Existence of strong traces for quasi-solutions of multidimensional scalar conservation laws

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Abstract

Abstract. We consider a conservation law in the domain $\Omega \subset \mathbb{R}^{n+1}$ with C^1 boundary $\partial\Omega$. For the wide class of functions including generalized entropy sub- and super-solutions we prove existence of strong traces for normal components of the entropy fluxes on $\partial\Omega$. No non-degeneracy conditions on the flux are required.

1 Introduction

In the open domain $\Omega \subset \mathbb{R}^{n+1}$ we consider a first-order conservation law

$$\mathrm{div}_x f(u) = 0, \tag{1.1}$$

$u = u(x)$, $x = (x_0, \dots, x_n) \in \Omega$; $f = (f_0, \dots, f_n) \in C(\mathbb{R}, \mathbb{R}^{n+1})$, where the flux functions are supposed to be only continuous: $f_i(u) \in C(\mathbb{R})$, $i = 0, \dots, n$. We assume that $\partial\Omega$ is a C^1 boundary that is for each point $x \in \partial\Omega$ there exists its neighborhood U and a C^1-diffeomorphism $\zeta : U\bar{\Omega} \to W_{rh}$, where $\bar{\Omega} = \mathrm{Cl}\,\Omega$ and $W_{rh} = [0, h) \times V_r$ is a cylinder in $\mathbb{R}^{n+1}$, $V_r = \{ y = (y_1, \dots, y_n) \in \mathbb{R}^n \mid |y|^2 = y_1^2 + \cdots + y_n^2 < r^2 \}$ is an open ball, $r, h > 0$. We shall also assume that derivatives of ζ and ζ^{-1} are bounded (this assumption can be always achieved by passing to a less neighborhood). Clearly, boundaries of the sets W_{rh} and $U \cap \bar{\Omega}$ must correspond each other under the map ζ: $\zeta(\partial\Omega \cap U) = \{0\} \times V_r$. We shall refer to triples (U, ζ, W_{rh}) as *boundary charts* on Ω. The corresponding neighborhood U will be called a coordinate neighborhood. For any point $\bar{x} \in \partial\Omega$ we can choose the boundary chart (U, ζ, W_{rh}), $\bar{x} \in U$ in the following special way. As is rather well-known, $\partial\Omega$ locally is a graph of some C^1 function, upon rotating and shift of coordinate axes if necessary. More precisely, making a shift of coordinate axes, we can assume that $\bar{x} = 0$. Then there exists a C^1 function $g = g(y)$ defined in some open ball V_r and a neighborhood U of the point $\bar{x} = 0$ such that, after possible rotating of coordinate axes

$$\bar{\Omega} \cap U = \{ x = (x_0, x') \in \mathbb{R}^{n+1} \mid 0 \le x_0 - g(x') < h, x' \in V_r \}$$

with some $h > 0$. In particular, $\partial\Omega \cap U$ is a graph $x_0 = g(x')$, $x' \in V_r$. Taking $\zeta(x) = (t(x), y(x))$ with $t(x) = x_0 - g(x')$, $y(x) = x'$ we obtain the chart (U, ζ, W_{rh}), which will be called *canonical*. Observe that the Jacobian of the map ζ equals identically 1, therefore ζ is a measure preserving map.

Denote by $\vec{\nu}(x)$ the outward normal vector at the point $x \in \partial\Omega$. Let (U, ζ, W_{rh}) be a chart, $\zeta(x) = (t(x), y(x))$ then $\partial\Omega \cap U$ is the level set $t(x) = 0$ and $t(x) > 0$ for

$x \in \Omega \cap U$. Therefore, $\vec{\nu}(x) = -\nabla t(x)$. Of course, this vector could be normalized to be unit but this is not necessary for the sequel. Notice also that if $t(x) = \tau > 0$ then $\vec{\nu}(x) = -\nabla t(x)$ is the normal vector to the hyper-surface $t(x) = \tau$.

Now, let us recall the definition of a generalized entropy solution of (1.1) in the sense of S.N. Kruzhkov [11].

Definition 1.1. A bounded measurable function $u = u(x) \in L^\infty(\Omega)$ is called a generalized entropy solution (briefly - g.e.s.) of the equation (1.1) in the domain Ω if $\forall k \in \mathbb{R}$

$$\operatorname{div}_x[\operatorname{sign}(u-k)(f(u)-f(k))] \le 0 \tag{1.2}$$

in the sense of distributions on Ω (in $\mathcal{D}'(\Omega)$).

In the case when (1.2) holds with equality sign: for all $k \in \mathbb{R}$

$$\operatorname{div}_x[\operatorname{sign}(u-k)(f(u)-f(k))] = 0 \ \text{ in } \mathcal{D}'(\Omega) \tag{1.3}$$

a g.e.s. $u(x)$ is called an *isentropic solutions* (briefly - i.s.)

Condition (1.2) means that for any test function $h = h(x) \in C_0^1(\Omega)$, $h \ge 0$

$$\int_\Omega \{\operatorname{sign}(u-k)(f(u)-f(k), \nabla_x h)\} dtdx \ge 0.$$

Here and below we use the notation $(\cdot,\cdot)$ for the scalar product in a finite-dimensional Euclidean space. We also denote by $|\cdot|$ the corresponding Euclidean norm.

Setting in (1.2) $k = \pm\|u\|_\infty$, one can easily derive that $\operatorname{div}_x f(u) = 0$ in $\mathcal{D}'(\Omega)$ and $u(x)$ satisfies (1.1) in the sense of distributions, i.e. $u(x)$ is a generalized solution (g.s.) of this equation.

If we replace the condition (1.2) by one of the following conditions

$$\operatorname{div}_x[\operatorname{sign}^+(u-k)(f(u)-f(k))] \le 0 \ \text{ in } \mathcal{D}'(\Omega), \tag{1.4}$$

$$\operatorname{div}_x[\operatorname{sign}^-(u-k)(f(u)-f(k))] \le 0 \ \text{ in } \mathcal{D}'(\Omega), \tag{1.5}$$

where $\operatorname{sign}^+(u-k) = \max(\operatorname{sign}(u-k), 0)$; $\operatorname{sign}^-(u-k) = \min(\operatorname{sign}(u-k), 0)$, we obtain notions of a generalized entropy sub-solution (g.e.sub-s.) and a generalized entropy super-solution (g.e.super-s.) respectively (see [12, 4, 20]). Obviously, a function $u(x)$ is a g.e.s. of (1.1) if and only if $u(x)$ is a g.e.sub-s. and g.e.super-s of this equation simultaneously.

The theory of g.e.sub-s. (g.e.super-s.) plays an important role in the study of conservation laws, especially in the case when the flux vector is only continuous (see [12, 4, 20]).

Denote by $\bar{M}_{loc}(\Omega)$ the space of Borel measures γ on Ω that are locally finite up to the boundary $\partial\Omega$, i.e. for any compact set $K \subset \mathbb{R}^{n+1}$

$$\operatorname{Var}\gamma(K \cap \Omega) < \infty. \tag{1.6}$$

Here $\mathrm{Var}\,\gamma(A)$ is the variation of γ on a Borel set A, so $\mathrm{Var}\,\gamma$ is a nonnegative Borel measure on Ω. We will consider $\bar{M}_{loc}(\Omega)$ as a locally convex space with topology generated by semi-norms $p_K(\gamma) = \mathrm{Var}\,\gamma(K \cap \Omega)$.

Now we introduce the notion of a *quasi-solution* of (1.1).

Definition 1.2. A function $u = u(x) \in L^\infty(\Omega)$ is called a quasi-solution (a quasi-s. for short) of equation (1.1) if for some dense set $F \subset \mathbb{R}$ of values k

$$\mathrm{div}_x \psi_k(u) = -\gamma_k \quad \text{in } \mathcal{D}'(\Omega), \tag{1.7}$$

where $\gamma_k = \gamma_k(x) \in \bar{M}_{loc}(\Omega)$, and $\psi_k(u) = \mathrm{sign}\,(u-k)(f(u)-f(k))$ is the Kruzhkov's entropy flux.

From (1.7) with $k > \|u\|_\infty$ it follows that

$$\mathrm{div}_x f(u) = -\gamma \quad \text{in } \mathcal{D}'(\Omega), \tag{1.8}$$

$\gamma \in \bar{M}_{loc}(\Omega)$. We underline that functions satisfying (1.7) generally are not g.s. of original equation (1.1).

Without loss of generality we can assume that the set F from Definition 1.2 is countable.

Remark 1.1. Definitions 1.1, 1.2 (as well as the definitions of g.e.sub-s. and g.e.super-s.) remain valid without changes also for the case when the flux vector in (1.1) depends on variables $x \in \Omega$, $f = f(x, u)$, and satisfies the assumption

$$\mathrm{div}_x f(x, k) = 0 \quad \text{in } \mathcal{D}'(\Omega)\ \forall k \in \mathbb{R}. \tag{1.9}$$

As we will demonstrate below, the class of quasi-solutions includes g.e.s. of (1.1) as well as g.e.sub-s. and g.e.super-s. of this equation.

Denote by $\mathcal{H}^n$ the n-dimensional Hausdorff measure on $\partial\Omega$. The main our result is the following theorem

Theorem 1.1. *Suppose a function $u(x)$ is a quasi-s. of (1.1). Then there exists a function $u_0 = u_0(x) \in L^\infty(\partial\Omega)$ such that for each $k \in \mathbb{R}$ the normal component of flux vector $(\psi_k(u(x)), \vec{\nu}(x))$ has the strong trace $(\psi_k(u_0(x)), \vec{\nu}(x))$ at the boundary $\partial\Omega$. This means that for any chart (U, ζ, W_{rh})*

$$\operatorname*{ess\,lim}_{t\to 0}(\psi_k(u(t,y)), \vec{\nu}(t,y)) = (\psi_k(u_0(y)), \vec{\nu}(y)) \quad \text{in } L^1(V_r), \tag{1.10}$$

where $u(t,y) = u(\zeta^{-1}(t,y))$, $\vec{\nu}(t,y) = \vec{\nu}(\zeta^{-1}(t,y))$, and $u_0(y) = u_0(\zeta^{-1}(0,y))$, $\vec{\nu}(y) = \vec{\nu}(\zeta^{-1}(0,y))$.

Besides, if for $\mathcal{H}^n$-a.e. $x \in \partial\Omega$ the function $u \to (f(u), \vec{\nu}(x))$ is not constant on non-degenerate intervals then $u_0(x)$ is the strong trace of $u(x)$.

Remark, that the first result on existence of strong traces for g.e.s. of the equation $u_t+f(u)_x = 0$ on initial line $t = 0$ was established in [7] under the condition that the flux function $f(u) \in C^1(\mathbb{R})$ is not affine on non-degenerate intervals. In multidimensional case existence of the strong traces for g.e.s. of equation $u_t + \mathrm{div}_x f(u) = 0$ was later proved in [26] under the assumptions that the flux $f(u) \in C^3(\mathbb{R}, \mathbb{R}^n)$ and satisfies the following non-degeneracy condition: $\forall \xi \in \mathbb{R}^n \setminus \{0\}$ the function $u \to (\xi, f'(u))$ is not constant on sets of positive Lebesgue measure. In [19] the existence of strong traces on initial hyperplane $t = 0$ was proved for quasi-solutions of the equation $u_t + \mathrm{div}_x\varphi(u) = 0$ without any restrictions on only continuous flux vector $\varphi(u) = (\varphi_1(u), \dots, \varphi_n(u))$. Recently in paper [13] the authors prove existence of strong traces for g.e.s. of the equation $u_t + f(u)_x = 0$ on the boundary $\partial\Omega$ of a plane domain $\Omega \subset \mathbb{R}^2$ without non-degeneracy restrictions but under the regularity assumption $f(u) \in C^2(\mathbb{R})$.

In the present paper we extend results of paper [19] to the case of an arbitrary boundary $\partial\Omega$. We keep the main scheme of the paper [19], utilizing the techniques of H-measures and induction on the spatial dimension. Recall that the flux vector is assumed to be only continuous. Our main result remains valid for more general case of a locally Lipschitz-deformable boundary $\partial\Omega$ (in the sense of [6]) with the same proof. We consider the case of C^1 boundaries only to simplify the notations.

We underline that Theorem 1.1 allows to formulate boundary value problems for (1.1) in the sense of Bardos, LeRoux & Nédélec [1]. For instance the Dirichlet problem $u|_{\partial\Omega} = u_b$ can be understood in the sense of the inequality: $\forall k \in \mathbb{R}$

$$(\mathrm{sign}\,(u-k) + \mathrm{sign}\,(k-u_b))(f(u) - f(k), \vec{\nu}) \geq 0 \quad \text{on } \partial\Omega,$$

which is well defined due to existence of the strong traces of
$\mathrm{sign}\,(u-k)(f(u) - f(k), \vec{\nu})$ and $(f(u), \vec{\nu})$.

1.1 Existence of the weak traces

Let $\vec{v}(x) \in L^\infty(\Omega, \mathbb{R}^n)$ be a bounded vector field on Ω. This field is called *a divergence measure field* if $\mathrm{div}\vec{v} = \gamma \in \bar{M}_{loc}(\Omega)$ in $\mathcal{D}'(\Omega)$. From results by G.-Q. Chen & H. Frid [6] it follows existence of the weak trace for normal component $(\vec{\nu}, \vec{v})$ of the field $\vec{v}$ at the boundary $\partial\Omega$.

Proposition 1.1. *Let $\vec{v}$ be a divergence measure field on Ω. Then there exists a function $v_0(x) \in L^\infty(\partial\Omega)$ such that for any chart (U, ζ, W_{rh})*

$$\operatorname*{ess\,lim}_{t\to 0}(\vec{v}(t,y), \vec{\nu}(t,y)) = v_0(y) \quad \textit{weakly-}* \textit{ in } L^\infty(V_r),$$

where $\vec{v}(t,y) = \vec{v}(\zeta^{-1}(t,y))$, $\vec{\nu}(t,y) = \vec{\nu}(\zeta^{-1}(t,y))$, and $v_0(y) = v_0(\zeta^{-1}(0,y))$.

The function $v(x)$ will be called a weak normal trace of $\vec{v}$. Applying this Theorem to the fields $\psi_k(u)$, where $u = u(x)$ is a quasi-s. of (1.1), we derive the following result.

Theorem 1.2. *Let $u(x)$ be a quasi-s. of* (1.1)*. Then for each $k \in \mathbb{R}$ there exists a function $v_{0k}(x) \in L^\infty(\partial\Omega)$ being the weak normal trace of the field $\psi_k(u(x))$.*

Proof. Let $u = u(x)$ be a quasi-s. of (1.1), $M = \|u\|_\infty$. If $k \in F$ then by Definition 1.2 $\psi_k(u(x))$ is a divergence measure field and by Proposition 1.1 there exists its weak normal trace $v_{0k}(x) \in L^\infty(\partial\Omega)$ that is for any boundary chart (U, ζ, W_{rh})

$$(\psi_k(u(t,y)), \vec{\nu}(t,y)) \to v_{0k}(y) \text{ weakly-}* \text{ in } L^\infty(V_r) \tag{1.11}$$

as $t \to 0$ running a set $E_k \subset (0,h)$ of full measure. Here $u(t,y) = u(\zeta^{-1}(t,y))$, $\vec{\nu}(t,y) = \vec{\nu}(\zeta^{-1}(t,y))$, $v_{0k}(y) = v_{0k}(\zeta^{-1}(0,y))$. Let $E = \bigcap\limits_{k\in F} E_k$. Since F is countable the set E has full measure in $(0,h)$ and limit relations (1.11) hold as $t \to 0$, $t \in E$ for all $k \in F$ simultaneously.

As easy to see, $\|\psi_k(u) - \psi_{k'}(u)\|_\infty \le 2\omega(|k-k'|)$, where

$$\omega(\delta) = \sup\{\ |f(u_1) - f(u_2)| \mid u_1, u_2 \in [-M, M],\ |u_1 - u_2| < \delta\ \}$$

is a continuity modulus of the vector $f(u)$ on the segment $[-M, M]$. This implies that for any $k, k' \in \mathbb{R}$

$$\begin{aligned} &|(\psi_k(u(t,y)), \vec{\nu}(t,y)) - (\psi_{k'}(u(t,y)), \vec{\nu}(t,y))| \le \\ &\qquad 2\omega(|k-k'|)|\vec{\nu}(t,y)| \le C\omega(|k-k'|), \end{aligned} \tag{1.12}$$

$C = \text{const}$. Relations (1.11), (1.12) implies in the limit as $t \to 0$, $t \in E$ that

$$\|v_{0k}(y) - v_{0k'}(y)\|_\infty \le C\omega(|k-k'|)\ \forall k, k' \in F \tag{1.13}$$

and since $\omega(\delta) \to 0$ as $\delta \to 0$, we find that the map $k \to v_{0k}(y) \in L^\infty(V_r)$ is uniformly continuous on F. Taking into account that F is dense in $\mathbb{R}$ the map $k \to v_{0k}$ is uniquely extended as a continuous map on the whole $\mathbb{R}$. Thus, the functions $v_{0k}(y)$ are defined for all real k and relation (1.13) remains valid already for all real k, k'. From limit relation (1.11) and estimates (1.12), (1.13) it readily follows that (1.11) is satisfied for all $k \in \mathbb{R}$. Clearly, the weak limit $v_{0k}(x) \equiv v_{0k}(\zeta(x))$ does not depend on the choice of a boundary chart. This allows to define $v_{0k}(x)$ on the whole boundary $\partial\Omega$ and by the construction $v_{0k}(x)$ is the weak normal trace of the field $\psi_{0k}(u(x))$. The proof is complete. □

1.2 Localization of the problem.

Since our result has local character we reformulate the problem in a coordinate neighborhood U of a boundary point $\bar{x} \in \partial\Omega$. More precisely, suppose that (U, ζ, W_{rh}) is a canonical boundary chart, i.e. after rotating and shift of coordinate axes if necessary $\bar{x} = 0$ and $(t, y) = \zeta(x) = (t(x), x')$, $t(x) = x_0 - g(x')$, $x' = (x_1, \ldots, x_n)$, where $g \in C^1(V_r)$, $|\nabla g| \le \text{const}$. Let $u = u(x) \in L^\infty(U \cap \Omega)$, and $u(t,y) \in L^\infty((0,h) \times V_r)$ be the same function written in the coordinates (t,y) that is $u(t,y) = u(\zeta^{-1}(t,y))$. We have the following statement.

Theorem 1.3. *If the function $u(x)$ is a g.e.s, an i.s., a g.e.sub-s., a g.e.super-s, a quasi-s. of equation (1.1) in Ω then the function $u(t,y)$ is, respectively, a g.e.s., an i.s., a g.e.sub-s, a g.e.super-s, a quasi-s. of the equation*

$$\varphi_0(y,u)_t + \mathrm{div}_y \varphi(u) = 0 \tag{1.14}$$

in the domain $W=(0,h)\times V_r$, where $\varphi(u)=\bar f(u)=(f_1(u),\dots,f_n(u))\in\mathbb{R}^n$ and

$$\varphi_0(y,u) = f_0(u) - (\nabla g(y), \bar f(u)) = -(\vec\nu(y), f(u)). \tag{1.15}$$

Proof. Let $h(x)\in C_0^1(U\cap\Omega)$ be a test function, $h(t,y)$ be the same function written in the variables $(t,y)\in W$. Then $h(t,y)\in C_0^1(W)$ and by the chain rule

$$h_{x_i}(x) = -\frac{\partial g}{\partial x_i}(y)h_t(t,y) + h_{y_i}(t,y),\ i=1,\dots,n,$$
$$h_{x_0}(x) = h_t(t,y),\ (t,y)=\zeta(x).$$

Using this relations and making the measure preserving change of variables $(t,y)=\zeta(x)$, we obtain the identity

$$\int_\Omega (\psi_k(u(x)),\nabla_x h(x))dx = \int_\Omega \mathrm{sign}\,(u(x)-k)(f(u)-f(k),\nabla_x h(x))dx =$$
$$\int_{W_{rh}} \mathrm{sign}\,(u(t,y)-k)[(\varphi_0(y,u(t,y))-\varphi_0(y,k))h_t(t,y) +$$
$$(\varphi(u(t,y))-\varphi(k),\nabla_y h(t,y))]dtdy. \tag{1.16}$$

Clearly, the functions $h(t,y)$ runs the whole space $C_0^1(W)$ while $h(x)\in C_0^1(U\cap\Omega)$. Therefore, $u=u(x)$ is g.e.s of (1) if and only if for all $k\in\mathbb{R}$

$$\frac{\partial}{\partial t}[\mathrm{sign}\,(u-k)(\varphi_0(y,u(t,y))-\varphi_0(y,k))] + \mathrm{div}_y[\mathrm{sign}\,(u-k)(\varphi(u(t,y))-\varphi(k))] \le 0$$

in $\mathcal{D}'(W)$, i.e. (see Remark 1.1) $u(t,y)$ is a g.e.s. of (1.14). Observe, that the condition (1.9) from this Remark

$$\varphi_0(y,k)_t + \mathrm{div}_y\varphi(k) = 0 \ \text{ in } \mathcal{D}'(W)\ \forall k\in\mathbb{R}$$

is evidently satisfied. If $u(x)$ is an i.s. of (1.1) then the same arguments yield that $u(t,y)$ is an i.s. of (1.14).

Finally, if $u(x)$ is a quasi-s. of (1.1) then, by relation (1.16) again, $\forall k\in F$, $h(t,y)\in C_0^1(W)$

$$\int_W \mathrm{sign}\,(u(t,y)-k)[(\varphi_0(y,u(t,y))-\varphi_0(y,k))h_t(t,y) +$$
$$(\varphi(u(t,y))-\varphi(k),\nabla_y h(t,y))]dtdy = \int_\Omega h(x)d\gamma_k(x) = \int_W h(t,y)d\tilde\gamma_k(t,y),$$

$\tilde{\gamma}_k(t,y)$ being the image $\zeta^*\gamma_k|_U$ under the map ζ. Thus, for all $k \in F$ in $\mathcal{D}'(W)$

$$\frac{\partial}{\partial t}[\operatorname{sign}(u-k)(\varphi_0(y,u(t,y))-\varphi_0(y,k))]+\operatorname{div}_y[\operatorname{sign}(u-k)(\varphi(u(t,y))-\varphi(k))] = -\tilde{\gamma}_k.$$

Clearly, $\tilde{\gamma}_k \in \bar{M}_{loc}(W)$ and therefore $u(t,y)$ is a quasi-s. of (1.14).

From identity (1.16) with sign replaced by sign^+, sign^- we derive that $u(t,y)$ is a g.e.sub-s or a g.e.super-s. together with $u(x)$. The proof is complete. $\square$

For our aims it is only essential behavior of quasi-s. near the boundary $\partial\Omega$. In particular, the space $\bar{M}_{loc}(W)$ appearing in the definition of quasi-s. can be defined as the space of Borel measures on W that are locally finite up to the "essential part" of the boundary laying in the hyperplane $t=0$. Thus, the condition (1.6) has the form:

$$\operatorname{Var}\gamma(K \cap W) < \infty \tag{1.17}$$

for each compact set $K \subset W_{rh} = [0,h) \times V_r$.

Remark also that the function $\varphi_0(y,u) = -(f(u), \vec{\nu}(y))$, where $\vec{\nu}(y)$ is the normal vector to the hyper-surface $t(x) = t$ written in the coordinate (t,y) (it does not depend on t). Thus, to prove Theorem 1.1 we have to establish that the functions

$$\psi_{0k}(u(t,y)) = \operatorname{sign}(u(t,y)-k)(\varphi_0(y,u(t,y)) - \varphi_0(y,k)),\ \ k \in \mathbb{R},$$

where $u(t,y)$ being a quasi-s. of (1.14), have strong traces at $t=0$.

For equation (1.14) relation (1.7) has the form

$$\psi_{0k}(y,u)_t + \operatorname{div}_y \psi_k(u) = -\gamma_k \ \text{ in } \mathcal{D}'(W), \tag{1.18}$$

where

$$\psi_{0k}(t,y,u) = \operatorname{sign}(u-k)(\varphi_0(y,u) - \varphi_0(y,k)),$$
$$\psi_k(u) = \operatorname{sign}(u-k)(\varphi(u) - \varphi(k)) \in \mathbb{R}^n,$$

and $u = u(t,y)$. Observe also that the analog of (1.8) is the relation

$$\varphi_0(y,u)_t + \operatorname{div}_y \varphi(u) = -\gamma \ \text{ in } \mathcal{D}'(W). \tag{1.19}$$

Now we demonstrate that g.e.s. as well as g.e.sub-s. and g.e.super-s. of (1.14) are quasi-s. of this equation. Let us firstly show that a g.e.s. $u(t,y)$ satisfies (1.18) for all $k \in \mathbb{R}$, moreover in this case $\gamma_k \ge 0$. Indeed, as it follows from the known representation of nonnegative distributions, $\psi_{0k}(y,u)_t + \operatorname{div}_y \psi_k(u) = -\gamma_k$ in $\mathcal{D}'(W)$, where γ_k is a nonnegative locally finite measure on W. We show that this measure is locally finite up to the initial hyperplane $t=0$, i.e. $\gamma_k \in \bar{M}_{loc}(W)$. For this, choose a function $\beta(t)\in C_0^1(\mathbb{R})$ such that $\operatorname{supp}\beta\subset[0,1]$, $\beta(t)\geq 0$, $\displaystyle\int\beta(t)dt=1$ and set for $\nu\in\mathbb{N}$

$$\delta_\nu(t)=\nu\beta(\nu t),\quad \theta_\nu(t)=\int_0^t\delta_\nu(s)ds, \tag{1.20}$$

so that $\theta'_\nu(t)=\delta_\nu(t)$. Clearly, the sequence $\delta_\nu(t)$ converges as $\nu\to\infty$ to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R})$ while the sequence $\theta_\nu(t)$ converges pointwise to the Heaviside function

$$\theta(t)=\begin{cases}0 & , \quad t\leq 0,\\ 1 & , \quad t>0.\end{cases}$$

Let $\rho(t,y)\in C_0^1(W_{rh})$, $\rho(t,y)\geq 0$. Applying the equality $\psi_{0k}(y,u)_t+\operatorname{div}_y\psi_k(u)=-\gamma_k$ to the test function $\rho(t,y)\theta_\nu(t-t_0)\in C_0^1(W)$, where $t_0>0$, we derive that

$$\int_W\rho(t,y)\theta_\nu(t-t_0)d\gamma_k(t,y)=\int_W\psi_{0k}(y,u)\rho(t,y)\delta_\nu(t-t_0)dtdy+$$
$$\int_W[\psi_{0k}(y,u)\rho_t(t,y)+(\psi_k(u),\nabla_y\rho)]\theta_\nu(t-t_0)dtdy.$$

Supposing that t_0 is a Lebesgue point of the function

$$t\to\int_{V_r}\psi_{0k}(y,u(t,y))\rho(t,y)dy,$$

we obtain, in the limit as $\nu\to\infty$, that

$$\int_{(t_0,h)\times V_r}\rho(t,y)d\gamma_k(t,y)=\int_{V_r}\psi_{0k}(y,u(t_0,y))\rho(t_0,y)dy+$$
$$\int_{(t_0,h)\times V_r}[\psi_{0k}(y,u)\rho_t(t,y)+(\psi_k(u),\nabla_y\rho)]dtdy\leq$$
$$\int_{V_r}|\psi_{0k}(y,u(t_0,y))|\rho(t_0,y)dy+$$
$$\int_W[|\psi_{0k}(y,u)|\cdot|\rho_t|+|\psi_k(u)|\cdot|\nabla_y\rho|]dtdy\leq C_\rho,$$

where the constant C_ρ does not depend on t_0 since $u\in L^\infty(W)$.

Passing to the limit as $t_0\to 0$ we derive the estimate

$$\int_W\rho(t,y)d\gamma_k(t,y)\leq C_\rho,$$

which implies, in view of arbitrariness of the nonnegative test function $\rho(t,y)\in C_0^1(W_{rh})$, that the measure γ_k satisfies (1.17).

The condition (1.18) is also satisfied for g.e.sub-s. (and g.e.super-s.) $u(t,y)$. Indeed, assume for definiteness that $u = u(t,y)$ is a g.e.sub-s. of (1.14). Then, in the same way as for g.e.s., we find that $\forall k \in \mathbb{R}$

$$[\text{sign}^+(u-k)(\varphi_0(y,u)-\varphi_0(y,k))]_t + \text{div}_y[\text{sign}^+(u-k)(\varphi(u)-\varphi(k))] = -\beta_k,$$

where $\beta_k \in \bar{M}_{loc}(W)$, $\beta_k \geq 0$.

In particular, taking in this relation $k < -\|u\|_\infty$, we obtain that

$$\varphi_0(y,u)_t + \text{div}_y\varphi(u) = (\varphi_0(y,u)-\varphi_0(y,k))_t + \text{div}_y(\varphi(u)-\varphi(k)) = -\beta$$

in $\mathcal{D}'(W)$, where $\beta \in \bar{M}_{loc}(W)$, $\beta \geq 0$. Taking into account that

$$\psi_{0k}(t,y,u) = (2\text{sign}^+(u-k)-1)(\varphi_0(y,u)-\varphi_0(y,k)),$$
$$\psi_k(u) = (2\text{sign}^+(u-k)-1)(\varphi(u)-\varphi(k)),$$

we derive the equality $\psi_{0k}(y,u)_t + \text{div}_y\psi_k(u) = -(2\beta_k - \beta)$ that is (1.18) is satisfied for all $k \in \mathbb{R}$.

Extending $\nabla g(y)$ from a ball $|y| \leq \rho < r$ as a continuous bounded nonzero vector on the whole space $\mathbb{R}^n$, we can consider equation (1.14) in the half space $\Pi = \mathbb{R}_+ \times \mathbb{R}^n$. If $u = u(t,y)$ is a quasi-s. of (1.14) in $W_1 = (0,h) \times V_\rho$ then we can extend this function as a quasi-s. in Π. For instance, one can set $u(t,y) = 0$ for $(t,y) \notin W_1$. Then, as easily follows from the generalized Gauss-Green formula (see [6]), for each $k \in F$, $g = g(t,y) \in C_0^1(\Pi)$

$$\int_\Pi [\psi_{0k}(y,u)g_t(\psi_k(u), \nabla_y g)]dtdy = \int g(t,y)d\gamma_k(t,y) + \int_{S'} \alpha_k(t,y)g(t,y)d\mathcal{H}^n(t,y),$$

where $S' = ((0,h) \times S_\rho) \cup (\{h\} \times V_\rho)$, $S_\rho = \partial V_\rho$ is a sphere $|y| = \rho$, $\alpha_k(t,y) = \alpha_{1k}(t,y) - \alpha_{0k}(t,y)$, where $\alpha_{1k}(t,y)$ is a weak normal trace of the divergence measure field $\vec{\psi}(y,u) = (\psi_{0k}(y,u), \psi_k(u))$, $u = u(t,y)$ at S' (in the sense of [6]), while $\alpha_{0k}(t,y) = (\vec{\psi}(y,0), \vec{n}(t,y))$, $\vec{n}$ being the outward unit normal vector on S'. This relation shows that for all $k \in F$

$$(\psi_{0k}(y,u))_t + \text{div}_y\psi_k(u) = -\gamma_k - \beta_k,$$

where the measure β_k, defined by the relation $\langle g, \beta_k\rangle = \int_{S'} \alpha_k(t,y)g(t,y)d\mathcal{H}^n(t,y)$, evidently lays in $\bar{M}_{loc}(\Pi)$, as well as the measure γ_k.

Hence the extended functions $u(t,y)$ is a quasi-s. of (1.14) in the whole half-space Π.

We have reduced our problem to the model case of equation (1.14) in Π. In this case our main result have the following form

Theorem 1.4. *If $u(t,y) \in L^\infty(\Pi)$ is a quasi-s. of (1.14) then there exists a function $u_0(y) \in L^\infty(\mathbb{R}^n)$ such that for all $k \in \mathbb{R}$*

$$\operatorname*{ess\,lim}_{t\to 0} \psi_{0k}(y,u(t,y)) = \psi_{0k}(y,u_0(y)) \quad in\ L^1_{loc}(\mathbb{R}^n).$$

By Theorem 1.2 there exist the weak traces $v_{0k}(y)$ of $\psi_{0k}(y,u(t,y))$ on the initial hyperplane $t=0$.

In the simple case of the plane boundary the proof of existence of the weak traces can be simplified. For completeness, we give below the proof, independent of results of Proposition 1.1.

Suppose $u = u(t,y) \in L^\infty(\Pi)$ is a quasi-s. of (1.14). We denote

$$E = \{\ t \in \mathbb{R}_+ \mid (t,y) \text{ is a Lebesgue point of } u(t,y) \text{ for a.e. } y \in \mathbb{R}^n\ \}. \tag{1.21}$$

Obviously, E is the set of full measure in $\mathbb{R}_+$. We have the following proposition:

Proposition 1.2. *There exists functions $v_{0k}(y) \in L^\infty(\mathbb{R}^n)$ such that $\psi_{0k}(\cdot,u(t,\cdot)) \to v_{0k}$ weakly-$*$ in $L^\infty(\mathbb{R}^n)$ as $t \to 0$, $t \in E$.*

Proof. Fix $k \in F$. Clearly, $\forall t \in E \quad \|\psi_{0k}(\cdot,u(t,\cdot))\|_\infty \le \text{const}$ and we can find a sequence $t_m \in E$, $m \in \mathbb{N}$, $t_m \underset{m\to\infty}{\to} 0$ and a function $v_{0k}(y) \in L^\infty(\mathbb{R}^n)$ such that $\psi_{0k}(y,u(t_m,y)) \to v_{0k}(y)$ weakly-$*$ in $L^\infty(\mathbb{R}^n)$ as $m \to \infty$.

If $\tau \in E$ then $t_m < \tau$ for large enough m. We fix such $t_m < \tau$ and set $\chi_\nu(t) = \theta_\nu(t-t_m) - \theta_\nu(t-\tau)$, where the sequence $\theta_\nu(t)$, $\nu \in \mathbb{N}$ was defined above in (1.20).

Let $h(y) \in C^1_0(\mathbb{R}^n)$. Then for large enough $\nu \in \mathbb{N}$ the function $h(y)\chi_\nu(t) \in C^1_0(\Pi)$. Applying relation (1.18) to this test function, we obtain, after simple transforms, that

$$\int_0^{+\infty}\left(\int_{\mathbb{R}^n}\psi_{0k}(y,u(t,y))h(y)dy\right)\delta_\nu(t-\tau)dt - \int_0^{+\infty}\left(\int_{\mathbb{R}^n}\psi_{0k}(y,u(t,y))h(y)dy\right)\delta_\nu(t-t_m)dt = \int_\Pi(\psi_k(u),\nabla_y h)\chi_\nu(t)dtdy - \int_\Pi h(y)\chi_\nu(t)d\gamma_k(t,y). \tag{1.22}$$

Passing in (1.22) to the limit as $\nu \to \infty$ and taking into account that $t_m, \tau \in E$ are Lebesgue points of the function $I(t) = \displaystyle\int_{\mathbb{R}^n}\psi_{0k}(y,u(t,y))h(y)dy$, and also that the sequence $\chi_\nu(t)$ converges pointwise to the indicator function of the interval $(t_m,\tau]$ and is bounded, we derive the equality

$$I(\tau) - I(t_m) = \int_{(t_m,\tau]\times\mathbb{R}^n}(\psi_k(u),\nabla_y h)dtdy - \int_{(t_m,\tau]\times\mathbb{R}^n}h(y)d\gamma_k(t,y),$$

which implies in the limit as $m \to \infty$ that

$$I(\tau) - I(0+) = \int_{\mathbb{R}^n} \psi_{0k}(y, u(\tau, y))h(y)dy - \int_{\mathbb{R}^n} v_{0k}(y)h(y)dy =$$

$$\int_{(0,\tau]\times\mathbb{R}^n} (\psi_k(u), \nabla_y h)dtdx - \int_{(0,\tau]\times\mathbb{R}^n} h(y)d\gamma_k(t, y) \underset{\tau\to 0}{\to} 0, \tag{1.23}$$

i.e. $\forall h(x) \in C_0^1(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \psi_{0k}(y, u(\tau, y))h(y)dy \underset{\tau\to 0}{\to} \int_{\mathbb{R}^n} v_{0k}(y)h(y)dy.$$

This, together with boundedness of $u(\tau, \cdot)$, $\tau \in E$ and density of $C_0^1(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$, implies that for each $k \in F$

$$\psi_{0k}(\cdot, u(t, \cdot)) \to v_{0k} \;\text{ weakly-}* \text{ in } L^\infty(\mathbb{R}^n) \;\text{ as } t \to 0, t \in E. \tag{1.24}$$

Since the map $k \to \psi_{0k}(y, u) \in C(\mathbb{R}^n \times I)$ is continuous, where $I = [-M, M]$, $M = \|u\|_\infty$, then repeating the arguments from the proof of Theorem 1.2, we conclude that (1.24) remains valid for all $k \in \mathbb{R}$. The proof is complete.

□

2 Preliminaries

2.1 Measure valued functions

Below we will frequently utilize results of the theory of measure valued functions.

Recall (see [9, 24]) that a measure valued function on a domain $\Omega \subset \mathbb{R}^N$ is a weakly measurable map $x \to \nu_x$ of the set Ω into the space of Borel probability measures having compact supports on $\mathbb{R}$. Weak measurability of ν_x means that for any continuous function $f(\lambda)$ the function

$$x \to \langle f(\lambda), \nu_x(\lambda)\rangle = \int f(\lambda)d\nu_x(\lambda)$$

is Lebesgue measurable on Ω.

A measure valued function ν_x is said to be bounded if there is a constant $M > 0$ such that $\operatorname{supp}\nu_x \subset [-M, M]$ for a.e. $x \in \Omega$. The minimal of such M will be denoted by $\|\nu_x\|_\infty$.

Finally, measure valued functions of the kind $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $\delta(\lambda - u) = \delta_u(\lambda)$ is the Dirac measure at the point u, are called regular and they are identified with the corresponding functions $u(x)$. Thus, the set $\mathrm{MV}(\Omega)$ of bounded measure valued functions on Ω includes the space $L^\infty(\Omega)$.

Remark 2.1. As was demonstrated in [15], for $\nu_x \in \mathrm{MV}(\Omega)$ measurability of the functions $x \to \langle f(\lambda), \nu_x(\lambda)\rangle$ remains valid for all Borel functions $f(\lambda)$.

We shall use also the following notions:

Definition 2.1. Suppose $\nu_x^m \in \mathrm{MV}(\Omega)$, $m \in \mathbb{N}$; $\nu_x \in \mathrm{MV}(\Omega)$. We shall say that

1) the sequence ν_x^m is bounded if for some constant $M > 0$ $\|\nu_x^m\|_\infty \le M$ $\forall m \in \mathbb{N}$;
2) the sequence ν_x^m converges to ν_x *weakly* if

$$\forall f(\lambda) \in C(\mathbb{R}) \quad \langle f(\lambda), \nu_x^m(\lambda)\rangle \underset{m\to\infty}{\to} \langle f(\lambda), \nu_x(\lambda)\rangle \quad \text{weakly-}* \text{ in } L^\infty(\Omega);$$

3) the sequence ν_x^m converges to ν_x *strongly* if

$$\forall f(\lambda) \in C(\mathbb{R}) \quad \langle f(\lambda), \nu_x^m(\lambda)\rangle \underset{m\to\infty}{\to} \langle f(\lambda), \nu_x(\lambda)\rangle \quad \text{in } L^1_{loc}(\Omega).$$

Remark 2.2. If a bounded sequence ν_x^m converges to ν_x weakly (strongly) then $\langle f(x,\lambda), \nu_x^m(\lambda)\rangle \underset{m\to\infty}{\to} \langle f(x,\lambda), \nu_x(\lambda)\rangle$ weakly-$*$ in $L^\infty(\Omega)$ (respectively, - in $L^1_{loc}(\Omega)$ for each function $f(x,\lambda) \in C(\Omega \times \mathbb{R})$, which is bounded on sets $\Omega \times I$ for each segment $I \subset \mathbb{R}$. This easily follows from the fact that $f(x,\lambda)$ can be uniformly on any compact approximated by functions of the form $\sum_{i=1}^{l} \alpha_i(x)\beta_i(u)$, $\alpha_i(x) \in C(\Omega)$, $\beta_i(u) \in C(\mathbb{R})$, $i = 1, \dots, l$.

Measure valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Omega)$ in the sense of the following theorem of Tartar (see [24]).

Theorem 2.1. *Let $u_m(x) \in L^\infty(\Omega)$, $m \in \mathbb{N}$ be a bounded sequence. Then there exist a subsequence $u_r(x)$ and a measure valued function $\nu_x \in \mathrm{MV}(\Omega)$ such that $u_r(x) \to \nu_x$ weakly. This means that*

$$\forall f(\lambda) \in C(\mathbb{R}) \quad f(u_r) \underset{m\to\infty}{\to} \langle f(\lambda), \nu_x(\lambda)\rangle \quad \textit{weakly-}* \textit{ in } L^\infty(\Omega)$$

that is the sequence of regular measure valued functions $\nu_x^r = \delta(\lambda - u_r(x))$, identified with u_r, converges weakly to ν_x.

Besides, ν_x is regular, i.e. $\nu_x = \delta(\lambda - u(x))$ if and only if $u_r(x) \underset{r\to\infty}{\to} u(x)$ in $L^1_{loc}(\Omega)$ that is $\nu_x^r \to \nu_x$ strongly.

More generally, the following statement on weak precompactness of bounded set of measure valued functions is valid (see [16]).

Theorem 2.2. *Let ν_x^m be a bounded sequence of measure valued functions. Then there exist a subsequence $\nu_x^r(x)$ and a measure valued function $\nu_x \in \mathrm{MV}(\Omega)$ such that $\nu_x^r \to \nu_x$ weakly as $r \to \infty$.*

Corollary 2.1. *Suppose a bounded sequence $\nu_x^m \in \mathrm{MV}(\Omega)$ weakly converges to $\nu_x \in \mathrm{MV}(\Omega)$, and a function $p(x,\lambda) \in C(\Omega \times \mathbb{R})$, bounded on sets $\Omega \times I$ for each segment $I \subset \mathbb{R}$, is such that $p(x,\lambda) \equiv u_m(x)$ on $\operatorname{supp}\nu_x^m$ for a.e. $x \in \Omega$. Then $u_m(x) \to u(x) = \langle p(x,\lambda), \nu_x(\lambda)\rangle$ weakly-$*$ in $L^\infty(\Omega)$, and $u_m(x) \underset{m\to\infty}{\to} u(x)$ in $L^1_{loc}(\Omega)$ if and only if $p(x,\lambda) \equiv \mathrm{const} = u(x)$ on $\operatorname{supp}\nu_x$ for a.e. $x \in \Omega$.*

Proof. Clearly, $u_m(x) = \langle p(x,\lambda), \nu_x^m(\lambda)\rangle \in L^\infty(\Omega)$. Denote by $p(x,\cdot)^*\nu_x$ the image of measure ν_x under the map $\lambda \to p(x,\lambda)$. Obviously, $x \to p(x,\cdot)^*\nu_x$ is a bounded measure valued function on Ω. From the relation (see Remark 2.2)

$$f(u_m(x)) = \langle f(p(x,\lambda)), \nu_x^m(\lambda)\rangle \underset{m\to\infty}{\to} \langle f(p(x,\lambda)), \nu_x(\lambda)\rangle = \langle f(\lambda), p(x,\cdot)^*\nu_x(\lambda)\rangle$$

$\forall f(\lambda) \in C(\mathbb{R})$ it follows that the the sequence $u_m(x)$ converges weakly to the measure valued function $p(x,\lambda)^*\nu_x$, and in particular, $u_m(x) \to u(x) = \langle p(x,\lambda), \nu_x(\lambda)\rangle$ weakly-$*$ in $L^\infty(\Omega)$. By Theorem 2.1 this convergence is strong if and only if $p(x,\cdot)^*\nu_x(\lambda) = \delta(\lambda - u(x))$ for a.e. $x \in \Omega$. Since the latter is equivalent to the condition $p(x,\lambda) \equiv u(x)$ on $\operatorname{supp}\nu_x$, the proof is complete. $\square$

We also need the following simple result

Lemma 2.1. *Let $\nu_x^m, \tilde\nu_x^m$, $m \in \mathbb{N}$ be bounded sequences on Ω, which converge weakly to the measure valued functions ν_x, $\tilde\nu_x$, respectively. Then, for each $p(\lambda,\mu) \in C(\mathbb{R}^2)$*

$$\iint p(\lambda,\mu)d\nu_x^m(\lambda)d\tilde\nu_y^m(\mu) \underset{m\to\infty}{\to} \iint p(\lambda,\mu)d\nu_x(\lambda)d\tilde\nu_y(\mu) \qquad (2.1)$$

weakly-$*$ in $L^\infty(\Omega\times\Omega)$.

Proof. Suppose firstly that $p(\lambda,\mu) = p_1(\lambda)p_2(\mu)$, $p_1, p_2 \in C(\mathbb{R})$. Since the variables x, y are independent we have

$$\iint p(\lambda,\mu)d\nu_x^m(\lambda)d\nu_y^m(\mu) = \langle p_1(\lambda), \nu_x^m(\lambda)\rangle\langle p_2(\mu), \tilde\nu_y^m(\mu)\rangle \underset{m\to\infty}{\to}$$
$$\langle p_1(\lambda), \nu_x(\lambda)\rangle\langle p_2(\mu), \tilde\nu_y(\mu)\rangle = \iint p(\lambda,\mu)d\nu_x(\lambda)d\tilde\nu_y(\mu)$$

weakly-$*$ in $L^\infty(\Omega\times\Omega)$, as required. Certainly, this relation remains true also for the functions $p(\lambda,\mu)$ represented as linear combinations of functions of the kind $p_1(\lambda)p_2(\mu)$. In particular (2.1) is valid for polynomials $p(\lambda,\mu)$. Since polynomials are dense in $C(\mathbb{R}\times\mathbb{R})$ we conclude that relation (2.1) is satisfied for general $p(\lambda,\mu)$. $\square$

2.2 Some properties of isentropic solutions

Fix some segment $I = [-M, M]$, $M > 0$ and denote by $\mathfrak{L} \subset C(I)$ the subspace of constant functions on I, and by $C(I)/\mathfrak{L}$ the corresponding factor space. Consider also the space $BV(I)$ consisting of continuous from the left functions, which have locally bounded variation. If $\eta(u) \in BV(I)$ then $\eta'(u) = d\eta(u)$ in the sense of distributions, where $d\eta(u)$ is a Radon measure of finite variation on I (the Stieltjes measure generated by $\eta(u)$), and $\operatorname{Var}\eta = \operatorname{Var} d\eta(I)$. Let us define the linear operator $T_\eta : C(I)/\mathfrak{L} \to C(I)/\mathfrak{L}$ such that, up to an additive constant (i.e. in the space $C(I)/\mathfrak{L}$)

$$T_\eta f(u) = f(u)\eta(u) - \int_{[-M,u)} f(s)d\eta(s) =$$
$$\int_{[-M,u)} (f(u) - f(s))d\eta(s) + f(u)\eta(-M). \tag{2.2}$$

From the representation $T_\eta f = \int_{[-M,u)}(f(u) - f(s))d\eta(s) + f(u)\eta(-M)$ it easily follows that the function $T_\eta f$ is continuous and $T_\eta f_1 - T_\eta f_2 = \text{const}$ whenever $f_1 - f_2 = \text{const}$. Thus, the operator T_η is well-defined on $C(I)/\mathfrak{L}$. It is easy to see that

$$\|T_\eta f(u)\|_\infty \le \|f(u)\|_\infty[|\eta(-M)| + \operatorname{Var}\eta],$$

which shows that the operator T_η is continuous. In the case $f(u) \in C^1(I)$, integrating by parts in (2.2), we obtain that $T_\eta f$ is uniquely defined in $C(I)/\mathfrak{L}$ by the equality

$$(T_\eta f)'(u) = \eta(u)f'(u) \ \text{ in } \mathcal{D}'((-M, M)). \tag{2.3}$$

If $\eta_1, \eta_2 \in BV(I)$ when the product $\eta_1\eta_2 \in BV(I)$ as well and

$$T_{\eta_1}T_{\eta_2} = T_{\eta_1\eta_2}. \tag{2.4}$$

Indeed, if $f \in C^1(I)$ when the equality $T_{\eta_1}T_{\eta_2}f = T_{\eta_1\eta_2}f$ directly follows from (2.3). Since $C^1(I)$ is dense in $C(I)$ and operators T_η are continuous we conclude that this equality holds for all $f \in C(I)$ (modulo $\mathfrak{L}$, of course) and identity (2.4) is satisfied.

Taking $\eta = \theta(u) = \begin{cases} -1 & , \quad u \le k, \\ 1 & , \quad u > k \end{cases}$ with $k \in I$ we derive that $T_\eta f = f_k(u) = \operatorname{sign}(u - k)(f(u) - f(k))$. The following Lemma shows that for general $\eta \in BV(I)$ the function $T_\eta f(u)$ is generated by the functions $f_k(u)$.

Lemma 2.2. *Let $\eta(u) \in BV(I)$. Then $\forall f(u) \in C(I)/\mathfrak{L}$*

$$T_\eta f(u) = \frac{1}{2}\int_{[-M,M)} \operatorname{sign}(u - k)(f(u) - f(k))d\eta(k) + Af(u), \tag{2.5}$$
$$A = \text{const} = (\eta(M) + \eta(-M))/2$$

on the segment $[-M, M]$, up to an additive constant.

Proof. Revealing the integral in the right-hand side of (2.5) and taking into account representation (2.2), we obtain

$$\int_{[-M,M)} \operatorname{sign}(u-k)(f(u)-f(k))d\eta(k) =$$

$$\int_{[-M,u)} (f(u)-f(k))d\eta(k) - \int_{[u,M)} (f(u)-f(k))d\eta(k) =$$

$$2\int_{[-M,u)} (f(u)-f(k))d\eta(k) - \int_{[-M,M)} (f(u)-f(k))d\eta(k) =$$

$$2\int_{[-M,u)} (f(u)-f(k))d\eta(k) - (\eta(M)-\eta(-M))f(u) + \text{const} =$$

$$2T_\eta f(u) - 2Af(u) + \text{const}$$

and (2.5) follows. □

Lemma 2.3. *Let $I = [-M, M]$, $\varphi(u) \in C(\mathbb{R})$, $\psi_k(u) = \operatorname{sign}(u-k)(\varphi(u)-\varphi(k))$, $v_k \in \mathbb{R}$, $k \in \mathbb{R}$. Then the set*

$$C = \{\ u \in I \mid \forall k \in \mathbb{R}\ \psi_k(u) = v_k\ \}$$

is closed and connected. In particular if $C \neq \emptyset$ then C is a segment $[a,b] \subset I$.

Proof. Since all functions $\psi_k(u)$ are continuous the set C is closed. To prove that this set is connected, take points $u_1, u_2 \in C$, $u_1 < u_2$. We have to show that $[u_1, u_2] \subset C$. Since for fixed $a \le -M$ $\varphi(u) = \psi_a(u) + \varphi(a)$ we see that $\varphi(u_1) = \varphi(u_2) = v = v_a + \varphi(a)$. Then $\forall k \in [u_1, u_2]$

$$v_k = \psi_k(u_1) = \varphi(k) - \varphi(u_1) = \psi_k(u_2) = \varphi(u_2) - \varphi(k),$$

which immediately implies that $\varphi(k) = v\ \forall k \in [u_1, u_2]$. Thus, $\varphi(u) \equiv v$ on $[u_1, u_2]$. From this it follows that $\psi_k(u) \equiv \psi_k(u_1) = v_k$ on the segment $[u_1, u_2]$ for all $k \in \mathbb{R}$. Hence, $[u_1, u_2] \subset C$ and the set C is connected. □

Now, we consider the equation

$$\varphi_0(u)_t + \operatorname{div}_y \varphi(u) = 0,\ u = u(t,y),\ (t,y) \in \Pi = \mathbb{R}_+ \times \mathbb{R}^n. \tag{2.6}$$

Here $\varphi_0(u) \in C(\mathbb{R})$, $\varphi(u) = (\varphi_1(u), \ldots, \varphi_n(u)) \in C(\mathbb{R}, \mathbb{R}^n)$.

We are going to prove that the statement of Theorem 1.4 is valid for isentropic solutions of (2.6), more precisely - for measure valued i.s.

Definition 2.2. A bounded measure valued function $\nu_{t,y}$ on the half-space Π is called a measure valued isentropic solution of (2.6) if for all $k \in \mathbb{R}$

$$\frac{\partial}{\partial t}\langle \psi_{0k}(\lambda), \nu_{t,y}(\lambda)\rangle + \operatorname{div}_y \langle \psi_k(\lambda), \nu_{t,y}(\lambda)\rangle = 0 \ \text{ in } \mathcal{D}'(\Pi), \tag{2.7}$$

where $\psi_{0k}(\lambda) = \operatorname{sign}(\lambda-k)(\varphi_0(\lambda)-\varphi_0(k))$, $\psi_k(\lambda) = \operatorname{sign}(\lambda-k)(\varphi(\lambda)-\varphi(k)) \in \mathbb{R}^n$ is the Kruzhkov entropy flux.

Obviously, a regular measure valued function $\nu_{t,y}(\lambda)=\delta(\lambda-u(t,y))$ is a measure valued i.s. of (2.6) if and only if the function $u(t,y)$ is a "usual" i.s. of this equation. Remark also that, as follows from (2.7) with $k>\|\nu_{t,y}\|_\infty$, any measure valued i.s. $\nu_{t,y}$ satisfies the equality

$$\frac{\partial}{\partial t}\langle\varphi_0(\lambda),\nu_{t,y}(\lambda)\rangle+\operatorname{div}_y\langle\varphi(\lambda),\nu_{t,y}(\lambda)\rangle=0 \ \text{ in } \mathcal{D}'(\Pi). \tag{2.8}$$

Proposition 2.1. *If $\nu_{t,y}\in \mathrm{MV}(\Pi)$ is a measure valued i.s. of (2.6), $\|\nu_{t,y}\|_\infty\le M$ then it is a measure valued i.s. of equations*

$$(T_\eta\varphi_0(u))_t+\operatorname{div}_y T_\eta\varphi(u)=0\quad \forall\eta(u)\in BV([-M,M]).$$

Proof. Let $I=[-M,M]$. By Lemma 2.2 $\forall\lambda\in I$

$$T_\eta\varphi_0(\lambda)=\frac{1}{2}\int_{[-M,M)}\psi_{0k}(\lambda)d\eta(k)+A\varphi_0(\lambda),$$

$$T_\eta\varphi(\lambda)=\frac{1}{2}\int_{[-M,M)}\psi_{k}(\lambda)d\eta(k)+A\varphi(\lambda),$$

where $A=$ const. Taking into account (2.7), (2.8), we see that in the sense of distributions

$$\begin{gathered}\langle T_\eta\varphi_0(\lambda),\nu_{t,y}(\lambda)\rangle_t+\operatorname{div}_y\langle T_\eta\varphi(\lambda),\nu_{t,y}(\lambda)\rangle=\\ \int_{[-M,M)}\big\{\langle\psi_{0k}(\lambda),\nu_{t,y}(\lambda)\rangle_t+\operatorname{div}_y\langle\psi_k(\lambda),\nu_{t,y}(\lambda)\rangle\big\}d\eta(k)+\\ A\big\{\langle\varphi_0(\lambda),\nu_{t,y}(\lambda)\rangle_t+\operatorname{div}_y\langle\varphi(\lambda),\nu_{t,y}(\lambda)\rangle\big\}=0\end{gathered} \tag{2.9}$$

for each $\eta(u)\in BV(I)$. Let $\eta(u)\in BV(I)$, $k\in\mathbb{R}$. Denote

$$\psi_0(\lambda)=T_\eta\varphi_0(\lambda),\quad \psi(\lambda)=T_\eta\varphi(\lambda)\in\mathbb{R}^n.$$

By identity (2.4), we have that, up to additive constants,

$$\operatorname{sign}(\lambda-k)(\psi_0(\lambda)-\psi_0(k))=T_{\theta\eta}\varphi_0(\lambda),\ \operatorname{sign}(\lambda-k)(\psi(\lambda)-\psi(k))=T_{\theta\eta}\varphi(\lambda),$$

where $\theta(u)=\begin{cases}-1 & , \quad u\le k,\\ 1 & , \quad u>k\end{cases}$. Therefore

$$\begin{aligned}\frac{\partial}{\partial t}\langle\operatorname{sign}(\lambda-k)(\psi_0(\lambda)-\psi_0(k)),\nu_{t,y}(\lambda)\rangle+\operatorname{div}_y\langle\operatorname{sign}(\lambda-k)(\psi(\lambda)-\psi(k)),\nu_{t,y}(\lambda)\rangle=\\ \langle T_{\theta\eta}\varphi_0(\lambda),\nu_{t,y}(\lambda)\rangle_t+\operatorname{div}_y\langle T_{\theta\eta}\varphi(\lambda),\nu_{t,y}(\lambda)\rangle=0\ \text{ in } \mathcal{D}'(\Pi)\end{aligned}$$

in view of (2.9). Since $k\in\mathbb{R}$ is arbitrary, $\nu_{t,y}$ is a measure valued i.s. of the equation

$$\psi_0(u)_t+\operatorname{div}_y\psi(u)=0,$$

as was to be proved. $\square$

No we can prove the following assertion on existence of strong traces for measure valued i.s.

Theorem 2.3. *Let $\nu_{t,y}$ be a measure valued i.s. of (2.6) such that for a.e. $(t,y)\in\Pi$ and all $k\in\mathbb{R}$ functions $\psi_{0k}(\lambda)\equiv v_k(t,y)$ on $\operatorname{supp}\nu_{t,y}$. Then there exist trace functions $v_{0k}(y)\in L^\infty(\mathbb{R}^n)$ such that*

$$\operatorname*{ess\,lim}_{t\to 0} v_k(t,\cdot)=v_{0k} \quad \text{in } L^1_{loc}(\mathbb{R}^n) \quad (\text{ strongly }).$$

Proof. We deduce from Lemma 2.3 that $\psi_{0k}(\lambda)\equiv v_k(t,y)$ also on the convex hull $\operatorname{Co}\operatorname{supp}\nu_{t,y}$ of $\operatorname{supp}\nu_{t,y}$. Define the set of full measure

$$E=\{\ t\in\mathbb{R}_+ \mid (t,y) \text{ is a Lebesgue point of the functions } \\ (t,y)\to\langle p(\lambda),\nu_{t,y}\rangle \ \text{ for a.e. } y\in\mathbb{R}^n \ \text{ and all } p(\lambda)\in C(\mathbb{R})\ \}.$$

From the fact that the space $C(\mathbb{R})$ is separable it easily follows that the set of common Lebesgue points of the functions $(t,y)\to\langle p(\lambda),\nu_{t,y}\rangle$ has full measure on Π, which directly implies that $E\subset\mathbb{R}_+$ is a set of full measure, as required. Taking $p(\lambda)=\varphi_0(\lambda),\psi_{0k}(\lambda)$, we conclude that for $t\in E$ for a.e. $y\in\mathbb{R}^n$ points (t,y) are Lebesgue points of functions $v(t,y)=\langle\varphi_0(\lambda),\nu_{t,y}(\lambda)\rangle$, $v_k(t,y)=\langle\psi_{0k}(\lambda),\nu_{t,y}(\lambda)\rangle$, $k\in\mathbb{R}$. From the relation

$$v_t+\operatorname{div}_y w=0 \ \text{ in } \mathcal{D}'(\Pi), \quad \text{where } w=w(t,y)=\langle\varphi(\lambda),\nu_{t,y}(\lambda)\rangle\in\mathbb{R}^n$$

it follows, in correspondence with Proposition 1.1, that the function $v(t,y)$ have the weak trace $v_0(y)\in L^\infty(\mathbb{R}^n)$, namely $v(t,\cdot)\to v_0$ weakly-$*$ in $L^\infty(\mathbb{R}^n)$ as $t\to 0$, $t\in E$.

Obviously, for each $\tau\in E$ the measure valued function $\nu_y^\tau=\nu_{\tau,y}\in\mathrm{MV}(\mathbb{R}^n)$ is well-defined, due to the relation

$$\langle p(\lambda),\nu_y^\tau(\lambda)\rangle=\langle p(\lambda),\nu_{\tau,y}(\lambda)\rangle$$

on the set of full measure consisting of $y\in\mathbb{R}^n$ such that (τ,y) is a common Lebesgue point of all functions $(t,y)\to\langle p(\lambda),\nu_{t,y}\rangle$. Moreover, $\|\nu_y^\tau\|_\infty\le M=\|\nu_{t,y}\|_\infty$. By Theorem 2.2 we can choose a sequence $t_m\in E$, $m\in\mathbb{N}$ such that $t_m\to 0$ and the bounded sequence of measure valued functions $\nu_y^m=\nu_{t_m,y}$ converges weakly as $m\to\infty$ to some measure valued function $\nu_{0,y}\in\mathrm{MV}(\mathbb{R}^n)$.

We have to prove that the trace v_0 is strong, i.e. $v(t,\cdot)\to v_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t\to 0$, $t\in E$. We firstly establish that $v(t_m,\cdot)\to v_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $m\to\infty$.

To do this, we apply the measure valued variant of the Kruzhkov method of doubling variables (like in [15], Theorem 2.3).

By relations (2.7) for all $\mu\in\mathbb{R}$

$$\frac{\partial}{\partial t}[\operatorname{sign}(v-\mu)(\varphi_0(v)-\varphi_0(\mu))]+\operatorname{div}_y\int\operatorname{sign}(\lambda-\mu)(\varphi(\lambda)-\varphi(\mu))d\nu_{t,y}(\lambda)=0 \ \text{ in } \mathcal{D}'(\Pi).$$

Integrating this relation applied to a test function $g = g(t, y; s, z) \in C_0^1(\Pi \times \Pi)$ firstly over the measure $\nu_{s,z}(\mu)$ and then over (s, z), we readily obtain that

$$\frac{\partial}{\partial t}[\operatorname{sign}(v(t,y) - v(s,z))(\varphi_0(v(t,y)) - \varphi_0(v(s,z)))] +$$
$$\operatorname{div}_y \iint \operatorname{sign}(\lambda - \mu)(\varphi(\lambda) - \varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{s,z}(\mu) = 0 \ \text{ in } \mathcal{D}'(\Pi \times \Pi). \quad (2.10)$$

Here we take into account that

$$\operatorname{sign}(v - \mu)(\varphi_0(v) - \varphi_0(\mu)) \equiv \operatorname{sign}(v - v(s,z))(\varphi_0(v) - \varphi_0(v(s,z)))$$

on $\operatorname{supp} \nu_{s,z}(\mu)$ for a.e. $(s, z) \in \Pi$.

Analogously, changing the places of the variables λ and μ, (t, y) and (s, z), we find

$$\frac{\partial}{\partial s}[\operatorname{sign}(v(t,y) - v(s,z))(\varphi_0(v(t,y)) - \varphi_0(v(s,z)))] +$$
$$\operatorname{div}_z \iint \operatorname{sign}(\lambda - \mu)(\varphi(\lambda) - \varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{s,z}(\mu) = 0 \ \text{ in } \mathcal{D}'(\Pi \times \Pi). \quad (2.11)$$

Combining (2.10) and (2.11) we arrive at the relation: in $\mathcal{D}'(\Pi \times \Pi)$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)[\operatorname{sign}(v(t,y) - v(s,z))(\varphi_0(v(t,y)) - \varphi_0(v(s,z)))] +$$
$$(\operatorname{div}_y + \operatorname{div}_z) \iint \operatorname{sign}(\lambda - \mu)(\varphi(\lambda) - \varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{s,z}(\mu) = 0. \quad (2.12)$$

Now we choose a function $\beta(t) \in C_0^1(\mathbb{R})$ such that $\operatorname{supp} \beta \subset [0, 1]$, $\beta(t) \geq 0$, $\int \beta(t)dt = 1$ and set, as in (1.20), for $\nu \in \mathbb{N}$ $\delta_\nu(t) = \nu\beta(\nu t)$, $\theta_\nu(t) = \int_0^t \delta_\nu(s)ds$, so that $\theta'_\nu(t) = \delta_\nu(t)$. Recall that the sequence $\delta_\nu(t)$ converges as $\nu \to \infty$ to the Dirac δ-measure in $\mathcal{D}'(\mathbb{R})$ while the sequence $\theta_\nu(t)$ converges pointwise to the Heaviside function. Let $f(t, y, z) \in C_0^1(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n)$, $g(t, y; s, z) = \delta_\nu(s - t)f(t, y, z)$, $\nu \in \mathbb{N}$. It is clear that $g(t, y; s, z) \in C_0^1(\Pi \times \Pi)$. Applying relation (2.12) to this test function and making simple transformations, we find

$$\int_{\Pi\times\Pi} \Bigg[\operatorname{sign}(v(t,y) - v(s,z))(\varphi_0(v(t,y)) - \varphi_0(v(s,z)))f_t +$$
$$\left(\iint \operatorname{sign}(\lambda - \mu)(\varphi(\lambda) - \varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{s,z}(\mu), (\nabla_y + \nabla_z)f\right)\Bigg] dydz \times$$
$$\delta_\nu(s - t)dsdt = 0$$

Passing in this relation to the limit as $\nu \to \infty$ we derive

$$\int_{\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n} \Big[\operatorname{sign}(v(t,y)-v(t,z))(\varphi_0(v(t,y))-\varphi_0(v(t,z)))f_t + \Big(\iint \operatorname{sign}(\lambda-\mu)(\varphi(\lambda)-\varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{t,z}(\mu), (\nabla_y+\nabla_z)f\Big)\Big]\, dydzdt=0 \quad (2.13)$$

for each $f(t,y,z) \in C_0^1(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n)$, i.e. in $\mathcal{D}'(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n)$

$$\frac{\partial}{\partial t}[\operatorname{sign}(v(t,y)-v(t,z))(\varphi_0(v(t,y))-\varphi_0(v(t,z)))] + (\operatorname{div}_y+\operatorname{div}_z)\iint \operatorname{sign}(\lambda-\mu)(\varphi(\lambda)-\varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{t,z}(\mu)=0.$$

Let $h(y,z) \in C_0^1(\mathbb{R}^n\times\mathbb{R}^n)$, $\tau \in E$, and $f(t,y,z) = \chi_\nu(t)h(y,z)$, where $\chi_\nu(t) = \theta_\nu(t-t_m)-\theta_\nu(t-\tau)$, $\nu, m \in \mathbb{N}$, $t_m < \tau$.

Taking in (2.13) the test function f, we obtain the relation

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^n\times\mathbb{R}^n} \operatorname{sign}(v(t,y)-v(t,z))(\varphi_0(v(t,y))-\varphi_0(v(t,z)))h(y,z)dydz \times (\delta_\nu(t-t_m)-\delta_\nu(t-\tau))dt + \int_{\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n}\Big(\iint \operatorname{sign}(\lambda-\mu)\times (\varphi(\lambda)-\varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{t,z}(\mu), (\nabla_y+\nabla_z)h\Big)\chi_\nu(t)dydzdt=0$$

Passing in this relation to the limit as $\nu \to \infty$ and taking into account that $t \in E$ are Lebesgue points of the function $t \to \int_{\mathbb{R}^n\times\mathbb{R}^n} \operatorname{sign}(v(t,y)-v(t,z))(\varphi_0(v(t,y)) - \varphi_0(v(t,z)))h(y,z)dydz$ and that $\chi_\nu(t)$ converges point-wise to the indicator function of $(t_m,\tau]$, we obtain

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \operatorname{sign}(v(t_m,y)-v(t_m,z))(\varphi_0(v(t_m,y))-\varphi_0(v(t_m,z)))h(y,z)dydz - \int_{\mathbb{R}^n\times\mathbb{R}^n} \operatorname{sign}(v(\tau,y)-v(\tau,z))(\varphi_0(v(\tau,y))-\varphi_0(v(\tau,x)))h(y,z)dydz + \int_{(t_m,\tau]\times\mathbb{R}^n\times\mathbb{R}^n}\Big(\iint \operatorname{sign}(\lambda-\mu)(\varphi(\lambda)-\varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{t,z}(\mu), (\nabla_y+\nabla_z)h\Big)\,dydzdt=0. \quad (2.14)$$

By Lemma 2.1

$$\operatorname{sign}(v(t_m,y)-v(t_m,z))(\varphi_0(v(t_m,y))-\varphi_0(v(t_m,z)))=$$
$$\iint \operatorname{sign}(\lambda-\mu)(\varphi_0(\lambda)-\varphi_0(\mu))d\nu_y^m(\lambda)d\nu_z^m(\mu)$$
$$\underset{m\to\infty}{\to}\iint \operatorname{sign}(\lambda-\mu)(\varphi_0(\lambda)-\varphi_0(\mu))d\nu_{0y}(\lambda)d\nu_{0z}(\mu)$$

weakly-$*$ in $L^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ and from (2.14) it follows, in the limit as $m\to\infty$, that

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(\iint \operatorname{sign}(\lambda-\mu)(\varphi_0(\lambda)-\varphi_0(\mu))d\nu_{0y}(\lambda)d\nu_{0z}(\mu)\right)h(y,z)dydz\\
&=\int_{\mathbb{R}^n\times\mathbb{R}^n}\operatorname{sign}(v(\tau,y)-v(\tau,z))(\varphi_0(v(\tau,y))-\varphi_0(v(\tau,z)))h(y,z)dydz\\
&\quad-\int_{(0,\tau]\times\mathbb{R}^n\times\mathbb{R}^n}\left(\iint \operatorname{sign}(\lambda-\mu)(\varphi(\lambda)-\varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{t,z}(\mu),\right.\\
&\qquad\qquad\left.(\nabla_y+\nabla_z)h\right)dydzdt. \qquad (2.15)
\end{aligned}$$

Now we choose the function h in the form $h(y,z)=\rho(y)\bar{\delta}_\nu(y-z)$, with $\bar{\delta}_\nu(y-z)=\prod_{i=1}^n\delta_\nu(z_i-y_i)$, $\rho(y)\in C_0^1(\mathbb{R}^n)$. Taking this function in (2.15), we find

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}^n}\left(\iint \operatorname{sign}(\lambda-\mu)(\varphi_0(\lambda)-\varphi_0(\mu))d\nu_{0y}(\lambda)d\nu_{0z}(\mu)\right)\rho(y)\bar{\delta}_\nu(y-z)dydz=\\
&\int_{\mathbb{R}^n\times\mathbb{R}^n}\operatorname{sign}(v(\tau,y)-v(\tau,z))(\varphi_0(v(\tau,y))-\varphi_0(v(\tau,z)))\rho(y)\bar{\delta}_\nu(y-z)dydz-\\
&\int_{(0,\tau]\times\mathbb{R}^n\times\mathbb{R}^n}\left(\iint \operatorname{sign}(\lambda-\mu)(\varphi(\lambda)-\varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{t,z}(\mu),\nabla_y\rho\right)\bar{\delta}_\nu(y-z)dydzdt.
\end{aligned}$$

Passing to the limit as $\nu\to\infty$, we derive that for all $\tau\in E$

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\iint \operatorname{sign}(\lambda-\mu)(\varphi_0(\lambda)-\varphi_0(\mu))d\nu_{0y}(\lambda)d\nu_{0y}(\mu)\right)\rho(y)dy=\\
&-\int_{(0,\tau]\times\mathbb{R}^n}\left(\iint \operatorname{sign}(\lambda-\mu)(\varphi(\lambda)-\varphi(\mu))d\nu_{t,y}(\lambda)d\nu_{t,y}(\mu),\nabla_y\rho\right)dydt
\end{aligned}$$

and in the limit at $\tau\to 0$ this implies that

$$\int_{\mathbb{R}^n}\left(\iint \operatorname{sign}(\lambda-\mu)(\varphi_0(\lambda)-\varphi_0(\mu))d\nu_{0y}(\lambda)d\nu_{0y}(\mu)\right)\rho(y)dy=0.$$

As $\rho(y)\in C_0^1(\mathbb{R}^n)$ is arbitrary, we conclude that for a.e. $y\in\mathbb{R}^n$

$$\iint \operatorname{sign}(\lambda-\mu)(\varphi_0(\lambda)-\varphi_0(\mu))d\nu_{0y}(\lambda)d\nu_{0y}(\mu)=0. \qquad (2.16)$$

In view of Proposition 2.1 one could replace equation (2.6) by the equation

$$(T_\eta\varphi_0(u))_t + \mathrm{div}_y T_\eta\varphi(u) = 0, \quad \eta(u) \in BV([-M,M])$$

and relation (2.16) yields

$$\iint \mathrm{sign}\,(\lambda-\mu)(T_\eta\varphi_0(\lambda) - T_\eta\varphi_0(\mu))d\nu_{0y}(\lambda)d\nu_{0y}(\mu) = 0 \ \text{ for a.e. } y \in \mathbb{R}^n. \quad (2.17)$$

Since the space $C(I\times I)$ is separable and integrands in (2.17) generate a subspace of $C(I\times I)$, we can choose a set of full measure $D\subset\mathbb{R}^n$ such that for $y\in D$ $\mathrm{supp}\,\nu_{0y}\subset[-M,M]$ and (2.17) is satisfied for all $\eta(u)\in BV(I)$. Fix $y\in D$ and denote $\nu=\nu_{0y}$, $[a,b]=\mathrm{Co}\,\mathrm{supp}\,\nu$. We are going to show that $\varphi_0(\lambda)\equiv\mathrm{const}$ on $[a,b]$. Assuming the contrary, we can find a point $c\in(a,b)$ such that $\varphi_0(\lambda)\not\equiv\mathrm{const}$ on any segment $[c,d]$, $c<d<b$ (otherwise, $\varphi_0(I)$ is at most countable and therefore $\varphi_0(\lambda)\equiv\mathrm{const}$ on I). Hence, there exists a sequence c_r, $r\in\mathbb{N}$ such that $c<c_{r+1}<c_r<b$ $\forall r\in\mathbb{N}$, $c_r\to c$ as $r\to\infty$, and

$$|\varphi_0(c_r)-\varphi_0(c)| = \max_{u\in[c,c_r]}|\varphi_0(u)-\varphi_0(c)| > 0. \quad (2.18)$$

Denote $h_r=\varphi_0(c_r)-\varphi_0(c)$ and set $\eta_r(u)=\dfrac{1}{h_r}\chi_{(c,c_r]}(u)$, where $\chi_{(c,c_r]}(u)$ is the indicator function of $(c,c_r]$. It is clear that $\eta_r\in BV(I)$ and

$$\psi_r(u) = T_{\eta_r}\varphi_0(u) = \begin{cases} 0 & , \quad u\le c, \\ (\varphi_0(u)-\varphi_0(c))/h_r & , \quad c<u\le c_r, \\ 1 & , \quad u>c_r. \end{cases} \quad (2.19)$$

Observe that, up to an additive constant, $\psi_r(u)=(\psi_{0c}(u)-\psi_{0c_r}(u))/(2h_r)$. As follows from (2.18), $|\psi_r(u)|\le 1$ and obviously the sequence ψ_r converges point-wise to the Heaviside function $\psi(u)=\theta(u-c)$. Taking in (2.17) $\eta=\eta_r$ and passing to the limit as $r\to\infty$, we find

$$0 = \iint \mathrm{sign}\,(\lambda-\mu)(\psi(\lambda)-\psi(\mu))d\nu(\lambda)d\nu(\mu) =$$

$$\iint_{\lambda\le c<\mu} d\nu(\lambda)d\nu(\mu) + \iint_{\mu\le c<\lambda} d\nu(\lambda)d\nu(\mu) = 2\nu([a,c])\nu((c,b]).$$

But, $[a,b]$ is the smallest segment containing $\mathrm{supp}\,\nu$, which implies that $\nu([a,c])\nu((c,b])>0$. The obtained contradiction shows that $\varphi_0(\lambda)\equiv C=\mathrm{const}$ on $[a,b]$. Thus, for a.e. $y\in\mathbb{R}^n$ $\varphi_0^*\nu_{0y}(\lambda)$ is the Dirac measure (as is easy to see, it is the Dirac measure at the point $v_0(y)$) and by Corollary 2.1, $v(t_m,\cdot)\underset{m\to\infty}{\to} v_0$ in $L^1_{loc}(\mathbb{R}^n)$, as was to be proved. By Proposition 2.1 we can replace $\varphi_0(u)$ by $\psi_{0k}(u)$ and thus conclude that the functions $\psi_{0k}(u(t_m,\cdot))\to v_{0k}$ in $L^1_{loc}(\mathbb{R}^n)$ as $m\to\infty$. Finally, since the limit functions v_{0k} do not depend on the choice of the sequence $t_m\in E$ with the prescribed above properties, this easily implies that $\psi_{0k}(u(t,\cdot))\to v_{0k}$ in $L^1_{loc}(\mathbb{R}^n)$ as $t\to 0$, $t\in E$. The prove is complete.

$\square$

2.3 One criterion of existence of the strong traces

As was shown in Proposition 1.2, there exist weak traces $v_{0k}(y) \in L^\infty(\mathbb{R}^n)$ of $\psi_{0k}(y, u(t, y)) = \operatorname{sign}(u - k)(\varphi_0(y, u) - \varphi_0(y, k))$, where $u = u(t, y)$ is a quasi-s. of (1.14). In the following Theorem we give a simple criterion for v_{0k} to be the strong traces. We formulate this theorem even for the following more general situation.

Theorem 2.4. *Let V be a bounded open set in $\mathbb{R}^n$, $W = (0, h) \times V \subset \Pi$, $0 < h < +\infty$, and $u = u(t, y) \in L^\infty(W)$, $M \geq \|u\|_\infty$. Suppose $\varphi_0(t, y, u) \in C(\operatorname{Cl} W \times \mathbb{R})$ and the functions $\psi_{0k}(t, y, u) = \operatorname{sign}(u - k)(\varphi_0(t, y, u) - \varphi_0(t, y, k))$ have the weak traces $v_{0k}(y)$ at $t = 0$ for each $k \in \mathbb{R}$. Then the traces v_{0k} are strong if and only if there exists a bounded function $u_0(y)$ such that $v_{0k}(y) = \psi_{0k}(0, y, u_0(y))$ a.e. on V. Moreover, if the function $u_0(y)$ exists then among such functions, satisfying the additional requirement $|u_0| \leq M$, there are unique minimal u_0^- and maximal u_0^+, and $u_0^\pm$ are measurable, i.e. $u_0^\pm \in L^\infty(V)$.*

If, in addition, for a.e. $y \in V$ the function $u \to \varphi(0, y, u)$ is not constant on non-degenerate intervals and $v_{0k}(y) = \psi_{0k}(0, y, u_0(y))$ a.e. on V then $u_0^- = u_0^+ = u_0$ (independently on M) and $u_0(y)$ is the strong trace of $u(t, y)$ at $t = 0$.

Proof. Suppose traces v_{0k} are strong. We define the set

$$E = \{\ t \in (0, h) \mid (t, y) \text{ is a Lebesgue point of } u(t, y) \text{ for a.e. } y \in V\ \} \tag{2.20}$$

similarly to (1.21). Then $\psi_{0k}(t, y, u(t, y)) \to v_{0k}(y)$ in $L^1(V)$ as $t \to 0$, $t \in E$. Taking into account also that the functions $\psi_{0k}(t, y, u)$ are continuous and therefore $\psi_{0k}(t, y, u(t, y)) - \psi_{0k}(0, y, u(t, y)) \underset{t \to 0}{\to} 0$ in $L^1(V)$, we see that

$$\psi_{0k}(0, y, u(t, y)) \to v_{0k}(y) \ \text{ in } L^1(V) \text{ as } t \to 0,\ t \in E. \tag{2.21}$$

In view of Theorem 2.1 we can choose a sequence $t_m \in E$, $t_m \to 0$ such that the sequence $u(t_m, y)$ converges weakly to some measure valued function $\nu_y \in \mathrm{MV}(V)$, and $\|\nu_y\|_\infty \leq \|u\|_\infty$. Taking $M \geq \|u\|_\infty$, we can suppose, without loss of generality, that $\operatorname{supp} \nu_y \subset I = [-M, M]$ for all $y \in V$. By Corollary 2.1 from strong convergence (2.21) it follows that for all $k \in \mathbb{R}$ $\psi_{0k}(0, y, \lambda) \equiv v_{0k}(y)$ on $\operatorname{supp} \nu_y$ for a.e. $y \in V$. Evidently, the set of full measure Y consisting of such y can be chosen common for all $k \in \mathbb{Q}$ and since the maps $k \to \psi_{0k}(0, y, \cdot) \in C(I)$, $k \to v_{0k}(y)$ are continuous, and the set $\mathbb{Q}$ is dense, we conclude that for $y \in Y$ $\psi_{0k}(0, y, \lambda) \equiv v_{0k}(y)$ on $\operatorname{supp} \nu_y$ for all $k \in \mathbb{R}$. By Lemma 2.3 the set C_y of $\lambda \in I$ such that for each $k \in \mathbb{R}$ $\psi_{0k}(0, y, \lambda) = v_{0k}(y)$ is a segment $[a(y), b(y)]$, which is nonempty due to the condition $\operatorname{supp} \nu_y \subset C_y$.

Taking a function $u_0(y) \in C_y$ we obtain the desired relation $\psi_{0k}(0, y, u_0(y)) = v_{0k}(y)$ for all $k \in \mathbb{R}$. Clearly, the functions

$$u_0^-(y) = a(y) = \min C_y, \quad u_0^+(y) = b(y) = \max C_y$$

are the minimal and respectively the maximal among such functions $u_0(y)$. Let us show that $u_0^\pm(x)$ are measurable on V. This follows from evident relations: for all $\lambda \in \mathbb{R}$

$$\{ \ y \in V \ | \ u_0^-(y) \ge \lambda \ \} = \bigcap_{\mu \in (-M,\lambda)\cap\mathbb{Q}} \ \bigcup_{k\in\mathbb{Q}} \{ \ y \in V \ | \ \psi_{0k}(0,y,\mu) \ne v_{0k}(y) \ \},$$

$$\{ \ y \in V \ | \ u_0^+(y) \le \lambda \ \} = \bigcap_{\mu \in (\lambda,M)\cap\mathbb{Q}} \ \bigcup_{k\in\mathbb{Q}} \{ \ y \in V \ | \ \psi_{0k}(0,y,\mu) \ne v_{0k}(y) \ \}$$

and measurability of $v_{0k}(y)$. Hence, $u_0^\pm \in L^\infty(V)$, $\|u_0^\pm\|_\infty \le M$.

Conversely, suppose that there exists a bounded function $u_0(y)$ such that $v_{0k}(y) = \psi_{0k}(0,y,u_0(y))$ a.e. on V for each $k \in \mathbb{R}$. We take $M \ge \max\{\|u\|_\infty, \sup|u_0(y)|\}$ and set $I = [-M,M]$. Since maps $k \to \psi_{0k}(0,y,\cdot) \in C(I)$, $k \to v_{0k}(y)$ are continuous, there is a set $Y \subset V$ of full measure such that $v_{0k}(y) = \psi_{0k}(0,y,u_0(y))$ for all $k \in \mathbb{R}$ if $y \in Y$ (see the arguments for the set Y in the first part of the proof). We should prove that for all $k \in \mathbb{R}$ $\psi_{0k}(t,y,u(t,y)) \to v_{0k}(y) = \psi_{0k}(0,y,u_0(y))$ in $L^1(V)$ as $t \to 0$, $t \in E$, where the set $E \subset (0,h)$ of full measure is defined by (2.20). As was shown above, this is equivalent to the convergence

$$\psi_{0k}(0,y,u(t,y)) \underset{t\to 0, t\in E}{\to} \psi_{0k}(0,y,u(t,y)) \ \text{ in } L^1(V) \ \forall k \in \mathbb{R}. \tag{2.22}$$

Assuming that (2.22) fails, we can find $k \in \mathbb{R}$ and a sequence $t_m \in E$, $t_m \to 0$, such that for some $\delta > 0$

$$\liminf_{m\to\infty} \int_V |\psi_{0k}(0,y,u(t_m,y)) - v_{0k}(y)|dy > \delta. \tag{2.23}$$

Without loss of generality we can suppose that $u(t_m,y)$ converges as $m \to \infty$ to some measure valued function ν_y. Clearly, $\|\nu_y\| \le M$, and by the weak-$*$ convergence $\psi_{0k}(0,y,u(t_m,y)) \underset{m\to\infty}{\to} v_{0k}(y)$ in $L^\infty(V)$ we have the relations

$$\psi_{0k}(0,y,u_0(y)) = v_{0k}(y) = \langle \psi_{0k}(0,y,\lambda), \nu_y(\lambda)\rangle = \int \psi_{0k}(0,y,\lambda)d\nu_y(\lambda) \ \forall k \in \mathbb{R}. \tag{2.24}$$

Clearly, the set of full measure $Y_1 \subset Y$ of values y, for which this relation holds, can be chosen common for all $k \in \mathbb{R}$. This easily follows from the fact that both sides of (2.24) depends continuously on k. Taking in (2.24) $k > M$, we derive

$$\varphi_0(0,y,u_0(y)) = v_0(y) = \langle \varphi_0(0,y,\lambda), \nu_y(\lambda)\rangle = \int \varphi_0(0,y,\lambda)d\nu_y(\lambda). \tag{2.25}$$

Fix $y \in Y_1$ and denote $\varphi_0(u) = \varphi_0(0,y,u)$, $u_0 = u_0(y)$, $\nu = \nu_y$. Let $\eta(u) \in BV(I)$. Integrating relation (2.24) over the measure $d\eta(k)$ and using Lemma 2.2 with $f(u) = \varphi_0(u)$, we find that for all $\eta \in BV(I)$

$$\langle T_\eta \varphi_0(\lambda), \nu(\lambda)\rangle = T_\eta \varphi_0(u_0). \tag{2.26}$$

Here we also take into account relation (2.25). Now, we are going to derive from (2.26) that $\varphi_0(u) \equiv \text{const}$ on $[a,b] = \operatorname{Co}\operatorname{supp}\nu$. Assuming the contrary, we can choose a value $c \in (a,b)$ and sequences c_r, h_r, $r \in \mathbb{N}$ such that the sequence $\psi_r = T_{\eta_r}\varphi_0(u)$ defined by (2.19) is bounded, and converges point-wise as $r \to \infty$ to the Heaviside function $\theta(u-c)$. Taking in (2.26) $\eta = \eta_r$ and passing to the limit as $r \to \infty$ we find that $\nu((c,b]) = \theta(u_0 - c) \in \{0,1\}$, which contradicts to the fact that $[a,b]$ is the minimal segment containing $\operatorname{supp}\nu$ and therefore $0 < \nu((c,b]) < 1$. Thus, $\varphi_0(u) = \varphi_0(u_0)$ for all $u \in [a,b]$. From the obtained relation it follows that $\psi_{0k}(u) \equiv \psi_{0k}(u_0)$ on $[a,b]$. Hence for a.e. $y \in V$ $\psi_{0k}(0,y,u) = \psi_{0k}(0,y,u_0(y)) = v_{0k}(y)$ on $\operatorname{supp}\nu_y$ and by Corollary 2.1 for all $k \in \mathbb{R}$ $\psi_{0k}(0,y,u(t_m,y))$ converges as $m \to \infty$ to $v_{0k}(y)$ in $L^1(V)$. But this contradicts (2.23). Therefore, (2.22) is satisfied, as was to be proved.

If, in addition, the function $\varphi_0(0,y,\cdot)$ is not constant on non-degenerate intervals for a.e. $y \in V$ then we take a sequence $t_m \in E$, $t_m \to 0$ such that $u(t_m,y)$ converges weakly to a measure valued function ν_y. Repeating the above arguments we find that for a.e. $y \in V$ $\operatorname{supp}\nu_y \subset [u^-(y), u^+(y)]$, $u_0(y) \in [u_0^-(y), u_0^+(y)]$, and $\varphi_0(0,y,u) \equiv \varphi_0(0,y,u_0(y))$ on $[u_0^-(y), u_0^+(y)]$. By our assumption the functions $\varphi_0(0,y,\cdot)$ are not constant on non-degenerate intervals for a.e. fixed $y \in V$. Therefore, for a.e. $y \in V$ $u_0(y) = u_0^-(y) = u_0^+(y)$, and $\nu_y(\lambda) = \delta(\lambda - u_0(y))$ is a regular measure valued function. By Theorem 2.1 the sequence $u(t_m,y) \to u_0(y)$ in $L^1(V)$ as $m \to \infty$ and since the limit function $u_0(y)$ does not depend on the choice of a vanishing sequence $t_m \in E$, we conclude that $u(t,\cdot) \to u_0$ in $L^1(V)$ as $t \to 0$, $t \in E$. The proof is complete. ☐

Corollary 2.2. *Suppose $u(t,y) \in L^\infty(\Pi)$ is a quasi-s. of (1.14), $E \subset \mathbb{R}_+$ is defined by (1.21), and there is a sequence $t_m \in E$ such that $t_m \underset{m\to\infty}{\to} 0$, and $\psi_{0k}(\cdot, u(t_m,\cdot)) \underset{m\to\infty}{\to} v_{0k}$ in $L^1_{loc}(\mathbb{R}^n)$. Then $\psi_{0k}(\cdot, u(t,\cdot)) \to v_{0k}$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0$, $t \in E$.*

Proof. Extracting a subsequence if necessary, we can assume that $u(t_m,y)$ converges weakly to a measure valued function $\nu_y \in \mathrm{MV}(\mathbb{R}^n)$. In view of strong convergence $\psi_{0k}(0,\cdot,u(t_m,\cdot)) \to v_{0k}$ from Corollary 2.1 it follows that for all $k \in \mathbb{R}$ $\psi_{0k}(0,y,\lambda) \equiv v_{0k}(y)$ on $\operatorname{supp}\nu_y$ for all $y \in Y$, where $Y \subset \mathbb{R}^n$ is a set of full measure, which does not depend on k (see the proof of Theorem 2.4). Thus, $\psi_{0k}(0,y,u_0(y)) = v_{0k}(y)$ for each $y \in Y$, where $u_0(y) \in \operatorname{supp}\nu_y$. By Theorem 2.4 we conclude that $v_{0k}(y)$ are the strong traces for $\psi_{0k}(t,y,u(t,y))$ for all $k \in \mathbb{R}$, as was to be proved. ☐

3 The "blow-up" procedure

Let $u(t,y) \in L^\infty(\Pi)$ be a quasi-s. of (1.14). We consider a sequence $\varepsilon_m > 0$, $m \in \mathbb{N}$ such that $\varepsilon_m \to 0$ as $m \to \infty$ and set

$$u^m = u^m(t,y;z) = u(\varepsilon_m t, \varepsilon_m y + z),$$

where $z \in \mathbb{R}^n$ is treated as a fixed parameter. By Proposition 1.2 there exists the weak traces $v_{0k}(y) \in L^\infty(\Pi)$ such that $\psi_{0k}(y, u(t,y)) \to v_{0k}(y)$ weakly-$*$ in $L^\infty(\mathbb{R}^n)$ as $t \to 0$, $t \in E$. We set also $v_{0k}^m(y;z) = v_{0k}(\varepsilon_m y + z)$.

The following lemma was essentially proved in [26] (Lemma 3) and in [19] (Lemma 3.1).

Lemma 3.1. *There exists a subsequence of ε_m (we keep the notation ε_m) such that for a.e. $z \in \mathbb{R}^n$ and all $k \in \mathbb{R}$ $v_{0k}^m(y;z) \to v_{0k}(z) = \mathrm{const}$ in $L^1_{loc}(\mathbb{R}^n)$ as $m \to \infty$.*

For completeness we give the proof.

Proof. Let $\rho(y) = e^{-|y|}$,

$$I_k^m(z) = \int_{\mathbb{R}^n} |v_{0k}(\varepsilon_m y + z) - v_{0k}(z)|\rho(y)dy.$$

We integrate this equality over z. Changing the order of integrating we obtain that

$$\int_{\mathbb{R}^n} I_k^m(z)\rho(z)dz = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |v_{0k}(\varepsilon_m y + z) - v_{0k}(z)|\rho(z)dz \right) \rho(y)dy. \tag{3.1}$$

By continuity of the average, for all $y \in \mathbb{R}^n$

$$J_k^m(y) = \int_{\mathbb{R}^n} |v_{0k}(\varepsilon_m y + z) - v_{0k}(z)|\rho(z)dz \underset{m\to\infty}{\to} 0,$$

and also $\|J_k^m(y)\|_\infty \le \mathrm{const}$. Using the Lebesgue theorem on dominated convergence, we conclude that the right-hand side of (3.1) converges to zero. Then, by (3.1)

$$\int_{\mathbb{R}^n} I_k^m(z)\rho(z)dz \underset{m\to\infty}{\to} 0.$$

Therefore, after possible extraction of a subsequence (we keep the previous notation for it), $I_k^m(z) \to 0$ as $m \to \infty$ for a.e. z and all $k \in \mathbb{Q}$. This means that for such values of y $v_{0k}^m(y;z) \to v_{0k}(z)$ in $L^1_{loc}(\mathbb{R}^n)$. Since the above functions depends on k continuously and $\mathbb{Q}$ is dense, we see that this limit relation is satisfied for all $k \in \mathbb{R}$. The proof is complete. $\square$

Evidently, $u^m(t,y;z)$ satisfies equalities like (1.18), namely in $\mathcal{D}'(\Pi)$

$$\psi_{0k}(\varepsilon_m y + z, u^m)_t + \mathrm{div}_y \psi_k(u^m) = -S_z^m(\gamma_k) = -\varepsilon_m \gamma_k(\varepsilon_m t, \varepsilon_m y + z). \tag{3.2}$$

The operator S_z^m is defined on the space $\bar{M}_{loc}(\Pi)$ by the equality $S_z^m(\gamma) = \varepsilon_m \gamma(\varepsilon_m t, \varepsilon_m y + z)$ understood in the sense of distributions. This means that $\forall h(t,y) \in C_0(\Pi)$

$$\langle S_z^m(\gamma), h \rangle = (\varepsilon_m)^{-n} \int_\Pi h(t/\varepsilon_m, (y-z)/\varepsilon_m) d\gamma(t,y).$$

The following lemma can be proved in just the same way as Lemma 2 in [26] (see also Lemma 3.2 in [19]).

Lemma 3.2. *If $\gamma \in \bar{M}_{loc}(\Pi)$ then, after possible extraction of a subsequence, for a.e. $z \in \mathbb{R}^n$ $S_z^m(\gamma) \underset{m\to\infty}{\to} 0$ in $\bar{M}_{loc}(\Pi)$.*

Proof. Firstly, observe that $\operatorname{Var} S_z^m(\gamma) = S_z^m(\operatorname{Var}\gamma)$. Therefore, without loss of generality we may assume that $\gamma \geq 0$. Let $r > 0$. Denote by B_r the ball $\{y \mid |y| \leq r\}$, and by $\chi(t, y)$ the indicator function of the cylinder $(0, r] \times B_r$. We set

$$I^m(z) = S_z^m(\gamma)((0,r] \times B_r) = (\varepsilon_m)^{-n} \int_\Pi \chi(t/\varepsilon_m, (y-z)/\varepsilon_m) d\gamma(t, y).$$

Now we integrate this equality over $z \in B_R$, where $R > 0$. Changing the order of integrating and making the change $x = (y - z)/\varepsilon_m$ we derive that

$$\int_{B_R} I^m(z)dz = (\varepsilon_m)^{-n} \int_\Pi \int_{B_R} \chi(t/\varepsilon_m, (y-z)/\varepsilon_m) dz d\gamma(t, y) =$$
$$\int_\Pi \int_{|y-\varepsilon_m x| \leq R} \chi(t/\varepsilon_m, x) dx d\gamma(t, y) \leq C_r \gamma((0, \varepsilon_m r] \times B_{R+\varepsilon_m r}), \tag{3.3}$$

where C_r is Lebesgue measure of the ball B_r. As $m \to \infty$

$$\gamma((0, \varepsilon_m r] \times B_{R+\varepsilon_m r}) \to 0,$$

and in view of (3.3), for all $R > 0$

$$\lim_{m\to\infty} \int_{B_R} I^m(z)dz = 0.$$

Therefore, we can choose a subsequence of ε_m (as usual, we keep the notation ε_m) such that $I^m(z) \to 0$ a.e. on $\mathbb{R}^n$. Using the diagonal extraction, we can chose the indicated subsequence and the set of full measure $z \in \mathbb{R}^n$, for which $I^m(z) \to 0$, being common for all rational values of r. Then, for such z

$$S_z^m(\gamma) \underset{m\to\infty}{\to} 0 \text{ in } \bar{M}_{loc}(\Pi),$$

as required. The proof is complete. □

Applying Lemma 3.2 to the measures γ_k, $k \in F$, we see that there exists a subsequence of the sequence ε_m (we keep the notation ε_m for this subsequence) and a set $Z \subset \mathbb{R}^n$ of full measure such that $S_z^m(\gamma_k) \to 0$ in $\bar{M}_{loc}(\Pi)$ as $m \to \infty$ $\forall z \in Z$. Recall that the dense set $F \subset \mathbb{R}$ is supposed to be countable. Then, using the standard diagonal extraction, we can choose the indicated subsequence being common for all $k \in F$.

Thus, we can assume that the sequence ε_m and the set $Z \subset \mathbb{R}^n$ of full measure are chosen in such way that the assertions of Lemma 3.1 and Lemma 3.2 for each $k \in F$ and $\gamma = \gamma_k$ hold.

Our interest to the sequence $u^m(t, y; z)$ is stipulated by the following theorem:

Theorem 3.1. *Let $v_k(t,y) = \psi_{0k}(y,u(t,y))$, $v_k^m(t,y;z) = v_k(\varepsilon_m t, \varepsilon_m y + z) = \psi_{0k}(\varepsilon_m y + z, u^m(t,y;z))$. Then*

(i) Existence of the strong traces $\operatorname*{ess\,lim}_{t\to 0} \psi_{0k}(y,u(t,y)) = v_{0k}(y)$ in $L^1_{loc}(\mathbb{R}^n)$ implies that, after possible extraction of a subsequence, for a.e. $z \in \mathbb{R}^n$ the sequences $v_k^m(t,y;z)$ converge as $m \to \infty$ to some functions $v_k(t,y;z)$ in $L^1_{loc}(\Pi)$;

(ii) Conversely, suppose that for a.e. $z \in \mathbb{R}^n$ and each $k \in F$ there is a subsequence of ε_m such that the sequence $v_k^m(t,y;z)$ is strongly convergent. Then there exist strong traces $\operatorname*{ess\,lim}_{t\to 0} \psi_{0k}(y,u(t,y)) = v_{0k}(y)$ in $L^1_{loc}(\mathbb{R}^n)$.

Proof. Suppose $\operatorname*{ess\,lim}_{t\to 0} v_k(t,y) = v_{0k}(y)$ in $L^1_{loc}(\mathbb{R}^n)$. We denote $\rho(z) = e^{-|z|}$. Then for a.e. $(t,y) \in \Pi$

$$\int |v_k^m(t,y;z) - v_{0k}(\varepsilon_m y + z)|\rho(z)dz =$$
$$\int |v_k(\varepsilon_m t, \varepsilon_m y + z) - v_{0k}(\varepsilon_m y + z)|\rho(z)dz =$$
$$\int |v_k(\varepsilon_m t, x) - v_{0k}(x)|\rho(x - \varepsilon_m y)dx \underset{m\to\infty}{\to} 0.$$

Integrating this relation over (t,y) weighted $p(t,y) = e^{-t-|y|}$, and using the Lebesgue dominated convergence theorem we derive the relation

$$\int \left(\int_\Pi |v_k(\varepsilon_m t, \varepsilon_m y + z) - v_{0k}(\varepsilon_m y + z)|p(t,y)dtdy \right) \rho(z)dz \underset{m\to\infty}{\to} 0.$$

From this it follows that, after possible extraction of a subsequence, $v_k^m(t,y;z) - v_{0k}^m(y;z) \to 0$ as $m \to \infty$ in $L^1_{loc}(\Pi)$ for a.e. $z \in Z$. By Lemma 3.1 we see also that $v_{0k}^m(y;z) \to v_{0k}(z)$ in $L^1_{loc}(\Pi)$ and we can conclude that for a.e. z $v_k^m(t,y;z) \to v_{0k}(z)$ in $L^1_{loc}(\Pi)$, as required. Observe also that the subsequence ε_m and the set of full measure of the values z, for which the above limit relation holds can be chosen to be common for all $k \in \mathbb{R}$. This easily follows from continuity $v_k^m(t,y;z) = \psi_{0k}(\varepsilon_m y + z, u^m(t,y;z))$ with respect to the parameter k.

Conversely, suppose that $\forall z \in Z_1$, where $Z_1 \subset \mathbb{R}^n$ is a set of full measure, there is a subsequence of ε_m such that $v_k^m(\cdot;z)$ converges in $L^1_{loc}(\Pi)$ to some function $v_k(\cdot;z) \in L^\infty(\Pi)$. Clearly, this subsequence can be chosen common for all $k \in \mathbb{R}$. Without loss of generality we can assume that $Z_1 \subset Z$. Fix $z \in Z_1$ and simplify the notations $u^m(t,y) = u^m(t,y;z)$, $v_k^m(t,y) = v_k^m(t,y;z)$, $v_k(t,y) = v_k(t,y;z)$. In correspondence with Theorem 2.1 we can suppose that the sequence $u^m(t,y)$ converges weakly to a measure valued function $\nu_{t,y} \in \mathrm{MV}(\Pi)$. Clearly, $\|\nu_{t,y}\|_\infty \le M = \|u\|_\infty$. Since the sequence $v_k^m(t,y) = \psi_{0k}(\varepsilon_m y + z, u^m(t,y))$ converges strongly while the function $\psi_{0k}(y,u)$ is continuous, we see that the sequence $\psi_{0k}(z, u^m(t,y))$ converges strongly to the limit $v_k(t,y) = \langle \psi_{0k}(z,\lambda), \nu_{t,y}(\lambda)\rangle$. By Corollary 2.1 we see that the functions $\psi_{0k}(z,\lambda) \equiv v_k(t,y)$ on $\operatorname{supp} \nu_{t,y}$ for a.e. $(t,y) \in \Pi$.

Passing to the limit as $m \to \infty$ in (3.2) and taking into account that for $k \in F$ $S_z^m(\gamma_k) \to 0$ in $\mathcal{D}'(\Pi)$, we obtain that $\forall k \in F$

$$(\langle \psi_{0k}(z,\lambda), \nu_{t,y}(\lambda)\rangle)_t + \operatorname{div}_y \langle \psi_k(\lambda), \nu_{t,y}(\lambda)\rangle = 0 \quad \text{in } \mathcal{D}'(\Pi). \tag{3.4}$$

Since the left-hand side of this equality is continuous with respect to k in $\mathcal{D}'(\Pi)$, while $F \subset \mathbb{R}$ is dense we conclude that (3.4) holds for all $k \in \mathbb{R}$, i.e. $\nu_{t,y}$ is a measure valued i.s. of the equation (2.6) with $\varphi_0(u) = \varphi_0(z,u)$. As was shown above, this i.s. satisfies the assumptions of Theorem 2.3 and, therefore, the functions $v_k(t,y)$ have strong traces at $t = 0$.

By the relation like (1.23) we see that for a.e. $\tau > 0$ $\forall h(y) \in C_0^1(\mathbb{R}^n)$

$$I_k^m(\tau) - I_k^m(0+) = \int_{\mathbb{R}^n} v_k^m(\tau,y)h(y)dy - \int_{\mathbb{R}^n} v_{0k}^m(y)h(y)dx =$$
$$\int_{(0,\tau]\times\mathbb{R}^n} (\psi_k(u_m), \nabla_y h)dtdy - \int_{(0,\tau]\times\mathbb{R}^n} h(x)d\gamma_k^m(t,y),$$

where $\gamma_k^m = S_z^m(\gamma_k)$. From this equality we derive the estimate

$$|I_k^m(\tau) - I_k^m(0+)| \le C_h(\tau + |\gamma_k^m|((0,T]\times K)), \tag{3.5}$$
$$C_h = \text{const}, \quad K = \operatorname{supp} h,$$

which holds for a.e. $\tau \in (0,T)$.

By our assumptions the conclusion of Lemma 3.2 holds, that is $|\gamma_k^m|((0,T]\times K) \to 0$ as $m \to \infty$. Further, for a.e. $\tau > 0$ $v_k^m(\tau,y) \to v_k(\tau,y)$ as $m \to \infty$ in $L^1_{loc}(\mathbb{R}^n)$ (after possible extraction of a subsequence), and also $v_{0k}^m(y) \to v_{0k}(z) = \text{const}$ in $L^1_{loc}(\mathbb{R}^n)$ (by Lemma 3.1). Therefore, from (3.5) it follows, in the limit as $m \to \infty$, that for a.e. $\tau > 0$, $\forall h(y) \in C_0^1(\mathbb{R}^n)$

$$\left| \int_{\mathbb{R}^n} v_k(\tau,y)h(y)dy - \int_{\mathbb{R}^n} v_{0k}h(y)dy \right| \le C_h\tau \to 0,$$

i.e. $\operatorname*{ess\,lim}_{t\to 0} v_k(t,\cdot) = v_{0k} = v_{0k}(z)$ weakly-$*$ in $L^\infty(\mathbb{R}^n)$.

Since the traces for $v_k(t,y)$ must be strong we find

$$\operatorname*{ess\,lim}_{t\to 0} \langle \psi_{0k}(z,\lambda), \nu_{t,y}(\lambda)\rangle = v_{0k}(z) \quad \text{in } L^1_{loc}(\mathbb{R}^n) \tag{3.6}$$

for each $k \in F$. Taking into account that the set F is dense and both sides in (3.6) are continuous with respect to k in $L^1_{loc}(\mathbb{R}^n)$, we derive that (3.6) holds for all $k \in \mathbb{R}$. As in the proof of Theorem 2.3 we can choose the sequence $t_r \to 0$ such that the corresponding sequence of measure valued functions $\nu_y^r = \nu_{t_r,y} \in MV(\mathbb{R}^n)$ is well-defined, converges weakly as $r \to \infty$ to a measure valued function ν_y, $\|\nu_y\|_\infty \le M$. In correspondence with (3.6) $\langle \psi_{0k}(z,\lambda), \nu_y^r(\lambda)\rangle \underset{r\to\infty}{\to} v_{0k}(z)$ in $L^1_{loc}(\mathbb{R}^n)$. Therefore, by Corollary 2.1, $\psi_{0k}(z,\lambda) \equiv v_{0k}(z)$ on $\operatorname{supp} \nu_y$ for all $k \in \mathbb{R}$ and almost all $y \in \mathbb{R}^n$. Fix such y and set $u_0 \in \operatorname{supp} \nu_y$. Then, $u_0 = u_0(z)$ satisfies the properties: $|u_0| \le M$ and $\psi_{0k}(z, u_0) = v_{0k}(z)$. Since here $z \in Z_1$ is arbitrary, we obtain the bounded function $u_0(z)$ such that $\psi_{0k}(z, u_0(z)) = v_{0k}(z)$ a.e. on $\mathbb{R}^n$. By Theorem 2.4 we conclude that the traces $v_{0k}(y)$ are strong, which completes the proof. □

4 Reduction of the space dimension

Suppose that $u(t, y)$ is a quasi-s. of (1.14) and one of the components of the flux vector, say φ_n, is absent, i.e. equation (1.14) has the form

$$\varphi_0(y, u)_t + \sum_{i=1}^{n-1} \varphi_i(u)_{y_i} = \varphi_0(y, u)_t + \operatorname{div}_{y'} \varphi(u) = 0,$$
$$y' = (y_1, \dots, y_{n-1}), \ \varphi(u) = (\varphi_1(u), \dots, \varphi_{n-1}(u)) \in \mathbb{R}^{n-1}.$$

It is natural to consider the equation for the fixed variable $y_n = \lambda$

$$\varphi_0(y', \lambda, u)_t + \operatorname{div}_{y'} \varphi(u) = 0, \ u = u(t, y') \tag{4.1}$$

in the half-space $\Pi' = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ with reduced space dimension.

Below, to justify the inductive jump in the proof of Theorem 1.4, we will need the following result.

Theorem 4.1. *For a.e. $y_n \in \mathbb{R}$ the function $\tilde{u}(t, y') = u(t, y', y_n)$ is a quasi-s. of equation (4.1).*

In the case $\varphi_0(y, u) = u$ Theorem 4.1 was proved in [19]. To prove this theorem we shall need the following lemma established in [19] (Lemma 4.2). For completeness we reproduce the proof.

Lemma 4.1. *Let $N \in \mathbb{N}$ and γ be a locally finite measure on $\mathbb{R}^N \times \mathbb{R}$. Then for a.e. $\lambda \in \mathbb{R}$ there exist a locally finite measure γ_λ on $\mathbb{R}^N$ such that $\forall h(y) \in C_0(\mathbb{R}^N)$*

$$\lim_{\nu \to \infty} \int_{\mathbb{R}^{N+1}} h(y) \delta_\nu(s - \lambda) d\gamma(y, s) = \int_{\mathbb{R}^N} h(y) d\gamma_\lambda(y).$$

Proof. We denote $|\gamma| = \operatorname{Var} \gamma$ and assume firstly that

$$|\gamma|(\mathbb{R}^N \times [a, b]) < +\infty \textit{ for any segment } [a, b] \subset \mathbb{R}. \tag{$*$}$$

Then we can define a projection $m = \operatorname{pr}^* |\gamma|$, which is the image of $|\gamma|$ under the map $\operatorname{pr}(y, s) = s$. It is clear that m is a locally finite nonnegative measure on $\mathbb{R}$. Further, for any function $h(y) \in C_0(\mathbb{R}^n)$ the correspondence

$$g \to \int_{\mathbb{R}^{N+1}} h(y) g(s) d\gamma(y, s), \quad g \in C_0(\mathbb{R})$$

defines a bounded linear functional l_h on $C_0(\mathbb{R})$, moreover

$$|\langle l_h, g\rangle| \le \|h\|_\infty \int_{\mathbb{R}^{N+1}} |g(s)|d|\gamma|(y,s) = \|h\|_\infty \int_{\mathbb{R}} |g(s)|dm(s). \tag{4.2}$$

In particular,

$$|\langle l_h, g\rangle| \le \|h\|_\infty m(K)\|g(s)\|_\infty,\ K = \operatorname{supp} g,$$

which implies that the functional l_h is in fact continuous. By the known representation theorem l_h is given as the integral

$$\langle l_h, g\rangle = \int_{\mathbb{R}} g(s)d\mu_h(s)$$

over some locally finite measure μ_h, and, as it follows from (4.2), its variation $|\mu_h| \le \|h\|_\infty \cdot m$. The latter implies that the measures μ_h are absolutely continuous with respect to the measure m and by the Radon-Nikodym theorem $\mu_h = \alpha_h(s)m$, where $\alpha_h(s)$ are Borel functions, $|\alpha_h| \le \|h\|_\infty$.

Now, we consider the Lebesgue decomposition $m = \beta(s)ds + \sigma$, where $\beta(s) \in L^1_{loc}(\mathbb{R})$, $\beta(s) \ge 0$, and σ is a singular measure. By known properties of measures (see for instance [23]) there exists a set of full Lebesgue measure $A \subset \mathbb{R}$, on which the measure σ has the null derivative over the Lebesgue measure ds that is $\forall\lambda \in A$

$$\lim_{\nu\to\infty} \int \delta_\nu(s-\lambda)d\sigma(s) = 0. \tag{4.3}$$

We choose a countable dense set $G \subset C_0(\mathbb{R}^N)$ and consider the set $A' \subset A$ consisting of common Lebesgue points for the function $\beta(s)$ and the countable family of functions $\rho_h(s) = \alpha_h(s)\beta(s)$ with $h \in G$. Since

$$|\rho_{h_1}(s) - \rho_{h_2}(s)| = |\alpha_{h_1-h_2}(s)|\beta(s) \le \|h_1 - h_2\|_\infty \beta(s) \quad \forall h_1, h_2 \in C_0(\mathbb{R}^N)$$

and G is dense in $C_0(\mathbb{R}^N)$ we conclude that $\lambda \in A'$ are Lebesgue points of all functions $\rho_h(s)$, $h \in C_0(\mathbb{R}^N)$. For $\lambda \in A'$ we have

$$\int_{\mathbb{R}^{N+1}} h(y)\delta_\nu(s-\lambda)d\gamma(y,s) = \int_{\mathbb{R}} \delta_\nu(s-\lambda)\alpha_h(s)dm(s) =$$
$$\int_{\mathbb{R}} \delta_\nu(s-\lambda)\rho_h(s)ds + \int_{\mathbb{R}} \delta_\nu(s-\lambda)\alpha_h(s)d\sigma(s). \tag{4.4}$$

Since λ is a Lebesgue point of the functions $\rho_h(s)$ then

$$\int_{\mathbb{R}} \delta_\nu(s-\lambda)\rho_h(s)ds \to \rho_h(\lambda) \ \text{ as } \nu \to \infty.$$

Further, in view of (4.3)

$$\left|\int_{\mathbb{R}}\delta_\nu(s-\lambda)\alpha_h(s)d\sigma(s)\right| \le \|\alpha_h\|_\infty \int_{\mathbb{R}}\delta_\nu(s-\lambda)d\sigma(s) \le \\ \|h\|_\infty \int_{\mathbb{R}}\delta_\nu(s-\lambda)d\sigma(s) \underset{\nu\to\infty}{\to} 0$$

and from (4.4) it follows that

$$\int_{\mathbb{R}^{N+1}} h(y)\delta_\nu(s-\lambda)d\gamma(y,s) \to \rho_h(\lambda) \quad \text{as } \nu\to\infty. \tag{4.5}$$

For a fixed $\lambda \in A'$ the correspondence $h \to \rho_h(\lambda)$ determines a bounded linear functional on $C_0(\mathbb{R}^N)$ and, taking into account the estimate $|\rho_h(\lambda)| = |\alpha_h(\lambda)\beta(\lambda)| \le \beta(\lambda)\|h\|_\infty$, we have the representation

$$\rho_h(\lambda) = \int_{\mathbb{R}^N} h(y)d\gamma_\lambda(y), \tag{4.6}$$

where γ_λ is a finite Borel measure such that $|\gamma_\lambda|(\mathbb{R}^N) \le \beta(\lambda)$, $\lambda \in A'$. From (4.5), (4.6) it directly follows the conclusion of the Lemma.

In the general case of arbitrary locally finite measure γ we introduce measures $\gamma^l = \gamma|_{B_l\times\mathbb{R}}$, where B_l is a ball $|y| \le l$, $l \in \mathbb{N}$. We see that the measures γ^l can be considered on the whole space $\mathbb{R}^N \times \mathbb{R}$ and satisfy the condition (*). As we have already proved, there are sets $A_l \subset \mathbb{R}$ of full Lebesgue measure and finite Borel measures γ^l_λ such that $\forall\lambda \in A_l$

$$\lim_{\nu\to\infty}\int_{\mathbb{R}^{N+1}} h(y)\delta_\nu(s-\lambda)d\gamma^l(y,s) = \int_{\mathbb{R}^N} h(y)d\gamma^l_\lambda(y). \tag{4.7}$$

It is clear that the set $A = \cap_{l\in\mathbb{N}}A_l$ has full measure and for $\lambda \in A$ relation (4.7) holds for all $l \in \mathbb{N}$. This, in particular, implies that measures γ^l_λ are compatible: $\gamma^{l'}_\lambda|_{B_l} = \gamma^l_\lambda$ for $l' > l$. Therefore, there exists a locally finite measure γ_λ on $\mathbb{R}^N$ such that $\gamma_\lambda|_{B_l} = \gamma^l_\lambda$ $\forall l \in \mathbb{N}$. From (4.7) it follows that for $\lambda \in A$ $\forall h(y) \in C_0(\mathbb{R}^N)$

$$\lim_{\nu\to\infty}\int_{\mathbb{R}^{N+1}} h(y)\delta_\nu(s-\lambda)d\gamma(y,s) = \int_{\mathbb{R}^N} h(y)d\gamma_\lambda(y),$$

as was to be proved. □

Proof of Theorem 4.1. Remark firstly that the measures γ_k from (1.18) can be extended on the whole space $\mathbb{R}^{n+1}$, so that the extended measure of a Borel set $A \subset \mathbb{R}^{n+1}$ equals to $\gamma_k(A\cap\Pi)$. As it is easy to see, the extended measures are locally finite and by Lemma 4.1 there exists a set $A \subset \mathbb{R}$ of full measure such that for $\lambda \subset A$ the following limit relation holds: $\forall k \in F, h(t,y') \in C_0(\Pi')$

$$\lim_{\nu\to\infty}\int_{\Pi} h(t,y')\delta_\nu(s-\lambda)d\gamma_k(t,y',s) = \int_{\Pi'} h(t,y')d\gamma_{k,\lambda}(t,y'), \tag{4.8}$$

where $\gamma_{k,\lambda}(t,y') \in \bar{M}_{loc}(\Pi')$. We also utilize the fact that the set F is countable, which allows us to choose the set A being common for all $k \in F$.

Let $A_1 \subset A$ be a set of $\lambda \in A$ such that for a.e. $(t,y') \in \Pi'$ (t,y',λ) is a Lebesgue point of the function $u(t,y) = u(t,y',s)$. Obviously, A_1 is a set of full measure on $\mathbb{R}$. Let $\lambda \in A_1$, $h(t,y') \in C_0^1(\Pi')$. We choose a test function $g(t,y) = h(t,y')\delta_\nu(y_n - \lambda)$. By condition (1.18), applying to the test function g, $\forall k \in F$

$$\int_\Pi [\psi_{0k}(y,u)h_t + (\psi_k(u), \nabla_{y'}h)]\delta_\nu(y_n - \lambda)dtdy'dy_n = \int_\Pi h(t,x')\delta_\nu(y_n - \lambda)d\gamma_k(t,y',y_n).$$

Passing in this equality to the limit as $\nu \to \infty$ and taking into account (4.8), we find that

$$\int_{\Pi'} [\psi_{0k}(y',\lambda,u(\cdot,\lambda))h_t + (\psi_k(u(\cdot,\lambda)), \nabla_{y'}h)]dtdy' = \int_{\Pi'} f(t,y')d\gamma_{k,\lambda}(t,y').$$

Since $h(t,y') \in C_0^1(\Pi')$ is arbitrary, we obtain that $\forall k \in F$

$$\frac{\partial}{\partial t}\psi_{0k}(\cdot,\lambda,u(\cdot,\lambda)) + \operatorname{div}_{y'}\psi_k(u(\cdot,\lambda)) = -\gamma_{k,\lambda} \quad \text{in } \mathcal{D}'(\Pi'),$$

i.e. the function $\tilde{u} = u(\cdot,\lambda)$ is a quasi-s. of equation (4.1). The proof is complete. $\square$

5 H-measures associated with sequences of measure valued functions

The notion of H-measure was introduced in [25, 10] and was further extended in [16] for the case of sequences of bounded measure valued functions.

Let $\nu_x^m \in \mathrm{MV}(\Omega)$, $m \in \mathbb{N}$ be a bounded sequence of measure valued functions weakly convergent to a measure valued function $\nu_x \in \mathrm{MV}(\Omega)$. For $x \in \Omega$ and $p \in \mathbb{R}$ we set

$$V_m(x,p) = \nu_x^m((p,+\infty)), \quad V_0(x,p) = \nu_x((p,+\infty)).$$

As was shown in [16], for $m \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{R}$ the functions $V_m(x,p) \in L^\infty(\Omega)$ and $0 \le V_m(x,p) \le 1$. Let

$$P = P(\nu_x) = \left\{ p_0 \in \mathbb{R} \,|\, V_0(x,p) \underset{p \to p_0}{\to} V_0(x,p_0) \text{ in } L^1_{loc}(\Omega) \right\}.$$

The following lemma was proved in [16]:

Lemma 5.1. *The complement $CP = \mathbb{R} \setminus P$ is at most countable and for $p \in P$ $V_m(x,p) \underset{m\to\infty}{\to} V_0(x,p)$ weakly-$*$ in $L^\infty(\Omega)$.*

Let $U_m^p(x) = V_m(x,p) - V_0(x,p)$. By Lemma 5.1 for $p \in P$ $U_m^p(x) \to 0$ as $m \to \infty$ weakly-$*$ in $L^\infty(\Omega)$. We introduce the following notations:
$F(u)(\xi)$, $\xi \in \mathbb{R}^N$ is the Fourier transform of $u(x) \in L^2(\mathbb{R}^N)$;
$S = S^{N-1} = \{\, \xi \in \mathbb{R}^N \mid |\xi| = 1 \,\}$ denotes the unit sphere in $\mathbb{R}^N$;
$u \to \bar{u}$, $u \in \mathbb{C}$, is complex conjugation.

Proposition 5.1 (see [16]). *1) There exists a family of complex-valued locally finite Borel measures $\{\mu^{pq}\}_{p,q\in P}$ on $\Omega \times S$ and a subsequence $U_r(x) = \{U_r^p(x)\}_{p\in P}$ such that $\forall \Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S)$*

$$\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle =$$
$$\lim_{r\to\infty} \int_{\mathbb{R}^N} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\psi(\xi/|\xi|)d\xi.$$

2) The map $(p,q) \to \mu^{pq}$ is continuous as a map from $P \times P$ into the space $M_{loc}(\Omega \times S)$ of locally finite Borel measures on $\Omega \times S$.

Definition 5.1. The family $\{\mu^{pq}\}_{p,q\in P}$ is called the H-measure corresponding to the subsequence ν_x^r.

In the following lemma some important properties of the H-measure are collected (see [16]):

Lemma 5.2. *1) $\forall p \in P$ $\mu^{pp} \geq 0$; 2) $\forall p,q \in P$ $\mu^{pq} = \overline{\mu^{qp}}$; 3) for $p_1, \dots, p_l \in P$ and for any bounded Borel set $C \subset \Omega \times S$ the matrix $a_{ij} = \mu^{p_i p_j}(C)$, $i,j = 1,\dots,l$ is positively definite; 4) $\mu^{pq} = 0$ $\forall p,q \in P$ if and only if the sequence ν_x^r is strongly convergent as $r \to \infty$.*

Remark that, as it directly follows from 3), $\forall p,q \in P$

$$\big(\mu^{pq}(C)\big)^2 \leq \mu^{pp}(C)\mu^{qq}(C) \tag{5.1}$$

for any bounded Borel set $C \subset \Omega \times S$.

We will need also the following result.

Lemma 5.3. *Let $\{\mu^{pq}\}_{p,q\in P}$ be the H-measure corresponding to a bounded sequence $\nu_x^r \in \mathrm{MV}(\Omega)$ and $C \subset P$ be a closed set such that $\mu^{pp} = 0$ for all $p \in C$. If $s(u)$ is a continuous from the left nondecreasing function, which is constant on connected components of $\mathbb{R} \setminus C$, then the sequence $s^*\nu_x^r$ converges strongly as $r \to \infty$.*

Proof. Suppose $\nu_x^r \to \nu_x$ as $r \to \infty$. Choose $M > \sup \|\nu_x^r\|_\infty$ and define

$$s^{-1}(\mu) = \inf\{\ \lambda \in [-M,M] \ \mid\ s(\lambda) > \mu\ \}.$$

We agree, as usual, that $s^{-1}(\mu) = M$ in the case when $s(\lambda) \leq \mu$ on $[-M,M]$. Let us demonstrate that $\lambda = s^{-1}(\mu) \in C \cup \{-M, M\}$. Indeed, if this is not true then $\lambda \in$

(a,b), where (a,b) is some connected component of the complement $(-M,M)\setminus C$. Clearly $s(u)>\mu$ in the interval $u\in(\lambda,b)$. But by our assumption $s(u)$ is constant on (a,b), which implies that $s(u)>\mu$ on (a,b). Therefore, $\lambda=s^{-1}(\mu)\le a$. The obtained contradiction yields the required relation $\lambda=s^{-1}(\mu)\in C\cup\{-M,M\}$. Observe that $p=\pm M\in P$ and for these values $\mu^{pp}=0$ (because $U_r^p(x)\equiv 0$ for $p=\pm M$). Hence, we can suppose that $\pm M\in C$. Taking into account that the function $s(\lambda)$ is continuous from the left, we find $s^*\nu_x^r((\mu,+\infty))=\nu_x^r((s^{-1}(\mu),+\infty))$, $r\in\mathbb{N}$; $s^*\nu_x((\mu,+\infty))=\nu_x((s^{-1}(\mu),+\infty))$, which implies the relation

$$\tilde{U}_r^\mu(x)=s^*\nu_x^r((\mu,+\infty))-s^*\nu_x((\mu,+\infty))=U_m^\lambda(x)=\nu_x^r((\lambda,+\infty))-\nu_x((\lambda,+\infty)),$$

where $\lambda=s^{-1}(\mu)$. Hence, the H-measure $\{\tilde{\mu}^{pq}\}_{p,q\in\mathbb{R}}$ corresponding the sequence $s^*\nu_x^r$ is well defined and $\tilde{\mu}^{pp}=\mu^{qq}$, $q=s^{-1}(p)$. Since in this relation $q=s^{-1}(p)\in C$ we conclude that $\tilde{\mu}^{pp}=0$ for all p. By (5.1) we find $\tilde{\mu}^{pq}\equiv 0$ and in correspondence with Lemma 5.2,4) the sequence $s^*\nu_x^r$ is strongly convergent.
The proof is complete. $\square$

Corollary 5.1. *Let $\{\mu^{pq}\}_{p,q\in P}$ be the H-measure corresponding to a bounded in $L^\infty(\Omega)$ sequence $u_r(x)$, $r\in\mathbb{N}$ (these functions are considered as regular measure valued functions). Suppose $u_r(x)$ converges weakly to a measure valued function ν_x and $[a(x),b(x)]=\mathrm{Co}\,\mathrm{supp}\,\nu_x$. Then for a.e. $x\in\Omega$ $\mu^{pp}\neq 0$ for all $p\in(a(x),b(x))\cap P$.*

Proof. Clearly for $\lambda\in\mathbb{R}$

$$\{\ x\in\Omega\ \mid\ b(x)\le\lambda\ \}=\{\ x\in\Omega\ \mid\ \nu_x((\lambda,+\infty))=0\ \},$$
$$\{\ x\in\Omega\ \mid\ a(x)\ge\lambda\ \}=\{\ x\in\Omega\ \mid\ \nu_x((-\infty,\lambda))=0\ \}$$

and since functions $x\to\nu_x((\lambda,+\infty))$, $x\to\nu_x((-\infty,\lambda))$ are measurable on Ω, by Remark 2.1, we see that $a(x)$, $b(x)$ are measurable as well. Hence $a(x),b(x)\in L^\infty(\Omega)$. Let x be a common Lebesgue point of the functions $a(x)$, $b(x)$. Let us show that $\mu^{pp}\neq 0$ for $p\in(a(x),b(x))\cap P$. Assuming the contrary, take a value $p\in P\cap(a(x),b(x))$ such that $\mu^{pp}=0$ and define $s(u)=\mathrm{sign}^+(u-p)$. This function satisfies the assumptions of Lemma 5.3 with $C=\{p\}$ and we conclude that the sequence $s(u_r)$ strongly converges to $s^*\nu_x$. By Theorem 2.1 $s^*\nu_x$ is regular. Therefore $p\notin(a(x),b(x))$ for a.e. $x\in\Omega$. But this contradicts to the fact that inequality $a(x)<p<b(x)$ remains valid on a set of positive measure. The proof is complete. $\square$

Now, let $\varphi(u)\in C(\mathbb{R},\mathbb{R}^N)$ be a continuous vector function. Suppose that the sequence ν_x^r satisfies the condition:

(C) $\forall p\in\mathbb{R}$ the sequence of distributions

$$\mathcal{L}_r^p=\mathrm{div}_x\int_{(p,+\infty)}(\varphi(\lambda)-\varphi(p))d\nu_x^r(\lambda)$$

is precompact in $H^{-1}_{loc}(\Omega)$.

Here, as usual, the space $H^{-1}_{loc}(\Omega)$ consists of distributions $u(x)$ such that for all $f(x) \in C_0^\infty(\Omega)$ the product $fu \in H_2^{-1}$. The topology in $H^{-1}_{loc}(\Omega)$ is generated by semi-norms $\|fu\|_{H_2^{-1}}$, $f \in C_0^\infty(\Omega)$.

From the results of [17] (see Lemma 2 with $q = p_0$ and the proof of Theorem 4) it directly follows the statement, which plays a key role in the proof of our main Theorems 1.4 and 1.1.

Theorem 5.1. *Suppose that $\{\mu^{pq}\}_{p,q\in P}$ is the H-measure corresponding to the sequence ν_x^r, which satisfies condition (C), and $\mu^{pp} \neq 0$ for some $p = p_0 \in P$. Then there exists a non-degenerate interval $I = (p_0 - \delta, p_0 + \delta)$ such that $(\xi, \varphi(u)) = \mathrm{const}$ on I.*

In particular from Theorem 5.1 it follows that under the non-degeneracy condition

$\forall \xi \in S$ *the function* $u \to (\xi, \varphi(u))$ *is not constant on non-degenerate intervals*

$\mu^{pq} \equiv 0$ and, therefore, the sequence ν_x^r is strongly convergent.

In [18] this result was generalized to the case when the flux vector φ depends also on the spatial variables x.

Observe that the Vasseur's result in [26] directly follows from the indicated precompactness property, with no regularity restrictions on the flux vector and the weaker non-degeneracy assumption.

6 Proof of Theorems 1.4 and 1.1

We shall use the method of mathematical induction on the spatial dimension n.

If $n = 0$ (the base) then by (1.18) $\psi_{0k}(u)' = -\gamma_k \in \bar{M}_{loc}(\mathbb{R}_+)$. This implies that $\psi_{0k}(u(t))$ has bounded variation on any interval $(0, T)$, $T > 0$. Therefore, we can find $v_{0k} \in \mathbb{R}$ such that $\lim\limits_{t\to 0+} \psi_{0k}(u(t)) = v_{0k}$. Clearly, $v_{0k} = \psi_{0k}(u_0)$, where u_0 is an arbitrary limit point of $u(t)$ at $t = 0$. We see that the statement of Theorem 1.4 is fulfilled.

Assuming that the conclusion of Theorem 1.4 is true for $n - 1$ space variables, we prove it for dimension n. So, suppose that the function $u = u(t, y)$ is a quasi-s. of equation (1.14).

We introduce the set $\mathfrak{I}$ of segments $I = [a, b]$, $a, b \in F$, such that for some nonzero vector $\xi = (\xi_0, \xi') \in \mathbb{R}^{n+1}$ the function $u \to (f(u), \xi)$ is constant on I. Here $f(u)$ is the flux vector of original equation (1.1). Since the set F is assumed to be countable, the set $\mathfrak{I}$ is also countable (or empty).

Let $I = [a, b] \in \mathfrak{I}$. We set $u_I = u_I(t, y) = \max(a, \min(u, b))$ so that $u_I(t, y) \in I$. Let us show that the function $u_I(t, y)$ is also a quasi-s. of (1.14). Take $k \in F$. Then

$k' = \max(a, \min(k, b)) \in F$ and one can easily verify that

$$\psi_{0k}(y, u_I) = \operatorname{sign}(u_I - k)(\varphi_0(y, u_I) - \varphi_0(y, k)) =$$
$$\psi_{0k'}(y, u) - (\psi_{0a}(y, u) + \psi_{0,b}(y, u))/2 + (\psi_{0a}(y, k) + \psi_{0b}(y, k))/2,$$
$$\psi_k(u_I) = \operatorname{sign}(u_I - k)(\varphi(u_I) - \varphi(k)) =$$
$$\psi_{k'}(u) - (\psi_a(u) + \psi_b(u))/2 + (\psi_a(k) + \psi_b(k))/2,$$

which implies that in $\mathcal{D}'(\Pi)$

$$\psi_{0k}(y, u_I)_t + \operatorname{div}_y \psi_k(u_I) = (\gamma_a + \gamma_b)/2 - \gamma_{k'} \in \bar{M}_{loc}(\Pi)$$

for all $k \in F$. Thus, $u_I(t, y)$ satisfies (1.18), i.e. it is a quasi-s. of (1.14).

Now, recall that $(f(u), \xi) = \text{const}$ for $u \in I$. Consider the following two cases.

1) $\xi_0 = 0$. In this case $\xi' = (\xi_1, \ldots, \xi_n) \neq 0$ and there exists a linear change $z = z(y)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ such that $z_n = (\xi', y)$. After this change (1.14) reduces to the form

$$\varphi_0(z, u)_t + \operatorname{div}_z \bar{\varphi}(u) = 0, \tag{6.1}$$

where $\bar{\varphi}_n = (\xi', \varphi(u)) = (\xi', \bar{f}(u)) = \text{const}$ on I. We consider the function u_I as a function of new variables (t, z). By Theorem 4.1, $u_I(t, z', z_n)$ is a quasi-s. of reduced equation (defined on the domain $\Pi' = \mathbb{R}_+ \times \mathbb{R}^{n-1}$)

$$u_t + \sum_{i=1}^{n-1} \tilde{\varphi}_i(u)_{z_i} = 0 \tag{6.2}$$

for a.e. z_n.

As is rather well-known (see for instance [23]), the set

$$\mathcal{M} = \{\ (t, z) = (t, z', z_n) \in \Pi \mid (t, z) \text{ is a Lebesgue point of } u,$$
$$(t, z') \text{ is a Lebesgue point of } u(\cdot, z_n)\ \}$$

has full measure. Therefore, for a set E_1 of full measure of values $t > 0$ the sections $\mathcal{M}_t = \{z \in \mathbb{R}^n \mid (t, z) \in \mathcal{M}\ \}$ have full measure on $\mathbb{R}^n$. Observe that $E_1 \subset E$, where the set E is defined by (1.21). Now, we choose a sequence $t_r \in E_1$, $t_r \underset{r\to\infty}{\to} 0$, and introduce the sets $A_r \subset \mathbb{R}$ consisting of values $s \in \mathbb{R}$ such that $(z', s) \in \mathcal{M}_{t_r}$ for a.e. $z' \in \mathbb{R}^{n-1}$. Clearly, A_r have full measure, consequently $A = \cap_{r\in\mathbb{N}} A_r$ is also a set of full measure. We see also that for $s \in A$ all $t_r \in E(s)$, where $E(s)$ is defined by (1.21) for the function $u(\cdot, s)$ on $\Pi' = \mathbb{R}_+ \times \mathbb{R}^{n-1}$.

As we have already established, for a.e. $z_n \in A$ $u_I(\cdot, z_n)$ is a quasi-s. of equation (6.2) on Π'. By the inductive assumption for all $k \in \mathbb{R}$ $\psi_{0k}(\cdot, z_n, u_I(t_r, \cdot, z_n)) \to v_{0k}(\cdot, z_n)$ in $L^1_{loc}(\mathbb{R}^{n-1})$ as $r \to \infty$, where $v_{0k} = v_{0k}(z', z_n) \in L^\infty(\mathbb{R}^n)$. Using the Lebesgue dominated convergence theorem, we conclude that as $r \to \infty$ also

$\psi_{0k}(z, u_I(t_r, z)) \to v_{0k}(z)$ in $L^1_{loc}(\mathbb{R}^n)$. This easily implies the same limit relation under original variables y: $\psi_{0k}(y, u_I(t_r, y)) \to v_{0k}(y)$ in $L^1_{loc}(\mathbb{R}^n)$. By Corollary 2.2 $\psi_{0k}(y, u_I(t, y))$ have the strong traces $v_{0k}(y)$ on the hyperspace $t = 0$, i.e. $\psi_{0k}(y, u_I(t, y)) \to v_{0k}(y)$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0$, $t \in E$.

2) $\xi_0 \neq 0$.

Let $C \subset \mathbb{R}^n$ be the set of y such that $\xi' + \xi_0 \nabla g(y) = 0$. Since by (1.15) $f_0(u) = \varphi_0(y, u) + (\nabla g(y), \varphi(u))$ we have the relation

$$\xi_0 \varphi_0(y, u) + (\xi' + \xi_0 \nabla g(y), \varphi(u)) = \text{const} \quad \forall u \in I.$$

Thus, for $y \in C$ $\varphi_0(y, u) = v_0(y) = \text{const}$ on I and consequently $\psi_{0k}(y, u)$ are constant on I as well: $\psi_{0k}(y, u) = v_{0k}(y)$. Therefore, for $u_0 \in I$ $\psi_{0k}(y, u_0) = v_{0k}(y)$ for every $y \in C$.

If $y \notin C$ then in a vicinity of the point $(0, y) \in \partial\Omega$ we can make the linear change of original variables x: $z_0 = \xi_0 x_0 + (\xi', x')$ (keeping the variables $x' = (x_1, \dots, x_n)$), which removes the component f_0 from equation (1.1). Clearly, the boundary $\partial\Omega$ is locally represented as the graph $z_0 = \tilde{g}(x') = \xi_0 g(x') + (\xi', x')$ and since $\nabla \tilde{g} = \xi' + \xi_0 \nabla g(x') \neq 0$ in a vicinity of our boundary point, we can express some variable x_i, $i = 1, \dots, n$ as a function of remaining variables on $\partial\Omega$. The corresponding canonical boundary chart transforms our equation to the form (6.2). As in the case 1), we deduce from the inductive assumption the strong trace property for $\psi_{0k}(\cdot, u_I)$. By Theorem 2.4 this implies that for some $u_0(y) \in I$ $v_{0k}(y) = \psi_{0k}(y, u_0(y))$ for a.e. $y \notin C$. Combining the cases $y \in C$, $y \notin C$, we find that the representation $v_{0k}(y) = \psi_{0k}(y, u_0(y))$ holds for a.e. $y \in \mathbb{R}^n$ and by Theorem 2.4 again we conclude that the strong trace property is satisfied on the whole hyperplane $t = 0$.

Thus, in the both cases the functions $\psi_{0k}(y, u_I)$ have strong traces at the hyperplane $t = 0$.

By Theorem 3.1 we can find a subsequence of the sequence ε_m such that the corresponding sequences $\psi_{0k}(\varepsilon_m y + z, u_I^m(t, y; z))$, with $u_I^m(t, y; z) = u_I(\varepsilon_m t, \varepsilon_m y + z)$, converge strongly (that is in $L^1_{loc}(\Pi)$) for a.e. $z \in \mathbb{R}^n$.

The established property remains valid for any choice of the segment $I \in \mathfrak{I}$. Since $\mathfrak{I}$ is countable then, using the diagonal extraction, we can choose the subsequence ε_m and the set $Z \subset \mathbb{R}^n$ of full measure to be common for all $I \in \mathfrak{I}$. More exactly, we can assume that the conclusions of Lemma 3.1 and Lemma 3.2 (for all $\gamma = \gamma_k$, $k \in F$) hold, and $\forall I \in \mathfrak{I}$, $z \in Z$ and $k \in \mathbb{R}$ the functions $\psi_{0k}(\varepsilon_m y + z, u_I^m(t, y; z))$ are strongly convergent as $m \to \infty$.

Now, we fix $z \in Z$ and $k \in \mathbb{R}$. According to Theorem 2.1 we can extract a subsequence ε_r such that the corresponding sequence $u^r(t, y) = u^r(t, y; z) = u(\varepsilon_r t, \varepsilon_r y + z)$ converges as $r \to \infty$ weakly to a measure valued function $\nu_{t,y}$, and for this sequence the H-measure

$$\mu^{pq} = \mu^{pq}(t, y, \xi) \in M_{loc}(\Pi \times S), \;\; p, q \in P$$

is determined, where $S = S^n = \{\, \xi = (\xi_0, \xi') \in \mathbb{R}^{n+1} \,|\, |\xi| = 1 \,\}$ is the unit sphere, and the set $P \subset \mathbb{R}$ is defined as in the previous section.

We consider the sequence of regular measure valued functions $\nu^r_{t,y}(\lambda) = \delta(\lambda - u^r(t, y; z))$ and introduce, as in the previous section,

$$V_r(t, y, p) = \nu^r_{t,y}((p, +\infty)) = \text{sign}^+(u^r(t, y) - p),$$
$$V_0(t, y, p) = \nu_{t,y}((p, +\infty)),\ U^p_r(t, y) = V_r(t, y, p) - V_0(t, y, p),$$

here $(t, y) \in \Pi$, $p \in P$. Let us show that the sequence $\nu^r_{t,y}$ satisfies to condition (C) applied to the vector $(\varphi_0(z, u), \varphi(u)) \in \mathbb{R}^{n+1}$. Indeed,

$$\mathfrak{L}^p_r = \frac{\partial}{\partial t}\int_{(p,+\infty)} (\varphi_0(z, \lambda) - \varphi_0(z, p))d\nu^r_{t,y}(\lambda) + \text{div}_y \int_{(p,+\infty)} (\varphi(\lambda) - \varphi(p))d\nu^r_{t,y}(\lambda) =$$
$$\frac{\partial}{\partial t}[\text{sign}^+(u^r - p)(\varphi_0(z, u^r) - \varphi_0(z, p))] + \text{div}_y[\text{sign}^+(u^r - p)(\varphi(u^r) - \varphi(p))] =$$
$$\frac{\partial}{\partial t}[\text{sign}^+(u^r - p)(\varphi_0(z + \varepsilon_r y, u^r) - \varphi_0(z + \varepsilon_r y, p))] +$$
$$\text{div}_y[\text{sign}^+(u^r - p)(\varphi(u^r) - \varphi(p))] + F^p_r,$$

where

$$F^p_r = \frac{\partial}{\partial t}\{\text{sign}^+(u^r - p)[\varphi_0(z, u^r) - \varphi_0(z + \varepsilon_r y, u^r) + \varphi_0(z + \varepsilon_r y, p) - \varphi_0(z, p)]\}$$

and from the identities

$$\text{sign}^+(u - p)(\varphi_0(y, u) - \varphi_0(y, p)) = (\psi_{0p}(y, u) + \varphi_0(y, u) - \varphi_0(y, p))/2,$$
$$\text{sign}^+(u - p)(\varphi(u) - \varphi(p)) = (\psi_p(u) + \varphi(u) - \varphi(p))/2,$$

relations (1.18), (1.19), and (3.2) it follows that for $p \in F$

$$\mathfrak{L}^p_r = F^p_r - S^r_z(\beta_p), \quad \text{where } \beta_p = (\gamma_p + \gamma)/2.$$

By Lemma 3.2 the sequence $S^r_z(\beta_p) \to 0$ as $r \to \infty$ in $\bar{M}_{loc}(\Pi)$ and, using, for instance, Murat's lemma ([14]), we conclude that $S^r_z(\beta_p) \to 0$ in H^{-1}_{loc}. As easy to see, $F^p_r \to 0$ in H^{-1}_{loc} as well and we conclude that $\mathfrak{L}^p_r \to 0$ in H^{-1}_{loc}. Further, if $h(t, y)$ is a function in the Sobolev space $H^1_2(\Pi)$ having compact support $K \subset \Pi$ then, as is easily verified, for $p > q$

$$|\langle \mathfrak{L}^p_r - \mathfrak{L}^q_r, h\rangle| \le C_K(\max_{u\in[q,p]} |\varphi_0(z, u) - \varphi_0(z, q)| \cdot \|h_t\|_2 +$$
$$\max_{u\in[q,p]} |\varphi(u) - \varphi(q)| \cdot \|\nabla_y h\|_2) \le$$
$$C_K(\max_{u\in[q,p]} |\varphi_0(z, u) - \varphi_0(z, q)| + \max_{u\in[q,p]} |\varphi(u) - \varphi(q)|)\|h\|_{H^1_2},$$

which implies that the sequence $\mathfrak{L}^p_r$ is equicontinuous over the parameter p in $H^{-1}_{loc}(\Pi)$ and, since $\mathfrak{L}^p_r \to 0$ in H^{-1}_{loc} for the dense set $p \in F$, this limit relation remains valid for all p. Thus, condition (C) is satisfied and we can use the conclusion of Theorem 5.1. By this Theorem, if $p \in P$ and $\mu^{pp} \neq 0$ then $p \in I$ for some $I \in \mathfrak{I}$ (we also take here into account that the correspondence between vectors $(\varphi_0(z,u), \varphi(u))$ and $f(u)$ is given by a linear isomorphism of $\mathbb{R}^{n+1}$). Therefore, the set $C = \mathbb{R} \setminus \bigcup_{I \in \mathfrak{I}} I$ satisfies the property:

$$\forall p \in C \cap P \quad \mu^{pp} = 0. \tag{6.3}$$

Now, suppose that $[a(t,y), b(t,y)]$ is the convex hull of $\operatorname{supp} \nu_{t,y}$. Since the sequences $\psi_{0k}(z, u_I^r(t,y;z))$ are strongly convergent we see that for a.e. $(t,y) \in \Pi$ all the functions $\psi_{0k}(z,u)$ must be constant on $[a(t,y), b(t,y)] \cap I$, $I \in \mathfrak{I}$. By Corollary 5.1 and (6.3) we see that for a.e. $(t,y) \in \Pi$ the set $[a(t,y), b(t,y)] \cap C$ lays in the complement $\mathbb{R} \setminus P$ and, therefore, is at most countable. Hence, the sets of values of the continuous functions $\psi_{0k}(z,u)$ on $[a(t,y), b(t,y)]$ are at most countable and we conclude that these functions must be constant on $[a(t,y), b(t,y)]$ for a.e. $(t,y) \in \Pi$. In correspondence with Corollary 2.1 this yields the strong convergence of $\psi_{0k}(z, u^r(t,y;z))$ and by Theorem 3.1,(ii) we conclude that there exist strong traces $\operatorname*{ess\,lim}_{t \to 0} \psi_{0k}(y, u(t,y)) = v_{0k}(y)$ in $L^1_{loc}(\mathbb{R}^n)$. The proof of Theorem 1.4 is complete.

To prove Theorem 1.1 remark that by Theorem 1.4 relation (1.10) has already proved for canonical boundary charts (U, ζ, W_{rh}). Since the corresponding neighborhoods U cover the boundary $\partial\Omega$ we derive from Theorem 2.4 that weak traces $v_{0k}(x) = (\psi_k(u_0(x)), \vec{\nu}(x))$ for some function $u_0(x) \in L^\infty(\partial\Omega)$. Taking an arbitrary boundary chart (U, ζ, W_{rh}), we have $u = u(t,y) \in L^\infty((0,h) \times V_r)$, and $\operatorname*{ess\,lim}_{t \to 0} \psi_{0k}(t,y,u(t,y)) = v_{0k}(y) = \psi_{0k}(0,y,u_0(y))$ weakly-$*$ in $L^\infty(V_r)$, where

$$\psi_{0k}(t,y,u) = \operatorname{sign}(u-k)(\varphi_0(t,y,u) - \varphi_0(t,y,k)), \ \varphi_0(t,y,u) =$$
$$(\vec{\nu}(x), f(u)) = (\nabla t(x), f(u)), \ x = \zeta^{-1}(t,y); \ v_{0k}(y) = v_{0k}(\zeta^{-1}(0,y)).$$

Again by Theorem 2.4 we conclude that $\operatorname*{ess\,lim}_{t \to 0} \psi_{0k}(t,y,u(t,y)) = v_{0k}(y)$ in $L^1(V_r)$, i.e. (1.10) is satisfied for any boundary charts. This completes the proof.

Remark 6.1. The presented results remain valid for more general case of unbounded quasi-s., which are understood in the following sense.

Definition 6.1. A measurable function $u(x)$ is called a quasi-s. of (1.1) if there exists a dense set $F \subset \mathbb{R}$ such that for each pair $a, b \in F$, $a < b$ the cut-off function $u_{a,b}(x) = \max(a, \min(u(x), b))$ satisfies the relation

$$\operatorname{div}_x f(u_{a,b}(x)) = -\gamma_{a,b} \ \text{ in } \mathcal{D}'(\Omega) \tag{6.4}$$

with a measure $\gamma_{a,b} \in \bar{M}_{loc}(\Omega)$.

Observe that renormalized entropy sub- and super-solutions of equation (1.1) in the sense of [3] are quasi-s. of this equation.

Let us demonstrate that Definition 6.1 is compatible with Definition 1.2. Indeed, if in Definition 6.1 $u(x) \in L^\infty(\Omega)$ and $M = \|u\|_\infty$ then taking in (6.4) $a < -M$, $b > M$ we derive $\operatorname{div}_x f(u) = -\gamma$, $\gamma \in \bar{M}_{loc}(\Omega)$. Further, taking $a = k$, $b > \max(k, M)$, we find

$$\operatorname{div}_x \psi_k(u) = \operatorname{div}_x[2f(u_{k,b}) - f(u) - f(k)] = -(2\gamma_{k,b} - \gamma) \text{ in } \mathcal{D}'(\Omega).$$

Thus, condition (1.7) is satisfied with $\gamma_k = 2\gamma_{k,b} - \gamma \in \bar{M}_{loc}(\Omega)$, and $u(x)$ is a quasi-s. of equation (1.1) in the sense of Definition 1.2. As is easily follows from the definition, if $u(x)$ is a quasi-s. of (1.1) then $u_{a,b}(x)$ is a bounded quasi-s. of this equation for each $a, b \in F$. Taking also into account that the set F is dense, we derive from Theorem 1.1 that for each $a, b \in \mathbb{R}$, $a < b$ the functions $(f(u_{a,b}(x)), \vec{\nu}(x))$ have strong traces at the boundary $\partial\Omega$. Moreover, as follows from Theorem 2.4, these traces can be represented as $(f(u^0_{a,b}(x)), \vec{\nu}(x))$, where $u^0_{a,b}(x) = \max(a, \min(u^0(x), b))$ and $u^0(x)$ is a measurable function on $\partial\Omega$ with values from $\mathbb{R} \cup \{\pm\infty\}$. Among such functions $u^0(x)$ there are the unique minimal $u^0_-(x)$ and the maximal $u^0_+(x)$ and $(f(u), \vec{\nu}(x)) = \text{const}$ on the intervals $(u^0_-(x), u^0_+(x))$ for a.e. $x \in \partial\Omega$. In particular, under the assumption that for a.e. $x \in \partial\Omega$ the function $u \to (f(u), \vec{\nu}(x)) \neq \text{const}$ on non-degenerate intervals, $u^0_-(x) = u^0_+(x) = u^0(x)$ a.e. on $\partial\Omega$ and $u^0(x)$ is the strong trace of the function $u(x)$ itself (in the sense of strong convergence of the cut-off functions).

Remark 6.2. For the general equation

$$\operatorname{div} f(x, u) = 0 \tag{6.5}$$

existence of the strong trace for quasi-s. $u(x)$ can be proved under the non-degeneracy condition:

for $\mathcal{H}^n$-a.e. $x \in \partial\Omega$ and all $\xi \in \mathbb{R}^{n+1} \setminus \{0\}$ the function $u \to (\xi, f(x, u))$ is not constant on non-degenerate intervals.

Indeed, under this condition Theorem 5.1 yields the strong convergence (for a.e. z) of the sequences $u^m(t, y; z)$ generated by the blow-up procedure, and existence of the strong trace follows from Theorem 3.1. Moreover, in view of locality of this result it remains true also in the case of equation (6.5) on a manifold Ω (in the sense of [21], see also forthcoming paper [2]).

Without non-degeneracy conditions existence of strong normal traces for entropy solutions of (6.5) is not generally true. For instance there are numerous examples (see [5, 8, 22]) of linear transport equations $\operatorname{div}(a(x)u) = 0$ with divergence free fields $a(x)$ (i.e. $\operatorname{div} a(x) = 0$ in the sense of distributions) which admits generalized solutions such that the vector $a(x)u$ has no strong normal traces on some hyperplane.

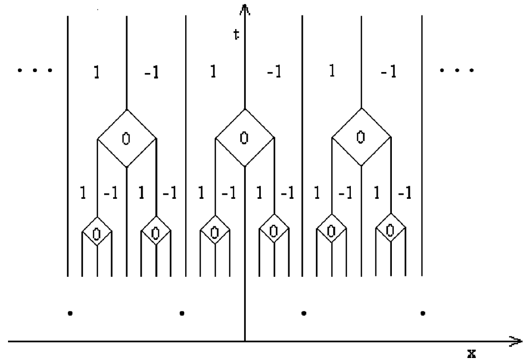

Figure 1:

7 Example of a weak solution without a strong trace

Existence of strong traces is not valid for g.s. of equation (1.1), which do not satisfy entropy condition (1.7). We confirm this fact by the following simple example. Let $n = 1$. We consider the Burgers equation $u_t + (u^2)_x = 0$. To construct the desired g.s. $u(t, x)$ we introduce the function

$$w(t,x) = \begin{cases} 0 & , \quad |x| + |t-6| \le 1 \\ -\operatorname{sign} x & , \quad |x| + |t-6| > 1, \ t \in (6,8] \\ \operatorname{sign}(1-x)\operatorname{sign} x & , \quad |x| + |t-6| > 1, \ t \in (4,6) \end{cases} ,$$

defined in the square $t \in (4, 8]$, $-2 \le x < 2$. We extend this function in the whole layer $t \in (4, 8]$, as a 4-periodic function v over the variable x so that for $y = x - 4[x/4] - 2$ (here $[a]$ denotes the integer part of a) $v(t, x) = u(t, y)$. In the half-space Π we define the piecewise constant function $u(t, x) = v(2^k t, 2^k x)$ if $t \in (4 \cdot 2^{-k}, 8 \cdot 2^{-k}]$, $k \in \mathbb{Z}$, see fig. 1. As is easily verified, on the discontinuity lines the Rankine-Hugoniot condition is satisfied and therefore $u(t, x)$ is a g.s. of the Burgers equation. As $t \to 0$ $u(t, \cdot) \to 0$ weakly-$*$ in $L^\infty(\mathbb{R})$ but there is no strong limit of $u(t_k, x)$ for any choice of a sequence $t_k \to 0$. Remark also that for the constructed g.s. condition (1.18) is satisfied with the measures $\gamma_k \in M_{loc}(\Pi)$ $\forall k \in \mathbb{R}$, but certainly $\gamma_k \notin \bar{M}_{loc}(\Pi)$.

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