On singularities of continuity equation solution

V. G. Danilov*

Abstract

Generalized solutions to the continuity equation in the one- and multidimensional cases are constructed in the case of discontinuous velocity field.

In this paper we construct a generalized solution to the continuity equation for the case in which the velocity field has jumps on smooth submanifolds of codimension one. Such a situation appears, in particular, in studying solutions of the system of pressureless gas dynamics equations (see, e.g., [1, 2, 10])

$$\frac{\partial u}{\partial t} + (u, \nabla)u = 0, \tag{0.1}$$

$$\frac{\partial \rho}{\partial t} + (\nabla, \rho u) = 0, \tag{0.2}$$

where $(\cdot, \cdot)$ is the scalar product in $\mathbb{R}^n$.

If the velocity field has jumps, then a δ-shock wave type solution appears, i.e., the solution component with the meaning of density contains the Dirac δ-function.

This leads to the following difficulty: in the formal substitution of the solution into the equation, it is required to define the derivatives of the product $\delta(x)H(x)$, where $H(x)$ is the Heaviside function. In general, the difficulties arising when the solution is directly substituted into the equation are well

1

known in the theory of conservation laws, where in the case of nonsmooth solutions, the solution is defined in the form of an integral identity that does not contain derivatives of nonsmooth functions.

In [3, 4], for a certain system of conservation laws, the solution is defined as an integral identity in which it is not required to define the abovementioned product $\delta \cdot H$ (although, *a posteriori*, the definition of this product, each own for each equation in general, can easily be obtained from this construction).

The integral identity given in [3, 4] is obtained as the limit of expressions obtained by substituting the weak asymptotic solutions (weak approximations of the solutions) into the equations.

Definition 0.1 ([3, 4]) A weak asymptotic solution $(u_\varepsilon, \rho_\varepsilon)$ of Eqs. (0.1), (0.2) is defined to be a family of functions u_ε, ρ_ε smooth for $\varepsilon > 0$ and such that

$$
\begin{aligned}
\frac{\partial u_\varepsilon}{\partial t} + (u_\varepsilon, \nabla)u_\varepsilon &= o_{\mathcal{D}'}(1), \\
\frac{\partial \rho_\varepsilon}{\partial t} + (\nabla, \rho_\varepsilon u_\varepsilon) &= o_{\mathcal{D}'}(1),
\end{aligned}
\tag{0.3}
$$

where $o_{\mathcal{D}'}(1)$ are infinitesimals as $\varepsilon \to 0$ in the weak sense, i.e., for any test function $\eta(x)$, we have the estimates

$$
\langle o_{\mathcal{D}'(\mathbb{R}^n)(1)}, \eta \rangle_{\mathbb{R}^n_x} = o(1), \qquad \varepsilon \to 0,
$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^n_x}$ means the action of the generalized function (here it is $o_{\mathcal{D}'}(1)$) on the test function $\eta(x)$, t, ε are parameters, and the estimate $o(1)$ is understood in the usual sense and has to be uniform in t.

Applying the left-and right-sides of Eqs. (0.3) to the test function, calculating the derivatives of the test function (i.e., integrating by parts), and passing to the limit as $\varepsilon \to 0$, we can define the required integral identity [3, 4].

In this paper, we construct a definition of the δ-shock type solution to continuity equation (0.2) without constructing a weak asymptotic solution and generalize this construction to the multidimensional case. As a consequence, we obtain the mass conservation law in the following form.

Let $\rho = \rho(x, t)$ be a δ-shock wave type solution of Eq. (0.2) whose support is a compact set in x for each t. Then, for any test function $\eta(x)$ such that

2

$\eta(x) = 1$ for $(x, t) \in \operatorname{supp} \rho$, we have the relation

$$\frac{d}{dt}\langle \rho, \eta \rangle = 0. \tag{0.4}$$

We note that (0.4) is an elementary corollary of the second equation in (0.3). Indeed, let us apply the left- and right-hand sides of the last equation in (0.3) to the function $\eta(x)$ from (0.4). Taking into account that the left-hand side of the last equation in (0.3) for $\varepsilon > 0$ is a regular generalized function, we obtain

$$\frac{\partial}{\partial t}\langle \rho_\varepsilon, \eta \rangle + \int_{\mathbb{R}^n} (\nabla, \rho_\varepsilon u_\varepsilon)\eta(x)\, dx = O(\varepsilon).$$

We assume that the family u_ε is bounded uniformly in $\varepsilon \geq 0$ and, in $\mathcal{D}'(\mathbb{R}'_x)$, for each t, the limit exists

$$\lim_{\varepsilon \to 0} \rho_\varepsilon(x, t) = \rho(x, t)$$

whose support, by assumption, is compact in x. Then, because of Eq. (0.2), we see that in $\mathcal{D}'(\mathbb{R}'_x)$, for each t, the limit $\rho_\varepsilon u_\varepsilon$ exists with the same support as that of the function $\rho(x, t)$.

Integrating by parts the second term in the left-hand side and passing to the limit as $\varepsilon \to 0$, we obtain (0.4).

But, in what follows, to justify the obtained formulas, we shall verify (0.4) independently of this argument.

We note that calculations concerning (0.4) were first considered in [5, 6] in a somewhat different situation.

The author is grateful to V. M. Shelkovich who pointed to several misprints in the text.

1 One-dimensional continuity equation

The appearance of the δ-shock type solutions to continuity equations can easily be explained.

For example, let $x \in \mathbb{R}^1$,

$$u = \begin{cases} 1, & x < 0, \\ 0, & x = 0, , \\ -1, & x < 0, \end{cases} \qquad \rho|_{t=0} = \rho_0 \in C(\mathbb{R}^1), \qquad \operatorname{supp} \rho_0 \in [-1, 1].$$

It is clear that for, $x \neq 0$, the solution of Eq. (0.2) with this initial data is defined for all $t > 0$ and $\rho|_{x \neq 0} = 0$ for $t > 1$. If the solution satisfied the conservation law (0.4), then it is clear that

$$\rho = C\delta(x) \qquad \text{for} \quad t > 1,$$

where $C = \int_{\mathbb{R}^1} \rho_0(x)$. The situation considered in this example is generalized in the following definition.

Definition 1.1 Let $u = u(x,t) \in L^\infty(\mathbb{R}^1 \times \mathbb{R}_-)$ be a given function. Let $\Gamma = \{\gamma_i, i \in I\}$ be a graph in the half-plane $\{(x,t); x \in \mathbb{R}^1, t \geq 0\}$ containing C^1 arcs γ_i, and let I be a finite set. By $I_0 \subset I$ we denoted the set of numbers of the arcs starting from the point $x_k^0 \in \mathbb{R}^1$. A distribution $\rho(x,t)$ and a graph Γ, where

$$\rho(x,t) = R(x,t) + E(t)\delta(\Gamma), \qquad E(t) = \sum_{i \in I} e_i(t)\delta(\gamma_i), \qquad (1.1)$$

$$e_i(t) \in L^\infty(\gamma_i), \qquad \gamma_i = \{x = \varphi_i(t)\},$$

is called a *generalized δ-shock wave type solution* to (0.2) if the integral identity

$$\int_0^\infty \int_{\mathbb{R}^1} (R\zeta_t + uR\zeta_x)\, dxdt + \sum_{i \in I} \int_{\gamma_i} e_i(t_i)\frac{d\zeta}{dt_i}\, dt$$

$$+ \int_{\mathbb{R}^1} R\zeta|_{t=0}\, dx + \sum_{k \in I_0} e_k(0)\zeta(\varphi(0), 0) = 0 \qquad (1.2)$$

holds for all test functions $\zeta(x,t) \in D(\mathbb{R}^1 \times \mathbb{R}_+^1)$, $\frac{d}{dt_i} = \frac{\partial}{\partial t} + \varphi_{it}'\frac{\partial}{\partial x}$.

The appearance of the summand

$$\sum_{i \in I} \int_{\gamma_i} e_i(t)\frac{d\zeta}{dt_i}\, dt$$

in (1.2) can easily be explained. Indeed, let ρ have the form (1.1), then differentiating in t, we obtain

$$\rho_t = \sum_{i \in I} e_i(t)(-\varphi_{it}')\delta'(\gamma_i) + \text{smoother summands.}$$

4

Hence it is clear that we must have

$$(\rho u)_x = -\sum_{i \in I} e_i(-\varphi'_{it})\delta'(\gamma_i) + \text{smoother summands.}$$

Now, for any test function $\zeta(x,t)$ such that $\zeta(x,0) = 0$, we have

$$
\begin{aligned}
\langle \rho_t + (\rho u)_x, \zeta(x,t) \rangle &= -\langle \rho, \zeta_t(x,t) \rangle - \langle \rho u, \zeta_x(x,t) \rangle \\
&= -\langle R, \zeta_t(x,t) \rangle - \langle E(t)\delta(\Gamma), \zeta_t(x,t) \rangle - \langle Ru, \zeta_x \rangle \\
&\quad + \sum_{i \in I} \langle e_i(t)(-\varphi'_{it})\delta(\gamma_i), \zeta_x(x,t) \rangle \\
&= -\langle R, \zeta_t \rangle - \langle Ru, \zeta_x \rangle - \sum_{i \in I} \int_{\gamma_i} e_i(t)(\zeta_t + \varphi'_{it}\zeta_x)\, dt.
\end{aligned}
$$

Of course, these calculations do not give any proof, but only provide a motivation.

Definition 1.1 gives a method for calculating the functions contained in (1.1).

Suppose that

$$u = u_0 + \sum_{i \in I} H(x - \varphi_i) u_i,$$

where $\varphi_i(t)$, $u_0(x,t)$, and $u_i(x,t)$ are smooth functions (i.e., the velocities have jumps on the given curves $x = \varphi_i$). Then, integrating by parts in (1.2), we obtain

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^1 \setminus \bigcup\{x = \varphi_i\}} (R_t + (uR)_x)\zeta\, dxdt \\
&\quad + \sum_{i \in I} \int_{x = \varphi_i} \{-[R]\varphi_{it} + [uR]\}\zeta\, dt + \sum_{i \in I} \int_{\gamma^i} e'_{it}\zeta\, dt = 0,
\end{aligned}
$$

where $[g] = [g]|_{x=\varphi_i} = g(\varphi_i + 0) - g(\varphi_i - 0)$.

This implies the system of equations

$$R_t + (uR_x)_x = 0 \qquad (x,t) \in \mathbb{R}^1 \times \mathbb{R}^1_+ \setminus \bigcup_{i \in I}\{x = \varphi_i\}, \qquad (1.3)$$

$$e'_{it} = \varphi_{it}[R]|_{x=\varphi_i} - [uR]|_{x=\varphi_i}, \qquad i \in I. \qquad (1.4)$$

5

In this case, at the nodes of the graph Γ lying above the axis $\{t = 0\}$, the following "Kirchhoff laws" must be satisfied:

$$\sum_{i \in \text{In } A_k} e_i(t_k^* - 0) = \sum_{i \in \text{Out } A_k} e_i(t_k^* + 0), \qquad (1.5)$$

where In A_k and Out A_k are the respective sets of incoming and outgoing arcs associated with a certain node $A_k = (x_k, t_k^*)$.

The following obvious statement holds.

Theorem 1.1 *Suppose that system* (1.3), (1.4), (1.5) *has a classical solution. Then the function* $\rho(x, t)$ (1.1) *is a generalized δ-shock wave type solution to* (0.2).

As an example, we construct the solution of the problem given at the beginning of this section. It is easy to verify that

$$R = \begin{cases} \rho_0(x + t), & x > 0, \\ \rho_0(x - t), & x < 0, \end{cases}$$

$$e = \int_0^t (\rho_0(t') + \rho_0(-t'))\, dt' = \int_{-t}^t \rho_0(t')\, dt'$$

and

$$e(t) = \text{const} = \int_{-1}^1 \rho_0(t')\, dt' \qquad \text{for} \quad t \geq 1.$$

Now let

$$u = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Then

$$R = \begin{cases} \rho_0(x - t), & 0 \leq t \leq x, \\ \rho_0(x + t), & 0 \leq t \leq -x. \end{cases}$$

In the domain $|x| < t$, the solution R is not determined by the initial data. In this case, as is known, the solution can be found by determining, for example, an arbitrary function $\rho_1(t)$ for $t \geq 0$. Then

$$R = \begin{cases} \rho_1(t - x), & x < t, \\ \rho_1(t + x), & x < -t. \end{cases}$$

6

Moreover, if $\rho_1(0) = \rho_0(0)$, then the function constructed in the t-plane is continuous. But if $\rho_1(0) \neq \rho_0(0)$, then the function R is discontinuous on the lines $x = \pm t$, but no δ-function appear on this discontinuity, since $[u]|_{x=\pm t} = 0$ and Eq. (1.4) becomes

$$e'_{\pm t} = -\psi'_{\pm t}[R] + u|_{x=\pm t}[R] = 0.$$

Here $\psi_{\pm} = x \pm t$ in the domains $x > 0$ and $x < 0$, respectively. Therefore, $\psi'_{\pm t} = u|_{x=\pm t}$ and $e'_{\pm t} = 0$.

Thus, if the R has a discontinuity on the line $\psi_t = u(\psi, t)$ in the domain of continuity of $u(x, t)$ (i.e., on the characteristic), then no δ-function is formed.

In other words, the graph Γ from Definition 1.1 does not contain arcs coinciding with trajectories of u.

We also note that, from the viewpoint of complete system (0.3), the first of the examples considered corresponds to the classical δ-shock wave type solution.

The second situation is impossible in the framework of system (0.3), because an unstable jump becomes a rarefaction wave and there is no concentration (no formation of a δ-function) (see [7]).

Now we prove relation (0.4). For this, we calculate the expression

$$\frac{d}{dt}\langle \rho, \eta \rangle = \frac{d}{dt} \int_{\mathbb{R}^1} R\,dx + \frac{d}{dt}\langle E\delta(\Gamma), \eta \rangle.$$

For all $t = \hat{t}$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^1} R\,dx = \frac{d}{dt} \sum_{i \in I'} \left(\int_{-\infty}^{\varphi_i} R\,dx + \int_{\infty}^{\varphi_i} R\,dx \right),$$

where I' is the set of arcs intersecting on the straight line $t = \hat{t}$ in the half-plane $\mathbb{R}^1_x \times [0, \infty)$. Hence we have

$$\frac{d}{dt} \int_{\mathbb{R}^1} R\,dx = \sum_{i \in I'} \left\{ -\varphi_{it}[R]|_{x=\varphi_i} + [Ru]|_{x=\varphi_i} \right\},$$

and by (1.4)

$$\frac{d}{dt} \int_{\mathbb{R}^1} R\,dx + \sum_{i \in I'} e'_{it} = 0$$

as was required to prove.

2 Conservation equation and concentration in multidimensional case

In this section, we generalize the results of the preceding section to the multidimensional case.

Let us recall several facts and formulas. We assume that an $n-1$-dimensional surface γ_t moving in $\mathbb{R}_x^n$ is determined by the equation

$$\gamma_t = \{x; t = \psi(x)\},$$

where $\psi(x) \in C^1(\mathbb{R}^n)$, $\nabla\psi \neq 0$, in a domain in $\mathbb{R}_x^n$, where we work.

Obviously, determining a surface by the equation $t = \psi$ is equivalent to determining a surface by an equation of the form

$$S(x,t) = 0$$

$(S(x,t) \in C^1$, $S(x,t) = 0$, $\nabla_{x,t}S|_{S=0} \neq 0)$ under the condition that

$$\frac{\partial S}{\partial t} \neq 0$$

The last relation shows that the implicit function theorem can be used and, moreover, the normal velocity of a point on the surface $S = 0$ does not change its direction with respect to the fixed direction of the normal to the surface $S = 0$. In turn, this means that the surface moves in the "same direction" all the time. But if $\frac{\partial S}{\partial t} = 0$ (the velocity of several points on the surface is zero), then we can make the change $x_i' = x_i - c_i t$ with appropriately chosen $c_1, \ldots, c_n$, solve the problem with moving boundary, and then return to the original variables. So, this restriction is not essential for the solutions of first-order equations whose construction is based on the method of characteristics.

Next, we assume that $\zeta(x,t) \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+^1)$. Then

$$\langle \delta(t - \psi(x)), \zeta(x,t) \rangle = \int_{\mathbb{R}^n} \zeta(x, \psi(x)) \, dx.$$

A more complicated construction appears if $\delta(t - \psi(x))$ is applied to the test function $\eta(x) \in C_0^\infty(\mathbb{R}^n)$. In this case,

$$\langle \delta(t - \psi(x)), \eta(x) \rangle = \int_{\gamma_t} \eta(x) \, d\omega_\psi,$$

8

where $d\omega$ is the Leray measure [8] on the surface $\{t = \psi(x)\}$ such that $-d\psi d\omega_\psi = dx_1 \ldots dx_n$. It is clear that the function $\Psi = (t - \psi)/|\nabla\psi|$ up to second-order infinitesimals is the Euclidean distance from a point x to the surface $\{t = \psi\}$.

Obviously, the relations $t = \psi(x)$ and $\Psi(x) = 0$ determine the same surface, and, by definition, the Leray forms (measures) corresponding to these functions satisfy the relation

$$\omega_\psi = \omega_\Psi \cdot \frac{1}{|\nabla\psi|}.$$

On the other hand, according to the relationship between the values of the function Ψ and the Euclidean distance to the surface $\{t = \psi\}$, the form ω_Ψ is an Euclidean element $d\sigma$ on the surface $\{t = \psi\}$. Finally, we have

$$\langle \delta(t - \psi(x)), \eta(x) \rangle = \int_{\gamma_t} \frac{\eta(x)}{|\nabla\psi|} \, d\sigma.$$

Now we formulate an integral identity, which is an analog of (1.2).

First, we assume that just as in the preceding section, the solution ρ has the form

$$\rho = R(x, t) + e(x)\delta(t - \psi(x)), \tag{2.1}$$

where $R(x,t) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^1_+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^1_+)$ and the function R has a discontinuity for $t = \psi(x)$, $R = R_0(x,t) + H(t - \psi)R_1(x,t)$, $e(x) \in L^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and has a compact support, $\psi(x) \in C^1$ and $\nabla\psi \neq 0$ for $x \in \operatorname{supp} e(x)$. About $u(x,t)$ we assume that $u(x,t) \in L^\infty(\mathbb{R}^n \times \mathbb{R}^1_+)$. In what follows, these assumptions are made more precise.

It is clear that, in differentiating the last term in (2.1) with respect to t, the term

$$e(x)\delta'(t - \psi(x)) \tag{2.2}$$

appears. Hence it is necessary to have

$$\langle \nabla, \rho u \rangle = -e(x)\delta'(t - \psi) + \text{more smooth summands},$$

since $\nabla\delta(t - \psi) = -\nabla\psi\delta'(t - \psi)$. Then we must have

$$\rho u = \frac{e\nabla\psi}{|\nabla\psi|^2}\delta(t - \psi) + \text{more smooth summands} \tag{2.3}$$

Of course, this argument only suggests the construction of the integral identity, but it is quite similar to the one-dimensional case.

We denote $\Gamma_t = \{(x,t); t = \psi(x)\}$; this is an n-dimensional surface in $\mathbb{R}^n \times \mathbb{R}^1_+$. Using the analogy, we can give the following definition.

Definition 2.1 Let

$$u(x,t) = u_0(x,t) + H(t - \psi)u_1(x,t),$$

where ψ is the same function as previously, and $u_0, u_1 \in L^\infty(\mathbb{R}^n \times \mathbb{R}^1_+)$. The function $\rho(x,t)$ determined by relation (2.1) is called a *generalized δ-shock wave type solution* to (0.2) on the surface $\{t = \psi(x)\}$ if the integral identity holds

$$\int_0^\infty \int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta)) \, dx \, dt + \int_{\Gamma_t} \frac{e}{|\nabla\psi|} \frac{d}{dn_\perp} \zeta(x,t) \, dx + \int_{\mathbb{R}^n} (R\zeta)|_{t=0} \, dx = 0$$

(2.4)

for all test functions $\zeta(x,t) \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^1_+)$, $\frac{d}{dn_\perp} = \left(\frac{\nabla\psi}{|\nabla\psi|}, \nabla\right) + |\nabla\psi|\frac{\partial}{\partial t}$.

We have the relation

$$\int_{\mathbb{R}^n} \frac{e}{|\nabla\psi|} \frac{d}{dn} \zeta(x,\psi) \, dx = \int_{\Gamma_t} \frac{e}{|\nabla\psi|} \frac{d}{dn_\perp} \zeta(x,t) \, dx,$$

where

$$\frac{d}{dn_\perp} = \left(\frac{\nabla\psi}{|\nabla\psi|}, \nabla\right) + |\nabla\psi|\frac{\partial}{\partial t}.$$

We note that the vector $n_\perp$ is orthogonal to the vector $(\nabla\psi, -1)$, which is the normal on the surface Γ_t, i.e., $\frac{d}{dn_\perp}$ lies in the plane tangent to Γ_t.

We can give a geometric definition of the field $\frac{d}{dn_\perp}$. The trajectories of this vector field are curves lying on the surface Γ_t, and they are orthogonal to all sections of this surface by the planes $t = \text{const}$. In this case, the vector tangent to the curves can be projected on $\mathbb{R}^n_x$ as the unit vector normal to the corresponding surface γ_t and keeping the same direction as the moving surface.

We show that the second summand in (2.4) containing $\int_{\mathbb{R}^n}$ is related to the above heuristic considerations.

Indeed, it follows from (2.1) and (2.3) that

$$\langle \rho, \zeta_t \rangle + \langle \rho u, \nabla\zeta \rangle = \int_0^\infty \int_{\mathbb{R}^n} R\zeta_t \, dx \, dt + \int_{\mathbb{R}^n} e(x)\zeta_t(x, \psi) \, dx$$

$$+ \int_0^\infty \int_{\mathbb{R}^n} (Ru, \nabla\zeta) \, dx \, dt + \int_{\mathbb{R}^n} \left(\frac{e\nabla\psi}{|\nabla\psi|^2}, \nabla\zeta(x, t) \right) \Bigg|_{t=\psi(x)} dx.$$

We consider the sum of integrals over $\mathbb{R}^n$. We obtain

$$\int_{\mathbb{R}^n} \left[e(x)\zeta_t(x, \psi) + \left(\frac{e\nabla\psi}{|\nabla\psi|^2}, \nabla\zeta(x, t) \right) \Bigg|_{t=\psi} \right] dx$$

$$= \int_{\mathbb{R}^n} \frac{e(x)}{|\nabla\psi|} \left[|\nabla\psi|\zeta_t(x, \psi) + \left(\frac{\nabla\psi}{|\nabla\psi|}, \nabla \right)\zeta \Bigg|_{t=\psi} \right] dx$$

$$= \int_{\mathbb{R}^n} \frac{e}{|\nabla\psi|} \left(\frac{\nabla\psi}{|\nabla\psi|}, \nabla \right)\zeta(x, \psi) \, dx.$$

Just as in the one-dimensional case, the above definition provides an algorithm for constructing δ-shock wave type solutions for Eq. (0.2).

Suppose that $R = R^+$ in the domain $\{t > \psi\}$ and $R = R^-$ in the domain $\{t < \psi\}$. Then we have the relation

$$R = R^- + H(t - \psi)(R^+ - R^-)$$

(here we assume that the function R^- is continued with smoothness preserved to the domain $t > \psi$). It is clear that the function $t - \psi$ increases while passing through Γ from the domain $\{t < \psi\}$ into the domain $\{t > \psi\}$. Therefore, the normal $(\nabla\psi, -1)$ is external for the domain $\{t > \psi\}$. For $t = 0$, the direction of the external normal is determined by the vector $(0, -1)$.

We let $[f]|_{t=\psi(x)}$ denote the jump of the function f across the surface Γ_t in the direction of the vector $(-\nabla\psi, 1)$. Integrating by parts in (2.4) by the Gauss–Ostrogradskii formula, we obtain

$$0 = \int_0^\infty \int_{\mathbb{R}^n} (R_t + (\nabla, Ru))\zeta \, dx \, dt$$

$$= \int_{(\mathbb{R}^n \times \mathbb{R}_+^1) \cap \{t > \psi\}} (R_t^+ + (\nabla, R^+ u))\zeta \, dx dt$$

$$+ \int_{(\mathbb{R}^n \times \mathbb{R}_+^1) \cap \{t < \psi\}} (R_t^- + (\nabla, R^- u)\zeta) \, dx dt$$

$$= \int_{\Gamma_t} \frac{-[R] + |\nabla\psi|[Ru_n]}{(1 + |\nabla\psi|^2)^{1/2}} \, d\sigma - \int \left(\frac{d}{dn} \right)^* \frac{e}{|\nabla\psi|} \, dx.$$

Here $u_n = (u, \frac{\nabla \psi}{|\nabla \psi|})$, $(\frac{d}{dn})^* = (\nabla, \frac{\nabla \psi}{|\nabla \psi|})$, and $(-\nabla \psi, 1)(1 + |\nabla \psi|^2)^{-1/2} = n$ is the unit vector of the normal on Γ_t. It is clear that

$$(1 + |\nabla \psi|^2)^{-1/2} = \cos \alpha,$$

where α is the angle between n and the t-axis and

$$d\sigma \cos \alpha = dx.$$

Finally, we obtain

$$0 = \int_0^\infty \int_{\mathbb{R}^n} (R_t + (\nabla, Ru)) \zeta \, dx \, dt$$
$$- \int_{\Gamma_t} \left\{ [R] - |\nabla \psi|[Ru_n] + \left(\frac{d}{dn}\right)^* \frac{e}{|\nabla \psi|} \right\} \zeta \, dx$$

It follows from the last relation that

$$R_t + (\nabla, Ru) = 0, \qquad (x, t) \notin \Gamma_t, \tag{2.5}$$

$$([R] - |\nabla \psi|[Ru_n]) + \left(\frac{d}{dn}\right)^* \frac{e}{|\nabla \psi|} = 0, \qquad (x, t) \in \Gamma_t, \tag{2.6}$$

and we have the following statement.

Theorem 2.1 *Suppose that the functions $R(x, t)$ and $\psi(x)$ satisfy system (2.5), (2.6), then the function $\rho(x, t)$ (2.1) is a δ-shock wave type solution to (0.2) on the surface Γ_t.*

We note that relation (2.6) can be rewritten in the form

$$\mathcal{K}E + \frac{d}{dn}E = [Ru_n]|\nabla \psi| - [R], \tag{2.7}$$

where $E = e/|\nabla \psi|$, the factor $\mathcal{K} = (\nabla, \frac{\nabla \psi}{|\nabla \psi|}) = \operatorname{div} \nu$ (ν is the normal on the surface $\{t = \psi(x)\}$) is, as is known, the mean curvature of the cross-section of the surface Γ_t by the plane $t = \operatorname{const}$.

Recall that, in the one-dimensional case, we saw that the δ-component of the solution is not formed or preserves its original value if the jump of ρ (the jump of R) occurs on the trajectory, i.e., on the line $\dot{X} = u(X, t)$. As follows from (2.7), this does not hold in the multidimensional case. Indeed, it is easy

to see that $\frac{1}{|\nabla\psi|} = V_n$ is the normal velocity of a point on the moving surface $\{t = \psi\} \subset \mathbb{R}^n$.

The relation

$$V_n = u_n, \tag{2.8}$$

where $u_n = (u, n)$, under the condition

$$[u_n]|_{t=\psi} = 0 \tag{2.9}$$

is an analog of the relation, which implies that there is no δ-component in the one-dimensional case.

In our case, under the condition that relations (2.8) and (2.9) are satisfied, the function E (or e) is equal to its initial value only if (see (2.7))

$$\mathcal{K} = 0.$$

Just as in the one-dimensional case, the solution of Eqs. (2.5), (2.7) is generally not unique, even if the equations are supplemented with initial conditions. The initial conditions to (2.5), (2.6) have the form

$$e|_{\psi=0} = e^0(x), \qquad R|_{t=0} = R^0.$$

If the trajectories of the velocity field are such that the right-hand of Eq. (2.7) is determined by the initial condition (the trajectories go from the plane $t = 0$ to the surface Γ_t), then, for a given function ψ, solving Eq. (2.7) is reduced to integrating ordinary differential equations.

Now we prove relation (0.4) in the case under study.

Theorem 2.2 *Suppose that $\rho(x, t)$ is a δ-shock wave type solution on the surface Γ_t with a compact support defined by (2.1). Then, for any test function $\eta(x)$, $\eta(x) \equiv 1$ for $(x, t) \in \operatorname{supp} \rho$, the following equality holds:*

$$\frac{d}{dt}\langle \rho, \eta \rangle_{\mathbb{R}_x^n} = 0.$$

Proof. By definition,

$$\langle \rho, \eta \rangle_{\mathbb{R}_x^n} = \int_{\mathbb{R}^n} R(x, t)\, dx + \int_{\{t=\psi\}} \frac{e}{|\nabla\psi|}\, d\sigma.$$

13

First, we calculate the derivative of the first integral

$$\frac{d}{dt} \int_{\mathbb{R}^n} R(x,t)\, dx = \int_{\psi \le t} R'_t\, dx + \int_{\psi \ge t} R'_t\, dx + \int_{\{t=\psi\}} \frac{[R]}{|\nabla \psi|}\, d\sigma$$

$$= -\int_{\psi \le t} (\nabla, Ru)\, dx - \int_{\psi \ge t} (\nabla, Ru)\, dx + \int_{\{t=\psi\}} \frac{[R]}{|\nabla \psi|}\, d\sigma.$$

The last integral in this relation appears because of the formula

$$R_t = R_t^- + H(t - \psi)(R_t^+ - R_t^-) + \delta(t - \psi)[R].$$

The first two integrals can easily be calculated by the Gauss–Ostrogradskii formula

$$-\int_{\psi \le t} (\nabla, R^+ u)\, dx - \int_{\psi \ge t} (\nabla, R^- u)\, dx = -\int_{\{t=\psi\}} [Ru_n]\, d\sigma.$$

We calculate

$$\frac{d}{dt} \int_{\{t=\psi\}} \frac{e}{|\nabla \psi|}\, d\sigma.$$

For this, we calculate the ratio $\frac{\delta F}{\delta t}$, where

$$F = \int_{\{t=\psi\}} \frac{e}{|\nabla \psi|}\, d\sigma.$$

Obviously, $e/|\nabla \psi| = (\frac{e\nabla \psi}{|\nabla \psi|^2}, \frac{\nabla \psi}{|\nabla \psi|})$. Therefore, by the Gauss–Ostrogradskii theorem, we have

$$\Delta F = \int_{\{t+\Delta t=\psi\}} \frac{e}{|\nabla \psi|}\, d\sigma - \int_{\{t=\psi\}} \frac{e}{|\nabla \psi|}\, d\sigma = \int_{\Delta} \left(\nabla, \frac{e\nabla \psi}{|\nabla \psi|^2} \right) dx,$$

where Δ is the strip between the surfaces $\{t + \Delta t = \psi\}$ and $\{t = \psi\}$. In this last formula, we also take into account that the vector $\nabla \psi$ on $\gamma_t = \{x; t = \psi(x)\}$ is directed inside the strip Δ. Let $\psi(x) = t$, $\psi(x + \delta x) = t + \Delta t$. It is clear that, up to infinitesimals of second order, we have

$$\left(\frac{\nabla \psi}{|\nabla \psi|}, \delta x \right) = \frac{\Delta t}{|\nabla \psi|}.$$

Therefore,

$$\lim \frac{\Delta F}{\Delta t} = \int_{\{t=\psi\}} \frac{1}{|\nabla \psi|} \left(\nabla, \frac{e\nabla \psi}{|\nabla \psi|^2} \right) d\sigma.$$

14

Finally, we obtain

$$\frac{d}{dt}\langle\rho,\eta\rangle_{\mathbb{R}^n_x} = \int_{\{t=\psi\}}\left\{\frac{[R]}{|\nabla\psi|} - [Ru_n] + \frac{1}{|\nabla\psi|}\left(\nabla,\frac{e\nabla\psi}{|\nabla\psi|^2}\right)\right\}d\sigma. \qquad (2.10)$$

It is easy to see that the integrand in the right-hand side of (2.10) multiplied by $|\nabla\psi|$ becomes the left-hand side of relation (2.6). The proof of Theorem 3 is complete.

Obviously, the graph Γ containing more than one arc in Definitions 1 and 2 corresponds to the problem of interaction (confluence) of generalized δ-shock waves. The smooth approximation of such a solution in the one-dimensional case was constructed in [9]. The construction given in Definition 2.2 permits studying the problem of interaction of generalized δ-shock waves in the multidimensional case.

Namely, we assume that there are two surfaces $\Gamma_t^{(i)} = \{(x,t); t = \psi_i(x)\}$ in $\mathbb{R}^n \times \mathbb{R}^1_+$, $i = 1, 2$, whose intersection is a smooth surface $\hat{\gamma} = \{(x,t); (t = \psi_1) \cap (t = \psi_2)\}$ belonging to the third surface $\Gamma_t^{(3)} = \{(x,t); t = \psi_3(x)\}$. Further, we assume that the surface $\Gamma_t^{(3)}$ is a continuation of the surfaces $\Gamma^{(i)}$ in the following sense. We let $n_\perp^{(i)}$ denote the curves on the surfaces $\Gamma_t^{(i)}$ and assume that each point $(\hat{x}, \hat{t})$ on the surface $\hat{\gamma}$ is assigned the graph consisting of the trajectories $n_\perp^{(1)}$ and $n_\perp^{(2)}$ entering $(\hat{x}, \hat{t})$ and the trajectory $n_\perp^{(3)}$ leaving this point (i.e., the trajectories $n_\perp^{(i)}$ fiber the surface $\Gamma^{(i)}$). We also assume that the surface $\Gamma_\cup = \Gamma^{(1)} \cup \Gamma^{(2)} \cup \Gamma^{(3)}$ consists of points belonging to these graphs. Next, we assume that $u(x,t)$ is a piecewise smooth vector field whose trajectories come to $\Gamma_\cup$.

Definition 2.2 Let

$$u(x,t) = u_0(x,t) + \sum_{i=1}^{3} H(t - \psi_i)u_{1i}(x,t),$$

where ψ is the same function as previously, and $u_0, u_{1i} \in C(\mathbb{R}^n \times \mathbb{R}^1_+)$. The function $\rho(x,t)$ determined by relation

$$\rho(x,t) = R(x,t) + \sum_{i=1}^{3} e_i(x)\delta(t - \psi_i(x)),$$

15

where $R(x, t) \in C^1(\mathbb{R}^n \times \mathbb{R}^1_+) \setminus \{\bigcup \Gamma^{(i)}_t\}$, is called a *generalized δ-shock wave type solution* to (0.2) on the graph $\Gamma_\cup$ if the integral identity holds

$$\int_0^\infty \int_{\mathbb{R}^n} (R\zeta_t + (uR, \nabla\zeta)) \, dx \, dt$$

$$+ \sum_{i=1}^3 \int_{\Gamma^{(i)}_t} \frac{e_i}{|\nabla\psi_i|} \frac{d}{dn^{(i)}_\perp} \zeta(x, t) \, dx + \int_{\mathbb{R}^n} (R\zeta)|_{t=0} \, dx = 0 \qquad (2.11)$$

for all test functions $\zeta(x, t) \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^1_+)$, $\frac{d}{dn^{(i)}_\perp} = \left(\frac{\nabla\psi_i}{|\nabla\psi_i|}, \nabla\right) + |\nabla\psi_i|\frac{\partial}{\partial t}$.

We can formulate an obvious analog of Theorem 2.1.

We assume that the system of equations

$$R_t + (\nabla, Ru) = 0, \qquad (x, t) \in \Gamma_\cup,$$

$$[R] - |\nabla\psi_i| [Ru_n] + \left(\frac{d}{dn_i}\right) \frac{e_i}{|\nabla\psi_i|} = 0, \qquad (x, t) \in \Gamma^{(i)}_t,$$

$$(e_1 + e_2)|_{\hat{\gamma}} = e_3|_{\hat{\gamma}}.$$

has a solution. Then the function $\rho(x, t)$ determined in Definition 2.2 is the generalized δ-shock wave type solution to (0.2).

In conclusion, we consider the case of collapse on a manifold of codimension > 1, namely, at a point on the plane.

In this example, it is convenient to use the polar coordinates. Let $u = (u_r, u_\varphi) = (-1, 0)$. In this case, as is known, the continuity equation has the form

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(ru_r\rho) = 0 \qquad (2.12)$$

(here the circular symmetry is taken into account).

We assume that ρ is compactly supported in r and calculate the rate of change in the "mass"

$$\frac{d}{dt}\int_0^\infty \int_0^{2\pi} \rho r \, dr \, d\varphi = \int_0^\infty \int_0^{2\pi} (\rho r)_t \, dr \, d\varphi$$

$$= -\int_0^\infty \int_0^{2\pi} (ru_r\rho)'_r \, dr \, d\varphi = 2\pi(ru_r\rho)\Big|_{r=0}. \qquad (2.13)$$

One can see that if the quantity on the right-hand side is different from zero, then a δ-component of the solution ensures the "mass" conservation law.

16

In the example under study, the function ρ has the form $\rho = R(r+t)r^{-1}$, where $R(z)$ is an arbitrary smooth function. Thus, the condition for the δ-component to appear in this example is the inequality

$$u_r\big|_{r=0} \neq 0,$$

which is satisfied in this case.

Definition 2.3 The function

$$\rho = \check{\rho}(r,t) + e(t)\delta(r),$$

where

$$\check{\rho}(r,t) \in C(\mathbb{R}^2 \setminus \{r = 0\}) \cap L^1(\mathbb{R}^2), \qquad e(t) \in C^1\mathbb{R}^1_+,$$

is called the δ-shock type solution to (2.11) if the relation

$$\int_0^\infty \int_0^\infty \int_0^{2\pi} (\check{\rho}\eta'_t + \rho u_r \eta'_r) r \, dr \, d\varphi \, dt + \int_0^\infty \int_0^{2\pi} (\rho\eta)\bigg|_{t=0} r \, dr \, d\varphi$$

$$- \int_0^\infty \int_0^{2\pi} e_t \eta\bigg|_{r=0} d\varphi \, dt = 0 \tag{2.14}$$

holds for any test function $\eta \in \mathcal{D}(\mathbb{R}^2)$.

In Definition 2.2, we use the Dirac δ-function $\delta(r)$ defined on test functions $\eta = \eta(x,y)$ from $\mathcal{D}(\mathbb{R}^2)$. For example, such a δ-function can be defined as

$$\delta(r) = \lim_{c \to 0} \delta(r - c),$$

where the limit is understood in the sense of $\mathcal{D}'$. The function $\delta(r-c)$ (with a singular support on the sphere, which is a manifold of codimension 1) can be defined according to the scheme presented above [8]. This leads to the formula

$$\langle \delta(r), \eta \rangle = \int_0^{2\pi} \eta\bigg|_{r=0} d\varphi = 2\pi\eta(0,0),$$

because $\eta|_{r=0}$ is independent of φ.

It is easy to see that here we use the same considerations as in the preceding sections, although the Heaviside function is absent here. Obviously, relation (2.13) implies Eq. (2.11) for $r = 0$ and the relation

$$e_t = (r u_r \rho)\big|_{r=0}$$

which, in view of (2.12) ensures the "mass" conservation law.

References

[1] F. Bouchut, On zero pressure gas dynamics, Advances in Math. for Appl. Sci., World Scientific, **22** (1994), 171–190.

[2] E. Weinan, Yu. Rykov, and Ya. G. Sinai, Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics, Comm. Math. Phys., **177** (1996), 349–380.

[3] V. G. Danilov and V. M. Shelkovich, Delta-shock wave type solution of hyperbolic systems of conservation laws, Quarterly in Appl. Math., **63** (2005), 401–427.

[4] V. G. Danilov and V. M. Shelkovich, Propagation and interaction of δ-shock waves to hyperbolic systems of conservation laws, Dokl. Ross. Akad. Nauk, **394** (2004), no. 1, 10–14; English transl. in Russian Dokl Math., **69** (2004), no. 1.

[5] V. M. Shelkovich, Delta-shock waves of a class of hyperbolic systems of conservation laws. In: "Patterns and Waves," A. Abramian, S. Vakulenko, and V. Volpert (eds.), Akadem Print, St.Petersburg, 2003, pp. 155–168.

[6] S. Albeverio and V. M. Shelkovich, On the delta-shock front problem. In: "Analytical Approaches to Multidimensional Balance Laws," Ch. 2, O. Rozanova (ed.), Nova Science Publ., 2005.

[7] V. G. Danilov, Remarks on vacuum state and uniqueness of concentration process, Conservation Laws Preprint 2006-033, http://www.math.ntnu.no/conservation/2006 (submitted to SIAM J. Math. Ann.).

[8] I. M. Gelfand and G. E. Shilov, Generalized Functions, Vol. 1, Academic Press, New York, 1964 (translated from the Russian).

[9] V. G. Danilov and V. M. Shelkovich, Dynamics of propagation and interaction of delta-shock waves in conservation law systems, J. Differential Equations, **211** (2005), pp. 333–381.

[10] Gui-Qiang Chen, Hailiang Liu, *Formation of δ-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isotropic fields*, SIAM J. Math. Anal., 34 (2003), no. 4, 925–938.