On delta-shocks and singular shocks

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1 Introduction

It is well known that there are “nonclassical” situations where, in contrast to Lax’s and Glimm’s results,systems of conservation laws may admit singular solutions (*δ-shocks and singular shocks*) such that their components contain delta functions [ASh05], [B94], [DSh03]– [LW02], [S02]– [Sh04], [TZZ94]. The exact structure of such type solutions is given below in (2), (7) and Definition 1. The theory of *δ-shocks and singular shocks* has been intensively developed in the last ten years. Moreover, in the recent papers [PSh06], [Sh06] the theory of *δ'-shocks* was established, and a concept of *$\delta^{(n)}$-shocks* was introduced, $n = 2, 3, \dots$. They are *new type singular solutions* such that their components contain delta functions and their derivatives. In the δ-shock (singular shock) and δ'-shock theories there are many open and complicated problems. One of them is connected with the concept of *singular shocks*.

Some problems related with singular shocks were studied in [K99]– [KK90], [S02], [S03]. A model system admitting a *singular shock* is the well-known Keyfitz-Kranzer system

$$u_t + (u^2 - v)_x = 0, \quad v_t + \Big(\frac{1}{3}u^3 - u\Big)_x = 0 \tag{1}$$

which was studied in [KK95], [KK90]. In the exellent paper [KK95], in order to construct *approximate solutions*, the Colombeau theory approach as well as the Dafermos–DiPerna regularization (under the assumption that Dafermos profiles exist) and the box approximations are used. However the notion of a *singular solution* has *not been defined*. Later, in [Sc04], the existence of Dafermos profiles for singular shocks was proved. But it was not clear in which sense a singular shock satisfies the system (1).

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In [KSS03], [KSZ04], [S02], [S03] for system of conservation laws $w_t + \big(q(w)\big)_x = 0$, $x \in \mathbf{R}$, $w(x,t) \in \mathbf{R}^n$, where $q : \mathbf{R}^n \to \mathbf{R}^n$ is a smooth function, a *singular shock solution* is a measure of the form

$$w(x,t) = \omega(x,t) + \sum_i M_i \chi_i(t) \delta(x - x_i(t)), \tag{2}$$

where ω is a classical weak solution away from the singularities, χ_i is the characteristic function of interval $[A_i, B_i)$; $M_i \in W^\infty$ and $x_i \in W^{1,\infty}$. The function w is the weak limit of a sequence w^ε with $w^\varepsilon(\cdot,t) \in L^1_{loc}$ uniformly with respect to ε, pointwise in t; satisfying $w^\varepsilon(\cdot,t) \to w(\cdot,t)w(x,t)$ and

$$\big(w^\varepsilon(\cdot,t)\big)_t + \big(q(w^\varepsilon(\cdot,t))\big)_x - \varepsilon(A(w^\varepsilon(\cdot,t))_x)_x \to 0, \quad \varepsilon \to 0, \tag{3}$$

weakly in the space of measures on $\mathbf{R}$, pointwise with respect to t, for some positive definite matrix A. In the above papers some modifications of this definition are also used. Note that since $w^\varepsilon \to w$ weakly, Definition (2), (3) can be used without the term $\varepsilon(A(w^\varepsilon(\cdot,t))_x)_x$ (this was done in [S02]). These authors ([K99]– [KSZ04], [S02], [S03]) *distinguish* between $\delta-$ and singular shocks. In fact, the main distinction of a *singular shock* is that its *flux function is not defined*. As said in [K99, p.106], "unlike the delta-shocks..., the singular shocks which are needed to solve (1) are truly nonlinear objects which cannot defined in the context of classical distribution theory." According to [K99]– [KSZ04], [S02], [S03], some model problems for δ-shocks are described in [B94], [ERS96], [LW02], [TZZ94]. Here for "zero-pressure gas dynamics" the *measure-valued solution* approach is used, and flux-functions ρu, ρu^2 are well-defined measures.

It is the author's opinion that Definition (2), (3) of a *singular shock* and the other ones from [KSS03], [KSZ04], [S02], [S03] are obscure. Namely, Definition (2), (3) does not connect the *limiting function* (2) with the system $w_t + \big(q(w)\big)_x = 0$; it only connects the *regularizing function* w^ε with the regularizing system (3). Thus it is not defined in which sense a singular shock (2) satisfies to nonlinear system. In this way only approximating (viscosity) solutions and their structure can be studied. Note that a more general and strict definition of the type (2), (3) was introduced in [DSh03].

In order to deal with δ- and δ'-shocks, the *weak asymptotics method* was developed in [DSh03]– [DSh06], [Sh03-1], [Sh04]. In [ASh05], [DSh03]– [DSh06], [Sh03], [Sh03-1] the definition of δ-shock type solutions to systems (8) (see Definition 1) and (9) were introduced, and the corresponding *δ-shock Rankine–Hugoniot conditions* derived. These definitions give *natural* generalizations of the classical definition of the weak L^∞-solutions. According to them, δ-shocks are *Schwartz distributional solutions.* In these papers some Cauchy problems admitting (exact) δ-shocks were solved. In particular, the Cauchy problems for the Keyfitz-Kranzer system (1) and its generalization

$$L_{21}[u,v] = u_t + \big(f(u) - v\big)_x = 0, \quad L_{22}[u,v] = v_t + \big(g(u)\big)_x = 0, \tag{4}$$

were first solved in [Sh03], [Sh03-1] (see also [ASh05]), where $f(u)$ and $g(u)$ are polynomials of degree n and $n+1$, respectively, n is even.

In this paper, by using our results [ASh05], [DSh05], [DSh06], [Sh03] – [Sh04], we show that both *singular shock* and *δ-shock are solutions of the same type* (in the sense of Definition 1). To prove our assertion we compare singular solutions which have δ-singularities for the systems (1), (4) and the system

$$L_{31}[u] = u_t + \big(f(u)\big)_x = 0, \quad L_{32}[u,v] = v_t + \big(g(u)v\big)_x = 0. \tag{5}$$

According to [K99]– [KSZ04], [S02], [S03], systems (1), (4) and (5) are model problems for *singular shocks* and δ-shocks, respectively. For these systems we consider the front-problem with the initial data of the form

$$\begin{aligned} u^0(x) &= u^0_+(x) + [u^0(x)]H(-x), \\ v^0(x) &= v^0_+(x) + [v^0(x)]H(-x) + e^0\delta(-x), \end{aligned} \tag{6}$$

where $[u^0] = u^0_- - u^0_+$, $[v^0] = v^0_- - v^0_+$, and $u^0_\pm$, $v^0_\pm$ are given smooth functions, e^0 is a given constant, $H(x)$ is the Heaviside function, $\delta(x)$ is the delta-function.

Our arguments are the following: (i) According to Theorems 2, 3 (from the papers [ASh05], [DSh05], [DSh06], [Sh03]– [Sh04]), δ-shock wave type solutions of the Cauchy problems (1), (6); (4), (6); (5), (6) have the form

$$\begin{aligned} u(x,t) &= u_+(x,t) + [u(x,t)]H(-x+\phi(t)), \\ v(x,t) &= v_+(x,t) + [v(x,t)]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)) \end{aligned} \tag{7}$$

(where $u_\pm(x,t)$, $v_\pm(x,t)$, $e(t)$, $\phi(t)$ are desired functions, $x = \phi(t)$ is the discontinuity curve) and satisfy corresponding systems of conservation laws in the sense of the same Definition 1. (ii) According to Theorem 1, the Rankine–Hugoniot conditions for the above δ-shock wave type solutions are given by the identical formula (12). (iii) For these problems the flux-functions of δ-shocks (15), (16) and (17), (18) are well-defined *Schwartz distributions.*

Nevertheless, flux-functions of δ-shocks for the Keyfitz-Kranzer system (1) and its generalization have some specific and "strange" properties. The point is that δ-shocks constitute the universe with unusual and "strange" properties, and the Keyfitz-Kranzer system is an excellent model example which demonstrates this. Note that it is *impossible* to construct δ-shocks for systems (1) and (4) by using the *nonconservative product* [DLM95] as well as the *measure-valued solutions approach*.

2 δ-Shocks and the Rankine–Hugoniot conditions

Consider two particular systems of conservation laws:

$$L_1[u,v] = u_t + \big(F(u,v)\big)_x = 0, \qquad L_2[u,v] = v_t + \big(G(u,v)\big)_x = 0, \tag{8}$$

$$L_1[u,v] = v_t + \big(G(u,v)\big)_x = 0, \qquad L_2[u,v] = (uv)_t + \big(H(u,v)\big)_x = 0, \quad (9)$$

where $F(u,v)$, $G(u,v)$, $H(u,v)$ are smooth functions, *linear* with respect to v; $u = u(x,t)$, $v = v(x,t) \in \mathbf{R}$; $x \in \mathbf{R}$. As far as we know, all one-dimensional systems of conservation laws admitting δ-shocks are particular cases of systems (8) and (9). Our model examples (1), (4), (5) are particular cases of (8); the "zero-pressure gas dynamics" is a particular case of (9).

Suppose that $\mathit{\Gamma} = \{\gamma_i : i \in I\}$ is a graph in the upper half-plane $\{(x,t) : x \in \mathbf{R},\ t \in [0,\infty)\} \in \mathbf{R}^2$ containing smooth arcs γ_i, $i \in I$, and I is a finite set. Arcs of $\mathit{\Gamma}$ have orientation corresponding to increasing of time t. By I_0 we denote a subset of I such that an arc γ_k for $k \in I_0$ starts from points of the x-axis. Let $\mathit{\Gamma}_0 = \{x_k^0 : k \in I_0\}$ be the set of initial points of arcs γ_k, $k \in I_0$.

Consider *δ-shock type initial data*

$$(u^0(x), v^0(x)), \quad v^0(x) = \widehat{v}^0(x) + e^0\delta(\mathit{\Gamma}_0), \qquad u^0, \widehat{v}^0 \in L^\infty\big(\mathbf{R};\mathbf{R}\big), \quad (10)$$

where $e^0\delta(\mathit{\Gamma}_0) \stackrel{def}{=} \sum_{k\in I_0} e_k^0\delta(x - x_k^0)$, e_k^0 are constants, $k \in I_0$.

Definition 1. ([DSh05], [DSh06]) A pair of distributions $(u(x,t), v(x,t))$ and a graph $\mathit{\Gamma}$, where $v(x,t)$ has the form of the sum

$$v(x,t) = \widehat{v}(x,t) + e(x,t)\delta(\mathit{\Gamma}), \qquad u, \widehat{v} \in L^\infty\big(\mathbf{R}\times(0,\infty);\mathbf{R}\big),$$

$e(x,t)\delta(\mathit{\Gamma}) \stackrel{def}{=} \sum_{i\in I} e_i(x,t)\delta(\gamma_i)$, $e_i(x,t) \in C(\mathit{\Gamma})$, $i \in I$, is called a *δ-shock wave type solution* of the Cauchy problem (8), (10) if the integral identities

$$\begin{gathered}
\int_0^\infty \int \Big(u\varphi_t + F(u,\widehat{v})\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx = 0, \\
\int_0^\infty \int \Big(\widehat{v}\varphi_t + G(u,\widehat{v})\varphi_x\Big)\,dx\,dt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl \\
+ \int \widehat{v}^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) = 0,
\end{gathered} \quad (11)$$

hold for all test functions $\varphi(x,t) \in \mathcal{D}(\mathbf{R}\times[0,\infty))$, where $\dfrac{\partial\varphi(x,t)}{\partial\mathbf{l}}$ is the tangential derivative on the $\mathit{\Gamma}$, $\int_{\gamma_i} \cdot\, dl$ is the line integral over the arc γ_i.

Suppose that the arcs of the graph $\mathit{\Gamma} = \{\gamma_i : i \in I\}$ have the form $\gamma_i = \{(x,t) : x = \phi_i\}$, $\phi_i(t) \in C^1(0,+\infty)$, $i \in I$. In this case $\mathbf{n} = (\nu_1,\nu_2) = \frac{(1,-\dot{\phi}_i(t))}{\sqrt{1+(\dot{\phi}_i(t))^2}}$ is the unit oriented normal to the curve γ_i, $\mathbf{l} = (-\nu_2,\nu_1)$. Here $\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\Big|_{\gamma_i} = \frac{\varphi_t(\phi_i(t),t)+\dot{\phi}_i(t)\varphi_x(\phi_i(t),t)}{\sqrt{1+(\dot{\phi}_i(t))^2}} = \frac{\frac{d\varphi(\phi_i(t),t)}{dt}}{\sqrt{1+(\dot{\phi}_i(t))^2}}$.

By using Definition 1 we derive the *δ-shock Rankine–Hugoniot conditions.*

Theorem 1. ([Sh03-1], [Sh04]) *Assume that $\Omega \subset \mathbf{R} \times (0, \infty)$ is some region cut by a curve $\Gamma = \{(x,t) : x = \phi(t)\}$, $\phi(t) \in C^1(0,+\infty)$ into left- and right-hand parts $\Omega_{\mp} = \{(x,t) : \pm(x-\phi(t)) > 0\}$; (u,v), and Γ is a δ-shock solution of the system* (8), *and (u,v) is smooth in $\Omega_{\pm}$ and have one-sided limits $u_{\pm}$, $\widehat{v}_{\pm}$, on Γ. Then the Rankine–Hugoniot conditions for the δ-shock*

$$\begin{aligned} \dot{\phi}(t) &= \tfrac{[F(u,v)]}{[u]}\Big|_{x=\phi(t)}, \\ \dot{e}(t) &= \Big([G(u,v)] - [v]\tfrac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)}, \end{aligned} \tag{12}$$

hold along Γ, where $\big[a(u,v)\big] = a(u_-,v_-) - a(u_+,v_+)$ is a jump of the function $a(u(x,t),v(x,t))$ across the discontinuity curve Γ, $e(t) \stackrel{def}{=} e(\phi(t),t)$.

The first equation in (12) is the *standard* Rankine–Hugoniot condition; the right-hand side of the second equation in (12) is the *Rankine–Hugoniot deficit* in v.

3 The Cauchy problems

The eigenvalues of the characteristic matrix of system (4) are $\lambda_{\pm}(u) = \frac{1}{2}\big(f'(u) \pm \sqrt{\big(f'(u)\big)^2 - 4g'(u)}\big)$, $\big(f'(u)\big)^2 \geq 4g'(u)$. For system (5) the eigenvalues of the characteristic matrix are $\lambda_-(u) = f'(u)$, $\lambda_+(u) = g(u)$. Let $f''(u) > 0$, $g'(u) > 0$, $f'(u) \leq g(u)$. We assume that the "overcompression" conditions are satisfied.

Theorem 2. ([Sh03]– [Sh04], see also [ASh05]) *Suppose that $\lambda_+(u^0_+(0)) \leq \frac{[f(u^0)]-[v^0]}{[u^0]}\Big|_{x=0} \leq \lambda_-(u^0_-(0))$. Then there exists $T > 0$ such that for $t \in [0, T)$ the Cauchy problem* (4), (6) *has a unique solution* (7) *which satisfies the integral identities* (11) *where $\Gamma = \{(x,t) : x = \phi(t),\, t \in [0, T)\}$, and functions $u_{\pm}(x,t)$, $v_{\pm}(x,t)$, $\phi(t)$, $e(t)$ are defined by the system*

$$\begin{aligned} L_{21}[u_{\pm}, v_{\pm}] &= 0, \quad \pm x > \pm\phi(t), \\ L_{22}[u_{\pm}, v_{\pm}] &= 0, \quad \pm x > \pm\phi(t), \\ \dot{\phi}(t) &= \tfrac{[f(u)]-[v]}{[u]}\Big|_{x=\phi(t)}, \\ \dot{e}(t) &= \Big([g(u)] - [v]\tfrac{[f(u)]-[v]}{[u]}\Big)\Big|_{x=\phi(t)}, \end{aligned} \tag{13}$$

with the initial data defined from (6), $\phi(0) = 0$.

Theorem 3. ([DSh05], [DSh06]) *Let $[u^0(0)] > 0$. Then there exists $T > 0$ such that for $t \in [0, T)$ the Cauchy problem* (5), (6) *has a unique solution* (7), *which satisfies the integral identities* (11), *where $\Gamma = \{(x,t) : x = \phi(t),\, t \in [0, T)\}$, and functions $u_{\pm}(x,t)$, $v_{\pm}(x,t)$, $\phi(t)$, $e(t)$ are defined by the system*

$$
\begin{aligned}
L_{31}[u_\pm] &= 0, \quad \pm x > \pm\phi(t), \\
L_{32}[u_\pm, v_\pm] &= 0, \quad \pm x > \pm\phi(t), \\
\dot\phi(t) &= \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \\
\dot e(t) &= \Big([vg(u)] - [v]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)},
\end{aligned} \tag{14}
$$

with the initial data defined from (6), $\phi(0) = 0$.

The last two equations in (13) and (14) give the corresponding Rankine–Hugoniot conditions. They are particular cases of (12).

Recall that $O_{\mathcal{D}'}(\varepsilon^\alpha)$, $\varepsilon \to +0$ ($\alpha \in \mathbf{R}$) is a collection of distributions (with respect to x) $f(x,t,\varepsilon) \in \mathcal{D}'(\mathbf{R}_x)$, $x \in \mathbf{R}$, $t \in [0,T]$, $\varepsilon > 0$ such that $\langle f(\cdot,t,\varepsilon), \psi(\cdot)\rangle = O(\varepsilon^\alpha)$, $\varepsilon \to +0$, for any test function $\psi(x) \in \mathcal{D}(\mathbf{R})$, $x \in \mathbf{R}$; $\langle f(\cdot,t,\varepsilon), \psi(\cdot)\rangle$ is a continuous function in t; the estimate $O(\varepsilon^\alpha)$ is understood in the standard sense, being uniform with respect to t. The notation $o_{\mathcal{D}'}(\varepsilon^\alpha)$ is understood in a corresponding way.

According to [DSh03]– [DSh06], a pair of functions $\big(u_\varepsilon(x,t), v_\varepsilon(x,t)\big)$ which are smooth as $\varepsilon > 0$, $t \in [0,T]$ is called a *weak asymptotic solution* of the Cauchy problem (8), (10) if $L_1[u_\varepsilon, v_\varepsilon] = o_{\mathcal{D}'}(1)$, $L_2[u_\varepsilon, v_\varepsilon] = o_{\mathcal{D}'}(1)$, $u_\varepsilon(x,0) = u^0(x) + o_{\mathcal{D}'}(1)$, $v_\varepsilon(x,0) = v^0(x) + o_{\mathcal{D}'}(1)$, $\varepsilon \to +0$, where the first two estimates are uniform in $t \in [0,T]$. Since within the *vanishing viscosity method* a *viscosity term* admits an estimate of the form $o_{\mathcal{D}'}(1)$, a *viscosity solution* can be considered as a *weak asymptotic solution*.

Within the framework of the *weak asymptotics method* [ASh05], [DSh03]–[DSh06], [Sh03] – [Sh04], we find a δ-shock wave type solution of the Cauchy problem (4), (6) or (5), (6) as a weak limit $u(x,t) = \lim_{\varepsilon\to+0} u_\varepsilon(x,t)$, $v(x,t) = \lim_{\varepsilon\to+0} v_\varepsilon(x,t)$ of the *weak asymptotic solution* to the Cauchy problem.

To prove Theorems 2, 3, constructing a *weak asymptotic solution* of the Cauchy problem, multiplying the relations $L_1[u_\varepsilon, v_\varepsilon] = o_{\mathcal{D}'}(1)$, $L_2[u_\varepsilon, v_\varepsilon] = o_{\mathcal{D}'}(1)$, by a test function $\varphi(x,t) \in \mathcal{D}(\mathbf{R} \times [0,\infty))$, integrating these relations by parts and then passing to the limit as $\varepsilon \to +0$, we will see that the pair limit distributions (u,v) of the form (7) satisfy the integral identities (11).

4 Flux-functions of δ-shocks

Using a *weak asymptotic solution* $(u_\varepsilon, v_\varepsilon)$ to the Cauchy problem (see Sec. 3) one can define flux-functions of δ-shocks, i.e., construct explicit unique formulas for the *"right" singular superpositions*: $F(u,v) \stackrel{def}{=} \lim_{\varepsilon\to+0} F(u_\varepsilon, v_\varepsilon)$, $G(u,v) \stackrel{def}{=} \lim_{\varepsilon\to+0} G(u_\varepsilon, v_\varepsilon)$ (see [ASh05], [DSh06],[Sh03-1], [Sh04]). For the solution (7) of the Cauchy problem (4), (6) we have

$$
f\big(u(x,t)\big) - v(x,t) \stackrel{def}{=} \lim_{\varepsilon\to+0} \big(f(u_\varepsilon) - v_\varepsilon\big)
$$

$$= f(u_+) - v_+ + \big[f(u) - v\big]H(-x + \phi(t)), \tag{15}$$

$$g\big(u(x,t)\big) \stackrel{def}{=} \lim_{\varepsilon\to+0}\big(g(u_\varepsilon)\big)$$

$$= g(u_+) + \big[g(u)\big]H(-x + \phi(t)) + e(t)\frac{\big[f(u)\big]}{[u]}\delta(-x + \phi(t)). \tag{16}$$

For the solution (7) of the Cauchy problem (5), (6) we have

$$f\big(u(x,t)\big) \stackrel{def}{=} \lim_{\varepsilon\to+0} f\big(u_\varepsilon\big) = f(u_+) + \big[f(u)\big]H(-x + \phi(t)), \tag{17}$$

$$v(x,t)g\big(u(x,t)\big) \stackrel{def}{=} \lim_{\varepsilon\to+0} v_\varepsilon g\big(u_\varepsilon\big)\big) = v_+ g(u_+)$$

$$+\big[vg(u)\big]H(-x + \phi(t)) + e(t)\frac{[f(u)]}{[u]}\delta(-x + \phi(t)). \tag{18}$$

In fact, by (18) we *define* the *unique "right" product* of the Heaviside function and the δ-function in the context of the Cauchy problem (5), (6). In contrast to system (5), formulas (15), (16) *do not define (!)* the *product of the Heaviside function and the δ-function.* Moreover, although (according to (7)), $u(x,t)$ *does not depend (!)* on the term $e(t)\delta(-x + \phi(t))$, the right-hand side of the *"right" singular superposition* (16) *does depend (!)* on this term. Thus one can say that the term $e(t)\delta(-x + \phi(t))$ "appears in (16) from nothing". Analogously, the left-hand side in (15) *depends* on $e(t)\delta(-x+\phi(t))$, while the right-hand side *does not depend* on this term. Nevertheless, in the context of solving the Cauchy problem, a flux-function is determined *uniquely.*

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