial \rho^2} &= \frac{\partial}{\partial q}\left(\frac{\partial W_+}{\partial \rho}\right)\frac{dq}{d\rho} = -\frac{c^2}{\rho q}\frac{\partial}{\partial q}\left(\frac{\partial W_+}{\partial \rho}\right) = -\frac{c^2}{\rho q}\frac{\partial}{\partial q}\left(\frac{c\sqrt{q^2-c^2}}{\rho q^2}\right)\\
&= -\frac{c\left((3-\gamma)q^4 + 2(\gamma-3)q^2c^2 + 4c^4\right)}{2\rho^2 q^4\sqrt{q^2-c^2}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\partial^2 W_+}{\partial \rho^2} - \frac{d\sigma_2}{d\rho}\frac{q^2}{q^2-c^2}\frac{\partial W_+}{\partial \rho}\\
&= -\frac{c}{2\rho^2 q^4\sqrt{q^2-c^2}}\left((3-\gamma)q^4 + 2(\gamma-3)q^2c^2 + 4c^4\right) + \frac{2c^2}{\rho q^4}\frac{q^2}{q^2-c^2}\frac{c\sqrt{q^2-c^2}}{\rho q^2}\\
&= \frac{c\left((\gamma-3)q^2 + 4c^2\right)}{2\rho^2 q^2\sqrt{q^2-c^2}},
\end{aligned}$$

and (4.10) becomes, in the case $\gamma > 1$,

$$\frac{qc}{\sqrt{q^2-c^2}}\left(\lambda_+\frac{\partial W_+}{\partial x} + \mu_+\frac{\partial W_+}{\partial y}\right) - \varepsilon\Delta W_+ = -\varepsilon c\frac{(\gamma-3)q^2+4c^2}{2\rho^2 q^2\sqrt{q^2-c^2}}|\nabla\rho|^2. \tag{4.12}$$

This equality is consistent with the case $\gamma = 1$, and this lemma is proved for W_+. The computation for W_- is done analogously. □

5. Boundary Conditions, Invariant Regions, and L^∞ Bounds

In this section, we consider the level sets of $W_\pm$ to assign appropriate boundary conditions and identify a family of invariant regions for the viscous problem (2.1) with (4.3)–(4.4).

First we discuss the isothermal case: $\gamma = 1$ and $c = 1$. From Proposition 4.1, we find that, when $q > \sqrt{2}$, $W_+(x,y) := W_+(q(x,y),\theta(x,y))$ cannot have an interior minimum since $\Delta W_+ < 0$. Similarly, $W_-(x,y) := W_-(q(x,y),\theta(x,y))$ cannot have an interior maximum since $\Delta W_- > 0$. If we assume for the moment that W_+ cannot have a minimum on the boundary of our domain, then a solution of the viscous problem which crosses from $q \le \sqrt{2}$ to $q > \sqrt{2}$ at (x_0,y_0) must satisfy

$$W_+(x,y) \ge W_+(x_0,y_0)|_{q=\sqrt{2}}.$$

Similarly, if we assume for the moment that W_- cannot have a maximum on the boundary of our domain, then a solution of the viscous problem which crosses from $q \le \sqrt{2}$ to $q > \sqrt{2}$ at (x_0,y_0) must satisfy

$$W_-(x,y) \le W_-(x_0,y_0)|_{q=\sqrt{2}}.$$

From the definition of $W_\pm$, we have

$$\frac{\partial W_\pm}{\partial q} = \mp \frac{\sqrt{q^2-1}}{q}.$$

Using the substitution $q = \sec t$, we can integrate to find

$$W_\pm = \theta \mp \sqrt{q^2-1} \pm \arccos(q^{-1}) + C_\pm = \theta \mp \left(\sqrt{q^2-1} - 1\right) \pm \left(\arccos(q^{-1}) - \frac{\pi}{4}\right),$$

where the constants $C_\pm$ is set to be $C_\pm = (\pm\sqrt{q^2-1} \mp \arccos(q^{-1}))|_{q=\sqrt{2}} = \pm(1-\frac{\pi}{4})$.

Along the level curves $W_\pm = const.$, we have

$$\frac{d\theta}{dq} = \pm \frac{\sqrt{q^2-1}}{q}.$$

Thus, on the level set of W_+, $\frac{d\theta}{dq} > 0$ and θ is increasing when q is increasing; while, on the level set of W_-, $\frac{d\theta}{dq} < 0$ and θ is decreasing when q is increasing.

If $\theta(x_0, y_0) = \theta_0$, $q(x_0, y_0) = \sqrt{2}$ and we leave $q = \sqrt{2}$, we have

$$\theta(x,y) - \theta_0 \geq \left(\sqrt{q^2-1} - 1\right) - \left(\arccos(q^{-1}) - \frac{\pi}{4}\right) \qquad \text{(stay below } W_+\text{)},$$

$$\theta(x,y) - \theta_0 \leq -\left(\sqrt{q^2-1} - 1\right) + \left(\arccos(q^{-1}) - \frac{\pi}{4}\right) \qquad \text{(stay above } W_-\text{)},$$

that is,

$$\left(\sqrt{q^2-1} - 1\right) - \left(\arccos(q^{-1}) - \frac{\pi}{4}\right) \leq |\theta(x,y) - \theta_0| \qquad \text{inside the “apple” shaped region.}$$

See Figure 1 with $q_{cav} = \infty$ when $\gamma = 1$.

The same situation occurs when the level set curves $W_\pm = W_\pm(q_0, \theta_0)$ for $\theta(x_0, y_0) = \theta_0$ and $q(x_0, y_0) = q_0 > \sqrt{2}$, for which we obtain similar invariant regions of “apple” shape past the point (q_0, θ_0) in the (u, v)-plane.

Now we return to the issue of boundary conditions. Denote the boundary of the bounded domain Ω by $\partial\Omega$, the boundary of the obstacle by $\partial\Omega_1$, and the far field boundary by $\partial\Omega_2$. Thus, $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. Since we do not want W_+ to have a minimum on boundary $\partial\Omega_1$ and W_- to have a maximum on boundary $\partial\Omega_1$, we require

$$\frac{\partial W_+}{\partial \mathbf{n}} < 0, \quad \frac{\partial W_-}{\partial \mathbf{n}} > 0 \qquad \text{at all boundary points},$$

where $\mathbf{n}$ is the unit normal into the flow region on $\partial\Omega$. Recall

$$\nabla W_\pm = \nabla\theta \mp \frac{q^2-1}{q}\nabla q.$$

Therefore, if we set

$$\nabla\theta \cdot \mathbf{n} = 0 \qquad \text{on} \quad \partial\Omega_1,$$

then

$$\operatorname{sign}(\nabla W_\pm \cdot \mathbf{n}) = \mp \operatorname{sign}(\nabla q \cdot \mathbf{n})$$

at those boundary points where $q > 1$, i.e. where the Riemann invariants are defined. Hence, one resolution of the boundary condition issue is to set

$$\varepsilon\sigma_2 \nabla\rho \cdot \mathbf{n} = -|\rho\,(u,v)\cdot\mathbf{n}| \qquad \text{on} \quad \partial\Omega_1,$$

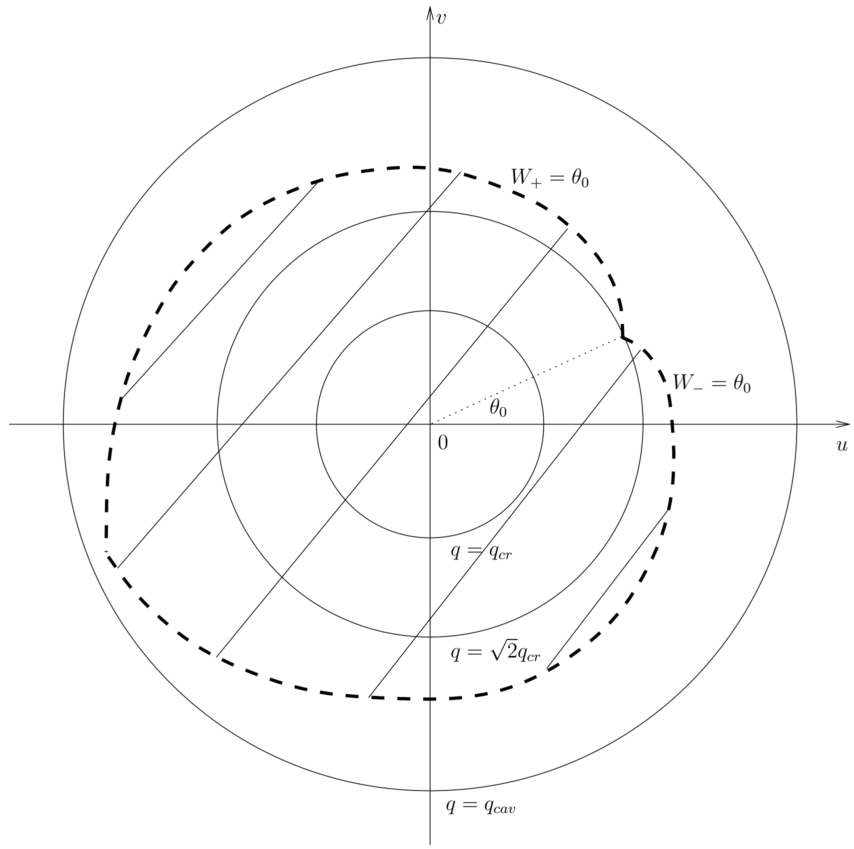

Figure 1: Invariant regions of "apple" shape.

and

$$(u, v) - (u_\infty, v_\infty) = 0 \qquad \text{on} \quad \partial\Omega_2,$$

with $q_\infty = |(u_\infty, v_\infty)| \in (0, q_{cav}) = (0, \infty)$. Trivially, $W_\pm$ are not even defined on $\partial\Omega_2$ and hence cannot have a maximum or minimum there. Since

$$\nabla\rho = \nabla q \frac{d\rho}{dq} = -\rho q \nabla q,$$

we have automatically

$$\text{sign}(\nabla W_\pm \cdot \mathbf{n}) = \pm\text{sign}(\nabla\rho \cdot \mathbf{n}) = \mp 1 \qquad \text{on all boundaries},$$

and our minimum principle for W_+ and maximum principle for W_- are indeed valid.

Secondly, they formally yield

$$\rho\,(u, v) \cdot \mathbf{n} = 0 \qquad \text{on} \quad \partial\Omega_1,$$

and

$$(u, v) - (u_\infty, v_\infty) \to 0 \qquad \text{as} \quad x^2 + y^2 \to \infty,$$

if $\varepsilon \to 0$. This is the case when any shock strength is zero at its intersection point with the boundary as conjectured for the shock formed in the supersonic bubble near the obstacle (see Morawetz [29]). Finally, we see that the relation between σ_1 and σ_2 is only necessary for $q \geq \sqrt{2}$. Any smooth continuations of σ_1 and σ_2 inside $q = \sqrt{2}$ with $\sigma_1, \sigma_2 > 0$ suffices.

With this, we conclude that (q, θ) stay inside the "apple" region for $\gamma = 1$. More generally, for $1 < \gamma < 3$, we have

Theorem 5.1. *Consider the viscous problem*

$$\begin{cases} v_x - u_y = \varepsilon \nabla \cdot (\sigma_1(\rho) \nabla \theta)\,, \\ (\rho u)_x + (\rho v)_y = \varepsilon \nabla \cdot (\sigma_2(\rho) \nabla \rho)\,, \end{cases} \tag{5.1}$$

with

$$\sigma_1 = 1, \qquad \sigma_2 = \frac{q^2 - c^2}{q^2} \quad \text{for } q \geq \sqrt{2} q_{cr},\ 1 \leq \gamma < 3,$$

and the boundary conditions:

$$\begin{cases} \nabla \theta \cdot \mathbf{n} = 0 \qquad \text{on} \quad \partial\Omega_1, \\ \varepsilon \sigma_2 \nabla \rho \cdot \mathbf{n} = -|\rho\,(u, v) \cdot \mathbf{n}| \qquad \text{on} \quad \partial\Omega_1, \\ (u, v) - (u_\infty, v_\infty) = 0 \qquad \text{on} \quad \partial\Omega_2 \text{ with} \quad q_\infty < q_{cav}, \end{cases} \tag{5.2}$$

Then all solutions to (5.1)*–*(5.2) *are bounded on the domain* Ω *staying uniformly in* ε *away from cavitation, i.e. there exists* $q^* < q_{cav}$ *such that* $q^\varepsilon \leq q^*$ *that is equivalent to* $\rho(q^\varepsilon) \geq \underline{\rho} > 0$ *for some* $\underline{\rho} = \underline{\rho}(q^*) > 0$*. More specifically, when* $1 \leq \gamma < 3$*, the viscous approximate solutions stay in a family of "apple" shaped invariant regions as shown in Figures* 1 *and* 3*–*5*.*

Proof. The theorem has been proved in the above for the isothermal case $\gamma = 1, c = 1$, and $q_{cr} = 1$.

For the case $1 < \gamma < 3$, from (4.5), we first require

$$(\gamma - 3) q^2 + 4c^2 < 0,$$

which implies $q^2 > \frac{4}{3-\gamma} c^2$. Then, from Bernoulli's law (2.4), we have

$$q^2 - q_{cr}^2 > \frac{2}{3-\gamma} c^2 = \frac{2}{3-\gamma} \big(1 - \frac{\gamma - 1}{2} q^2\big),$$

thus $q^2 > 2 q_{cr}^2$, i.e. $q > \sqrt{2} q_{cr}$.

We first consider the case that the far field speed $q_\infty \leq \sqrt{2} q_{cr}$. Then, from the definition of the Riemann invariants (4.1), we follow Landau-Lifshitz [22], page 446, and have

$$W_\pm = \theta \mp \Big(W(q) - W(\sqrt{2} q_{cr})\Big)\,,$$

where

$$W(q) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arcsin \sqrt{\frac{\gamma-1}{2} \Big(\frac{q^2}{q_{cr}^2} - 1\Big)} - \arcsin \sqrt{\frac{\gamma+1}{2} \Big(1 - \frac{q_{cr}^2}{q^2}\Big)}.$$

The two level set curves of $W_\pm$ intersect to form a invariant region of "apple" shape past $(\sqrt{2} q_{cr}, \theta_0)$ as in Figure 1 if

$$W(q_{cav}) - W(\sqrt{2} q_{cr}) > \pi,$$

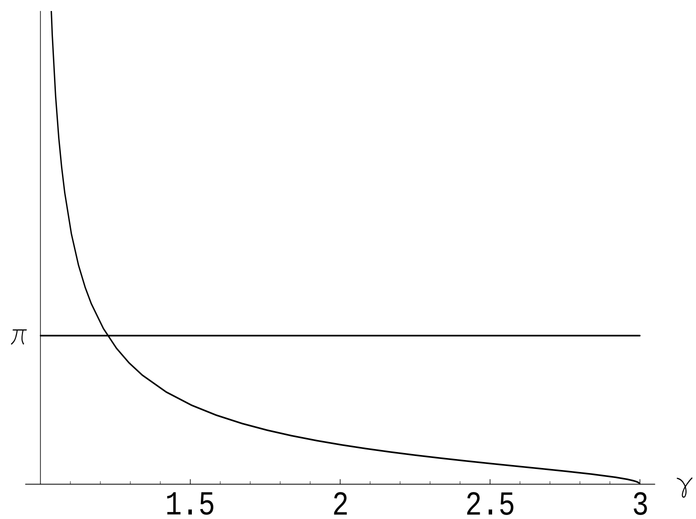

Figure 2: The graph of $a(\gamma)$

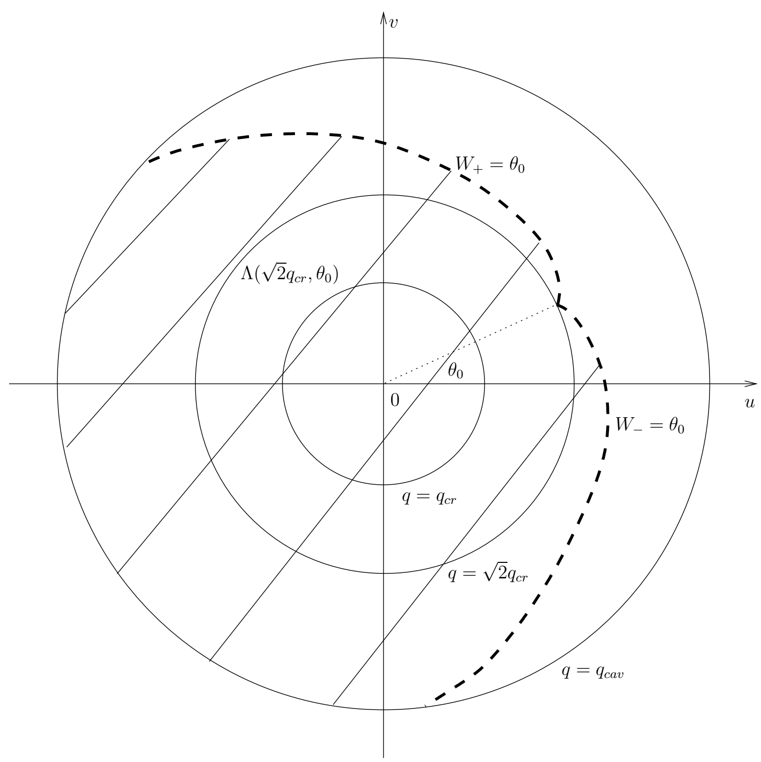

Figure 3: Round "apple" shaped region $\Lambda(\sqrt{2}q_{cr}, \theta_0)$ when $a(\gamma) < \pi$

that is,

$$\begin{aligned} a(\gamma) &:= W(q_{cav}) - W(\sqrt{2}q_{cr}) \\ &= \Big(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\Big)\frac{\pi}{2} - \Big(\sqrt{\frac{\gamma+1}{\gamma-1}}\arcsin\sqrt{\frac{\gamma-1}{2}} - \arcsin\sqrt{\frac{\gamma+1}{4}}\Big) > \pi. \end{aligned}$$

The graph of the function $a(\gamma)$ is shown in Figure 2, which shows that $a(\gamma*) = \pi$ for some $\gamma* \approx 1.224$ and $a(\gamma) > \pi$ for $1 \leq \gamma < \gamma*$.

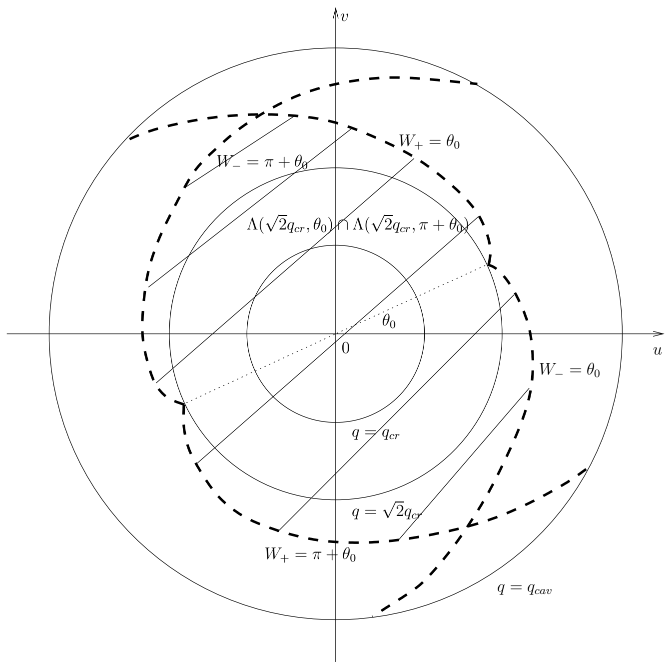

Figure 4: Invariant regions $\Lambda(\sqrt{2}q_{cr}, \theta_0) \cap \Lambda(\sqrt{2}q_{cr}, \pi + \theta_0)$ when $a(\gamma) \in (\frac{\pi}{2}, \pi)$

For general $\gamma \in [\gamma^*, 3)$, the level set curves of $W_\pm = \theta_0$ past $(\sqrt{2}q_{cr}, \theta_0)$ end at some points on the cavitation circle $q = q_{cav}$ and do not intersect each other. We denote $\Lambda(\sqrt{2}q_{cr}, \theta_0)$ the round "apple" shaped region formed by $W_+ > \theta_0$, $W_- < \theta_0$, and the cavitation circle $q = q_{cav}$; see Figure 3.

When $a(\gamma) \in (\frac{\pi}{2}, \pi]$, we find that, for any $\theta_0 \in [0, 2\pi)$,

$$\Lambda(\sqrt{2}q_{cr}, \theta_0) \cap \Lambda(\sqrt{2}q_{cr}, \pi + \theta_0)$$

forms an invariant region for the viscous solutions; this follows from application of our minimum and maximum principle for W_+ and $W-$, respectively, at the crossing point on the new level set curves. Such invariant regions stay away from the cavitation circle (see Figure 4).

When $a(\gamma) \in (\frac{\pi}{4}, \frac{\pi}{2}]$, then, for any $\theta_0 \in [0, 2\pi)$,

$$\Lambda(\sqrt{2}q_{cr}, \theta_0) \cap \Lambda(\sqrt{2}q_{cr}, \frac{\pi}{2} + \theta_0) \cap \Lambda(\sqrt{2}q_{cr}, \pi + \theta_0) \cap \Lambda(\sqrt{2}q_{cr}, \frac{3\pi}{2} + \theta_0),$$

forms an invariant region for the viscous solutions, staying away from cavitation; see Figure 5.

In general, when $a(\gamma) \in (\frac{\pi}{2^n}, \frac{\pi}{2^{n-1}})$, $n = 1, 2, \ldots$, for each $\theta_0 \in [0, 2\pi)$, it requires an intersection of at least $2n$ round "apple" shaped regions including $\Lambda(\sqrt{2}q_{cr}, \theta_0)$ to form an invariant region staying away from the cavitation circle.

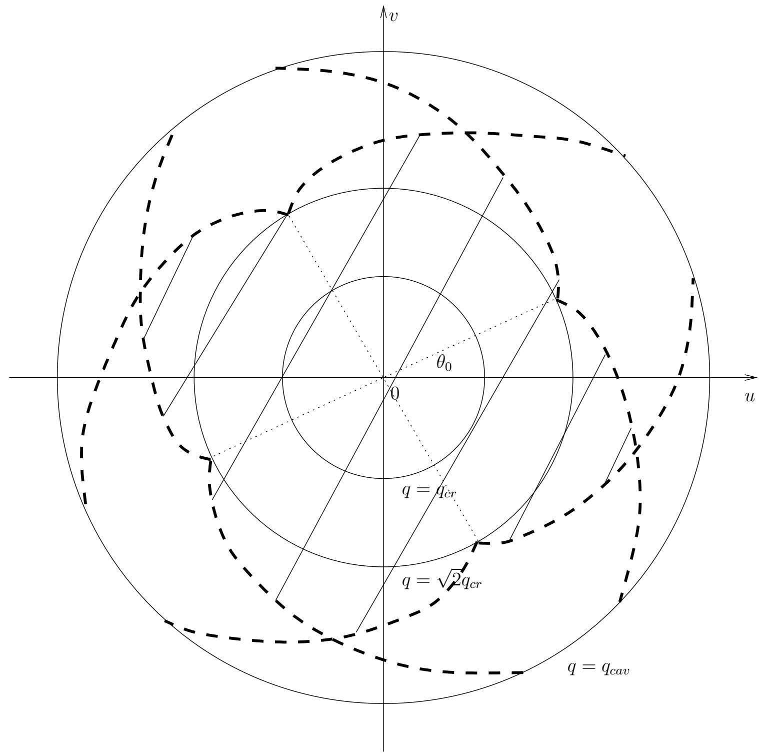

Figure 5: Invariant regions $\Lambda(\sqrt{2}q_{cr}, \theta_0) \cap \Lambda(\sqrt{2}q_{cr}, \frac{\pi}{2}+\theta_0) \cap \Lambda(\sqrt{2}q_{cr}, \pi+\theta_0) \cap \Lambda(\sqrt{2}q_{cr}, \frac{3\pi}{2}+\theta_0)$ when $a(\gamma) \in (\frac{\pi}{4}, \frac{\pi}{2}]$

When the far field speed $q_\infty \in (\sqrt{2}q_{cr}, q_{cav})$, then the level set curves of $W_\pm = W_\pm(q_0, \theta_0)$, for some $q_0 \in (q_\infty, q_{cav})$ and $\theta_0 = arctan(\frac{v_\infty}{u_\infty})$, past the point (q_0, θ_0) either intersects each other or end at some points on the cavitation circle $q = q_{cav}$. As before, we denote $\Lambda(q_0, \theta_0)$ the "apple" shaped region formed by $W_+ > W_+(q_0, \theta_0), W_- < W_-(q_0, \theta_0)$, and the cavitation circle $q = q_{cav}$ similar to either Figure 1 or Figure 3. Then we can similarly obtain the invariant regions which consist of either a single "apple" shaped region or some unions of certain number round "apple" shaped regions including $\Lambda(q_0, \theta_0)$, which stay away from the cavitation but include the state (u_∞, v_∞) (cf. Figures 5–6). $\square$

6. Compensated Compactness Framework for Steady Flow

Let a sequence of functions $w^\varepsilon(x, y) = (u^\varepsilon, v^\varepsilon)(x, y)$, defined on open subset $\Omega \subset \mathbb{R}^2$, satisfy the following Set of Conditions (A):

(A.1) $q^\varepsilon(x, y) = |w^\varepsilon(x, y)| \le q_*$ a.e. in Ω, for some positive constant $q_* < q_{cav} < \infty$;

(A.2) $\partial_x Q_{1\pm}(w^\varepsilon) + \partial_y Q_{2\pm}(w^\varepsilon)$ are confined in a compact set in $H^{-1}_{loc}(\Omega)$, for any entropy-entropy flux pairs (Q_1, Q_2) so that $(Q_{1\pm}(w^\varepsilon), Q_{2\pm}(w^\varepsilon))$ are confined in a bounded set uniformly in $L^\infty_{loc}(\Omega)$.

Then, by the Div-Curl Lemma of Tartar [36] and Murat [30] and the Young measure representation theorem for a uniformly bounded sequence of functions (cf. Tartar [36];

also Ball [1]), we have the following commutation identity:

$$
\begin{aligned}
&< \nu(w), Q_{1+}(w)Q_{2-}(w) - Q_{1-}(w)Q_{2+}(w) > \\
&\quad =< \nu(w), Q_{1+}(w) >< \nu(w), Q_{2-}(w) > - < \nu(w), Q_{1-}(w) >< \nu(w), Q_{2+}(w) >,
\end{aligned} \tag{6.1}
$$

where $\nu = \nu_{x,y}(w), w = (u, v)$, is the associated family of Young measures (probability measures) for the sequence $w^\varepsilon(x, y) = (u^\varepsilon, v^\varepsilon)(x, y)$. This is equivalent to

$$
< \nu(w) \otimes \nu(w'),\ I(w, w') >= 0, \tag{6.2}
$$

where $\nu(w) \otimes \nu(w')$ is a product measure for $(w, w') \in \mathbb{R}^2 \times \mathbb{R}^2$ and

$$
I(w, w') = (Q_{1+}(w) - Q_{1+}(w'))(Q_{2-}(w) - Q_{2-}(w')) - (Q_{2+}(w) - Q_{2+}(w'))(Q_{1-}(w) - Q_{1-}(w')).
$$

The main point for the compensated compactness framework is to prove that ν is in fact a Dirac measure by using entropy pairs, which implies the compactness of the sequence $w^\varepsilon(x, y) = (u^\varepsilon, v^\varepsilon)(x, y)$ in $L^1_{loc}(\Omega)$. Some additional references on compensated compactness method include Chen [5, 6], Dafermos [10], DiPerna [11], Evans [13], and Serre [34]. Theorem 5.1 shows that (A.1) is satisfied for our viscous solution sequence $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$. In the next two sections, we show how (A.2) can be achieved.

7. Entropy Pairs via Entropy Generators

We first construct all mathematical entropy pairs for the potential flow system.

For some function $V(\rho, \theta)$ to be determined, multiply (3.3) from left by (V_θ, V_ρ) to get the entropy equality

$$
Q_{1x} + Q_{2y} = -V_\theta R_1 + \frac{q^2}{c^2 - q^2} V_\rho R_2, \tag{7.1}
$$

where (Q_1, Q_2) is defined by

$$
\begin{aligned}
&\frac{\partial Q_1}{\partial \rho} = \frac{c^2}{\rho q} \sin\theta\, V_\theta - q\cos\theta\, V_\rho, \quad \frac{\partial Q_1}{\partial \theta} = -q\cos\theta\, V_\theta - \frac{\rho q^3 \sin\theta}{c^2 - q^2} V_\rho, \\
&\frac{\partial Q_2}{\partial \rho} = -\frac{c^2}{\rho q} \cos\theta\, V_\theta - q\sin\theta\, V_\rho, \quad \frac{\partial Q_2}{\partial \theta} = -q\sin\theta\, V_\theta + \frac{\rho q^3 \cos\theta}{c^2 - q^2} V_\rho.
\end{aligned} \tag{7.2}
$$

We note that, the appearance of the term $c^2 - q^2$ in the denominator of (7.2) is only a consequence of our formalism. It will cancel out as we proceed. Using $\frac{\partial^2 Q_i}{\partial\theta\partial\rho} = \frac{\partial^2 Q_i}{\partial\rho\partial\theta}$, $i = 1, 2$, from (7.2), we see that V satisfies the Tricomi type equation of mixed type:

$$
\frac{c^2}{\rho q} V_{\theta\theta} + q V_\rho + \left(\frac{\rho q^3}{c^2 - q^2} V_\rho \right)_\rho = 0. \tag{7.3}
$$

Thus, we have defined an entropy pair (Q_1, Q_2) generated by V. Alternatively, we can define an entropy pair (Q_1, Q_2) generated by H, where H and V are related by

$$
\rho H_{\mu\theta} - H_\theta = -V_\theta, \qquad H_\mu + \frac{1}{\rho} H_{\theta\theta} = \frac{q^2}{c^2 - q^2} V_\rho, \tag{7.4}
$$

with $\mu = \mu(\rho)$ defined by $\mu'(\rho) = c^2/q^2$, and H is determined by the generalized Tricomi equation:

$$
H_{\mu\mu} + \frac{1}{\rho^2}(1 - M^2) H_{\theta\theta} = 0, \tag{7.5}
$$

and $M = q/c$ is the Mach number.

Lemma 7.1. *The entropy pairs* (Q_1, Q_2) *are given by the Loewner-Morawetz relation:*

$$Q_1 = \rho q H_\mu \cos\theta - qH_\theta \sin\theta, \qquad Q_2 = \rho q H_\mu \sin\theta + qH_\theta \cos\theta, \tag{7.6}$$

where the generators H *are all solutions of* (7.5).

This can be seen by differentiation of (7.6) with respect to (ρ, θ) and comparison with (7.2).

A prototype of the generators H, as suggested in [32], is

$$H^* = \frac{\theta^2}{2} + \int^\mu \int^{\mu'} \frac{1}{\rho^2}(M^2 - 1)d\mu' d\mu, \tag{7.7}$$

which is a trivial solution of the generalized Tricomi equation (7.5). Notice that H^* is strictly convex in (μ, θ) in the supersonic region. We denote by (Q_1^*, Q_2^*) the corresponding entropy pair generated by the convex generator H^*. With this, we introduce the notion of entropy solutions.

Definition 7.1 (Notion of Entropy Solutions)**.** A bounded, measurable vector function $w(x,y) := (u,v)(x,y), q(x,y) \le q_{cav}$, is called an entropy solution of the potential flow system in a domain Ω:

$$\begin{cases} v_x - u_y = 0, \\ (\rho u)_x + (\rho v)_y = 0, \end{cases} \tag{7.8}$$

if $w(x,y)$ satisfies (7.8) and the following entropy inequality:

$$Q^*_{1x} + Q^*_{2y} \le 0 \tag{7.9}$$

in the sense of distributions in Ω, in addition to the corresponding boundary conditions in the trace or asymptotic sense on $\partial\Omega$.

The physical correctness of the entropy inequality (7.9) is provided by Theorem 2.1 of Osher-Hafez-Whitlow [32].

8. H^{-1} Compactness of Entropy Dissipation Measures

We now limit ourselves to the case $v_\infty = 0$ and the two types of domains Ω in Figure 6(a)-(b), where $\partial\Omega_1$ (the solid curve in (a) and the solid closed curve in (b)) is the boundary of the obstacle, $\partial\Omega_2$ (dashed line segments in both (a) and (b)) is the far field boundary, and Ω is the domain bounded by $\partial\Omega_1$ and $\partial\Omega_2$. This assumption implies $\theta = 0$ on $\partial\Omega_2$. In case (b), the circulation about the boundary $\partial\Omega_2$ is zero.

Proposition 8.1. *Let* $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$ *be a solution to* (5.1)–(5.2) *with* $u_\infty > 0$ *and* $v_\infty = 0$ *on either domain* Ω *in Figure* 6(a)-(b)*, satisfying that* $q^\varepsilon \le q^* < q_{cav}$ *and* θ^ε *is bounded. Then the integral*

$$\varepsilon \int_\Omega \left(\sigma_1(\rho^\varepsilon)|\nabla\theta^\varepsilon|^2 + \sigma_2(\rho^\varepsilon)\frac{c^2(\rho^\varepsilon)}{(\rho^\varepsilon q^\varepsilon)^2}|\nabla\rho^\varepsilon|^2 \right) dxdy$$

is bounded uniformly in all $\varepsilon > 0$.

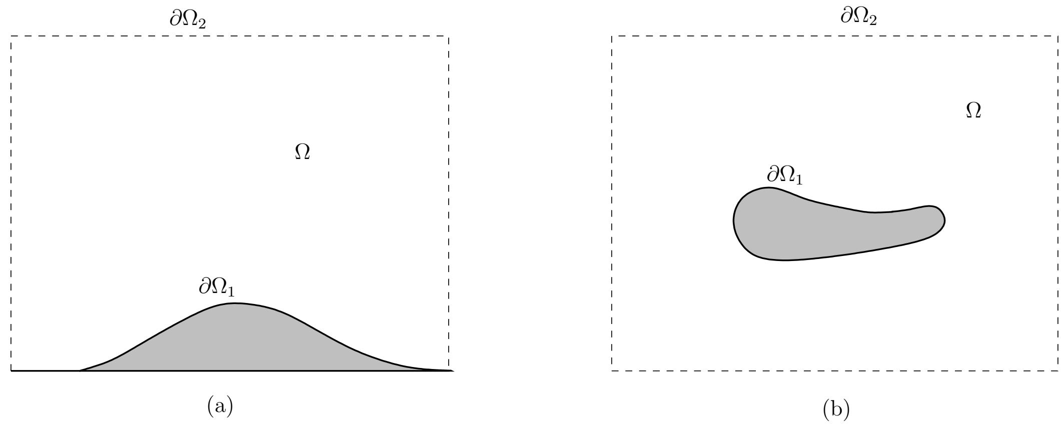

Figure 6: Domains

Proof. We choose the special generator H^* in (7.7) which yields V^* of the form:

$$V^* = \frac{\theta^2}{2} + P(\rho), \qquad V^*_\rho = P'(\rho) = \frac{c^2 - q^2}{q^2} \int_{\bar q}^{q} \frac{dq}{\rho q}, \qquad V^*_\theta = \theta. \tag{8.1}$$

and (7.1) becomes

$$Q^*_1(w^\varepsilon)_x + Q^*_2(w^\varepsilon)_y = -\varepsilon\theta^\varepsilon \nabla(\sigma_1(\rho^\varepsilon)\,\nabla\theta^\varepsilon) + \varepsilon \int_{\bar q}^{q(\rho^\varepsilon)} \frac{dq}{\rho q}\ \nabla(\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon), \tag{8.2}$$

where

$$Q^*_1(w) = \rho q H^*_\mu \cos\theta - qH^*_\theta \sin\theta, \qquad Q^*_2(w) = \rho q H^*_\mu \sin\theta + qH^*_\theta \cos\theta.$$

Then

$$\begin{aligned} Q^*_1(w^\varepsilon)_x + Q^*_2(w^\varepsilon)_y &= \varepsilon \mathrm{div} \left(-\sigma_1(\rho^\varepsilon)\theta^\varepsilon\nabla\theta^\varepsilon + \sigma_2(\rho^\varepsilon)\big(\int_{\bar q}^{q^\varepsilon} \frac{dq}{\rho q}\big)\ \nabla\rho^\varepsilon \right) \\ &\quad + \varepsilon\sigma_1(\rho^\varepsilon)|\nabla\theta^\varepsilon|^2 - \varepsilon\sigma_2(\rho^\varepsilon)\frac{|\nabla\rho^\varepsilon|^2}{\rho^\varepsilon q^\varepsilon}\frac{dq(\rho^\varepsilon)}{d\rho^\varepsilon} \\ &= \varepsilon \mathrm{div} \left(-\sigma_1(\rho^\varepsilon)\theta^\varepsilon\nabla\theta^\varepsilon + \sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon \int_{\bar q}^{q^\varepsilon} \frac{dq}{\rho q} \right) \\ &\quad + \varepsilon\sigma_1(\rho^\varepsilon)|\nabla\theta^\varepsilon|^2 + \varepsilon\sigma_2(\rho^\varepsilon)\frac{c^2(\rho^\varepsilon)}{(\rho^\varepsilon q^\varepsilon)^2}|\nabla\rho^\varepsilon|^2. \end{aligned} \tag{8.3}$$

Integrating (8.3) over Ω and using the divergence theorem, we obtain with $\bar q = u_\infty > 0$ that

$$\begin{aligned} \int_{\partial\Omega} (Q^*_1(w^\varepsilon), Q^*_2(w^\varepsilon)) \cdot \mathbf{n}\, ds =& \varepsilon \int_{\partial\Omega} \Big(-\sigma_1(\rho^\varepsilon)\theta^\varepsilon\nabla\theta^\varepsilon\cdot\mathbf{n} + \sigma_2(\rho^\varepsilon)\big(\int_{u_\infty}^{q^\varepsilon}\frac{dq}{\rho q}\big)\nabla\rho^\varepsilon\cdot\mathbf{n}\Big) ds \\ &+ \varepsilon \int_\Omega \left(\sigma_1(\rho^\varepsilon)|\nabla\theta^\varepsilon|^2 + \sigma_2(\rho^\varepsilon)\frac{c^2(\rho^\varepsilon)}{(\rho^\varepsilon q^\varepsilon)^2}|\nabla\rho^\varepsilon|^2 \right) dxdy, \\ :=& I_1 + I_2, \end{aligned} \tag{8.4}$$

where Q_1^*, Q_2^* depend only on $(\rho^\varepsilon, \theta^\varepsilon)$ and are independent of their derivatives. Thus, from the L^∞ bound, the left hand side of (8.4) is uniformly bounded for all $\varepsilon > 0$. More specifically, from (7.6) and the formula of H^*, we have the following:

On $\partial\Omega_1$, $(Q_1^*, Q_2^*) \cdot \mathbf{n} = q^\varepsilon \theta^\varepsilon$ which is uniformly bounded in $\epsilon > 0$;

On the horizontal part of $\partial\Omega_2$, $\theta^\varepsilon = 0$ and $|(Q_1^*, Q_2^*) \cdot \mathbf{n}| = |Q_2^*| = |q^\varepsilon \theta^\varepsilon| = 0$;

On the vertical parts of $\partial\Omega_2$, $\theta^\varepsilon = 0$ and $|(Q_1^*, Q_2^*) \cdot \mathbf{n}| = |Q_1^*| = |\rho(u_\infty) u_\infty H_\mu^*(u_\infty)|$ is uniformly bounded in $\varepsilon > 0$.

Using the boundary conditions (5.2), one has

$$\begin{aligned} I_1 =& \varepsilon \int_{\partial\Omega_1} \sigma_2\Big(\int_{u_\infty}^{q^\varepsilon} \frac{dq}{\rho q} \Big) \nabla\rho^\varepsilon \cdot \mathbf{n}\, ds + \varepsilon \int_{\partial\Omega_2} \sigma_2\Big(\int_{u_\infty}^{u_\infty} \frac{dq}{\rho q} \Big) \nabla\rho^\varepsilon \cdot \mathbf{n}\, ds \\ =& -\varepsilon \int_{\partial\Omega_1} |\rho^\varepsilon (u^\varepsilon, v^\varepsilon) \cdot \mathbf{n}| \Big(\int_{u_\infty}^{q^\varepsilon} \frac{dq}{\rho q} \Big)\, ds, \end{aligned}$$

which implies that I_1 is uniformly bounded due to the L^∞ bound, since the integrand is like $q \ln q$ near $q = 0$ and is well behaved. Therefore, (8.4) implies that I_2 is uniformly bounded. □

In terms of a general generator H, equation (7.1) becomes

$$Q_1(w^\varepsilon)_x + Q_2(w^\varepsilon)_y = \varepsilon \nabla(\sigma_1(\rho^\varepsilon)\nabla\theta^\varepsilon)(\rho^\varepsilon H_{\mu\theta} - H_\theta) + \varepsilon \nabla(\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon)\big(H_\mu + \frac{1}{\rho} H_{\theta\theta}\big),$$

or

$$\begin{aligned} Q_1(w^\varepsilon)_x + Q_2(w^\varepsilon)_y =& \varepsilon \mathrm{div}\Big(\sigma_1(\rho^\varepsilon)\nabla\theta^\varepsilon(\rho^\varepsilon H_{\mu\theta} - H_\theta) + \sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon\big(H_\mu + \frac{1}{\rho} H_{\theta\theta}\big)\Big) \\ & - \varepsilon\sigma_1(\rho^\varepsilon)\nabla\theta^\varepsilon \cdot \nabla\big(\rho^\varepsilon H_{\mu\theta} - H_\theta\big) - \varepsilon\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon \cdot \nabla\big(H_\mu + \frac{1}{\rho^\varepsilon} H_{\theta\theta}\big). \end{aligned} \tag{8.5}$$

The entropy pair (Q_1, Q_2) satisfies (7.2) which can be written in terms of H,

$$\begin{aligned} \frac{\partial Q_1}{\partial\rho} &= -\frac{c^2}{\rho q} \sin\theta\, (\rho H_{\mu\theta} - H_\theta) - q\cos\theta\, \frac{c^2 - q^2}{q^2}\big(H_\mu + \frac{1}{\rho} H_{\theta\theta}\big), \\ \frac{\partial Q_1}{\partial\theta} &= q\cos\theta\, (\rho H_{\mu\theta} - H_\theta) - \rho q \sin\theta\, \big(H_\mu + \frac{1}{\rho} H_{\theta\theta}\big), \\ \frac{\partial Q_2}{\partial\rho} &= \frac{c^2}{\rho q} \cos\theta\, (\rho H_{\mu\theta} - H_\theta) - q\sin\theta\, \frac{c^2 - q^2}{q^2}\big(H_\mu + \frac{1}{\rho} H_{\theta\theta}\big), \\ \frac{\partial Q_2}{\partial\theta} &= q\sin\theta\, (\rho H_{\mu\theta} - H_\theta) + \rho q \cos\theta\, \big(H_\mu + \frac{1}{\rho} H_{\theta\theta}\big). \end{aligned} \tag{8.6}$$

Proposition 8.2. *Assume that $q^\varepsilon(x,y) \geq \alpha_U$ on $U \subset \Omega$ for some constant $\alpha_U > 0$, in addition $q^\varepsilon(x,y) \leq q^* < q_{cav}$ ensured by the invariant regions. Then*

$$\partial_x Q_{1\pm}(w^\varepsilon) + \partial_y Q_{2\pm}(w^\varepsilon) \quad \textit{are confined in a compact set in } H^{-1}(U),$$

for any entropy flux pair (Q_1, Q_2) generated by $H \in C^3$. That is, hypothesis (A.2) *of the compensated compactness framework is satisfied for such entropy pair (Q_1, Q_2) generated by $H \in C^3$.*

Proof. For such an entropy pair (Q_1, Q_2) generated by $H \in C^3$, equation (8.5) is satisfied. By Proposition 8.1, we have

$$J_1^\varepsilon := \varepsilon \mathrm{div}\Big(\sigma_1(\rho^\varepsilon)(\rho^\varepsilon H_{\mu\theta} - H_\theta)\nabla\theta^\varepsilon + \sigma_2(\rho^\varepsilon)\big(H_\mu + \frac{1}{\rho^\varepsilon}H_{\theta\theta}\big)\Big)\nabla\rho^\varepsilon \to 0$$

in $H^{-1}(U)$ as $\varepsilon \to 0$, and

$$J_2^\varepsilon := -\varepsilon\sigma_1(\rho^\varepsilon)\nabla\theta^\varepsilon \cdot \nabla\left(\rho^\varepsilon H_{\mu\theta} - H_\theta\right) - \varepsilon\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon \cdot \nabla\big(H_\mu + \frac{1}{\rho^\varepsilon}H_{\theta\theta}\big)$$

is in $L^1(U)$ uniformly in ε. On the other hand, $(Q_1(w^\varepsilon), Q_2(w^\varepsilon))$ are uniformly bounded which yield that $J_1^\varepsilon + J_2^\varepsilon$ is bounded in $W^{-1,\infty}(U)$. Then Murat's lemma [31] implies that $J_1^\varepsilon + J_2^\varepsilon$ are confined in a compact subset of $H^{-1}(U)$. □

As a corollary of Proposition 8.2, we conclude

Proposition 8.3. *Let the viscous viscosity fields $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$ have the speed $q^\varepsilon(x,y)$ uniformly bounded in $\varepsilon > 0$ away from zero when (x,y) is away from the obstacle boundary $\partial\Omega_1$, i.e. there exists a positive ε-independent $\alpha = \alpha(\delta) \to 0$ as $\delta \to 0$ such that $q^\varepsilon(x,y) \geq \alpha(\delta)$ for any $(x,y) \in \Omega_\delta = \{(x,y) \in \Omega : dist((x,y), \partial\Omega_1) \geq \delta > 0\}$. Then*

$$\partial_x Q_{1\pm}(w^\varepsilon) + \partial_y Q_{2\pm}(w^\varepsilon) \quad \textit{are confined in a compact set in } H^{-1}_{loc}(\Omega),$$

for any entropy pair (Q_1, Q_2) generated by $H \in C^3$.

9. Convergence of the Vanishing Viscosity Solutions

As noted earlier, the reduction of the support of the Young measure ν is accomplished via application of the commutation identity (6.2). Here we follow a technique of Morawetz [28] and use the entropy generators obtained via classical separation variables first given by Loewner [25]. Specifically, we look for H as the following two forms:

$$H_n(\mu,\theta) = F_n(\mu)e^{\pm in\theta},$$

and

$$H_n(\mu,\theta) = K_n(\mu)e^{\pm n\theta}.$$

Hence, from the generalized Tricomi equation (7.5) for H, we see

$$\ddot{F}_n + n^2\frac{M^2-1}{\rho}F_n = 0, \qquad \ddot{K}_n - n^2\frac{M^2-1}{\rho}K_n = 0,$$

where $\dot{} = \frac{d}{d\mu}$. Substitution into the representation for Q_1, Q_2 as given in Proposition 7.1 generates an infinite sequence of entropy pairs $(Q_{1\pm}^{(n)}, Q_{2\pm}^{(n)})$ associated with F_n; and a similar construction can be done for K_n.

First we apply the commutation identity to $(Q_{1\pm}^{(n)}, Q_{2\pm}^{(n)})$ associated with F_n. Set

$$I = (Q_{1+}^{(n)} - Q_{1+}^{(n)'})(Q_{2-}^{(n)} - Q_{2-}^{(n)'}) - (Q_{2+} - Q_{2+}^{(n)'})(Q_{1-} - Q_{1-}^{(n)'}).$$

Then

$$\begin{aligned}
I = &\Big((\dot F_n\rho q\cos\theta + inF_nq\sin\theta)e^{in\theta} - (\dot F_n'\rho' q'\cos\theta' + inF_n'q'\sin\theta')e^{in\theta'}\Big)\\
&\times\Big((\dot F_n\rho q\sin\theta + inF_nq\cos\theta)e^{-in\theta} - (\dot F_n'\rho' q'\sin\theta' + inF_n'q'\cos\theta')e^{-in\theta'}\Big)\\
&-\Big((\dot F_n\rho q\sin\theta - inF_nq\cos\theta)e^{in\theta} - (\dot F_n'\rho' q'\sin\theta' - inF_n'q'\cos\theta')e^{in\theta'}\Big)\\
&\times\Big((\dot F_n\rho q\cos\theta - inF_nq\sin\theta)e^{-in\theta} - (\dot F_n'\rho' q'\cos\theta' - inF_n'q'\sin\theta')e^{-in\theta'}\Big).
\end{aligned}$$

With a tedious calculation, we obtain

$$\begin{aligned}
&< \nu\otimes\nu', \quad I >\\
&= \Big\langle \nu\otimes\nu', \quad 2in\rho q^2\Big(F_n\dot F_n + F_n\dot F_n\Big)\\
&\qquad + 2e^{in(\theta-\theta')}\Big(-\dot F_n\rho q\cos\theta\dot F_n'\rho' q'\sin\theta' + n^2F_nq\sin\theta F_n'q'\cos\theta'\\
&\qquad\qquad +\dot F_n\rho q\sin\theta\dot F_n'\rho' q'\cos\theta' - n^2F_nq\cos\theta F_n'q'\sin\theta'\Big)\\
&\qquad + 2e^{in(\theta-\theta')}in\Big(-\dot F_n\rho q\cos\theta\dot F_n'q'\cos\theta' - F_nq\sin\theta F_n'\rho' q'\sin\theta'\\
&\qquad\qquad -\dot F_n\rho q\sin\theta\dot F_n'q'\sin\theta' - F_nq\cos\theta F_n'\rho' q'\sin\theta'\Big)\Big\rangle,
\end{aligned}$$

and then

$$\begin{aligned}
&\Big\langle \nu\otimes\nu', \quad -\frac{i}{4}I\Big\rangle\\
&= \Big\langle \nu\otimes\nu', \quad n\rho q^2F_n\dot F_n + \frac12 F_nF_n'qq'\sin(n(\theta-\theta'))\sin(\theta-\theta')(\rho\rho'+n^2)\\
&\qquad - n\dot F_nF_n'\rho qq'\cos(n(\theta-\theta'))\cos(\theta-\theta')\Big\rangle\\
&= \Big\langle \nu\otimes\nu', \quad \frac{n}{2}(qF_n - q'F_n')(\rho q\dot F_n - \rho' q'\dot F_n')\\
&\qquad + \frac12\sin^2\Big(\frac{n+1}{2}(\theta-\theta')\Big)\Big(2n\dot F_nF_n'\rho qq' + F_nF_n'qq'(\rho\rho'+n^2)\Big)\\
&\qquad + \frac12\sin^2\Big(\frac{n-1}{2}(\theta-\theta')\Big)\Big(2n\dot F_nF_n'\rho qq' - F_nF_n'qq'(\rho\rho'+n^2)\Big)\Big\rangle.
\end{aligned}$$

Thus, from (6.2),

$$\begin{aligned}
< \nu\otimes\nu', \quad &\frac{n}{2}(qF_n - q'F_n')(\rho q\dot F_n - \rho' q'\dot F_n')\\
&+ \frac12\sin^2\Big(\frac{n+1}{2}(\theta-\theta')\Big)\Big(2n\dot F_nF_n'\rho qq' + F_nF_n'qq'(\rho\rho'+n^2)\Big)\\
&+ \frac12\sin^2\Big(\frac{n-1}{2}(\theta-\theta')\Big)\Big(2n\dot F_nF_n'\rho qq' - F_nF_n'qq'(\rho\rho'+n^2)\Big) >= 0.
\end{aligned}$$

Now interchange n and $-n$ to get the n^2-terms equal to zero. Hence

$$
\begin{aligned}
< \nu \otimes \nu', \quad & (qF_n - q'F_n')(\rho q\dot{F}_n - \rho' q' \dot{F}_n') \\
& + 2\left(\sin^2\left(\frac{n+1}{2}(\theta-\theta')\right) + \sin^2\left(\frac{n-1}{2}(\theta-\theta')\right)\right)\dot{F}_n F_n' \rho q q' >= 0.
\end{aligned} \tag{9.1}
$$

Then we follow verbatim from the argument of Morawetz [28] to lead the following convergence theorem.

Theorem 9.1. *Let $v_\infty = 0$, $|u_\infty| < q_{cav}$, $1 \le \gamma < 3$. Assume that there exists a positive ε-independent $\alpha(\delta) \to 0$ as $\delta \to 0$ such that $q^\varepsilon(x,y) \ge \alpha(\delta)$ for any $(x,y) \in \Omega_\delta = \{(x,y) \in \Omega : dist((x,y), \partial\Omega_1) \ge \delta > 0\}$. Then the following hold:*

- (i) *The support of the Young measure $\nu_{x,y}$ strictly excludes the stagnation point $q = 0$ and reduces to a point for a.e. $(x,y) \in \Omega_\delta$, hence the Young measure is a Dirac mass;*
- (ii) *The sequence $(u^\varepsilon, v^\varepsilon)$ has a subsequence converging strongly in $L^2_{loc}(\Omega)^2$ to an entropy solution in the sense of Definition 7.1;*
- (iii) *The boundary condition $(u,v)\cdot\mathbf{n} \ge 0$ on $\partial\Omega_1$ is satisfied in the sense of normal trace in [8].*

Proof. First, we choose $F_n \sim q^{-n}$, with n sufficiently large and $q < q_{cr}$, to show that the support of ν cannot lie in both $q < q_{cr}$ and $q > q_{cr}$ as in Morawetz [27]. If the support of ν lies inside $q \le q_{cr}$, equation (9.1) or the argument of Chen-Dafermos-Slemrod-Wang [7] applies to show that ν is a Dirac mass. If the support of ν is in $q \ge q_{cr}$, Morawetz's reconstruction of DiPerna's argument in [27] by using K_n associated entropies $(Q_{1\pm}, Q_{2\pm})$ works to show again that ν is a Dirac mass. The boundary condition on $\partial\Omega_1$ follows the weak form of the second equation in (2.1) with $R_2 = \varepsilon\nabla(\sigma_2(\rho)\nabla\rho)$. The entropy inequality follows from (8.3). □

Remark 9.1. From the boundary condition (5.2) for the viscous problem, we find that, if one can show that $\varepsilon\sigma_2(\rho^\varepsilon)\nabla\rho^\varepsilon\cdot\mathbf{n} \to 0$ in the weak sense on $\partial\Omega_1$ as $\varepsilon \to 0$, then we can conclude the desired boundary condition $(u,v)\cdot\mathbf{n} = 0$ on $\partial\Omega_1$ satisfied in the weak sense. This is the case when the limit solution has certain regularity along the boundary, i.e. any shock strength is zero at its intersection point with the boundary, as conjectured for the shock formed in the supersonic bubble near the obstacle (cf. Morawetz [26, 29]).

Remark 9.2. For the first boundary value problem in Figure 6(a), if $\partial\Omega_1$ is sufficiently smooth, one expects that the transonic flow is strictly away from stagnation, which would expect that the viscous solutions stay uniformly away from cavitation in the whole domain Ω. However, for the second boundary value problem in Figure 6(b), one expects that the transonic flow generally has a stagnation point on the boundary near the head of the obstacle since the circulation about the boundary $\partial\Omega_2$ is zero. The assumption that there exists an ε-independent positive $\alpha(\delta) \to 0$ as $\delta \to 0$ such that $q^\varepsilon(x,y) \ge \alpha(\delta)$, for any $(x,y) \in \Omega_\delta$, is physically motivated and designed in order to fit both cases.

10. Resolution of the Viscous Problem

As we have seen in the previous sections, it is crucial that our viscous problem possesses smooth solutions $w^\varepsilon = (u^\varepsilon, v^\varepsilon)$. We address the issue in this section. Consider the viscous problem:

$$\begin{cases} v_x - u_y = \varepsilon \nabla(\sigma_1(\rho)\nabla\theta), \\ (\rho u)_x + (\rho v)_y = \varepsilon \nabla(\sigma_2(\rho)\nabla\rho), \end{cases} \tag{10.1}$$

with

$$\sigma_1 = 1, \quad \sigma_2(\rho) = \frac{q^2 - c^2}{q^2} \qquad \text{for} \quad q \geq \sqrt{2}\, q_{cr}, \tag{10.2}$$

and σ_1, σ_2 any smooth bounded, positive continuation for $q \leq \sqrt{2} q_{cr}$, and with the boundary conditions on the obstacle $\partial\Omega$:

$$\begin{cases} \nabla\theta \cdot \mathbf{n} = 0 \qquad \text{on} \quad \partial\Omega_1, \\ \varepsilon\sigma_2 \nabla\rho \cdot \mathbf{n} - |\rho\,(u, v) \cdot \mathbf{n}| = 0 \qquad \text{on} \quad \partial\Omega_1, \\ (u, v) - (u_\infty, v_\infty) = 0 \qquad \text{on} \quad \partial\Omega_2, \quad \text{with} \quad q_\infty < q_{cr}, \end{cases} \tag{10.3}$$

where ρ is a function of q given by Bernoulli's law (2.2) for $\gamma > 1$ and (2.5) for $\gamma = 1$, and $\mathbf{n}$ is the unit normal pointing into the flow region on $\partial\Omega$.

Introduce a new variable

$$\sigma(\rho) := \int_1^\rho \sigma_2(\xi) d\xi.$$

Since $\sigma'(\rho) = \sigma_2(\rho) > 0$, we can always invert to write $\rho = \rho(\sigma)$, as well as $q = q(\sigma)$. The advantage of this change of dependent variable is that we now have the viscous terms to be $\varepsilon\Delta\theta$ and $\varepsilon\Delta\sigma$.

We use the polar velocity notation

$$u = q(\sigma)\cos\theta, \quad v = q(\sigma)\sin\theta,$$

and write the viscous system (10.1) as

$$\begin{cases} (q(\sigma)\sin\theta)_x - (q(\sigma)\cos\theta)_y = \varepsilon\Delta\theta, \\ (\rho(\sigma)q(\sigma)\cos\theta)_x + (\rho(\sigma)q(\sigma)\sin\theta)_y = \varepsilon\Delta\sigma, \end{cases} \tag{10.4}$$

which yields

$$\begin{cases} q'(\sigma)\sin\theta\,\sigma_x + q(\sigma)\cos\theta\,\theta_x - q'(\sigma)\cos\theta\,\sigma_y + q(\sigma)\sin\theta\,\theta_y = \varepsilon\Delta\theta, \\ (\rho(\sigma)q(\sigma))'\cos\theta\,\sigma_x - \rho(\sigma)q(\sigma)\sin\theta\,\theta_x + (\rho(\sigma)q(\sigma))'\sin\theta\,\sigma_y + \rho(\sigma)q(\sigma)\cos\theta\,\theta_y = \varepsilon\Delta\sigma, \end{cases} \tag{10.5}$$

with the boundary conditions on $\partial\Omega_1$:

$$\begin{cases} \nabla\theta \cdot \mathbf{n} = 0, \\ \varepsilon\nabla\sigma \cdot \mathbf{n} - |\rho(\sigma)q(\sigma)(\cos\theta, \sin\theta) \cdot \mathbf{n}| = 0, \end{cases} \tag{10.6}$$

and the boundary conditions on $\partial\Omega_2$:

$$\sigma = \sigma_\infty, \quad \theta = \theta_\infty, \tag{10.7}$$

where $\sigma_\infty = \sigma(\rho_\infty)$ and θ_∞ are constants. It is convenient to have the homogeneous boundary conditions on $\partial\Omega_2$:

$$\bar{\sigma} := \sigma - \sigma_\infty, \quad \bar{\theta} := \theta - \theta_\infty; \qquad \bar{q}(\bar{\sigma}) := q(\bar{\sigma} + \sigma_\infty), \quad \bar{\rho}(\bar{\sigma}) := \rho(\bar{\sigma} + \sigma_\infty).$$

Hence, we have the following boundary value problem:

$$\begin{cases}\varepsilon\Delta\bar\theta = & \bar q'(\bar\sigma)\sin(\bar\theta+\theta_\infty)\,\bar\sigma_x+\bar q(\bar\sigma)\cos(\bar\theta+\theta_\infty)\,\bar\theta_x\\ & -\bar q'(\bar\sigma)\cos(\bar\theta+\theta_\infty)\,\bar\sigma_y+\bar q(\bar\sigma)\sin(\bar\theta+\theta_\infty)\,\bar\theta_y,\\ \varepsilon\Delta\bar\sigma = & (\bar\rho(\bar\sigma)\bar q(\bar\sigma))'\cos(\bar\theta+\theta_\infty)\,\bar\sigma_x-\bar\rho(\bar\sigma)\bar q(\bar\sigma)\sin(\bar\theta+\theta_\infty)\,\bar\theta_x\\ & +(\bar\rho(\bar\sigma)\bar q(\bar\sigma))'\sin(\bar\theta+\theta_\infty)\,\bar\sigma_y+\bar\rho(\bar\sigma)\bar q(\bar\sigma)\cos(\bar\theta+\theta_\infty)\,\bar\theta_y,\end{cases} \tag{10.8}$$

with the boundary conditions on $\partial\Omega_1$:

$$\begin{cases}\varepsilon\nabla\bar\theta\cdot\mathbf{n}=0,\\ \varepsilon\nabla\bar\sigma\cdot\mathbf{n}-|\bar\rho(\bar\sigma)\bar q(\bar\sigma)(\cos(\bar\theta+\theta_\infty),\sin(\bar\theta+\theta_\infty))\cdot\mathbf{n}|=0,\end{cases} \tag{10.9}$$

and $(\bar\sigma,\bar\theta)=(0,0)$ on $\partial\Omega_2$.

Consider the classical equation

$$(\bar\theta,\bar\sigma)=\lambda\Gamma(\bar\theta,\bar\sigma),\quad 0\le\lambda\le 1,$$

which corresponds to the boundary value problem:

$$\begin{aligned}&\varepsilon\Delta\bar\theta=\lambda\bar q'(\bar\sigma)\sin(\bar\theta+\theta_\infty)\,\bar\sigma_x+\lambda\bar q(\bar\sigma)\cos(\bar\theta+\theta_\infty)\,\bar\theta_x\\ &\qquad\quad-\lambda\bar q'(\bar\sigma)\cos(\bar\theta+\theta_\infty)\,\bar\sigma_y+\lambda\bar q(\bar\sigma)\sin(\bar\theta+\theta_\infty)\,\bar\theta_y\qquad\text{in }\ \Omega,\\ &\varepsilon\Delta\bar\sigma=\lambda(\bar\rho(\bar\sigma)\bar q(\bar\sigma))'\cos(\bar\theta+\theta_\infty)\,\bar\sigma_x-\lambda\bar\rho(\bar\sigma)\bar q(\bar\sigma)\sin(\bar\theta+\theta_\infty)\,\bar\theta_x\\ &\qquad\quad+\lambda(\bar\rho(\bar\sigma)\bar q(\bar\sigma))'\sin(\bar\theta+\theta_\infty)\,\bar\sigma_y+\lambda\bar\rho(\bar\sigma)\bar q(\bar\sigma)\cos(\bar\theta+\theta_\infty)\,\bar\theta_y\qquad\text{in }\ \Omega,\\ &\varepsilon\nabla\bar\theta\cdot\mathbf{n}=0\qquad\text{on }\ \partial\Omega_1,\\ &\varepsilon\nabla\bar\sigma\cdot\mathbf{n}=\lambda|\bar\rho(\bar\sigma)\bar q(\bar\sigma)(\cos(\bar\theta+\theta_\infty),\sin(\bar\theta+\theta_\infty))\cdot\mathbf{n}|\qquad\text{on }\ \partial\Omega_1,\\ &(\bar\sigma,\bar\theta)=(0,0)\qquad\text{on }\ \partial\Omega_2,\end{aligned} \tag{10.10}$$

which is the original system with ε replaced by $\varepsilon\lambda^{-1}$.

Now we recall some results from the linear elliptic theory. Consider the mixed Dirichlet-Neumann problem

$$\begin{cases}\Delta w=f_1 & \text{in }\ \Omega,\\ \frac{\partial w}{\partial\mathbf{n}}=f_2 & \text{on }\ \partial\Omega_1,\\ w=f_3 & \text{on }\ \partial\Omega_2,\end{cases} \tag{10.11}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with the boundary $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$. We now introduce weighted Hölder norms. For an arbitrary open set S and an arbitrary $a>0$, we have the usual Hölder norms $\|\cdot\|_{a,S}$. If $\delta>0$ and $a+b\ge 0$, we define

$$\Omega_\delta=\{x\in\Omega:\ \mathrm{dist}(x,\partial\Omega_2)>\delta\},$$

and

$$\|w\|_a^{(b)}=\sup_{\delta>0}\delta^{a+b}\|w\|_{a,\Omega_\delta},$$

and then denote $C^a_{(b)}$ the set of all functions with $\|w\|_a^{(b)}<\infty$, and $C^*(\Omega)=C^2(\Omega\cup\partial\Omega_1)\cap C(\bar\Omega)$.

Lemma 10.1 (Lieberman[24], Theorem 2(b)). *Suppose $\partial\Omega_1 \in C^{2+\alpha}$, $\partial\Omega_2$ is nonempty, and Ω satisfies the global Σ-wedge condition and the uniform interior and exterior cone condition on $\partial\Omega$. Then there is a unique solution to the linear elliptic problem* (10.11) *and*

$$\|w\|_{2+\alpha}^{(\beta)} \le C\left(\|f_1\|_{\alpha}^{(2+\beta)} + \|f_2\|_{1+\alpha}^{(1+\beta)} + \|f_3\|_{\beta}\right)$$

for some $\beta > 0$ and constant $C > 0$.

Remark 10.1. Note that Lieberman's theorem is given for the boundary condition on $\partial\Omega_1$: $Mw := \beta^i D_i w + d\,w = f_2$, where $d < 0$. However, as noted by Lieberman on page 429 in [24], the Fredholm alternative holds, and if $\partial\Omega_2$ is nonempty, the restriction $d < 0$ may be relaxed to $d \le 0$. The domains shown in Figure 6 satisfy the above conditions.

We define $\Gamma : (\bar\Theta, \bar\Sigma) \to (\bar\theta, \bar\sigma)$ to be the solution map associated with solving the decoupled mixed Dirichlet-Neumann boundary value problem obtained from substitution of $(\bar\Theta, \bar\Sigma)$ into the right hand side of (10.10). For $(\bar\Theta, \bar\Sigma) \in (C^{1+\alpha}_{(2+\beta)})^2$, the associated functions f_1 and f_2 give finite values on the right-hand side of the Schauder estimates of Lieberman's theorem and $f_3 = 0$ in our problem. Therefore, $(\bar\theta, \bar\sigma)$ stays in a bounded subset of $(C^{2+\alpha}_{(\beta)})^2$ which is a compact subset of $(C^{1+\alpha}_{(2+\beta)})^2$, and Γ is a compact map of the Banach space $B = (C^{1+\alpha}_{(2+\beta)})^2$ into itself.

We recall one popular version of the Leray-Schauder fixed point theorem.

Lemma 10.2 (Leray-Schauder Fixed Point Theorem). *Let Γ be a compact mapping of a Banach space B into itself, and suppose that there exists a constant M such that $\|W\|_B \le M$ for all $W \in B$ and all $\lambda \in [0,1]$ satisfying $W = \lambda\Gamma W$. Then Γ has a fixed point.*

To apply the above theorem to our example, we first note that the only solution when $\lambda = 0$ is $(\bar\theta, \bar\sigma) = (0,0)$, hence it is certainly bounded in B. For $\lambda \ne 0$, we study the solutions to our original viscous system with $\bar\varepsilon = \varepsilon\lambda^{-1} \ge \varepsilon$. The invariant region argument we have given provides the uniform $L^\infty(\bar\Omega)$ bound independent of $\bar\varepsilon$. The gradient estimates from the entropy equality yield that $\sqrt{\bar\varepsilon}\nabla(\bar\theta, \bar\sigma)$ are in a bounded set of $(L^2(\Omega))^2$, and hence $\sqrt{\varepsilon}\nabla(\bar\theta, \bar\sigma)$ are in the same bounded set of $(L^2(\Omega))^2$ for all $0 < \lambda \le 1$.

Now we examine the equations themselves:

$$\varepsilon\Delta\bar\theta = \lambda f_1, \qquad \varepsilon\Delta\bar\sigma = \lambda f_2,$$

where f_1, f_2 are in a bounded set of $L^2(\Omega)$. Hence, for $0 < \lambda \le 1$, $\varepsilon\Delta(\bar\theta, \bar\sigma)$ are in a bounded set of $L^2(\Omega)$ and, by the Calderon-Zygmund inequality (Theorem 9.9 in Gilbarg-Trudinger [21]), $(\bar\theta, \bar\sigma) \in W^{2,2}(\Omega)$.

Next we differentiate the both sides of our equations with respect to x, y, and we get the equations of the form

$$\varepsilon\Delta\bar\theta_x = \lambda f_{1x}, \qquad \varepsilon\Delta\bar\theta_y = \lambda f_{1y}.$$

Since the right-hand sides are in a bounded set of $L^2(\Omega)$, so are the left-hand sides. We continue in this fashion to see that there is a constant M_1 (independent of m and λ) such that

$$\|(\bar\theta, \bar\sigma)\|_\infty + \|\nabla(\bar\theta, \bar\sigma)\|_{(H^m(\Omega))^2} \le M_1,$$

for all $1 \le m < \infty$.

Then Morrey's embedding theorem applied to our case yields $\nabla(\bar\theta, \bar\sigma)$ bounded by constant $M_1 = M$ in $(C^s(\bar\Omega))^2$ for any $s \ge 1$ and $(\bar\theta, \bar\sigma)$ bounded in B for any $\tau > 0$ by the constant $M_1 = M$, where the constant is independent of λ. Thus, all possible solutions of $(\bar{\theta}, \bar{\sigma}) = \lambda\,\Gamma(\bar{\theta}, \bar{\sigma})$, $0 \le \lambda \le 1$, are bounded by constant $M_1 = M$ with M independent of λ. Therefore, the hypotheses of the Leray-Schauder fixed point theorem are satisfied and the viscous problem has a solution, i.e. we have the following theorem.

Theorem 10.1. *The viscous problem has a solution (u^ε, v^e) for every $\varepsilon > 0$ such that $q^\varepsilon \le q^* < q_{cav}$ for $\gamma \in [1, 3)$, where q^* is a constant independent of $\varepsilon > 0$.*

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G.-Q. Chen, Department of Mathematics, Northwestern University, Evanston, IL 60208.
E-mail address: gqchen@math.northwestern.edu

M. Slemrod, Department of Mathematics, University of Wisconsin, Madison, WI 53706.
E-mail address: slemrod@math.wisc.edu

D. Wang, Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260.
E-mail address: dwang@math.pitt.edu