A MULTIWAVE APPROXIMATE RIEMANN SOLVER FOR IDEAL MHD BASED ON RELAXATION I - THEORETICAL FRAMEWORK

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Abstract. We present a relaxation system for ideal MHD that is an extension of the Suliciu relaxation system for the Euler equations of gas dynamics. From it one can derive approximate Riemann solvers with three or seven waves, that generalize the HLLC solver for gas dynamics. Under some subcharacteristic conditions, the solvers satisfy discrete entropy inequalities, and preserve positivity of density and internal energy. The subcharacteristic conditions are nonlinear constraints on the relaxation parameters relating them to the initial states and the intermediate states of the approximate Riemann solver itself. The 7-wave version of the solver is able to resolve exactly all material and Alfven isolated contact discontinuities. Practical considerations and numerical results will be provided in another paper.

1. Introduction

The equations of ideal magnetohydrodynamics (MHD) give a continuum description of a charged gas interacting with a magnetic field. They may be formulated as conservation laws for mass density, energy, momentum and magnetic field strength. If the state is a function of time t, and only one spatial dimension x, the equations are

$$\rho_t + (\rho u)_x = 0, \tag{1.1}$$

$$(\rho u)_t + (\rho u^2 + p + \frac{1}{2}|B_\perp|^2 - \frac{1}{2}B_x^2)_x = 0, \tag{1.2}$$

$$(\rho u_\perp)_t + (\rho u u_\perp - B_x B_\perp)_x = 0, \tag{1.3}$$

$$E_t + [(E + p + \frac{1}{2}|B_\perp|^2 - \frac{1}{2}B_x^2)u - B_x(B_\perp \cdot u_\perp)]_x = 0, \tag{1.4}$$

$$(B_\perp)_t + (B_\perp u - B_x u_\perp)_x = 0, \tag{1.5}$$

where ρ is the mass density, p the pressure, and the velocity is split into its longitudinal and transversal component u and $u_\perp$, as is the magnetic field into B_x and $B_\perp$. Hence $u_\perp$ and $B_\perp$ are two-dimensional vectors. Since the divergence of the magnetic field is zero at all times, we may assume that B_x is constant for one-dimensional data. Finally E is the total energy, $E = \frac{1}{2}\rho(u^2 + |u_\perp|^2) + \rho e + \frac{1}{2}(B_x^2 + |B_\perp|^2)$, with e denoting the specific internal energy.

The system is closed by an equation of state connecting p to ρ and e. For an ideal gas, $p = (\gamma - 1)\rho e$ with $\gamma > 1$, but we consider here a more general setting: the specific physical entropy $s = s(\rho, e)$ must be well-defined and satisfy

$$de + pd(\frac{1}{\rho}) = Tds \tag{1.6}$$

for some temperature $T(\rho, e) > 0$. Then, to ensure the hyperbolicity of the system, we assume

$$p' \equiv \left(\frac{\partial p}{\partial \rho}\right)_s > 0, \tag{1.7}$$

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where the subscript s means that the partial derivative is taken with s constant. We shall also make the classical assumption that

$$-s \text{ is a convex function of } (\frac{1}{\rho}, e). \tag{1.8}$$

To ensure the dissipativity of shocks, we need some additional constraints, and the second law of thermodynamics implies the entropy inequalities

$$(\rho\phi(s))_t + (\rho u\phi(s))_x \leq 0 \tag{1.9}$$

for all smooth, nonincreasing, convex functions ϕ, the assumption (1.8) ensuring that $\rho\phi(s)$ is convex with respect to the conservative variable. For an isentropic gas on the other hand, one would still solve (1.1)-(1.5) with $s = cst$, except that from the second law of thermodynamics, the energy equation (1.4) is replaced by an inequality

$$E_t + [(E + p + \frac{1}{2}|B_\perp|^2 - \frac{1}{2}B_x^2)u - B_x(B_\perp \cdot u_\perp)]_x \leq 0, \tag{1.10}$$

so that E becomes a mathematical entropy for the system.

The eigenvalues of system (1.1)-(1.5) are given by

$$\begin{aligned} &u,\ u \pm \sqrt{\frac{1}{2}\left(p' + \frac{|B|^2}{\rho} - \sqrt{\left(p' + \frac{|B|^2}{\rho}\right)^2 - 4p'\frac{B_x^2}{\rho}}\right)},\ u \pm \frac{|B_x|}{\sqrt{\rho}},\\ &u \pm \sqrt{\frac{1}{2}\left(p' + \frac{|B|^2}{\rho} + \sqrt{\left(p' + \frac{|B|^2}{\rho}\right)^2 - 4p'\frac{B_x^2}{\rho}}\right)}. \end{aligned} \tag{1.11}$$

The associated waves are called respectively material wave, slow magnetosonic waves, Alfven waves, and fast magnetosonic waves. Some of these waves will have the same speed when either B_x or $B_\perp$ vanishes, which means the system is nonstrictly hyperbolic. The system has three types of contact discontinuities corresponding to linearly degenerate eigenvalues: the material contacts associated to the eigenvalue u, the left Alfven contacts associated to $u - \frac{|B_x|}{\sqrt{\rho}}$, and the right Alfven contacts associated to $u + \frac{|B_x|}{\sqrt{\rho}}$. The jump relations associated to these contact discontinuities are as follows. Across a material contact, the quantities u, $u_\perp$, $p + \frac{1}{2}|B_\perp|^2 - \frac{1}{2}B_x^2$, $B_xB_\perp$ are constant. Across an Alfven contact, the quantities ρ, u, p, $|B_\perp|^2$ are constant, and moreover for a left Alfven contact we have $\Delta B_\perp = \text{sign}(B_x)\sqrt{\rho}\Delta u_\perp$, while for a right Alfven contact $\Delta B_\perp = -\text{sign}(B_x)\sqrt{\rho}\Delta u_\perp$ (where Δ denotes the jump).

1.1. **Conservative schemes and stability.** Let us consider a general system of conservation laws

$$U_t + F(U)_x = 0. \tag{1.12}$$

The MHD system (1.1)-(1.5) can be written under the form (1.12), with $U = (\rho, \rho u, \rho u_\perp, E, B_\perp)$ and $F(U) = (\rho u, \rho u^2 + p + |B_\perp|^2/2 - B_x^2/2, \rho u u_\perp - B_xB_\perp, (E + p + |B_\perp|^2/2 - B_x^2/2)u - B_xB_\perp \cdot u_\perp, B_\perp u - B_x u_\perp)$. The general system (1.12) may be approximated by the Godunov scheme, which consists of the following steps. Let the initial data be given as constants U_i^n over intervals $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$ partitioning $\mathbf{R}$, and evolve this by (1.12) for a time interval Δt small enough that the waves emerging from the cell boundaries do not interact. Then take U_i^{n+1} as the averages of the obtained solution over the cells, and restart the process. One iteration may be written as

$$U_i^{n+1} - U_i^n + \frac{\Delta t}{h_i}[F^c(U_i^n, U_{i+1}^n) - F^c(U_{i-1}^n, U_i^n)] = 0, \quad h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \tag{1.13}$$

where $F^c(U_i^n, U_{i+1}^n)$ is the numerical flux, given via the solution to the so called Riemann problem, that is the interaction of initially two constant states U_i^n, U_{i+1}^n separated by a single jump. More generally, if we take some numerical flux F^c such that $F^c(U, U) = F(U)$, (1.13) is consistent to first-order accuracy, and we call it a conservative scheme.

If the flux F has an entropy flux pair (η, G) (meaning that η is a smooth convex function and G is such that $G'(U) = \eta'(U)F'(U)$), we also prescribe an entropy inequality

$$\eta(U)_t + G(U)_x \leq 0. \tag{1.14}$$

In our case we have a family of entropy inequalities (1.9) for all convex nonincreasing ϕ. In this situation it is desirable to look for conservative schemes that satisfy a discrete entropy inequality

$$\eta(U_i^{n+1}) - \eta(U_i^n) + \frac{\Delta t}{h_i}[G^c(U_i^n, U_{i+1}^n) - G^c(U_{i-1}^n, U_i^n)] \leq 0, \tag{1.15}$$

with $G^c(U, U) = G(U)$. Such inequalities in fact play a central role when rigorous convergence analysis is possible, for example for scalar equations and two-by-two systems. In any case, such an inequality provides an a priori bound, and ensures that the computed shocks are physically relevant.

A problem when solving gas dynamics problems numerically is that unphysical states may occur, more specifically density or internal energy may become negative. In addition to these irrelevant values, this often ruins computer simulations when it occurs. It is therefore desirable to have schemes such that if $\rho^n > 0$ and $e^n > 0$, then $\rho^{n+1} > 0$ and $e^{n+1} > 0$. This means that we want

$$\rho > 0 \text{ and } \rho e = E - \frac{1}{2}\rho(u^2 + |u_\perp|^2) - \frac{1}{2}(B_x^2 + |B_\perp|^2) > 0 \tag{1.16}$$

at all times also for the numerical computation. However, it is well-known that positivity of density and entropy inequalities (1.15) for $\eta = \rho\phi(s)$ for all ϕ imply positivity of internal energy.

Since the Riemann problem is often very complicated to solve, and generally contains a lot of detail that is averaged over before the next timestep, simpler ways of determining the numerical flux F^c are often preferred. The main method to do that is to replace the exact Riemann solution with an approximate one, by defining a self-similar function $R(\frac{x}{t}, U_l, U_r)$, called an approximate Riemann solver. This provides a consistent conservative numerical flux if $R(\frac{x}{t}, U, U) = U$, and

$$F(U_l) - \int_{-\infty}^0 (R(\xi, U_l, U_r) - U_l)\, d\xi = F(U_r) + \int_0^\infty (R(\xi, U_l, U_r) - U_r)\, d\xi, \tag{1.17}$$

with the left or right-hand side defining the numerical flux $F^c(U_l, U_r)$. It yields an entropy inequality (1.15) for an entropy pair (η, G) if it is entropy consistent, meaning that

$$G(U_l) - \int_{-\infty}^0 (\eta(R(\xi, U_l, U_r)) - \eta(U_l))\, d\xi \geq G(U_r) + \int_0^\infty (\eta(R(\xi, U_l, U_r)) - \eta(U_r))\, d\xi, \tag{1.18}$$

and if a suitable CFL condition is satisfied, see [4]. For the Euler and MHD equations, if $R(\frac{x}{t}, U_l, U_r)$ has positive density and internal energy, then so will U_i^{n+1}.

The simplest approximate Riemann solver is the HLL solver [17], which consists of two discontinuities separating a constant intermediate state. Conservativity (1.17) implies

$$R_{HLL}(\xi, U_l, U_r) = \begin{cases} U_l, & \xi < \sigma_1, \\ \frac{\sigma_2 U_r - \sigma_1 U_l - F(U_r) + F(U_l)}{\sigma_2 - \sigma_1}, & \sigma_1 < \xi < \sigma_2, \\ U_r, & \sigma_2 < \xi, \end{cases} \tag{1.19}$$

where the signal velocities σ_1 and σ_2 must be chosen properly.

Conditions of stability, like positivity or entropy inequalities, are usually much more subtle to prove than consistency and conservativity. For the HLL solver, finding good signal velocities σ_1 and σ_2 is crucial for stability. They must be chosen larger than the characteristic speeds over a certain subset of state space, typically a subset containing the exact solution. However, the sizes of these signal speeds control the amount of artificial diffusion applied by the scheme. If the signal speeds are too large, the scheme will not have optimal accuracy. The behaviour of more complex solvers is governed by similar conditions. The main weakness of the HLL solver is that it is too dissipative, because it approximates the solution with only two waves, instead of seven in the true solver for the MHD system. It is therefore important to find approximate Riemann solvers with more waves, that can in particular well resolve the contact discontinuities, which are the most diffused waves.

1.2. **The Suliciu relaxation scheme.** For the gas dynamics system (i.e. (1.1)-(1.5) with $B \equiv 0$ and $u_\perp \equiv 0$), the Suliciu relaxation system is obtained as follows. We observe first that for smooth solutions, one has

$$(\rho p)_t + (\rho u p)_x + \rho^2 p' u_x = 0. \tag{1.20}$$

Then, the idea of relaxation is to replace the pressure $p = p(\rho, e)$ by an independent variable π, that will be an approximation to it, and solve for π an additional equation which is preferably an approximation to (1.20). This motivates the Suliciu relaxation system

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + \pi)_x &= 0, \\ E_t + [(E + \pi)u]_x &= 0, \\ (\rho \pi)_t + (\rho \pi u)_x + c^2 u_x &= 0, \end{aligned} \tag{1.21}$$

where $E = \frac{1}{2}\rho u^2 + \rho e$, and c is a constant replacing the Lagrangian sound speed $\rho\sqrt{p'}$. We say that the system is at equilibrium whenever $\pi = p(\rho, e)$. In order for π to be an approximation to $p(\rho, e)$, one needs to include a procedure of relaxation to equilibrium. A classical way of doing this is to put a right-hand side $\rho(p - \pi)/\varepsilon$ in the right-hand side of the last equation of (1.21). In the isentropic case this relaxation approximation has been shown to converge as $\varepsilon \to 0$ in [25].

In the time discrete case, which is our interest here, the relaxation procedure is performed at each timestep, this is the so called transport-projection method introduced in [5]. It works as follows. We start from initial data at equilibrium, that is to say knowing values of ρ, u, E we set $\pi = p(\rho, e)$ to complete the data. Then we solve (1.21) over a timestep, and in the solution at the next time level we keep only the conserved variables $\rho, \rho u, E$. In this way the timestep Δt takes the role of the relaxation parameter ε, as can be seen from a Chapman-Enskog analysis. The algorithm can also be interpreted within the Godunov approach. Indeed, starting from piecewise constant data and averaging the obtained solution over the cell, one sees that the method is equivalent to an approximate Riemann solver $R(x/t, U_l, U_r)$ obtained by taking only the components $\rho, \rho u, E$ of the solution to the Riemann problem associated to (1.21) when starting from initial data at equilibrium (i.e. we complete U_l, U_r by setting $\pi_l = p(\rho_l, e_l)$, $\pi_r = p(\rho_r, e_r)$), see [4]. Since the resolution of (1.21) is exact, the numerical flux $F^c(U_l, U_r)$ associated to the method is then given by the first components of the flux of (1.21) evaluated at the interface $x/t = 0$.

The system (1.21) has characteristic speeds $u - \frac{c}{\rho}, u$ and $u + \frac{c}{\rho}$ with the intermediate speed having multiplicity 2. All of the characteristic fields are linearly degenerate, hence the Riemann problem is easy to solve. Note that the constant c in (1.21) represents the signal speed of the corresponding approximate Riemann solver. Hence it is not surprising that it plays a crucial role in the convergence behaviour of relaxation systems as well as for the approximate Riemann solver. In the context of relaxation systems, a lower bound on c that is sufficient for stability is called a subcharacteristic condition.

In [8] a general framework for relaxation of conservation laws was presented. One may consider relaxation systems of the form

$$\psi_t + A(\psi)_x = \frac{Q(\psi)}{\epsilon}, \tag{1.22}$$

with an equilibrium mapping $\psi = M(U)$, and a linear operator L such that $LM(U) = U$. One requires also that $LQ(\psi) = 0$, and that $Q(\psi) = 0$ if and only if $\psi = M(U)$ for some U. The fluxes are connected by the relation $LA(M(U)) = F(U)$. One can show that such systems define an approximate Riemann solver, and hence a conservative scheme, by the same procedure as described for the Suliciu solver, see [3] or [4]. If the resulting approximate Riemann solver is a simple solver, which means that it only consists of constant states separated by discontinuities, the numerical flux is $LA(\psi)$ evaluated at the cell interface (Note that $A(\psi)$ is always continuous here by the Rankine-Hugoniot condition).

We can also formalize the entropy stability of (1.22) with respect to an entropy pair (η, G) for F. Let A have an entropy pair $(\mathcal{H}, \mathcal{G})$, such that $\mathcal{H}(M(U)) = \eta(U)$, $\mathcal{G}(M(U)) = G(U)$, and the minimization principle $\mathcal{H}(M(L\psi)) \le \mathcal{H}(\psi)$ holds for any ψ. Then we say that (1.22) has an entropy extension relative to η, and if so is the case, the deduced approximate Riemann solver will be entropy consistent with respect to η. The relaxation system (1.21) has an entropy extension in the isentropic case under the subcharacteristic condition $\rho^2 p'(\rho) \le c^2$. We explain below how this can be used to create entropy satisfying schemes for full gas dynamics.

1.3. **Some previous results on approximate Riemann solvers.**

In this section we summarize some results on approximate Riemann solvers for the Euler equations and for ideal MHD.

First, for the HLL solver, methods to choose the signal speeds have been given for example in [10], [14] and [24]. Entropy inequalities for the HLL solver may be found in [10], or in [18] where a relaxation interpretation is employed.

It was remarked already in [17] that the HLL solver is very diffusive on contact waves, especially for nearly stationary contact discontinuities. To improve this, they suggested adding a third wave inside the approximate Riemann fan. This was carried out in [23] by assigning a constant value u^* to u across the whole Riemann fan, and let u^* be the speed of the middle wave, defining the HLLC approximate Riemann solver. The choice of signal velocities for HLLC is addressed in [1]. The speeds of [14] for HLL ensure positivity and sharpness at shocks also for HLLC, but they may underestimate shock speeds for shocks emerging from a Riemann problem. In [13] it was shown that a linearized solver can not be positive, but that for HLLC it is enough that $C_l < u^* < C_r$, $C_l < u_l - \sqrt{\frac{\gamma-1}{2\gamma}}\sqrt{\frac{\gamma p_l}{\rho_l}}$, and $C_r > u_r + \sqrt{\frac{\gamma-1}{2\gamma}}\sqrt{\frac{\gamma p_r}{\rho_r}}$ for an ideal gas. The last two conditions were also given in [14] for HLL.

The HLLC solver can indeed be interpreted as the approximate Riemann solver deduced from the relaxation system (1.21), but with a nonconstant c solving $c_t + uc_x = 0$. This gives two independent signal speeds $u_l - \frac{c_l}{\rho_l}$ and $u_r + \frac{c_r}{\rho_r}$. This is presented in more detail in [3] and [4], where an entropy inequality is proved to hold under the following subcharacteristic condition. Let the left and right intermediate values of ρ be given as ρ_l^* and ρ_r^*, and assume that they are positive. Then entropy consistency is implied by

$$
\begin{aligned}
&\rho^2 p'(\rho, s_l) \le c_l^2, \quad \text{for } \rho \in [\rho_l, \rho_l^*], \\
&\rho^2 p'(\rho, s_r) \le c_r^2, \quad \text{for } \rho \in [\rho_r, \rho_r^*].
\end{aligned} \tag{1.23}
$$

Note that the condition does not refer to the exact solution, but only to the approximate one. From this one can derive explicit estimates on the signal speeds such that the entropy inequality holds and positivity of ρ and e is maintained, see [4].

In [12] a flux vector splitting method is given that is entropy consistent under some unspecified CFL-condition for Lagrangian gas dynamics. This method can indeed be identified with the Suliciu relaxation solver. An extension to MHD is given in [2], with a proof of asymptotic entropy inequalities when the sound speeds tend to infinity. Both a 7-wave and a 3-wave solver are suggested for MHD in [15] with a proof of entropy stability for large enough sound speeds. Moreover, the 7-wave solver exactly solves isolated Alfven contacts.

Generalized HLLC solvers have been proposed for MHD in [16] and [19]. They present tests and both observe increased resolution at material contact discontinuities compared to the HLL-solver. In addition, [16] uses a modified solver whenever $B_x = 0$ such that so called tangential discontinuities are exactly resolved also, but $u_\perp$ and B are otherwise taken to be constant across the approximate Riemann fan. An Einfeldt type speed is used and shown to lead to exact resolution of isolated fast shocks. This idea was taken further with the 5-wave solver of [20], which can exactly resolve isolated Alfven contacts. A positivity condition is given there, but otherwise no stability results are known.

Concerning other approaches to derive numerical fluxes, we mention that a kinetic flux vector splitting scheme for MHD was derived and tested in [22] and [26]. A Roe-solver was derived and tested in [6] for ideal gases with $\gamma = 2$, and this was extended to general values of γ in [7].

2. The relaxation system

In order to get a relaxation system for MHD that corresponds to (1.21), we first observe that for a smooth solution to (1.1)-(1.5), we have

$$p_t + up_x + \rho p' u_x = 0, \tag{2.1}$$

$$\left(\frac{|B_\perp|^2}{2}\right)_t + u\left(\frac{|B_\perp|^2}{2}\right)_x + |B_\perp|^2 u_x - B_x B_\perp \cdot (u_\perp)_x = 0, \tag{2.2}$$

and

$$(-B_x B_\perp)_t + u(-B_x B_\perp)_x - B_x B_\perp u_x + B_x^2 (u_\perp)_x = 0. \tag{2.3}$$

Replacing $p + \frac{1}{2}|B_\perp|^2 - \frac{1}{2}B_x^2$ by an independent variable π, and $-B_x B_\perp$ by an independent variable $\pi_\perp$, we obtain the following relaxation system,

$$\rho_t + (\rho u)_x = 0, \tag{2.4}$$

$$(\rho u)_t + (\rho u^2 + \pi)_x = 0, \tag{2.5}$$

$$(\rho u_\perp)_t + (\rho u u_\perp + \pi_\perp)_x = 0, \tag{2.6}$$

$$E_t + [(E + \pi)u + \pi_\perp \cdot u_\perp]_x = 0, \tag{2.7}$$

$$(B_\perp)_t + (B_\perp u - B_x u_\perp)_x = 0, \tag{2.8}$$

with still $E = \frac{1}{2}\rho(u^2 + |u_\perp|^2) + \rho e + \frac{1}{2}(B_x^2 + |B_\perp|^2)$, and with the relaxation pressures π and $\pi_\perp$ evolved by

$$(\rho\pi)_t + (\rho\pi u)_x + (|b|^2 + c_b^2)u_x - c_a b \cdot (u_\perp)_x = 0, \tag{2.9}$$

$$(\rho\pi_\perp)_t + (\rho\pi_\perp u)_x - c_a b\, u_x + c_a^2 (u_\perp)_x = 0. \tag{2.10}$$

The parameters $c_a \geq 0$, $c_b \geq 0$, and $b \in \mathbf{R}^2$ play the role of approximations of $\sqrt{\rho}|B_x|$, $\rho\sqrt{p'}$ and $\mathrm{sign}(B_x)\sqrt{\rho}B_\perp$ respectively. Indeed, c_a, c_b, b are not taken constant, but are evolved with

$$(c_a)_t + u(c_a)_x = 0, \quad (c_b)_t + u(c_b)_x = 0, \quad b_t + ub_x = 0. \tag{2.11}$$

The equilibrium is defined by

$$\pi = p + \frac{1}{2}|B_\perp|^2 - \frac{1}{2}B_x^2 \quad \text{and} \quad \pi_\perp = -B_x B_\perp. \tag{2.12}$$

As in the gas dynamics case, the approximate Riemann solver associated to the relaxation system is obtained as follows. We start with left and right states U_l, U_r, and we complete them with left and right values of π and $\pi_\perp$ at equilibrium, i.e. $\pi_{l/r} = (p + \frac{1}{2}|B_\perp|^2 - \frac{1}{2}B_x^2)_{l/r}$ and $(\pi_\perp)_{l/r} = -(B_x B_\perp)_{l/r}$. We also have to provide values for $(c_a)_{l/r}$, $(c_b)_{l/r}$, $b_{l/r}$. Then we solve the Riemann problem for (2.4)-(2.11), and in the solution (that depend only on x/t) we retain only the components ρ, ρu, $\rho u_\perp$, E, $B_\perp$. This gives the approximate Riemann solver. By construction this automatically gives a consistent conservative scheme, and the numerical flux is given by $F^c(U_l, U_r) = (\rho u, \rho u^2 + \pi, \rho u u_\perp + \pi_\perp, (E + \pi)u + \pi_\perp \cdot u_\perp, B_\perp u - B_x u_\perp)$ evaluated at $x/t = 0$. Of course, this is true provided that the solution to the Riemann problem for (2.4)-(2.11) exists and takes physically relevant values. In order to get this property, and also entropy inequalities, we have to make a good choice of the parameters $(c_a)_{l/r}$, $(c_b)_{l/r}$, $b_{l/r}$.

2.1. **Treating nonsolenoidal magnetic fields.**

For multidimensional applications, it can be useful to allow B_x to vary, and a convenient technique to facilitate this consists of augmenting the system with a term depending on $\nabla \cdot B$, an idea introduced in [21]. Here we propose a different system than in [21], which consists in letting $\nabla \cdot B$ be transported by the flow, by adding the term $u\nabla \cdot B$ to the induction equation. Hence in one dimension we get

$$(B_\perp)_t + (B_\perp u - B_x u_\perp)_x + u_\perp (B_x)_x = 0, \tag{2.13}$$

$$(B_x)_t + u(B_x)_x = 0. \tag{2.14}$$

Equation (2.13) simply replaces (2.8) in the relaxation system, while (2.14) is added. It allows for left and right values of B_x. A related approach can be found in [11], and this would also easily fit into (2.4)-(2.10). The source terms suggested there do not violate conservation, and could be treated by an operator splitting. There are several other approaches to this issue, but they are not directly connected to the design of one-dimensional Riemann solvers.

2.2. **Exact resolution of contact discontinuities.** The approximate Riemann solver derived from the relaxation system (2.4)-(2.11) has the property of being able to solve exactly isolated contact discontinuities of the MHD system.

Indeed, consider data U_l, U_r corresponding to an isolated contact with speed λ. Then the exact resolution property is true as soon as the solution to the MHD Riemann problem (i.e. $U(x/t, U_l, U_r) = U_l$ if $x/t < \lambda$, $U(x/t, U_l, U_r) = U_r$ if $x/t > \lambda$) when completed with π, $\pi_\perp$ at equilibrium, is a solution to the relaxation system (2.4)-(2.10). Since we are at equilibrium and $U(x/t)$ is a solution to the MHD system, equations (2.4)-(2.8) hold, and it only remains to check (2.9)-(2.10). Consider first the case of a material contact. Then the jump relations ensure that u, $u_\perp$, π, $\pi_\perp$ are constant, and since $\lambda = u$, the equations (2.9)-(2.10) hold obviously. Consider then the case of a left Alfven discontinuity, $\lambda = u - |B_x|/\sqrt{\rho}$ with the jump relations written in the introduction. Then by a simple computation, (2.9)-(2.10) hold as soon as $c_{al} = \sqrt{\rho}|B_x|$, and b_l colinear to $(B_{\perp l} + B_{\perp r})/2$ (no condition is needed on c_{ar} and b_r, nor on c_{bl}, c_{br}). For a right Alfven discontinuity $\lambda = u + |B_x|/\sqrt{\rho}$, we get the conditions $c_{ar} = \sqrt{\rho}|B_x|$, and b_r colinear to $(B_{\perp l} + B_{\perp r})/2$.

Therefore, one would like to derive choices of the parameters c_a, c_b, b on the left and on the right, in such a way that the previous conditions are satisfied whenever the data are those of an isolated Alfven contact.

2.3. **Chapman-Enskog analysis.** The Chapman-Enskog expansion provides a stability condition for a relaxation system when the solution is sufficiently smooth. Consider our MHD relaxation system (2.4)-(2.11) completed with BGK relaxation terms, i.e. (2.9)-(2.10) is replaced by

$$(\rho\pi)_t + (\rho\pi u)_x + (|b|^2 + c_b^2)u_x - c_a b\cdot(u_\perp)_x = \rho\,\frac{p + |B_\perp|^2/2 - B_x^2/2 - \pi}{\varepsilon}, \tag{2.15}$$

$$(\rho\pi_\perp)_t + (\rho\pi_\perp u)_x - c_a b\, u_x + c_a^2(u_\perp)_x = \rho\,\frac{-B_xB_\perp - \pi_\perp}{\varepsilon}. \tag{2.16}$$

We perform an expansion in ε, keeping only the first term, proportional to ε. From (2.15)-(2.16) we deduce that $\pi = p + \frac{1}{2}|B_\perp|^2 - \frac{1}{2}B_x^2 + O(\varepsilon)$ and $\pi_\perp = -B_xB_\perp + O(\varepsilon)$. Inserting this in (2.4)-(2.8), we get the MHD system (1.1)-(1.5), up to terms in ε. In order to get second-order expansions of π and $\pi_\perp$, we write down the values of π and $\pi_\perp$ obtained from the right-hand side of (2.15)-(2.16), and express the left-hand side with the first-order expansion of π and $\pi_\perp$. This gives

$$\begin{aligned}
\pi = p + \frac{|B_\perp|^2}{2} - \frac{B_x^2}{2} - \frac{\varepsilon}{\rho}\Bigg[\bigg(\rho(p + \frac{|B_\perp|^2}{2} - \frac{B_x^2}{2})\bigg)_t + \bigg(\rho u(p + \frac{|B_\perp|^2}{2} - \frac{B_x^2}{2})\bigg)_x \\
+ (|b|^2 + c_b^2)u_x - c_a b\cdot(u_\perp)_x\Bigg] + O(\varepsilon^2), \\
\pi_\perp = -B_xB_\perp - \frac{\varepsilon}{\rho}\Bigg[(\rho(-B_xB_\perp))_t + (\rho(-B_xB_\perp u))_x - c_a b u_x + c_a^2(u_\perp)_x\Bigg] + O(\varepsilon^2).
\end{aligned} \tag{2.17}$$

But since the MHD system (1.1)-(1.5) is resolved up to terms in ε, the identities (2.1)-(2.3) hold true up to terms in ε, and using this in (2.17), we get

$$\begin{aligned}
\pi = p + \frac{|B_\perp|^2}{2} - \frac{B_x^2}{2} - \frac{\varepsilon}{\rho}\Bigg[(|b|^2 + c_b^2 - \rho(\rho p' + |B_\perp|^2))u_x + (\rho B_xB_\perp - c_a b)\cdot(u_\perp)_x\Bigg] + O(\varepsilon^2), \\
\pi_\perp = -B_xB_\perp - \frac{\varepsilon}{\rho}\Bigg[(\rho B_xB_\perp - c_a b)u_x + (c_a^2 - \rho B_x^2)(u_\perp)_x\Bigg] + O(\varepsilon^2).
\end{aligned} \tag{2.18}$$

Putting this in (2.4)-(2.8) we obtain

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0,\\
&(\rho u)_t + (\rho u^2 + p + \frac{|B_\perp|^2}{2} - \frac{B_x^2}{2})_x = \varepsilon\Bigg[\left(\frac{|b|^2+c_b^2}{\rho} - (\rho p' + |B_\perp|^2)\right)u_x\\
&\qquad + (B_x B_\perp - \frac{c_a b}{\rho})\cdot(u_\perp)_x\Bigg]_x + O(\varepsilon^2),\\
&(\rho u_\perp)_t + (\rho u u_\perp - B_x B_\perp)_x = \varepsilon\left[(B_x B_\perp - \frac{c_a b}{\rho})u_x + (\frac{c_a^2}{\rho} - B_x^2)(u_\perp)_x\right]_x + O(\varepsilon^2),\\
&E_t + [(E + p + \frac{|B_\perp|^2}{2} - \frac{B_x^2}{2})u - B_x B_\perp\cdot u_\perp]_x\\
&= \varepsilon\Bigg[u\left(\frac{|b|^2+c_b^2}{\rho} - (\rho p' + |B_\perp|^2)\right)u_x + u(B_x B_\perp - \frac{c_a b}{\rho})\cdot(u_\perp)_x\\
&\qquad + u_\perp\cdot(B_x B_\perp - \frac{c_a b}{\rho})u_x + u_\perp\cdot(\frac{c_a^2}{\rho} - B_x^2)(u_\perp)_x\Bigg]_x + O(\varepsilon^2),\\
&(B_\perp)_t + (B_\perp u - B_x u_\perp)_x = 0.
\end{aligned}
\tag{2.19}
$$

Now, up to ε^2 we have a system of the form

$$U_t + F(U)_x = \varepsilon(D(U)U_x)_x. \tag{2.20}$$

The entropy is then evolved according to

$$\eta(U)_t + G(U)_x - \varepsilon[\eta'(U)D(U)U_x]_x = -\varepsilon D(U)^t\eta''(U)\cdot U_x\cdot U_x. \tag{2.21}$$

A natural stability condition is to ensure entropy dissipation by enforcing $D(U)^t\eta''(U)$ to be symmetric nonnegative. Computing the matrix $D(U)$ from (2.19) one can check that the symmetry holds for all entropies $\eta(U) = \rho\phi(s)$ (ϕ convex nonincreasing), while nonnegativity means that

$$
\begin{aligned}
&\frac{c_a^2}{\rho} - B_x^2 \geq 0, \qquad \frac{|b|^2+c_b^2}{\rho} - (\rho p' + |B_\perp|^2) \geq 0,\\
&\left|B_x B_\perp - \frac{c_a b}{\rho}\right|^2 \leq \left(\frac{|b|^2+c_b^2}{\rho} - (\rho p' + |B_\perp|^2)\right)\left(\frac{c_a^2}{\rho} - B_x^2\right).
\end{aligned}
\tag{2.22}
$$

Developing the last inequality and factorizing it differently, we can rewrite it to get finally the stability conditions

$$
\begin{aligned}
&\frac{1}{\rho} - \frac{B_x^2}{c_a^2} \geq 0, \qquad c_b^2 - \rho^2 p' \geq 0,\\
&\left|B_\perp - \frac{B_x b}{c_a}\right|^2 \leq \left(c_b^2 - \rho^2 p'\right)\left(\frac{1}{\rho} - \frac{B_x^2}{c_a^2}\right).
\end{aligned}
\tag{2.23}
$$

We observe that the reference values $c_a = \sqrt{\rho}|B_x|$, $c_b = \rho\sqrt{p'}$, $b = \mathrm{sign}(B_x)\sqrt{\rho}B_\perp$ give equalities in (2.23). Indeed, for these optimal values, the dissipation matrix $D(U)$ vanishes. However, the above analysis is valid only for smooth solutions, and the inequalities (2.23) involve only a single state U. What we are going to do in the next sections is to analyze the entropy inequalities for the Riemann problem. Then we shall derive a discrete version of (2.23), involving U_l, U_r and the intermediate values of the Riemann solver.

2.4. **Relations with other solvers.** The approximate Riemann solver obtained with our relaxation approach has a priori nothing to do with other proposed MHD solvers, like those of [2, 15, 16, 19, 20]. However, a few links exist.

At first, it is easy to see that if we take c_a, c_b, b constant, then writing the relaxation system in Lagrange coordinates gives a linear system, leading to a Lagrange numerical flux of flux vector splitting type. Thus it is somehow related to [2].

The main difference between what we do here and the solvers of [2, 15] is that for our 7-wave solver, the entropy inequality is here obtained for consistent values of the relaxation speeds c_a, c_b, b (instead of "sufficiently large" values). This means that for data U_l, U_r being sufficiently close to a reference state U, the speeds of the solver tend to the true eigenvalues of the system (evaluated at U), and the jumps in the intermediate states tend to have the direction of the true eigenvectors. This accuracy property is related to the fact that the viscosity in the Chapman-Enskog expansion (2.20) vanishes identically for these consistent values of c_a, c_b, b.

Finally, one can check that the approximate Riemann solver of [20] can be interpreted as the solution to a (partial) relaxation system, where only the longitudinal pressure π is relaxed while the orthogonal pressure $\pi_\perp$ is kept to equilibrium,

$$\rho_t + (\rho u)_x = 0, \tag{2.24}$$

$$(\rho u)_t + (\rho u^2 + \pi)_x = 0, \tag{2.25}$$

$$(\rho u_\perp)_t + (\rho u u_\perp - B_x B_\perp)_x = 0, \tag{2.26}$$

$$E_t + [(E + \pi)u - B_x B_\perp \cdot u_\perp]_x = 0, \tag{2.27}$$

$$(B_\perp)_t + (B_\perp u - B_x u_\perp)_x = 0, \tag{2.28}$$

with still $E = \frac{1}{2}\rho(u^2 + |u_\perp|^2) + \rho e + \frac{1}{2}(B_x^2 + |B_\perp|^2)$, and with the relaxation pressure π evolved by

$$(\rho\pi)_t + (\rho\pi u)_x + (|b|^2 + c_b^2)u_x - c_a b \cdot (u_\perp)_x = 0. \tag{2.29}$$

The equilibrium is still defined by

$$\pi = p + \frac{1}{2}|B_\perp|^2 - \frac{1}{2}B_x^2. \tag{2.30}$$

However, a Chapman-Enskog expansion from (2.24)-(2.29) gives instability, unfortunately.

2.5. **The approximate Riemann solver.** In order to get the approximate Riemann solver, we have to solve the Riemann problem for (2.4)-(2.11). This system is a quasilinear system of dimension 14. We shall not give the details of the computation, but one can check that its eigenvalues are u with multiplicity 8, $u \pm c_a/\rho$, and $u + X/\rho$ where X is a root of the polynomial

$$P(X) = X^4 - (|b|^2 + c_b^2 + c_a^2)X^2 + c_a^2 c_b^2. \tag{2.31}$$

Since $P(c_a) \le 0$ (and also $P(c_b) \le 0$), there are two real roots with respect to X^2. They are both nonnegative since their sum $|b|^2 + c_b^2 + c_a^2$ and their product $c_a^2 c_b^2$ are both nonnegative, thus P has two nonnegative and two nonpositive roots (which are opposite). Now, define $0 \le c_s \le c_f$ to be the two nonnegative roots of P, i.e.

$$c_s^2 + c_f^2 = |b|^2 + c_b^2 + c_a^2, \quad c_s^2 c_f^2 = c_a^2 c_b^2. \tag{2.32}$$

Then since $P(c_a) \le 0$ and $P(c_b) \le 0$, we have

$$c_s \le c_a \le c_f, \qquad c_s \le c_b \le c_f. \tag{2.33}$$

Notice that if $b = 0$ we get $c_s = \min(c_a, c_b)$, $c_f = \max(c_a, c_b)$. The eigenvalues of the relaxation system are finally

$$u, u \pm \frac{c_s}{\rho}, u \pm \frac{c_a}{\rho}, u \pm \frac{c_f}{\rho}, \tag{2.34}$$

the central one u having multiplicity 8. For further reference we notice the identity

$$c_a^2|b|^2 = (c_f^2 - c_a^2)(c_a^2 - c_s^2). \tag{2.35}$$

We notice this very nice property of the relaxation system: when taking for c_a, c_b, b their reference values ($c_a = \sqrt{\rho}|B_x|$, $c_b = \rho\sqrt{p'}$, $b = \mathrm{sign}(B_x)\sqrt{\rho}B_\perp$), the eigenvalues (2.34) of the relaxation system reduce to the ones of the MHD system (1.11). One can check the hyperbolicity of the relaxation system, and also that all the eigenvalues (2.34) are linearly degenerate. Thus one needs not specify the sense of the nonconservative products in (2.9), (2.10), (2.11), and the solution to the Riemann problem is made of constant states separated by discontinuities, one for each eigenvalue. This solution is characterized by the relations at each discontinuity, saying that $14-m$ independent weak Riemann invariants attached to the eigenvalue do not jump (m being the multiplicity). However there can be a collapse between the eigenvalues if either $b = 0$, $c_a = 0$ or $c_b = 0$. This leads to several formulas for the solution to the Riemann problem according these limit cases. We have to mention that since c_s and c_f are functions of b, c_a and c_b which are advected according to (2.11), c_s and c_f are also advected, $(c_s)_t + u(c_s)_x = 0$, $(c_f)_t + u(c_f)_x = 0$. Thus we have left and right values for all these parameters, namely $c_{sl}, c_{al}, c_{bl}, c_{fl}, b_l$ and $c_{sr}, c_{ar}, c_{br}, c_{fr}, b_r$.

2.6. **The 3-wave solver.** A simple choice we can make for the parameters is to take $b = 0$ and $c_s = c_a = c_b = c_f = c$, which leads to only two parameters c_l, c_r. Then the eigenvalues of the relaxation system are $u - c/\rho$, u, $u + c/\rho$, and it gives an approximate Riemann solver with three waves. This can be understood as a generalization of the HLLC solver for gas dynamics, except that here there remains quite a lot of diffusion. Indeed, the stability condition (2.23) from the Chapman-Enskog analysis gives here that c must be greater than the fast speed of the MHD system, and thus the diffusion matrix $D(U)$ is not small. Also, the solver cannot exactly solve isolated Alfven contact waves, because the stability condition prevents c to be taken $\sqrt{\rho}|B_x|$ in this case (see Subsection 2.2). Indeed, only the fast waves are resolved with good accuracy, while the Alfven and slow waves are diffused.

For the 3-wave solver, the left and right waves have multiplicity 3. There are 8 strong Riemann invariants associated to the central wave (i.e. quantities that lie in the kernel of $\partial_t + u\partial_x$), which are c_a, c_b, b, and

$$\frac{1}{\rho} + \frac{\pi}{c^2}, \qquad \frac{B_\perp}{\rho} + \frac{B_x}{c^2}\pi_\perp, \qquad e + \frac{|B|^2}{2\rho} - \frac{\pi^2}{2c^2} - \frac{|\pi_\perp|^2}{2c^2}. \tag{2.36}$$

These quantities are thus weak Riemann invariants for the left and right waves. They must be completed with 3 weak Riemann invariants, that are found to be $\pi + cu$, $\pi_\perp + cu_\perp$ for the left wave, and $\pi - cu$, $\pi_\perp - cu_\perp$ for the right wave. For the central wave, 6 weak Riemann invariants are u, $u_\perp$, π, $\pi_\perp$. We deduce that the solution has two intermediate states denoted $l*$ and $r*$ separated by speeds $\sigma_1 < \sigma_2 < \sigma_3$,

$$\sigma_1 = u_l - \frac{c_l}{\rho_l}, \quad \sigma_2 = u_l^* = u_r^* \equiv u^*, \quad \sigma_3 = u_r + \frac{c_r}{\rho_r}. \tag{2.37}$$

The values of c_a, c_b, b are the left values for the $l*$ state, and the right values for the $r*$ state. The intermediate values for ρ, $B_\perp$, e are deduced from the fact that the quantities in (2.36) do not jump through the left and right waves. It remains to determine the values u^*, $u_\perp^*$, π^*, $\pi_\perp^*$ (which are common for the $l*$ and $r*$ states). They are determined by the relations

$$\begin{aligned} (\pi + cu)_l^* &= (\pi + cu)_l, \qquad & (\pi - cu)_r^* &= (\pi - cu)_r, \\ (\pi_\perp + cu_\perp)_l^* &= (\pi_\perp + cu_\perp)_l, \qquad & (\pi_\perp - cu_\perp)_r^* &= (\pi_\perp - cu_\perp)_r. \end{aligned} \tag{2.38}$$

Hence we get the intermediate values

$$\begin{aligned} u^* &= \frac{c_l u_l + c_r u_r + \pi_l - \pi_r}{c_l + c_r}, \\ \pi^* &= \frac{c_r \pi_l + c_l \pi_r - c_l c_r (u_r - u_l)}{c_l + c_r}, \end{aligned} \tag{2.39}$$

$$\begin{aligned} u_\perp^* &= \frac{c_l u_\perp^l + c_r u_\perp^r + \pi_\perp^l - \pi_\perp^r}{c_l + c_r}, \\ \pi_\perp^* &= \frac{c_r \pi_\perp^l + c_l \pi_\perp^r - c_l c_r (u_\perp^r - u_\perp^l)}{c_l + c_r}. \end{aligned} \tag{2.40}$$

Notice the relations $\sigma_2 - \sigma_1 = c_l/\rho_l^*$, $\sigma_3 - \sigma_2 = c_r/\rho_r^*$, which show that to have the right ordering $\sigma_1 < \sigma_2 < \sigma_3$ is equivalent to having positivity of the intermediate densities ρ_l^*, ρ_r^*. We remark that the characteristic speeds and the intermediate values of ρ, u and π are formally the same as for the case of Euler equations (the equilibria of course differ, since $\pi_{l/r}$ and $(\pi_\perp)_{l/r}$ are initialized according to (2.12)). Notice also the symmetry between u, π on one hand, and $u_\perp$, $\pi_\perp$ on the other hand.

2.7. **The 7-wave solver.** We now consider the general case with 7 waves. Thus we assume that $b \neq 0$ and $c_a, c_b > 0$. Still, 6 weak Riemann invariants attached to the central wave are u, $u_\perp$, π, $\pi_\perp$, while 8 strong Riemann invariants (i.e. quantities that lie in the kernel of $\partial_t + u\partial_x$) are c_a, c_b, b, and

$$\begin{gathered}\frac{1}{\rho}+\frac{1}{c_b^2}(\pi+\frac{b}{c_a}\cdot\pi_\perp),\\ \frac{B_\perp}{\rho}+\frac{B_x}{c_a^2}\pi_\perp+\frac{1}{c_b^2}(\pi+\frac{b}{c_a}\cdot\pi_\perp)\frac{b}{c_a}B_x,\\ e+\frac{|B|^2}{2\rho}-\frac{1}{2c_b^2}(\pi+\frac{b}{c_a}\cdot\pi_\perp)^2-\frac{|\pi_\perp|^2}{2c_a^2}.\end{gathered}\tag{2.41}$$

These are consequently weak Riemann invariants for all the noncentral waves, and for each noncentral wave one has to complete them with 5 more weak Riemann invariants, that are obtained from the following list of six W_j by eliminating the one attached to the wave considered:

$$\begin{aligned}W_s&=\pi+c_su+\frac{c_a}{c_a^2-c_s^2}b\cdot(\pi_\perp+c_su_\perp),\\ W_{-s}&=\pi-c_su+\frac{c_a}{c_a^2-c_s^2}b\cdot(\pi_\perp-c_su_\perp),\\ W_f&=\pi+c_fu-\frac{c_a}{c_f^2-c_a^2}b\cdot(\pi_\perp+c_fu_\perp),\\ W_{-f}&=\pi-c_fu-\frac{c_a}{c_f^2-c_a^2}b\cdot(\pi_\perp-c_fu_\perp),\\ W_a&=\pi_\perp+c_au_\perp-(\pi_\perp+c_au_\perp)\cdot b\frac{b}{|b|^2},\\ W_{-a}&=\pi_\perp-c_au_\perp-(\pi_\perp-c_au_\perp)\cdot b\frac{b}{|b|^2}.\end{aligned}\tag{2.42}$$

Note that W_a and W_{-a} are two-dimensional vectors, but each one represents only one independent scalar function since they are orthogonal to b (their components are not independent). It is useful to write the inverse relations from (2.42),

$$\begin{aligned}\pi&=\frac{c_a^2-c_s^2}{2(c_f^2-c_s^2)}(W_s+W_{-s})+\frac{c_f^2-c_a^2}{2(c_f^2-c_s^2)}(W_f+W_{-f}),\\ u&=\frac{c_a^2-c_s^2}{2c_s(c_f^2-c_s^2)}(W_s-W_{-s})+\frac{c_f^2-c_a^2}{2c_f(c_f^2-c_s^2)}(W_f-W_{-f}),\\ \pi_\perp&=\frac{1}{2}(W_a+W_{-a})+\frac{(c_f^2-c_a^2)(c_a^2-c_s^2)}{2c_a(c_f^2-c_s^2)}(W_s+W_{-s}-W_f-W_{-f})\frac{b}{|b|^2},\\ u_\perp&=\frac{1}{2c_a}(W_a-W_{-a})+\frac{(c_f^2-c_a^2)(c_a^2-c_s^2)}{2c_a(c_f^2-c_s^2)}(\frac{W_s-W_{-s}}{c_s}-\frac{W_f-W_{-f}}{c_f})\frac{b}{|b|^2}.\end{aligned}\tag{2.43}$$

In order to find the intermediate states, we can argue as follows. First, the values of c_a, c_b, b are taken left or right respectively on the left and on the right of the central wave. Next, observe that given the intermediate values of u, $u_\perp$, π, $\pi_\perp$, the intermediate values of ρ, $B_\perp$, e are obtained by writing that the quantities (2.41) only jump through the central wave. Thus we only need to find the intermediate values of u, $u_\perp$, π, $\pi_\perp$. They are determined with the identities (2.42) or (2.43), knowing that for each noncentral $j-$wave, only W_j jumps; together with the fact that u, $u_\perp$, π, $\pi_\perp$ do not jump through the central wave, and thus have common values u^*, $u_\perp^*$, π^*, $\pi_\perp^*$ on each side. More explicitly, two methods are possible for this resolution.

The first method to solve it, is to write the relations

$$\begin{gathered}(W_{-s})^{r*}=(W_{-s})^r,\quad (W_{-f})^{r*}=(W_{-f})^r,\quad (W_{-a})^{r*}=(W_{-a})^r,\\ (W_s)^{l*}=(W_s)^l,\quad (W_f)^{l*}=(W_f)^l,\quad (W_a)^{l*}=(W_a)^l,\end{gathered}\tag{2.44}$$

where $(W_{-s})^{r*}$ represents W_{-s} evaluated to the right of the middle wave, etc. This gives six linear equations in the six unknowns u^*, $u_\perp^*$, π^*, $\pi_\perp^*$. Once the linear system (2.44) is solved, all the values of W_j at $l*$ and $r*$ are deduced from (2.42), and the values of u, $u_\perp$, π, $\pi_\perp$ follow from (2.43).

The second method is to define the main jumps as

$$\begin{aligned}
\Delta W_{-s} = (W_{-s})^{l*} - (W_{-s})^l, \quad \Delta W_{-f} = (W_{-f})^{l*} - (W_{-f})^l, \quad \Delta W_{-a} = (W_{-a})^{l*} - (W_{-a})^l,\\
\Delta W_s = (W_s)^{r*} - (W_s)^r, \quad \Delta W_f = (W_f)^{r*} - (W_f)^r, \quad \Delta W_a = (W_a)^{r*} - (W_a)^r.
\end{aligned} \tag{2.45}$$

Then from (2.43) and taking into account (2.44), we can express the values of u, $u_\perp$, π, $\pi_\perp$ on the states $l*$ and $r*$, linearly in terms of the ΔW_j. Writing the equality between the $l*$ and $r*$ values, we get a system of six linear equations in the six unknowns ΔW_j,

$$\begin{aligned}
&\pi_l + \frac{c_{fl}^2 - c_{al}^2}{2(c_{fl}^2 - c_{sl}^2)}\Delta W_{-f} + \frac{c_{al}^2 - c_{sl}^2}{2(c_{fl}^2 - c_{sl}^2)}\Delta W_{-s}\\
&= \pi_r + \frac{c_{fr}^2 - c_{ar}^2}{2(c_{fr}^2 - c_{sr}^2)}\Delta W_f + \frac{c_{ar}^2 - c_{sr}^2}{2(c_{fr}^2 - c_{sr}^2)}\Delta W_s,\\
&u_l - \frac{c_{fl}^2 - c_{al}^2}{2c_{fl}(c_{fl}^2 - c_{sl}^2)}\Delta W_{-f} - \frac{c_{al}^2 - c_{sl}^2}{2c_{sl}(c_{fl}^2 - c_{sl}^2)}\Delta W_{-s}\\
&= u_r + \frac{c_{fr}^2 - c_{ar}^2}{2c_{fr}(c_{fr}^2 - c_{sr}^2)}\Delta W_f + \frac{c_{ar}^2 - c_{sr}^2}{2c_{sr}(c_{fr}^2 - c_{sr}^2)}\Delta W_s,\\
&\pi_{\perp l} - \frac{c_{al}}{2(c_{fl}^2 - c_{sl}^2)}\Delta W_{-f} b_l + \frac{1}{2}\Delta W_{-a} + \frac{c_{al}}{2(c_{fl}^2 - c_{sl}^2)}\Delta W_{-s} b_l\\
&= \pi_{\perp r} - \frac{c_{ar}}{2(c_{fr}^2 - c_{sr}^2)}\Delta W_f b_r + \frac{1}{2}\Delta W_a + \frac{c_{ar}}{2(c_{fr}^2 - c_{sr}^2)}\Delta W_s b_r,\\
&u_{\perp l} + \frac{c_{al}}{2c_{fl}(c_{fl}^2 - c_{sl}^2)}\Delta W_{-f} b_l - \frac{1}{2c_{al}}\Delta W_{-a} - \frac{c_{al}}{2c_{sl}(c_{fl}^2 - c_{sl}^2)}\Delta W_{-s} b_l\\
&= u_{\perp r} - \frac{c_{ar}}{2c_{fr}(c_{fr}^2 - c_{sr}^2)}\Delta W_f b_r + \frac{1}{2c_{ar}}\Delta W_a + \frac{c_{ar}}{2c_{sr}(c_{fr}^2 - c_{sr}^2)}\Delta W_s b_r.
\end{aligned} \tag{2.46}$$

Once it is solved, the values of u, $u_\perp$, π, $\pi_\perp$ follow from (2.43).

Finally, the wave speeds σ_{-f}, σ_{-a}, σ_{-s}, σ_0, σ_s, σ_a, σ_f of the Riemann solution (corresponding to the eigenvalues $u - c_f/\rho$, $u - c_a/\rho$, $u - c_s/\rho$, u, $u + c_s/\rho$, $u + c_a/\rho$, $u + c_f/\rho$) can be computed using the relations

$$\begin{aligned}
\sigma_{-f} &= (u - c_f/\rho)_l = (u - c_f/\rho)_{*afl},\\
\sigma_{-a} &= (u - c_a/\rho)_{*afl} = (u - c_a/\rho)_{*asl},\\
\sigma_{-s} &= (u - c_s/\rho)_{*asl} = (u - c_s/\rho)_{*l},\\
\sigma_0 &= u_{l*} = u_{r*},\\
\sigma_s &= (u + c_s/\rho)_{*r} = (u + c_s/\rho)_{*asr},\\
\sigma_a &= (u + c_a/\rho)_{*asr} = (u + c_a/\rho)_{*afr},\\
\sigma_f &= (u + c_f/\rho)_{*afr} = (u + c_f/\rho)_r,
\end{aligned} \tag{2.47}$$

where the intermediate states are denoted from left to right by l, $*afl$, $*asl$, $l*$, $r*$, $*asr$, $*afr$, r. Noticing that $\rho^{*afl} = \rho^{*asl} \equiv \rho^{*al}$ and $\rho^{*afr} = \rho^{*asr} \equiv \rho^{*ar}$, we deduce the identities

$$\begin{aligned}
\sigma_{-a} - \sigma_{-f} &= \frac{c_{fl} - c_{al}}{\rho^{*al}},\\
\sigma_{-s} - \sigma_{-a} &= \frac{c_{al} - c_{sl}}{\rho^{*al}},\\
\sigma_0 - \sigma_{-s} &= \frac{c_{sl}}{\rho^{*l}},\\
\sigma_s - \sigma_0 &= \frac{c_{sr}}{\rho^{*r}},\\
\sigma_a - \sigma_s &= \frac{c_{ar} - c_{sr}}{\rho^{*ar}},\\
\sigma_f - \sigma_a &= \frac{c_{fr} - c_{ar}}{\rho^{*ar}}.
\end{aligned} \tag{2.48}$$

Therefore, again, to have the right ordering $\sigma_{-f} < \sigma_{-a} < \sigma_{-s} < \sigma_0 < \sigma_s < \sigma_a < \sigma_f$ is equivalent to having positive intermediate densities.

3. Entropy analysis

In this section we analyze the entropy stability of the approximate Riemann solver defined by the relaxation system (2.4)-(2.11).

3.1. **Local entropy condition.** We use first an argument introduced in [9] for gas dynamics, which is based on switching the role of the energy equation and the entropy inequality, thus reducing to the isentropic case. It leads to a condition written for each intermediate state.

Let us extend the system (2.4)-(2.11) with an additional unknown $\hat{s}$ solving

$$(\rho\hat{s})_t + (\rho u\hat{s})_x = 0, \tag{3.1}$$

which initial data at equilibrium, $\hat{s} = s(\rho, e)$. In other words, $\hat{s}$ is advected, and in the Riemann solution, $\hat{s}$ just takes the left and right values $s_l = s(\rho_l, e_l)$, $s_r = s(\rho_r, e_r)$ on each side of the central wave.

Proposition 3.1. *Assume that in the Riemann solution to the relaxation system* (2.4)-(2.11), *each intermediate state* $U^* \equiv (\rho^*, \rho^* u^*, \rho^* u_\perp^*, \rho^*((u^*)^2 + (u_\perp^*)^2)/2 + \rho^* e^* + B_x^2/2 + (B_\perp^*)^2/2, B_\perp^*)$ *has positive density* $\rho^* > 0$*, and satisfies*

$$e^* \geq e(\rho^*, \hat{s}^*). \tag{3.2}$$

Then the approximate Riemann solver preserves the positivity of density and internal energies, and satisfies all entropy inequalities related to the entropies $\rho\phi(s)$ *with* ϕ *convex nonincreasing.*

Proof. The positivity of internal energy is obvious from (3.2) since $e(\rho^*, \hat{s}^*) \geq 0$. Then, consider an entropy $\eta = \rho\phi(s)$, which has entropy flux $G = \rho u\phi(s)$. Because of (3.1), one has

$$(\rho\phi(\hat{s}))_t + (\rho u\phi(\hat{s}))_x = 0, \tag{3.3}$$

and let us denote $G^c(U_l, U_r) = (\rho u\phi(\hat{s}))_{x/t=0}$. In order to get the entropy inequality (1.18), we are going to prove that

$$G_r(U_l, U_r) \leq G^c(U_l, U_r) \leq G_l(U_l, U_r), \tag{3.4}$$

where G_l and G_r denote respectively the left-hand side and the right-hand side of (1.18). This will not only prove (1.18), but also that $G^c(U_l, U_r)$ can be used as numerical entropy flux. Denoting by $\xi = x/t$ the self-similar variable, we notice that

$$\begin{aligned} G^c(U_l, U_r) &= G(U_l) - \int_{-\infty}^{0} (\rho\phi(\hat{s})(\xi) - \eta(U_l))\, d\xi \\ &= G(U_r) + \int_{0}^{\infty} (\rho\phi(\hat{s})(\xi) - \eta(U_r))\, d\xi. \end{aligned} \tag{3.5}$$

Therefore, in order to get (3.4), it is enough to prove that for a.e. ξ, $\eta(U(\xi)) \leq \rho\phi(\hat{s})(\xi)$. This means equivalently that for any intermediate state U^*,

$$\rho^*\phi(s(\rho^*, e^*)) \leq \rho^*\phi(\hat{s}^*). \tag{3.6}$$

But since $\rho^* > 0$ and ϕ is nonincreasing, we thus only have to prove that $s(\rho^*, e^*) \geq \hat{s}^*$. Recalling that according to (1.6), at ρ fixed, $e(\rho, s)$ is an increasing function of s, this inequality is equivalent to $e^* \geq e(\rho^*, \hat{s}^*)$, which proves the claim. □

3.2. **Sufficient stability conditions for a fixed intermediate state.** In this subsection we derive sufficient conditions for (3.2) to hold, for a fixed intermediate state U^*. The state U^* is described by ρ^*, u^*, $u_\perp^*$, e^*, $B^* = (B_x, B_\perp^*)$, and we also have the associated relaxation pressures π^* and $\pi_\perp^*$. We shall denote by $U_{l/r}$ the initial state on the same side as U^* with respect to the central wave, and we shall use the same convention for $s_{l/r}$ (indeed $s_{l/r} = \hat{s}^*$ with the notation of the previous paragraph). The values of c_s, c_a, c_b, c_f, b, are evaluated also locally, i.e. on the same side as U^* (even if we do not write explicitly the index l/r), in accordance with (2.11). We use finally the short-hand notations

$$e(\rho^*) \equiv e(\rho^*, s_{l/r}), \qquad p(\rho^*) \equiv p(\rho^*, s_{l/r}). \tag{3.7}$$

The desired inequality (3.2) then becomes $e^* \geq e(\rho^*)$.

We first write a decomposition into elementary entropy dissipation terms, similarly as in [3]. The main one D_0 is related to the central wave, and we just group the ones related to the other waves into a longitudinal part and a transverse part.

Lemma 3.2. *We have the identity*

$$\begin{aligned} e(\rho^*) - e^* = D_0(U^*, U_{l/r}) &- \frac{1}{2c_b^2}\left(p(\rho^*) + \frac{|B_\perp^*|^2}{2} - \frac{B_x^2}{2} - \pi^* + \frac{b}{c_a}\cdot(-B_x B_\perp^* - \pi_\perp^*)\right)^2 \\ &- \frac{1}{2c_a^2}\left|-B_x B_\perp^* - \pi_\perp^*\right|^2, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} D_0(U^*, U_{l/r}) = {} & e(\rho^*) - e(\rho^{l/r}) + p(\rho^*)\left(\frac{1}{\rho^*} - \frac{1}{\rho^{l/r}}\right) \\ & + \frac{1}{2c_b^2}\left(p(\rho^*) + \frac{|B_\perp^*|^2}{2} - B_x\frac{b}{c_a}\cdot B_\perp^* - p(\rho^{l/r}) - \frac{|B_\perp^{l/r}|^2}{2} + B_x\frac{b}{c_a}\cdot B_\perp^{l/r}\right)^2 \\ & - \left(\frac{1}{\rho^{l/r}} - \frac{B_x^2}{c_a^2}\right)\frac{1}{2}|B_\perp^* - B_\perp^{l/r}|^2. \end{aligned} \tag{3.9}$$

This identity can be verified using that the weak Riemann invariants (2.41) take the same value at the intermediate state and on the initial data l/r. Note that this is true also for the 3-wave solver since the weak Riemann invariants (2.36) are obtained formally as the one in (2.41) where we set $c_s = c_a = c_b = c_f = c$, $b = 0$.

In order to analyze $D_0(U^*, U_{l/r})$, let us recall the following inequality, that was proved in [3] or [4].

Lemma 3.3. *As soon as $\rho^* > 0$, one has*

$$e(\rho^*) - e(\rho^{l/r}) + p(\rho^*)\left(\frac{1}{\rho^*} - \frac{1}{\rho^{l/r}}\right) + \frac{1}{2}\frac{1}{(\rho^2 p')_{*,l/r}}\left(p(\rho^*) - p(\rho^{l/r})\right)^2 \leq 0, \tag{3.10}$$

with

$$(\rho^2 p')_{*,l/r} \equiv \sup_\rho \rho^2 p'(\rho, s_{l/r}), \tag{3.11}$$

where the supremum is taken over all ρ between $\rho_{l/r}$ and ρ^*.

Proof. Since in the inequality, the specific entropy s takes a fixed value $s_{l/r}$, one can consider that e and p are functions of ρ only. Recall that according to (1.6), one has then $e'(\rho) = p(\rho)/\rho^2$.

Consider an interval $I \subset (0, \infty)$ and a constant $c > 0$ such that for all $\rho \in I$, one has $\rho^2 p'(\rho) \leq c^2$. Then, for a fixed $\rho^{l/r} \in I$, define for $\rho^* \in I$

$$\Phi(\rho^*) = e(\rho^*) - e(\rho^{l/r}) + p(\rho^*)\left(\frac{1}{\rho^*} - \frac{1}{\rho^{l/r}}\right) + \frac{1}{2c^2}\left(p(\rho^*) - p(\rho^{l/r})\right)^2. \tag{3.12}$$

One computes

$$\Phi'(\rho^*) = p'(\rho^*)\left(\frac{1}{\rho^*} - \frac{1}{\rho^{l/r}} + \frac{p(\rho^*) - p(\rho^{l/r})}{c^2}\right). \tag{3.13}$$

Now, since $p' > 0$ and by assumption $1/\rho + p(\rho)/c^2$ is a nonincreasing function of $\rho \in I$, we deduce that $\Phi'(\rho^*)$ has the sign of $\rho^{l/r} - \rho^*$, and therefore that Φ has a maximum at $\rho^{l/r}$. Thus for all $\rho^* \in I$, $\Phi(\rho^*) \leq \Phi(\rho^{l/r}) = 0$. Finally, for any given $\rho^{l/r}, \rho^* > 0$, one can take for I the closed interval $[\rho^{l/r}, \rho^*]$ and $c^2 = (\rho^2 p')_{*,l/r}$. This gives the result. □

The main estimate on $D_0(U^*, U_{l/r})$ is the following.

Lemma 3.4. *If $\rho^* > 0$, then*

$$
\begin{aligned}
&D_0(U^*, U_{l/r}) - \frac{1}{2c_b^2}\left(p(\rho^*) + \frac{|B_\perp^*|^2}{2} - \frac{B_x^2}{2} - \pi^* + \frac{b}{c_a}\cdot(-B_x B_\perp^* - \pi_\perp^*)\right)^2 \\
&\leq -\frac{1}{2}\left(c_b^2 - (\rho^2 p')_{*,l/r}\right)\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^*}\right)^2 \\
&+\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^*}\right)\left(\frac{B_\perp^{l/r} + B_\perp^*}{2} - B_x\frac{b}{c_a}\right)\cdot(B_\perp^* - B_\perp^{l/r}) - \left(\frac{1}{\rho^{l/r}} - \frac{B_x^2}{c_a^2}\right)\frac{1}{2}|B_\perp^* - B_\perp^{l/r}|^2.
\end{aligned}
\tag{3.14}
$$

Proof. Since $1/\rho + \frac{1}{c_b^2}(\pi + \frac{b}{c_a}\cdot\pi_\perp)$ is a strong Riemann invariant for the central wave, it has the same value at U^* and $U_{l/r}$. Substituting the equilibrium values for $\pi^{l/r}$ and $\pi_\perp^{l/r}$ gives

$$
\pi^* + \frac{b}{c_a}\cdot\pi_\perp^* = p(\rho^{l/r}) + \frac{|B_\perp^{l/r}|^2}{2} - \frac{B_x^2}{2} - B_x\frac{b}{c_a}\cdot B_\perp^{l/r} + c_b^2\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^*}\right).
\tag{3.15}
$$

Therefore, the left-hand side of (3.14) can be rewritten as

$$
\begin{aligned}
LHS = D_0(U^*, U_{l/r}) - \frac{1}{2c_b^2}\Bigg(&p(\rho^*) + \frac{|B_\perp^*|^2}{2} - B_x\frac{b}{c_a}\cdot B_\perp^* \\
&-p(\rho^{l/r}) - \frac{|B_\perp^{l/r}|^2}{2} + B_x\frac{b}{c_a}\cdot B_\perp^{l/r} - c_b^2\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^*}\right)\Bigg)^2.
\end{aligned}
\tag{3.16}
$$

Subtracting the last term from the second line in (3.9) and using the identity $\alpha^2/2 - \beta^2/2 = (\alpha - \beta)(\alpha + \beta)/2$, we deduce

$$
\begin{aligned}
LHS = &\, e(\rho^*) - e(\rho^{l/r}) + p(\rho^*)\left(\frac{1}{\rho^*} - \frac{1}{\rho^{l/r}}\right) \\
&+\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^*}\right)\Bigg(p(\rho^*) + \frac{|B_\perp^*|^2}{2} - B_x\frac{b}{c_a}\cdot B_\perp^* \\
&\qquad -p(\rho^{l/r}) - \frac{|B_\perp^{l/r}|^2}{2} + B_x\frac{b}{c_a}\cdot B_\perp^{l/r} - \frac{c_b^2}{2}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^*}\right)\Bigg) \\
&-\left(\frac{1}{\rho^{l/r}} - \frac{B_x^2}{c_a^2}\right)\frac{1}{2}|B_\perp^* - B_\perp^{l/r}|^2.
\end{aligned}
\tag{3.17}
$$

Combining the inequality

$$
\begin{aligned}
&\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^*}\right)\left(p(\rho^*) - p(\rho^{l/r})\right) \\
&\leq \frac{1}{2}\frac{1}{(\rho^2 p')_{*,l/r}}\left(p(\rho^*) - p(\rho^{l/r})\right)^2 + \frac{1}{2}(\rho^2 p')_{*,l/r}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^*}\right)^2
\end{aligned}
\tag{3.18}
$$

with (3.17), and then applying (3.10) gives the Lemma. □

We now introduce the following notation. For $\theta \in \mathbf{R}$, take

$$
B_\perp^\theta = \frac{1-\theta}{2}B_\perp^* + \frac{1+\theta}{2}B_\perp^{l/r}, \qquad \frac{1}{\rho_\theta} = \frac{1-\theta}{\rho^{l/r}} + \frac{\theta}{\rho^*}.
\tag{3.19}
$$

By this definition,

$$\left(\frac{B_\perp^{l/r}+B_\perp^*}{2}-B_\perp^\theta\right)\cdot(B_\perp^*-B_\perp^{l/r})=\theta\frac{1}{2}|B_\perp^*-B_\perp^{l/r}|^2. \tag{3.20}$$

Therefore, since $1/\rho_\theta-1/\rho^{l/r}=\theta(1/\rho^*-1/\rho^{l/r})$, we have

$$\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)\left(\frac{B_\perp^{l/r}+B_\perp^*}{2}-B_\perp^\theta\right)\cdot(B_\perp^*-B_\perp^{l/r})=\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho_\theta}\right)\frac{1}{2}|B_\perp^*-B_\perp^{l/r}|^2. \tag{3.21}$$

This identity enables us to express the last line in (3.14) in terms of $B_\perp^\theta$ and ρ_θ,

$$\begin{aligned}&\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)(\frac{B_\perp^{l/r}+B_\perp^*}{2}-B_x\frac{b}{c_a})\cdot(B_\perp^*-B_\perp^{l/r})-\left(\frac{1}{\rho^{l/r}}-\frac{B_x^2}{c_a^2}\right)\frac{1}{2}|B_\perp^*-B_\perp^{l/r}|^2\\ &=\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)(B_\perp^\theta-B_x\frac{b}{c_a})\cdot(B_\perp^*-B_\perp^{l/r})-\left(\frac{1}{\rho_\theta}-\frac{B_x^2}{c_a^2}\right)\frac{1}{2}|B_\perp^*-B_\perp^{l/r}|^2.\end{aligned} \tag{3.22}$$

We deduce the following stability criterion.

Proposition 3.5. *In order to have $e(\rho^*)-e^*\le 0$ (ensuring the discrete entropy inequality), it is enough that $\rho^*>0$ and that there exists some $\theta\in\mathbf{R}$ such that*

$$(\rho^2p')_{*,l/r}\le c_b^2,\qquad \frac{1}{\rho_\theta}-\frac{B_x^2}{c_a^2}\ge 0, \tag{3.23}$$

and

$$\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)(B_\perp^\theta-B_x\frac{b}{c_a})\cdot(B_\perp^*-B_\perp^{l/r})\le\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)|B_\perp^*-B_\perp^{l/r}|\Upsilon, \tag{3.24}$$

for some Υ satisfying

$$\Upsilon^2\le\left(c_b^2-(\rho^2p')_{*,l/r}\right)\left(\frac{1}{\rho_\theta}-\frac{B_x^2}{c_a^2}\right). \tag{3.25}$$

Proof. Starting from (3.8), we neglect the last term and use Lemma 3.4 for the two first terms, and also use the identity (3.22). Then we use (3.24), and apply the estimate

$$\begin{aligned}\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)|B_\perp^*-B_\perp^{l/r}|\Upsilon\le{}&\frac{1}{2}\left(c_b^2-(\rho^2p')_{*,l/r}\right)\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)^2\\ &+\frac{1}{2}\frac{1}{c_b^2-(\rho^2p')_{*,l/r}}|B_\perp^*-B_\perp^{l/r}|^2\Upsilon^2.\end{aligned} \tag{3.26}$$

With (3.25) this gives the result. □

Remark Taking $\theta=0$ gives $B_\perp^\theta=(B_\perp^*+B_\perp^{l/r})/2$ and $\rho_\theta=\rho^{l/r}$. Another special choice is $\theta=1$, that gives $B_\perp^\theta=B_\perp^{l/r}$ and $\rho_\theta=\rho^*$. This yields our most simple sufficient condition for entropy stability, that is a discrete version of (2.23).

Proposition 3.6. *The approximate Riemann solver defined by the relaxation system* (2.4)-(2.11) *is positive and satisfies all discrete entropy inequalities whenever for all intermediate states U^*, one has $\rho^*>0$ and*

$$\begin{gathered}(\rho^2p')_{*,l/r}\le c_b^2,\qquad \frac{1}{\rho^*}-\frac{B_x^2}{c_a^2}\ge 0,\\ \left|B_\perp^{l/r}-B_x\frac{b}{c_a}\right|^2\le\left(c_b^2-(\rho^2p')_{*,l/r}\right)\left(\frac{1}{\rho^*}-\frac{B_x^2}{c_a^2}\right),\end{gathered} \tag{3.27}$$

where $(\rho^2p')_{*,l/r}$ is defined by (3.11).

This condition is useful for the 3-wave solver, as it will be shown in a follow-up paper. However, it does not allow exact resolution of isolated Alfven waves since by the discussion of Subsection 2.2, in this case one should have a vanishing right-hand side in (3.27), which sets a value of b colinear to $B_\perp^{l/r}$, which is not colinear to $B_{\perp l}+B_{\perp r}$ in general. Therefore, we provide a more precise analysis, adapted to the 7-wave solver, that allows the exact resolution of isolated Alfven waves.

Lemma 3.7. *Assume $b\neq 0$ and define the projections parallel and orthogonal to b*

$$P^\parallel X=\frac{X\cdot b}{|b|^2}\,b,\qquad P^\perp X=X-\frac{X\cdot b}{|b|^2}\,b. \tag{3.28}$$

Consider again $B_\perp^\theta$ and ρ_θ as in (3.19) for any $\theta\in\mathbf{R}$. Then for any $\theta^\parallel$, $\theta^\perp\in\mathbf{R}$ we have

$$\begin{aligned}
&D_0(U^*,U_{l/r})-\frac{1}{2c_b^2}\left(p(\rho^*)+\frac{|B_\perp^*|^2}{2}-\frac{B_x^2}{2}-\pi^*+\frac{b}{c_a}\cdot(-B_xB_\perp^*-\pi_\perp^*)\right)^2\\
&\leq-\frac12\left(c_b^2-(\rho^2p')_{*,l/r}\right)\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)^2\\
&\quad+\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)P^\parallel(B_\perp^{\theta^\parallel}-B_x\frac{b}{c_a})\cdot P^\parallel(B_\perp^*-B_\perp^{l/r})-\left(\frac{1}{\rho_{\theta^\parallel}}-\frac{B_x^2}{c_a^2}\right)\frac12|P^\parallel(B_\perp^*-B_\perp^{l/r})|^2\\
&\quad+\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)P^\perp(B_\perp^{\theta^\perp}-B_x\frac{b}{c_a})\cdot P^\perp(B_\perp^*-B_\perp^{l/r})-\left(\frac{1}{\rho_{\theta^\perp}}-\frac{B_x^2}{c_a^2}\right)\frac12|P^\perp(B_\perp^*-B_\perp^{l/r})|^2.
\end{aligned}\tag{3.29}$$

Proof. We use Lemma 3.4, and decompose the vectors in their components parallel and orthogonal to b,

$$\begin{aligned}
&\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)(\frac{B_\perp^{l/r}+B_\perp^*}{2}-B_x\frac{b}{c_a})\cdot(B_\perp^*-B_\perp^{l/r})-\left(\frac{1}{\rho^{l/r}}-\frac{B_x^2}{c_a^2}\right)\frac12|B_\perp^*-B_\perp^{l/r}|^2\\
&=\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)P^\parallel(\frac{B_\perp^{l/r}+B_\perp^*}{2}-B_x\frac{b}{c_a})\cdot P^\parallel(B_\perp^*-B_\perp^{l/r})-\left(\frac{1}{\rho^{l/r}}-\frac{B_x^2}{c_a^2}\right)\frac12|P^\parallel(B_\perp^*-B_\perp^{l/r})|^2\\
&\quad+\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)P^\perp(\frac{B_\perp^{l/r}+B_\perp^*}{2}-B_x\frac{b}{c_a})\cdot P^\perp(B_\perp^*-B_\perp^{l/r})-\left(\frac{1}{\rho^{l/r}}-\frac{B_x^2}{c_a^2}\right)\frac12|P^\perp(B_\perp^*-B_\perp^{l/r})|^2.
\end{aligned}\tag{3.30}$$

We have an identity similar to (3.21),

$$\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^*}\right)P\left(\frac{B_\perp^{l/r}+B_\perp^*}{2}-B_\perp^\theta\right)\cdot P(B_\perp^*-B_\perp^{l/r})=\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho_\theta}\right)\frac12|P(B_\perp^*-B_\perp^{l/r})|^2, \tag{3.31}$$

for any θ and any projection $P=P^\parallel$ or $P=P^\perp$. In the part parallel to b of the right-hand side of (3.30), use (3.31) with $P=P^\parallel$ and $\theta=\theta^\parallel$, while in the part orthogonal to b, use (3.31) with $P=P^\perp$ and $\theta=\theta^\perp$. This gives (3.29). □

We shall use Lemma 3.7 in the following way. Assume that $(B_\perp^{l/r}-B_x\frac{b}{c_a})\cdot b=0$. Then take $\theta^\parallel=1$. Provided $1/\rho^*-B_x^2/c_a^2\geq 0$, the second line on the right-hand side of (3.29) gives a nonpositive contribution. To the remaining first and last line we can apply the Cauchy-Schwarz inequality as in Proposition 3.5, to deduce that we only need

$$\left|P^\perp(B_\perp^{\theta^\perp}-B_x\frac{b}{c_a})\right|^2\leq\left(c_b^2-(\rho^2p')_{*,l/r}\right)\left(\frac{1}{\rho_{\theta^\perp}}-\frac{B_x^2}{c_a^2}\right) \tag{3.32}$$

for some $\theta^\perp$.

4. Stability conditions on each intermediate state

Let us now examine more precisely the stability conditions for each intermediate state U^*. We shall give sufficient conditions for the seven wave solver, thus we assume that $b\neq 0$ and $c_a,c_b>0$. We recall that we use the same convention as in the previous section: the index l/r mean that we take l if U^* is on the left of the central wave, and r if U^* is on the right of the central wave. The parameters c_a, c_b, b are evaluated in the same way (according to (2.11)).

We shall assume that

$$B_\perp^{l/r} - B_x \frac{b}{c_a} = \mu Z, \tag{4.1}$$

where $\mu \equiv \mu_{l/r} \in \mathbf{R}$, $Z \equiv Z_{l/r}$ satisfies

$$Z \cdot b = 0, \tag{4.2}$$

and Z is an approximation of the strength of the Alfven wave. At least, one should have that when the data are that of a left isolated Alfven wave one has $Z_l = -(B_\perp^r - B_\perp^l)/2$, $\mu_l = 1$, and when the data are that of a right isolated Alfven wave, $Z_r = (B_\perp^r - B_\perp^l)/2$, $\mu_r = 1$.

Since we aim to resolve isolated Alfven waves, we shall consider as 'small' any term proportional to $\Delta W_{\mp f}$, $\Delta W_{\mp s}$, and

$$\frac{1}{4}\frac{\rho^{l/r} B_x}{c_a^2}\Delta W_{\mp a} - Z, \tag{4.3}$$

where the notation $\Delta W_{\mp a}$ means that we take ΔW_{-a} if U^* is on the left of the central wave, and ΔW_a if it is on the right. Indeed, from the assumptions on Z, this term vanishes on the left for an isolated left Alfven wave (not necessarily for a right Alfven wave), and on the right for an isolated right Alfven wave.

Example 1. Define

$$\begin{aligned} W_l &= \frac{1}{4}\frac{\rho_l B_{xl}}{c_{al}^2}\frac{2c_{al}}{c_{al}+c_{ar}}\Big[\pi_{\perp r} - \pi_{\perp l} + c_{ar}(u_{\perp l} - u_{\perp r})\Big],\\ W_r &= \frac{1}{4}\frac{\rho_r B_{xr}}{c_{ar}^2}\frac{2c_{ar}}{c_{al}+c_{ar}}\Big[\pi_{\perp l} - \pi_{\perp r} + c_{al}(u_{\perp l} - u_{\perp r})\Big], \end{aligned} \tag{4.4}$$

and

$$\mu_l = \min\left(1, \frac{|B_\perp^l|}{|W_l|}\right), \qquad \mu_r = \min\left(1, \frac{|B_\perp^r|}{|W_r|}\right). \tag{4.5}$$

In other words, $\mu_l W_l = \operatorname*{proj}_{|B_\perp^l|} W_l$, $\mu_r W_r = \operatorname*{proj}_{|B_\perp^r|} W_r$ with

$$\operatorname*{proj}_{\nu} X = \begin{cases} X & \text{if } |X| \le \nu,\\ \dfrac{X}{|X|}\nu & \text{if } |X| > \nu. \end{cases} \tag{4.6}$$

Then, define

$$\begin{aligned} V_l &= B_\perp^l - \mu_l W_l,\\ V_r &= B_\perp^r - \mu_r W_r, \end{aligned} \tag{4.7}$$

and assume that $V_l \neq 0$, $V_r \neq 0$ (otherwise one should take $b_l = 0$ or $b_r = 0$). This implies $V_l \cdot B_\perp^l > 0$, and $V_r \cdot B_\perp^r > 0$. Thus we can define

$$\frac{B_x b}{c_a} = V + \mu\frac{W \cdot V}{|V|^2}V = \frac{B_\perp^{l/r} \cdot V}{|V|^2}V. \tag{4.8}$$

We have that b is colinear to V, and

$$\frac{B_x b}{c_a} = B_\perp^{l/r} - \mu W + \mu\frac{W \cdot b}{|b|^2}b, \tag{4.9}$$

i.e. (4.1)-(4.2) hold with $Z = W - \frac{W\cdot b}{|b|^2}b$.

In this example, all $\Delta W_{\mp f}$, $\Delta W_{\mp s}$ and (4.3) are expressed linearly in terms of

$$\pi_r - \pi_l,\ u_r - u_l,\ b_l \cdot [\pi_{\perp l} - \pi_{\perp r} + c_{ar}(u_{\perp r} - u_{\perp l})],\ b_r \cdot [\pi_{\perp l} - \pi_{\perp r} - c_{al}(u_{\perp r} - u_{\perp l})], \tag{4.10}$$

which are small for any left or right isolated Alfven wave.

Example 2. Colinear b_l and b_r. Assume that $B_\perp^l \neq 0$, $B_\perp^r \neq 0$, and that $B_\perp^l/|B_\perp^l|+B_\perp^r/|B_\perp^r| \neq 0$. Take $\mu_l = \mu_r = 1$,

$$\begin{aligned} Z_l &= \frac{1}{2}B_\perp^l - \frac{1}{2}\frac{B_\perp^r}{|B_\perp^r|}|B_\perp^l|, \\ Z_r &= \frac{1}{2}B_\perp^r - \frac{1}{2}\frac{B_\perp^l}{|B_\perp^l|}|B_\perp^r|, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \frac{B_{xl}b_l}{c_{al}} &= \frac{1}{2}B_\perp^l + \frac{1}{2}\frac{B_\perp^r}{|B_\perp^r|}|B_\perp^l|, \\ \frac{B_{xr}b_r}{c_{ar}} &= \frac{1}{2}B_\perp^r + \frac{1}{2}\frac{B_\perp^l}{|B_\perp^l|}|B_\perp^r|. \end{aligned} \tag{4.12}$$

Then conditions (4.1) and (4.2) are satisfied, and b_l and b_r are colinear with the same direction. The interest of this choice is that it simplifies the calculation of the intermediate states, since the system (2.44) (or (2.46)) decouples into a part colinear to b and a part normal to b, leading to a linear system of four equations and a system of two equations instead of one of six equations.

4.1. **Fast intermediate states.** We first consider the state between the waves corresponding to c_a and c_f, which we denote with the superscript '*af' or with '*a' for quantities that are constant across the c_a-wave. From the Riemann invariants relations we get

$$\begin{aligned} \frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}} &= \frac{1}{2}\frac{c_f^2-c_a^2}{c_f^2(c_f^2-c_s^2)}\Delta W_{\mp f}, \\ \frac{B_\perp^{*af}}{\rho^{*a}} = \frac{B_\perp^{l/r}}{\rho^{l/r}} + \frac{1}{2}\frac{c_a}{c_f^2(c_f^2-c_s^2)}\Delta W_{\mp f}B_x b &= \frac{B_\perp^{l/r}}{\rho^{l/r}} + \left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)\frac{c_a^2}{c_f^2-c_a^2}\frac{B_x b}{c_a}, \end{aligned} \tag{4.13}$$

and

$$B_\perp^{*af} - B_\perp^{l/r} = \rho^{*a}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)\left[B_\perp^{l/r} + \frac{c_a^2}{c_f^2-c_a^2}\frac{B_x b}{c_a}\right]. \tag{4.14}$$

We use Lemma 3.7 with $\theta^\| = 1$ and also $\theta^\perp = 1$. The second line gives a nonpositive contribution as soon as $1/\rho^{*a} - B_x^2/c_a^2 \geq 0$. For the third line, we compute

$$\begin{aligned} P^\perp(B_\perp^{*af} - B_\perp^{l/r}) &= \rho^{*a}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)\mu Z, \\ P^\perp(B_\perp^{\theta^\perp} - \frac{B_x b}{c_a}) &= \mu Z. \end{aligned} \tag{4.15}$$

Here we take also into account the last term in the decomposition (3.8), which involves

$$B_x B_\perp^{*af} + \pi_\perp^{*af} = B_x\rho^{*a}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)\left(B_\perp^{l/r} + \frac{c_a^2}{c_f^2-c_a^2}\frac{B_x b}{c_a}\right) - \left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)\frac{c_f^2}{c_f^2-c_a^2}c_a b. \tag{4.16}$$

Its orthogonal projection is given by

$$P^\perp(B_x B_\perp^{*af} + \pi_\perp^{*af}) = B_x\rho^{*a}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)\mu Z. \tag{4.17}$$

In order to get $e(\rho^*) - e^* \leq 0$, it is enough to estimate the first and last line in the right-hand side of (3.29), to which we add $-|P^\perp(B_x B_\perp^{*af} + \pi_\perp^{*af})|^2/(2c_a^2)$. Thus the inequality reduces to

$$\begin{aligned} &-\frac{1}{2}\left(c_b^2 - (\rho^2 p')_{*a,l/r}\right)\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)^2 \\ &+ \left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)^2 \rho^{*a}|\mu Z|^2 - \left(\frac{1}{\rho^{*a}} - \frac{B_x^2}{c_a^2}\right)\frac{1}{2}(\rho^{*a})^2\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)^2|\mu Z|^2 \\ &-\frac{B_x^2(\rho^{*a})^2}{2c_a^2}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)^2|\mu Z|^2 \\ &\leq 0. \end{aligned} \tag{4.18}$$

Dividing by $(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}})^2$, this gives the sufficient condition

$$c_b^2 - (\rho^2 p')_{*a,l/r} - \rho^{*a}|\mu Z|^2 \geq 0. \tag{4.19}$$

4.2. **Middle intermediate states.** Next, we move on to the states between the waves associated with c_s and c_a, which we denote by the superscript '*as', or just '*a' if there is no jump at the c_a-wave. We have $B_\perp^{*as} = B_\perp^{*af} - \frac{1}{2}\frac{\rho^{*a}B_x}{c_a^2}\Delta W_{\mp a}$, thus

$$\begin{aligned}B_\perp^{*as} - B_\perp^{l/r} &= B_\perp^{*af} - B_\perp^{l/r} - \frac{1}{2}\frac{\rho^{*a}B_x}{c_a^2}\Delta W_{\mp a}\\ &= \rho^{*a}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)\left(B_\perp^{l/r} + \frac{c_a^2}{c_f^2 - c_a^2}\frac{B_x b}{c_a}\right) - \frac{1}{2}\frac{\rho^{*a}B_x}{c_a^2}\Delta W_{\mp a}.\end{aligned} \tag{4.20}$$

Next, we have for any θ

$$\begin{aligned}&B_\perp^\theta - \frac{B_x b}{c_a}\\ &= \frac{1-\theta}{2}(B_\perp^{*as} - B_\perp^{l/r}) + B_\perp^{l/r} - \frac{B_x b}{c_a}\\ &= \frac{1-\theta}{2}\rho^{*a}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)\left(B_\perp^{l/r} + \frac{c_a^2}{c_f^2 - c_a^2}\frac{B_x b}{c_a}\right)\\ &\qquad + (1-\theta)\frac{\rho^{*a}}{\rho^{l/r}}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right) + \left(\mu - (1-\theta)\frac{\rho^{*a}}{\rho^{l/r}}\right)Z\\ &= \frac{1-\theta}{2}\rho^{*a}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)\frac{c_f^2}{c_f^2 - c_a^2}\frac{B_x b}{c_a}\\ &\qquad + (1-\theta)\frac{\rho^{*a}}{\rho^{l/r}}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\\ &\qquad + \left(\mu - (1-\theta)\frac{\rho^{*a}}{\rho^{l/r}} + \frac{1-\theta}{2}\mu\left(\frac{\rho^{*a}}{\rho^{l/r}} - 1\right)\right)Z.\end{aligned} \tag{4.21}$$

We apply Lemma 3.7 with $\theta^\| = 1$. For $\theta^\perp$, a useful choice is to make the last term in (4.21) vanish, since it is large for isolated Alfven wave data, hence

$$1 - \theta^\perp = \frac{2\mu}{\frac{\rho^{*a}}{\rho^{l/r}}(2-\mu) + \mu}. \tag{4.22}$$

This gives

$$P^\perp\left(B_\perp^{\theta^\perp} - \frac{B_x b}{c_a}\right) = \frac{2\mu\frac{\rho^{*a}}{\rho^{l/r}}}{\frac{\rho^{*a}}{\rho^{l/r}}(2-\mu) + \mu}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right), \tag{4.23}$$

to which we need only to require the inequality (3.32). As soon as $0 \leq \mu \leq 1$ and $\rho^{*a}/\rho^{l/r} \geq \mu/(2-\mu)$ this gives the natural bounds $0 \leq \theta^\perp \leq 1$, and we get the sufficient condition

$$\left|\frac{2\mu}{2-\mu}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|^2 \leq \left(c_b^2 - (\rho^2 p')_{*a,l/r}\right)\left(\frac{1}{\rho_{\theta^\perp}} - \frac{B_x^2}{c_a^2}\right). \tag{4.24}$$

Otherwise, still for $0 \leq \mu \leq 1$, another possible choice is

$$\theta^\perp = \frac{4(1-\mu)\left(\frac{\rho^{*a}}{\rho^{l/r}}\right)^2}{\left(\frac{\rho^{*a}}{\rho^{l/r}}(2-\mu) + \mu\right)^2} \tag{4.25}$$

which satisfies $0 \leq \theta^\perp \leq 1$, and

$$\begin{aligned}P^\perp\left(B_\perp^{\theta^\perp} - \frac{B_x b}{c_a}\right) &= (1-\theta^\perp)\frac{\rho^{*a}}{\rho^{l/r}}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\\ &\quad + \frac{1}{2}\frac{1 - \left(\frac{\rho^{*a}}{\rho^{l/r}}\right)^2}{\frac{\rho^{*a}}{\rho^{l/r}}(2-\mu) + \mu}\mu^2 Z.\end{aligned} \tag{4.26}$$

Then, the last term in (4.26) can be grouped in (3.29) with the term in $(1/\rho^{l/r} - 1/\rho^*)^2$. Using the Cauchy-Schwarz inequality this gives the sufficient condition

$$
\begin{aligned}
&\left|(1-\theta^\perp)\frac{\rho^{*a}}{\rho^{l/r}}\left(Z-\frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|^2\\
&\leq \left(c_b^2-(\rho^2p')_{*a,l/r}+\frac{1+\frac{\rho^{*a}}{\rho^{l/r}}}{\frac{\rho^{*a}}{\rho^{l/r}}(2-\mu)+\mu}\rho^{*a}\mu^2 Z\cdot P^\perp(B_\perp^{*as}-B_\perp^{l/r})\right)\left(\frac{1}{\rho_{\theta^\perp}}-\frac{B_x^2}{c_a^2}\right),
\end{aligned}
\tag{4.27}
$$

where

$$
\begin{aligned}
P^\perp(B_\perp^{*as}-B_\perp^{l/r}) &= \rho^{*a}\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^{*a}}\right)\mu Z-\frac{1}{2}\frac{\rho^{*a}B_x}{c_a^2}\Delta W_{\mp a}\\
&= -\left(\frac{\rho^{*a}}{\rho^{l/r}}(2-\mu)+\mu\right)Z+2\frac{\rho^{*a}}{\rho^{l/r}}\left(Z-\frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right).
\end{aligned}
\tag{4.28}
$$

This is especially interesting when $\rho^{*a}/\rho^{l/r} \leq \mu/(2-\mu)$, where we get the sufficient condition

$$
\begin{aligned}
&\left|\frac{\mu}{2-\mu}\left(Z-\frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|^2\\
&\leq \left(c_b^2-(\rho^2p')_{*a,l/r}-\left(1+\frac{\rho^{*a}}{\rho^{l/r}}\right)\rho^{*a}\mu^2\left(|Z|^2+\left|Z\cdot\left(Z-\frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|\right)\right)\\
&\times\left(\frac{1}{\rho_{\theta^\perp}}-\frac{B_x^2}{c_a^2}\right).
\end{aligned}
\tag{4.29}
$$

4.3. **Internal intermediate states.** The intermediate states between the middle wave and the slow wave will be denoted by '*i'. We have similarly as for fast intermediate states

$$
\begin{aligned}
\frac{1}{\rho^{*a}}-\frac{1}{\rho^{*i}} &= \frac{1}{2}\frac{c_a^2-c_s^2}{c_s^2(c_f^2-c_s^2)}\Delta W_{\mp s},\\
\frac{B_\perp^{*i}}{\rho^{*i}} = \frac{B_\perp^{*as}}{\rho^{*a}}-\frac{1}{2}\frac{c_a}{c_s^2(c_f^2-c_s^2)}\Delta W_{\mp s}B_x b &= \frac{B_\perp^{*as}}{\rho^{*a}}-\left(\frac{1}{\rho^{*a}}-\frac{1}{\rho^{*i}}\right)\frac{c_a^2}{c_a^2-c_s^2}\frac{B_x b}{c_a},\\
B_\perp^{*i}-B_\perp^{*as} &= \rho^{*i}\left(\frac{1}{\rho^{*a}}-\frac{1}{\rho^{*i}}\right)\left[B_\perp^{*as}-\frac{c_a^2}{c_a^2-c_s^2}\frac{B_x b}{c_a}\right].
\end{aligned}
\tag{4.30}
$$

Using that $\frac{B_\perp}{\rho}+\frac{B_x}{c_a^2}\pi_\perp-\frac{B_x b}{c_a}\frac{1}{\rho}$ is a Riemann invariant for the central wave, we have

$$
\left(B_\perp^{*i}-\frac{B_x b}{c_a}\right)\frac{1}{\rho^{*i}}+\frac{B_x}{c_a^2}\pi_\perp^{*i} = \left(B_\perp^{l/r}-\frac{B_x b}{c_a}\right)\frac{1}{\rho^{l/r}}+\frac{B_x}{c_a^2}\pi_\perp^{l/r}.
\tag{4.31}
$$

Then, we decompose

$$
\begin{aligned}
\pi_\perp^{*i}-\pi_\perp^{l/r} &= (\pi_\perp^{*i}-\pi_\perp^{*as})+(\pi_\perp^{*as}-\pi_\perp^{*af})+(\pi_\perp^{*af}-\pi_\perp^{l/r})\\
&= \frac{c_s^2}{c_a^2-c_s^2}\left(\frac{1}{\rho^{*a}}-\frac{1}{\rho^{*i}}\right)c_a b+\frac{1}{2}\Delta W_{\mp a}-\frac{c_f^2}{c_f^2-c_a^2}\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^{*a}}\right)c_a b,
\end{aligned}
\tag{4.32}
$$

and from (4.31) we get

$$
\begin{aligned}
B_\perp^{*i}-\frac{B_x b}{c_a} = \left(B_\perp^{l/r}-\frac{B_x b}{c_a}\right)\frac{\rho^{*i}}{\rho^{l/r}}+\rho^{*i}\Bigg[&-\frac{c_s^2}{c_a^2-c_s^2}\left(\frac{1}{\rho^{*a}}-\frac{1}{\rho^{*i}}\right)\frac{B_x b}{c_a}\\
&-\frac{1}{2}\frac{B_x}{c_a^2}\Delta W_{\mp a}+\frac{c_f^2}{c_f^2-c_a^2}\left(\frac{1}{\rho^{l/r}}-\frac{1}{\rho^{*a}}\right)\frac{B_x b}{c_a}\Bigg].
\end{aligned}
\tag{4.33}
$$

Then, we compute

$$
\begin{aligned}
B_\perp^\theta - \frac{B_x b}{c_a} &= \frac{1-\theta}{2}\left(B_\perp^{*i} - \frac{B_x b}{c_a}\right) + \frac{1+\theta}{2}\left(B_\perp^{l/r} - \frac{B_x b}{c_a}\right)\\
&= \frac{1-\theta}{2}\rho^{*i}\left[-\frac{c_s^2}{c_a^2-c_s^2}\left(\frac{1}{\rho^{*a}} - \frac{1}{\rho^{*i}}\right) + \frac{c_f^2}{c_f^2-c_a^2}\left(\frac{1}{\rho^{l/r}} - \frac{1}{\rho^{*a}}\right)\right]\frac{B_x b}{c_a}\\
&\quad + (1-\theta)\frac{\rho^{*i}}{\rho^{l/r}}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\\
&\quad + \left[\mu - (1-\theta)\frac{\rho^{*i}}{\rho^{l/r}} + \frac{1-\theta}{2}\mu\left(\frac{\rho^{*i}}{\rho^{l/r}} - 1\right)\right] Z.
\end{aligned}
\tag{4.34}
$$

As for the middle intermediate states, apply Lemma 3.7 with $\theta^{\|} = 1$. Then, the first choice of $\theta^\perp$ is to make the last term in (4.34) vanish,

$$
1 - \theta^\perp = \frac{2\mu}{\frac{\rho^{*i}}{\rho^{l/r}}(2-\mu) + \mu}. \tag{4.35}
$$

This gives

$$
P^\perp\left(B_\perp^{\theta^\perp} - \frac{B_x b}{c_a}\right) = \frac{2\mu\frac{\rho^{*i}}{\rho^{l/r}}}{\frac{\rho^{*i}}{\rho^{l/r}}(2-\mu) + \mu}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right), \tag{4.36}
$$

to which we need only to require the inequality (3.32). As soon as $0 \le \mu \le 1$ and $\rho^{*i}/\rho^{l/r} \ge \mu/(2-\mu)$ this gives the natural bounds $0 \le \theta^\perp \le 1$, and we get the sufficient condition

$$
\left|\frac{2\mu}{2-\mu}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|^2 \le \left(c_b^2 - (\rho^2 p')_{*i,l/r}\right)\left(\frac{1}{\rho_{\theta^\perp}} - \frac{B_x^2}{c_a^2}\right). \tag{4.37}
$$

Otherwise, still for $0 \le \mu \le 1$, the other possible choice is

$$
\theta^\perp = \frac{4(1-\mu)\left(\frac{\rho^{*i}}{\rho^{l/r}}\right)^2}{\left(\frac{\rho^{*i}}{\rho^{l/r}}(2-\mu) + \mu\right)^2} \tag{4.38}
$$

which satisfies $0 \le \theta^\perp \le 1$, and

$$
\begin{aligned}
P^\perp\left(B_\perp^{\theta^\perp} - \frac{B_x b}{c_a}\right) &= (1-\theta^\perp)\frac{\rho^{*i}}{\rho^{l/r}}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\\
&\quad + \frac{1}{2}\frac{1 - \left(\frac{\rho^{*i}}{\rho^{l/r}}\right)^2}{\frac{\rho^{*i}}{\rho^{l/r}}(2-\mu) + \mu}\mu^2 Z.
\end{aligned}
\tag{4.39}
$$

Again, the last term in (4.39) can be grouped in (3.29) with the term in $(1/\rho^{l/r} - 1/\rho^*)^2$. Using the Cauchy-Schwarz inequality this gives the sufficient condition

$$
\begin{aligned}
&\left|(1-\theta^\perp)\frac{\rho^{*i}}{\rho^{l/r}}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|^2\\
&\le \left(c_b^2 - (\rho^2 p')_{*i,l/r} + \frac{1 + \frac{\rho^{*i}}{\rho^{l/r}}}{\frac{\rho^{*i}}{\rho^{l/r}}(2-\mu) + \mu}\rho^{*i}\mu^2 Z \cdot P^\perp(B_\perp^{*i} - B_\perp^{l/r})\right)\left(\frac{1}{\rho_{\theta^\perp}} - \frac{B_x^2}{c_a^2}\right),
\end{aligned}
\tag{4.40}
$$

where according to (4.33)

$$
\begin{aligned}
P^\perp(B_\perp^{*i} - B_\perp^{l/r}) &= \left(\frac{\rho^{*i}}{\rho^{l/r}} - 1\right)\mu Z - \frac{1}{2}\frac{\rho^{*i} B_x}{c_a^2}\Delta W_{\mp a}\\
&= -\left(\frac{\rho^{*i}}{\rho^{l/r}}(2-\mu) + \mu\right) Z + 2\frac{\rho^{*i}}{\rho^{l/r}}\left(Z - \frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right).
\end{aligned}
\tag{4.41}
$$

This gives when $\rho^{*i}/\rho^{l/r} \leq \mu/(2-\mu)$ the sufficient condition

$$\begin{aligned}&\left|\frac{\mu}{2-\mu}\left(Z-\frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|^2\\ &\leq\left(c_b^2-(\rho^2p')_{*i,l/r}-\left(1+\frac{\rho^{*i}}{\rho^{l/r}}\right)\rho^{*i}\mu^2\left(|Z|^2+\left|Z\cdot\left(Z-\frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|\right)\right)\\ &\times\left(\frac{1}{\rho_{\theta^\perp}}-\frac{B_x^2}{c_a^2}\right).\end{aligned} \tag{4.42}$$

4.4. **Summary of sufficient conditions.** From subsections 4.1-4.3 we deduce the following sufficient conditions for entropy inequalities on each side (left or right) for the 7-wave solver.

Proposition 4.1. *The approximate Riemann solver is entropy stable if all intermediate densities are positive,* (4.1)-(4.2) *hold,*

$$0 \leq \mu \leq 1, \tag{4.43}$$

and

$$\begin{aligned}&\left|\frac{2\mu}{2-\mu}\left(Z-\frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|^2\\ &\leq\left(c_b^2-(\rho^2p')_{*a,l/r}-\frac{2}{2-\mu}\rho^{*a}\mu^2\left(|Z|^2+\left|Z\cdot\left(Z-\frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|\right)\right)\\ &\times\left(\frac{1}{\max(\rho^{*a},\rho^{l/r})}-\frac{B_x^2}{c_a^2}\right),\end{aligned} \tag{4.44}$$

$$\begin{aligned}&\left|\frac{2\mu}{2-\mu}\left(Z-\frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|^2\\ &\leq\left(c_b^2-(\rho^2p')_{*i,l/r}-\frac{2}{2-\mu}\rho^{*i}\mu^2\left(|Z|^2+\left|Z\cdot\left(Z-\frac{1}{4}\rho^{l/r}\frac{B_x}{c_a^2}\Delta W_{\mp a}\right)\right|\right)\right)\\ &\times\left(\frac{1}{\max(\rho^{*i},\rho^{l/r})}-\frac{B_x^2}{c_a^2}\right),\end{aligned} \tag{4.45}$$

where in both inequalities the two factors on the right-hand side must be nonnegative.

Remark. If we want to exactly resolve an isolated, say left, Alfven discontinuity, the above conditions impose that either b_l and b_r are not colinear, or $c_{al} \neq c_{ar}$. Indeed, if $c_{al} = c_{ar} = |B_x|\sqrt{\rho}$, the right-hand sides of (4.44)-(4.45) vanish. However, since $\Delta W_a = 0$, we deduce that $\mu_r Z_r = 0$, and from (4.1) that $B_x b_r/c_a = B_\perp^r$, which is not colinear to $B_\perp^l + B_\perp^r$ in general.

Even in the Euler case, the nonlinearity of the subcharacteristic condition is too complicated to directly give values of the relaxation parameters. One has to make a bit of analysis to find them, see [4]. In contrast to that case where there is only one speed c, here there are four parameters c_a, c_b, $b \in \mathbf{R}^2$ to be chosen (on each left and right side), and the simplifications we can make are limited due to the previous remark. The issue of finding good relaxation velocities for this scheme can nevertheless be rather well resolved, using Proposition 4.1 and lower bounds for $1/\rho^*$ for any intermediate density ρ^*. This will be presented in a follow-up paper.

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