ON THE DELTA-SHOCK FRONT PROBLEM

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Abstract. In this paper the *δ-shock front problem* is studied. For some classes of hyperbolic systems of conservation laws (in several space dimension, too) we introduce the definitions of a *δ-shock wave type solution* relevant to the front problem. The Rankine–Hugoniot conditions for δ-shocks are analyzed from both geometrical and physical points of view. δ-Shock *balance relations* connected with *area and mass transportation* are derived. The geometric aspect of δ-shock formation from sufficiently smooth compactly supported initial data is considered. We study the propagation of δ-shocks in two hyperbolic systems of conservation laws. In the one-dimensional case, we consider the system

$$u_t + \big(f(u) - v\big)_x = 0, \qquad v_t + \big(g(u)\big)_x = 0,$$

where $f(u)$ and $g(u)$ are polynomials of degree n and $n+1$, respectively, n is even. The well-known Keyfitz–Kranzer system

$$u_t + (u^2 - v)_x = 0, \qquad v_t + (u^3/3 - u)_x = 0$$

is a particular case of the last system. In the multidimensional case a non-conservative form of zero-pressure gas dynamics system

$$\rho_t + \nabla\cdot(\rho U) = 0, \qquad U_t + (U\cdot\nabla)U = 0,$$

is studied. This system has been used to describe the formation of large-scale structures of the universe. Both systems have several “bad” properties (see below). As far as we know, δ-shock wave type solutions for them have never been constructed.

1. Introduction

1.1. **Singular solutions of hyperbolic systems.** Consider the following hyperbolic systems of conservation laws

$$\begin{aligned} L_1[u,v] &= u_t + \Big(F(u,v)\Big)_x = 0,\\ L_2[u,v] &= v_t + \Big(G(u,v)\Big)_x = 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned} L_1[u,v] &= v_t + \Big(G(u,v)\Big)_x = 0,\\ L_2[u,v] &= (uv)_t + \Big(H(u,v)\Big)_x = 0, \end{aligned} \tag{1.2}$$

where $F(u,v)$, $G(u,v)$, $H(u,v)$ are smooth functions, *linear* with respect to v; $u = u(x,t),\ v = v(x,t) \in \mathbb{R};\ x \in \mathbb{R}$.

As is well known, hyperbolic systems of conservation laws, even in the case of smooth (and, certainly, in the case of discontinuous) initial data $(u^0(x), v^0(x))$, may have discontinuous solutions. In this case, it is said that a pair of functions

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$\big(u(x,t), v(x,t)\big) \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}^2\big)$ is a *generalized solution* of the Cauchy problem (1.1) with the initial data $\big(u^0(x), v^0(x)\big)$ if the integral identities

$$\begin{array}{rcl}\displaystyle\int_0^\infty\!\!\int \Big(u\varphi_t + F(u,v)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &=& 0,\\ \displaystyle\int_0^\infty\!\!\int \Big(v\varphi_t + G(u,v)\varphi_x\Big)\,dx\,dt + \int v^0(x)\varphi(x,0)\,dx &=& 0\end{array} \tag{1.3}$$

hold for all compactly supported test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $\int\,\cdot\,dx$ denotes an improper integral $\int_{-\infty}^{\infty}\,\cdot\,dx$. A definition of a *generalized solution* of system (1.2) can be introduced in the same way as for system (1.1).

The theory of nonlinear hyperbolic systems usually assumes systems to be strictly hyperbolic with genuinely nonlinear or linear degenerate characteristic field, and to be in conservative form. General results on the existence of entropy weak solutions are obtained only for initial values with small total variation [20], [28]. On the other hand, it is recognized that most of the physical systems do not fit into the standard theory of conservation laws [25], [29]. The Riemann problem in this "nonclassical" situation does not possess a weak L^∞-solution except for some *particular* initial data even if they are assumed to be small [29]. In contrast to the standard cases, here the second (*linear*) component v may contain Dirac measures and must be sought in the space of measures, while the first component u has bounded variation. That is the reason to introduce a *new type of generalized solutions called δ-shocks*.

In particular, it is well known (see below), that for some cases of systems (1.1), (1.2) the Cauchy problem with the initial data

$$u^0(x) = u_0 + u_1H(-x), \quad v^0(x) = v_0 + v_1H(-x), \tag{1.4}$$

where u_0, u_1, v_0, v_1 are constants and $H(\xi)$ is the Heaviside function, may admit a *δ-shock wave type solution*, i.e., a generalized solution of the form

$$\begin{array}{rcl} u(x,t) &=& u_0 + u_1H(-x+ct),\\ v(x,t) &=& v_0 + v_1H(-x+ct) + e(t)\delta(-x+ct),\end{array} \tag{1.5}$$

where $e(t)$ is a smooth function such that $e(0)=0$ and $\delta(\xi)$ is the Dirac delta function.

Recently, the theory of δ-shock type solutions for systems of conservation laws has attracted intensive attention. In particular, there are large number of papers where the system of zero-pressure gas dynamics is studied.

Several approaches to constructing *δ-shock type solutions* are known. An apparent difficulty in defining such solutions arises due to the fact that, to introduce a definition of the δ-shock type solution, we need to define *singular superpositions* of distributions (for example, the *product of the Heaviside function and the δ-function*). We also need to define *in which sense* a distributional solution (for example, (1.5)) satisfies nonlinear systems.

In what follows, we present a short review of well-known methods used to solve problems close to those studied in this paper.

In [23], a δ-shock wave type solution of the system

$$u_t + (u^2/2)_x = 0, \qquad v_t + (uv)_x = 0$$

(here $F(u,v)=u^2/2$, $G(u,v)=vu$) with the initial data (1.4), is defined as a weak limit of the solution $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ of the parabolic regularization

$$u_t + (u^2/2)_x = \varepsilon u_{xx}, \quad v_t + (uv)_x = \varepsilon v_{xx}$$

with the initial data (1.4), as $\varepsilon\to+0$.

In [21], in order to construct a δ-shock wave type solution of the system

$$u_t + \big(f(u)\big)_x = 0, \qquad v_t + \big(g(u)v\big)_x = 0, \tag{1.6}$$

(here $F(u,v) = f(u)$, $G(u,v) = vg(u)$) it is reduced to a system of Hamilton–Jacobi equations, and then the Lax formula is used. In [18], a δ-shock wave type solution of system (1.6) is constructed as self-similar viscosity limit.

In [29], to construct a δ-shock wave type solution of system (1.6) for the case $g(u) = f'(u)$, the problem of multiplication of distributions is solved by using the definition of Volpert's averaged superposition [58]. In [42], a general framework for *nonconservative product*

$$g(u)\frac{du}{dx} \tag{1.7}$$

was introduced, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded Borel function and $u : (a,b) \to \mathbb{R}^n$ is a discontinuous function of bounded variation. In the framework of this approach the Cauchy problems for nonlinear hyperbolic systems in non-conservative form can be considered [29], [30], [31]. Note that in [30], [31], for non-conservative systems the notion of generalized solution *does depend on the specific family of paths*, which *can not be derived from the hyperbolic system only*.

The system

$$u_t + (u^2 - v)_x = 0, \qquad v_t + \Big(\frac{1}{3}u^3 - u\Big)_x = 0 \tag{1.8}$$

(here $F(u,v) = u^2 - v$, $G(u,v) = \frac{1}{3}u^3 - u$) with the initial data (1.4) is studied in [27], [26]. In [26], in order to construct *approximate solutions*, the Colombeau theory approach, as well as the Dafermos–DiPerna regularization (under the assumption that Dafermos profiles exist), and the box approximations are used. But the notion of a *singular solution* has *not been defined*. It is unclear in which sense δ-shock solution (1.5) satisfies the system (1.8). In [48], the existence of Dafermos profiles for singular shocks is proved. A generalization of Keyfitz–Kranzer system ($\frac{1}{3}$ replace by $\frac{\gamma}{3}$, $0 < \gamma \le 1$) is discussed in [47]. In [50], a class of problems for which the lowest-order asymptotic approximations to Dafermos profiles can be constructed is identified. System (1.8) is an example of a system satisfying general hypotheses of paper [50].

In [56], for the system

$$u_t + (u^2)_x = 0, \qquad v_t + (uv)_x = 0,$$

in [7] for the system of "zero-pressure gas dynamics"

$$v_t + \big(vu\big)_x = 0, \qquad (vu)_t + \big(vu^2\big)_x = 0, \tag{1.9}$$

(here $G(u,v) = uv$, $H(u,v) = vu^2$), in [60] for the system

$$v_t + \big(vf(u)\big)_x = 0, \qquad (vu)_t + \big(vuf(u)\big)_x = 0, \tag{1.10}$$

(here $G(u,v) = vf(u)$, $H(u,v) = vuf(u)$) with the initial data (1.4), the δ-shock wave type solution is defined as a *measure-valued solution*.

Recall the definition of a *measure-valued solution*. Let $BM(\mathbb{R})$ be the space of bounded Borel measures. A pair (u,v), where $u(x,t) \in L^\infty\big(L^\infty(\mathbb{R}), [0,\infty)\big)$, $v(x,t) \in C\big(BM(\mathbb{R}), [0,\infty)\big)$, and u is measurable with respect to v at almost all $t \ge 0$, is said to be a *measure-valued solution* of the Cauchy problem (1.10), (1.4) if the integral identities

$$\begin{aligned} \int_0^\infty \int \Big(\varphi_t + f(u)\varphi_x\Big)\, v(dx,t) &= 0, \\ \int_0^\infty \int u\Big(\varphi_t + f(u)\varphi_x\Big)\, v(dx,t) &= 0, \end{aligned} \tag{1.11}$$

hold for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0, \infty))$. Within the framework of this definition the following formula for δ-shock wave type solution was derived

$$(u(x,t),\ v(x,t)) = \left\{ \begin{array}{rcl} \big(u^-,\ v^-\big), & x & < \phi(t), \\ \big(u_\delta,\ w(t)\delta(x-\phi(t))\big), & x & = \phi(t), \\ \big(u^+,\ v^+\big), & x & > \phi(t). \end{array} \right. \tag{1.12}$$

Here u^-, u^+ and u_δ are the velocities before the discontinuity, after the discontinuity, and at the point of discontinuity, respectively, and $\phi(t) = \sigma_\delta t$ is the equation for the discontinuity line.

The same type of definition of δ-shock wave type solution is used in [32], [33], [34] [55], to solve the Riemann problem for multidimensional system of "zero-pressure gas dynamics"

$$\rho_t + \nabla \cdot (\rho U) = 0, \qquad (\rho U)_t + \nabla \cdot (\rho U \otimes U) = 0, \tag{1.13}$$

where $\rho = \rho(x,t) \geq 0$ is the density, $U = (u_1(x,t), \dots, u_n(x,t)) \in \mathbb{R}^n$ is the velocity, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\nabla = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$, $\cdot$ is the scalar product of vectors, $\otimes$ is the usual tensor product of vectors. Here the case of planar multidimensional δ-shock was only considered.

In [17], for system (1.9) the global δ-shock wave type solution in the sense of Radon measures was obtained. In [22], for this system the uniqueness of the weak solution is proved for the case when the initial value is a Radon measure.

In [45], [46], for a 2-D system of "zero-pressure gas dynamics" the notion of generalized solutions in terms of Radon measures is introduced, and the problem of the propagation of δ-shock waves is considered. The existence of a global weak solution for the multidimensional system of "zero-pressure gas dynamics" is obtained in [49].

There is the *singular-front problem*: for a system of conservation laws (or for nonlinear equation) to describe the propagation and interaction of singular fronts starting from the initial positions. We recall that the *classical singular-front problem* for shocks was solved by A. Majda [36]– [38] (see also G. Métivier [43]). Note that physically interesting processes usually occur on the wave front.

In [9], [10]– [15], [53], [54], a new approach to solving the *singular-front problem* was developed. This approach is called the *weak asymptotics method*. The *key role* in the method is played by the definition of a *weak asymptotic solution* of the Cauchy problem, which admits passing to the limit in the *weak sense* as $\varepsilon \to 0$, where ε is the regularization parameter. Using V. P. Maslov's idea, this method permits to derive the Rankine–Hugoniot conditions directly from the differential equations *considered in the weak sense*. V. P. Maslov's *algebras of singularities* are also contained in the basis of our method [39], [40], [41], [8], [52].

By using the techniques of the *weak asymptotics method* in the above mentioned papers the *dynamics of propagation and interaction* of different nonlinear waves (infinitely narrow δ-solitons, shocks, δ-shocks) of nonlinear equations and hyperbolic systems of conservation laws is studied.

In the framework of the *weak asymptotics method*, in [13]– [15] new Definitions 2.1, 2.2 of a δ-shock wave type solution for systems (1.1), (1.2) were introduced. These definitions are *close* to the standard Definition (1.3) of a weak L^∞-solution and *relevant* to the *δ-shock front problem*. Using the *weak asymptotics method*, the propagation of δ-shock waves in systems (1.8), (1.9) is described. Formulas describing the propagation and interaction of δ-shock waves are constructed for system (1.6).

1.2. **Contents of the paper.** In Subsec. 1.3 a brief sketch scheme of the *weak asymptotics method* is given. Definitions 2.1, 2.2 of a *δ-shock wave type solution* for systems (1.1), (1.2) are given, and corresponding Rankine–Hugoniot conditions for δ-shocks are derived in Subsec. 2.1. We stress that the cases of systems (1.1) and (1.2) are studied separately. The Cauchy problem and the Rankine–Hugoniot conditions for system (1.2) are *essentially different* from the Cauchy problem and the Rankine–Hugoniot conditions for system (1.1) (see Remark 2.1 and [14]).

Next, the geometrical and physical sense of the Rankine–Hugoniot conditions for systems (1.1), (1.2) and the geometric aspect of δ-shock formation from sufficiently smooth compactly supported initial data are considered in Subsec. 2.2. We recall that the geometric aspect of shock formation was considered in [59, 2.8.]. In Subsec. 2.2 δ-shock *balance relations* for the *area and mass transportation* are also derived.

In Subsec. 2.3 we introduce the notion of a *weak asymptotic solution* of the Cauchy problem, which is one of the most important notions in the *weak asymptotics method*.

In Sec. 3 we study the problem of the propagation of a δ-shock in system

$$\begin{array}{rcccl} L_{11}[u,v] & = & u_t + \big(f(u) - v\big)_x & = & 0, \\ L_{12}[u,v] & = & v_t + \big(g(u)\big)_x & = & 0, \end{array} \tag{1.14}$$

where $f(u) = \sum_{k=0}^{n} A_k u^k$, $A_n \neq 0$, $g(u) = \sum_{k=0}^{n+1} B_k u^k$, $B_{n+1} \neq 0$, are polynomials, n is an even integer, $u = u(x,t), v = v(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$. The Keyfitz–Kranzer system (1.8) is a well known particular case of system (1.14). We solve the Cauchy problem for system (1.14) with the *δ-shock front initial data*

$$\begin{array}{rcl} u^0(x) & = & u_0^0(x) + u_1^0(x)H(-x), \\ v^0(x) & = & v_0^0(x) + v_1^0(x)H(-x) + e^0\delta(-x), \end{array} \tag{1.15}$$

where $u_k^0(\mathrm{x})$, $v_k^0(x)$, $k = 0, 1$ are given smooth functions, e^0 is a given constant.

Remark 1.1. The system (1.14) and its particular case (1.8) differ from above systems (1.6), (1.9) and have a *specific "strange"* property. Namely, they have *no balance* of singularities. Let (u,v) be a δ-shock type solution (1.5) of system (1.8). Hence, u contains the Heaviside function H, and v contains the Heaviside function H and δ-function (see (1.5). Thus, $u^2 - v$ contains the distributions H, δ, and $\frac{1}{3}u^3 - u$ contains the distribution H. It is easily seen that the term $(u^2 - v)_x$ contains the distributions H, δ, δ', while the term u_t contains *only* the distributions H and δ. Analogously, the term v_t contains the distributions H, δ, δ', but the term $(u^3/3 - u)_x$ contains *only* the distributions H, δ. Seemingly, it is impossible to obtain δ-shock type solutions for systems (1.14), (1.8). Nevertheless, we prove by Theorems 3.2, 3.3 that *there are exact solutions of this type*.

δ-Shock wave type solutions for *specific "strange"* systems (1.14), (1.8) were first constructed in [53] for piecewise constant initial data. Namely, in this paper we prove that δ-shock wave type solutions satisfy the integral identities (2.1).

We shall seek a *δ-shock wave type solution* of the Cauchy problems (1.14), (1.15) and (1.8), (1.15) in the form

$$\begin{array}{rcl} u(x,t) & = & u_0(x,t) + u_1(x,t)H(-x+\phi(t)), \\ v(x,t) & = & v_0(x,t) + v_1(x,t)H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{array} \tag{1.16}$$

where $u_0(x,t)$, $u_1(x,t)$, $v_0(x,t)$, $v_1(x,t)$, $e(t)$, $\phi(t)$ are desired functions.

As in [18], [26], [56], we use the "overcompression" condition (see [35])

$$\begin{array}{rcccl} \lambda_1(u_+,v_+) & \leq & \dot{\phi}(t) & \leq & \lambda_1(u_-,v_-), \\ \lambda_2(u_+,v_+) & \leq & \dot{\phi}(t) & \leq & \lambda_2(u_-,v_-) \end{array} \tag{1.17}$$

as the admissibility condition for the δ-shocks. Here $\lambda_1(u,v)$, $\lambda_2(u,v)$ are eigenvalues of the characteristic matrix of a hyperbolic system of conservation laws, $\dot{\phi}(t)$ is the velocity of propagation of δ-shock wave, i.e., the velocity of motion of the δ-shock front, and u_-, v_- and u_+, v_+ are the respective left- and right-hand values of u, v on the discontinuity curve. It means that all characteristics on both sides of the discontinuity are in-coming.

A *δ-shock wave type solution* (1.16) is defined as a weak limit of a *weak asymptotic solution* of the Cauchy problem (1.14), (1.15). In the framework of our approach, we will construct a *weak asymptotic solution* as a sum of the singular ansatz regularized *with respect to singularities* $H(-x+\phi(t))$ and $\delta(-x+\phi(t))$, and *corrections*:

$$\begin{aligned} u(x,t,\varepsilon) &= \widetilde{u}(x,t,\varepsilon) + R_u(x,t,\varepsilon), \\ v(x,t,\varepsilon) &= \widetilde{v}(x,t,\varepsilon) + R_v(x,t,\varepsilon), \end{aligned}$$

where a pair of functions $\big(\widetilde{u}(x,t,\varepsilon), \widetilde{v}(x,t,\varepsilon)\big)$ is a *regularization* of the singular ansatz (1.16), and the *corrections* $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$ are the desired functions, which must admit the estimates:

$$R_j(x,t,\varepsilon) = o_{\mathcal{D}'}(1), \quad \frac{\partial R_j(x,t,\varepsilon)}{\partial t} = o_{\mathcal{D}'}(1), \quad \varepsilon \to +0, \qquad j = u, v. \tag{1.18}$$

In order to construct a regularization $f(x,\varepsilon)$ of the distribution $f(x) \in \mathcal{D}'(\mathbb{R})$ we use the representation

$$f(x,\varepsilon) = f(x) * \frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0, \tag{1.19}$$

where $*$ is a convolution, and a mollifier $\omega(\eta)$ has the following properties: (a) $\omega(\eta) \in C^\infty(\mathbb{R})$, (b) $\omega(\eta)$ has a compact support or decreases sufficiently rapidly as $|\eta| \to \infty$, (c) $\int \omega(\eta)\,d\eta = 1$, (d) $\omega(\eta) \ge 0$, (e) $\omega(-\eta) = \omega(\eta)$. We have $\lim\limits_{\varepsilon\to+0} \big\langle f(\xi,\varepsilon),\ \phi(\xi)\big\rangle = \big\langle f,\ \phi\big\rangle$ for all $\phi \in \mathcal{D}(\mathbb{R})$.

Thus, we will seek a *weak asymptotic solution* in the form

$$\begin{aligned} u(x,t,\varepsilon) &= u_0(x,t) + u_1(x,t)H_u(-x+\phi(t),\varepsilon) \\ &\qquad\qquad + R_u(x,t,\varepsilon), \\ v(x,t,\varepsilon) &= v_0(x,t) + v_1(x,t)H_v\big(-x+\phi(t),\varepsilon\big) \\ &\qquad + e(t)\delta\big(-x+\phi(t),\varepsilon\big) + R_v(x,t,\varepsilon), \end{aligned} \tag{1.20}$$

where the *corrections* are defined by (3.1), and according to (1.19),

$$\delta(x,\varepsilon) = \frac{1}{\varepsilon}\omega_\delta\big(x/\varepsilon\big) \tag{1.21}$$

is a regularization of the δ-function,

$$H_j(x,\varepsilon) = \omega_{0j}\Big(\frac{x}{\varepsilon}\Big) = \int_{-\infty}^{x/\varepsilon} \omega_j(\eta)\,d\eta, \quad j = u, v \tag{1.22}$$

are regularizations of the Heaviside function $H(x)$. Here the mollifiers $\omega_u(\eta)$, $\omega_v(\eta)$, $\omega_\delta(\eta)$ have properties (a)–(e). It is clear that $\omega_{0j}(\eta) \in C^\infty(\mathbb{R})$, $\lim_{\eta\to+\infty}\omega_{0j}(\eta) = 1$, $\lim_{\eta\to-\infty}\omega_{0j}(\eta) = 0$, $j = u, v$.

A *weak asymptotic solution* of the Cauchy problem (1.14), (1.15) is constructed in Theorem 3.1. Note, if $e^0 = 0$, and the initial data are piecewise constant, according to Corollary 3.2 and Remark 3.1, our results on a *weak asymptotic solution* of system (1.8) coincide with the main statements of [26]. In particular, the Rankine–Hugoniot deficit $\dot{e}(t) = \frac{[u^3]}{3} - [u] - [v]\frac{[u^2]-[v]}{[u]}$ is positive.

By Theorems 3.2, 3.3 a weak limit of a *weak asymptotic solution* (1.20) of the Cauchy problem (1.14), (1.15) satisfies the integral identities (2.1). Thus a *δ-shock wave type solution* of the Cauchy problem (1.14), (1.15) is constructed.

The results of Secs. 2 on the geometrical and physical sense of the Rankine–Hugoniot conditions for δ-shocks were first published in [54]. The results of Secs. 3 on the propagation of δ-shocks in systems (1.14), (1.8) were first published in [53], [54].

In Secs. 4 and 5 we study the problem of propagation of δ-shock waves for a multidimensional system of "zero-pressure gas dynamics" in *non-conservative* form

$$\begin{array}{rcccl} \mathcal{L}_1[\rho,U] &=& \rho_t+\nabla\cdot(\rho U) &=& 0,\\ \mathcal{L}_2[U] &=& U_t+(U\cdot\nabla)U &=& 0. \end{array} \tag{1.23}$$

Thus, we solve the Cauchy problem for system (1.23) with the *δ-shock front initial data*

$$\begin{array}{rcl} \rho^0(x) &=& \rho_0^0(x)+\rho_1^0(x)H\big(-S^0(x)\big)+\widehat{e}^{\,0}(x)\delta\big(S^0(x)\big),\\ U^0(x) &=& U_0^0(x)+U_1^0(x)H\big(-S^0(x)\big), \end{array} \tag{1.24}$$

where $U^0=(u_1^0,\dots,u_n^0)$, $U_k^0=(u_{k1}^0,\dots,u_{kn}^0)$, $\rho_k^0\ge 0$, $k=0,1$, $u_j^0=u_{0j}^0+u_{1j}^0H\big(S^0(x)\big)$, $\widehat{e}^{\,0}\ge 0$, S^0 are given smooth functions, $j=1,\dots,n$, $x\in\Omega_0$, Ω_0 is a compact in $\mathbb{R}^n$; $H(S^0)$ is the Heaviside function, $\delta(S^0)$ is the Dirac delta function. The facts related to distributions concentrated on the surfaces are fully explained in Sec. 6.2.

We assume that $\nabla S^0(x)\big|_{S^0=0}\ne 0$, i.e., $\Gamma_0=\big\{x: S^0(x)=0\big\}$ is a smooth compact initial hypersurface of codimension 1 in the space $\mathbb{R}^n$. Denote by $\Omega_0^-=\{x: S^0(x)<0\}$ and $\Omega_0^+=\{x: S^0(x)>0\}$ the domains on the one side and on the other side of the hypersurface Γ_0. Here $\rho^0=\rho^{0-}=\rho_0^0+\rho_1^0$, $U^{0-}=U_0^0+U_1^0$ if $x\in\Omega_0^-$, and $\rho^0=\rho^{0+}=\rho_0^0$, $U^{0+}=U_0^0$ if $x\in\Omega_0^+$. In a neighborhood of any point of the surface Γ_0, one can introduce local coordinates $(\tau,\tilde\tau)$, where $\tau=S^0(x)$ and the other coordinates $\tilde\tau=(\tilde\tau_2,\dots,\tilde\tau_n)$ can be chosen so that the formulas relating x and $(\tau,\tilde\tau)$ are determined by infinitely differentiable functions with positive Jacobian. It is clear that in local coordinates the function $\widehat{e}^{\,0}(x)$ depends *only* on the variable $\tilde\tau$.

In addition to (1.24), we assume that the geometric entropy condition

$$U^{0+}(x)\cdot\nu\Big|_{\Gamma_0}<U^{0-}(x)\cdot\nu\Big|_{\Gamma_0} \tag{1.25}$$

is satisfied for the initial data, where $\nu=\frac{\nabla S^0(x)}{|\nabla S^0(x)|}$ is the unit space normal of Γ_0 oriented from Ω_0^- to Ω_0^+. Thus, $U_0^1(x)\cdot\nu\Big|_{\Gamma_0}>0$. Condition (1.24) implies that all characteristics on both sides of initial discontinuity Γ_0 must overlap (see below).

Thus, we shall solve the *classical multidimensional singular-front problem* for *δ-shocks*.

For smooth solutions, system (1.23) can be rewritten in conservative form (1.13). Systems (1.23), (1.13) are obtained from the isentropic Euler equations

$$\begin{array}{rcl} \rho_t+\nabla\cdot(\rho U) &=& 0,\\ (\rho U)_t+\nabla\cdot(\rho U\otimes U)+\nabla p(\rho) &=& 0, \end{array}$$

where the pressure term $p(\rho)$ is set to be equal to zero.

Let us mention that system (1.13) has its origin in the theory of kinetic equations $f_t+V\cdot\nabla f=0$. If we look for solutions of the form

$$f(x,t,V)=\rho(x,t)\delta\big(V-U(x,t)\big),$$

using Lemma 6.3, we obtain (1.13) for (ρ,U) (see [17], [32]).

The second equation $U_t+(U\cdot\nabla)U=0$ in system (1.23) is the inviscid Burgers equation which was used in [61], [1] to describe the formation of large-scale structures of the universe. Further, in [51], the whole system (1.23) was used as a model to describe the formation of large-scale structures of the universe. The models of "zero-pressure gas dynamics" (1.23) and (1.13) (called the "sticky particle dynamics") can be described at a discrete level by a finite collection of particles. These models are used to describe the motion of free particles which stick under collision. In [16], propagation chaos for the multidimensional viscous "zero-pressure gas dynamics" is studied.

In Sec. 4, the Definition of a δ-shock wave type solution of *non-conservative* system (1.23) is given.

In Sec. 5, we shall seek a *δ-shock wave type solution* of the Cauchy problem (1.23), (1.24), (1.25) in the form

$$\begin{array}{rcl} \rho(x,t) &=& \rho_0(x,t)+\rho_1(x,t)H\big(-S(x,t)\big)+\widehat{e}(x,t)\delta\big(S(x,t)\big), \\ U(x,t) &=& U_0(x,t)+U_1(x,t)H\big(-S(x,t)\big), \end{array} \tag{1.26}$$

where vector-functions $U=(u_1,\dots,u_n)$, $U_k=(u_{k1},\dots,u_{kn})$ and functions $\rho_k\ge 0$, $\widehat{e}\ge 0$, S, $k=0,1$, are to be found, $\Gamma_t=\big\{x: S(x,t)=0\big\}$ is the δ-shock wave front.

We shall construct a *weak asymptotic solution* of the Cauchy problem (1.23), (1.24) in the form of the *smooth ansatz*:

$$\rho(x,t,\varepsilon)=\widetilde{\rho}(x,t,\varepsilon), \qquad U(x,t,\varepsilon)=\widetilde{U}(x,t,\varepsilon),$$

where a pair $\big(\widetilde{\rho}(x,t,\varepsilon),\widetilde{U}(x,t,\varepsilon)\big)$ is a *regularization* of the singular ansatz (1.26) *with respect to the singularities* $H(x)$ and $\delta(x)$. Thus a *weak asymptotic solution* has the form

$$\begin{array}{rcl} \rho(x,t,\varepsilon) &=& \rho_0(x,t)+\rho_1(x,t)H_\rho\big(-S(x,t),\varepsilon\big)+\widehat{e}(x,t)\delta\big(S(x,t),\varepsilon\big), \\ u_j(x,t,\varepsilon) &=& u_{0j}(x,t)+u_{1j}(x,t)H_j\big(-S(x,t),\varepsilon\big), \quad j=1,\dots,n, \end{array} \tag{1.27}$$

where a regularization of the δ-function $\delta(\xi,\varepsilon)$ is given by formula (1.21), and regularizations of the Heaviside function $H(\xi)$

$$H_\rho(\xi,\varepsilon)=\omega_{0\rho}\Big(\frac{\xi}{\varepsilon}\Big)=\int_{-\infty}^{\frac{x}{\varepsilon}}\omega_\rho(\eta)\,d\eta, \quad H_j(\xi,\varepsilon)=\omega_{0j}\Big(\frac{\xi}{\varepsilon}\Big)=\int_{-\infty}^{\frac{x}{\varepsilon}}\omega_j(\eta)\,d\eta \tag{1.28}$$

are given by formula (1.22). Here functions $\omega_{0j}\in C^\infty(\mathbb{R})$, $\lim_{z\to+\infty}\omega_{0j}(z)=1$, $\lim_{z\to-\infty}\omega_{0j}(z)=0$, and mollifiers ω_ρ, ω_j, ω_δ have properties (a)–(e), $j=1,\dots,n$.

In Theorem 5.1, a *weak asymptotic solution* of the Cauchy problem (1.23), (1.24) is constructed. In Theorem 5.2, we construct a *δ-shock type solution* of the Cauchy problem as a weak limit of a *weak asymptotic solution* (1.27).

In Theorem 5.3, the δ-shock *balance relation* for the *mass transportation* is derived. According to Theorems 5.2, 5.3, on the δ-shock wave front Γ_t the *concentration process* is going on.

By Corollary 5.1 we obtain a solution of the Cauchy problem in the case of piecewise constant initial data. The Cauchy problem (1.23), (1.24), (1.25) has *only* a δ-shock wave type solution (see Remark 5.2).

Just like system (1.14), our system (1.23) has "bad" properties. It is non-conservative linear degenerate hyperbolic system with repeated eigenvalues which has n linearly independent corresponding eigenvectors.

Note that the initial data (1.15) and (1.24) *may contain* a δ-function, but as a rule, in the well-known papers on δ-shocks, initial data without a δ-function is considered, because the technical base of these papers is connected with self-similar solutions.

1.3. **The scheme of the weak asymptotics method.** We solve the above mentioned Cauchy problems using the *weak asymptotics method*.

a. To study the propagation of a *solitary* δ-shock wave, we seek a *δ-shock wave type solution* of the Cauchy problem (1.14), (1.15) or (1.23), (1.24), (1.25) in the form of the *singular ansatz* (1.16) or (1.26), which *preserves* the structure of the initial data (1.15) or (1.24).

b. Next, we construct a *weak asymptotic solution* of the problem in the form of the *smooth ansatz* (1.20) or (1.27). This *smooth ansatz* is the sum of the singular ansatz regularized *with respect to singularities* and *corrections*. Let us note that *choosing the corrections* is an *essential* part of the "right" construction of the *weak asymptotic solution* [12]– [15], [53], [54].

c. The next step is to substitute the smooth ansatz (1.20) or (1.27) into system (1.14) or (1.23) and calculate the weak asymptotics of the left-hand side of this system up to $o_{\mathcal{D}'}(1)$, as $\varepsilon \to +0$. The definition of $o_{\mathcal{D}'}(1)$ is introduced in Subsec. 2.3 and Sec. 4. We stress that in the framework of the weak asymptotics method, the discrepancy is assumed to be small in the sense of the space of functionals $\mathcal{D}'_x$ over test functions depending only on the "space" variable x. The weak asymptotics of the left-hand side of the system can be represented as *linear combinations* of the singularities H, δ, δ' with smooth coefficients. That is why we can "separate" the singularities and find a system of equations (in particular, the Rankine–Hugoniot conditions), which describes the dynamics of singularities and defines the desired functions. The *weak asymptotic solutions* of our problems are constructed in Theorems 3.1, 5.1.

d. *Generalized δ-shock wave type solutions* of the Cauchy problems are constructed in Theorems 3.2, 5.2 by using *weak asymptotic solutions*. We stress that *generalized solutions* are independent of either mollifiers or *corrections*.

The problem of defining *δ-shock wave type solutions* is connected with the construction of *singular superpositions* (*products*) of distributions. In this paper we omit the algebraic aspects of our technique which are given in detail in [8], [9], [52]. The *"right" singular superpositions of distributions* can be obtained *only in the context of constructing a weak asymptotic solution* to the Cauchy problems. The explicit formulas for the *"right" singular superpositions* are given and discussed in Subsec. 3.4 and Subsec. 5.4.

If we knew in advance the *"right" singular superpositions* constructed by (3.22), (3.23) and (5.43)–(5.45) then Theorem 3.2 and Theorem 5.2 could be proved explicitly by substituting these superpositions into systems (1.14) and (1.23), respectively.

2. One-dimensional δ-shock wave type solutions

2.1. **Generalized solutions. Rankine–Hugoniot conditions.** Suppose that $\Gamma = \{\gamma_i : i \in I\}$ is a connected graph in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\} \in \mathbb{R}^2$ containing smooth arcs γ_i, $i \in I$, and I is a finite set. By I_0 we denote a subset of I such that an arc γ_k for $k \in I_0$ starts from the points of the x-axis; $\Gamma_0 = \{x_k^0 : k \in I_0\}$ is the set of initial points of arcs γ_k, $k \in I_0$.

Let $(u^0(x), v^0(x))$ be *δ-shock wave type initial data*, where

$$v^0(x) = V^0(x) + e^0\delta(\Gamma_0),$$

$u^0, V^0 \in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, and $e^0\delta(\Gamma_0) \stackrel{def}{=} \sum_{k\in I_0} e_k^0\delta(x - x_k^0)$, e_k^0 are constants, $k \in I_0$.

Let us introduce the definition of a *δ-shock wave type solution* for system (1.1).

Definition 2.1. ([13]– [15]) A pair of distributions $\big(u(x,t), v(x,t)\big)$ and graph Γ, where $v(x,t)$ is represented in the form of the sum

$$v(x,t) = V(x,t) + e(x,t)\delta(\Gamma),$$

$u, V \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}\big)$, $e(x,t)\delta(\Gamma) \stackrel{def}{=} \sum_{i\in I} e_i(x,t)\delta(\gamma_i)$, $e_i(x,t) \in C^1(\Gamma)$, $i \in I$, is called a *generalized δ-shock wave type solution* of system (1.1) with the initial data

$(u^0(x), v^0(x))$ if the integral identities

$$
\begin{array}{rcl}
\displaystyle\int_0^{\infty}\int \Big(u\varphi_t + F(u,V)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &=& 0,\\
\displaystyle\int_0^{\infty}\int \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl && \\
\displaystyle + \int V^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) &=& 0,
\end{array}
\tag{2.1}
$$

hold for all test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}$ is the tangential derivative on the graph Γ, $\int_{\gamma_i}\,\cdot\,dl$ is a line integral over the arc γ_i.

Theorem 2.1. *Let us assume that $\Omega\subset\mathbb{R}\times(0,\,\infty)$ is some region cut by a smooth curve Γ into a left- and right-hand parts $\Omega_{\mp}$, $(u(x,t),v(x,t))$ and Γ is a generalized δ-shock wave type solution of system* (1.1) *and $(u(x,t),v(x,t))$ is smooth in $\Omega_{\pm}$. Then the Rankine–Hugoniot conditions for δ-shocks*

$$
\begin{array}{rcl}
\big[F(u,v)\big]_{\Gamma}\nu_1 + \big[u\big]_{\Gamma}\nu_2 &=& 0,\\
\big[G(u,v)\big]_{\Gamma}\nu_1 + \big[v\big]_{\Gamma}\nu_2 &=& \frac{\partial e(x,t)|_{\Gamma}}{\partial\mathbf{l}},
\end{array}
\tag{2.2}
$$

hold along Γ, where $\mathbf{n}=(\nu_1,\nu_2)$ is the unit normal to the curve Γ pointing from Ω_- into Ω_+,

$$
\big[h(u,v)\big]\Big|_{\Gamma} = \Big(h(u_-,v_-) - h(u_+,v_+)\Big)\Big|_{\Gamma}
$$

is a jump in function $h(u(x,t),v(x,t))$ across the discontinuity curve Γ, $(u_{\mp},v_{\mp})$ are respective left- and right-hand values of (u,v) on the discontinuity curve.

If $\Gamma=\{(x,t): x=\phi(t)\}$, $\Omega_{\pm}=\{(x,t): \pm(x-\phi(t))>0\}$ then relations (2.2) *can be rewritten as*

$$
\begin{array}{rcl}
\dot\phi(t) &=& \frac{[F(u,v)]}{[u]}\Big|_{x=\phi(t)},\\
\dot e(t) &=& \Big([G(u,v)] - [v]\frac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)},
\end{array}
\tag{2.3}
$$

where $e(t)\stackrel{def}{=}e(x,t)\big|_{x=\phi(t)}$, and $\dot{(\cdot)}=\frac{d}{dt}(\cdot)$.

Proof of Theorem 2.1. Selecting the test function $\varphi(x,t)$ with compact support in $\Omega_{\pm}$, we deduce from (2.1) that (1.1) hold in $\Omega_{\pm}$, respectively. Now, choosing a test function $\varphi(x,t)$ with support in Ω, we deduce from the second identity (2.1) that

$$
\begin{aligned}
0 &= \int_0^{\infty}\int \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt\\
&= \int\int_{\Omega_-}\Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt + \int\int_{\Omega_+}\Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt.
\end{aligned}
$$

Next, integrating by parts, we obtain

$$
\begin{aligned}
&\int\int_{\Omega_{\pm}}\Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt\\
&= -\int\int_{\Omega_{\pm}}\Big(V_t + \big(G(u,V)\big)_x\Big)\varphi\,dx\,dt \mp \int_{\Gamma}\Big(\nu_2 v_{\pm} + \nu_1 G(u_{\pm},v_{\pm})\Big)\varphi\,dl\\
&= \mp\int_{\Gamma}\Big(\nu_2 v_{\pm} + \nu_1 G(u_{\pm},v_{\pm})\Big)\varphi\,dl,
\end{aligned}
$$

owing to (1.1). Adding the last relations, we have

$$
\int_0^{\infty}\int \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt = \int_{\Gamma}\Big(\big[G(u,v)\big]\nu_1 + \big[v\big]\nu_2\Big)\varphi(x,t)\,dl \tag{2.4}
$$

for all $\varphi(x,t)\in\mathcal{D}(\Omega)$.

Now, integrating by parts, we can easily see that

$$\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl=-\int_\Gamma\frac{\partial e(x,t)}{\partial\mathbf{l}}\varphi(x,t)\,dl, \tag{2.5}$$

where

$$\frac{\partial}{\partial\mathbf{l}}e(x,t)|_\Gamma=\frac{\partial}{\partial t}e(x,t)\big|_\Gamma\nu_1-\frac{\partial}{\partial x}e(x,t)\big|_\Gamma\nu_2,\quad \mathbf{l}=(-\nu_2,\nu_1).$$

Adding (2.4) and (2.5), we deduce

$$\int_\Gamma\Big(\big[G(u,v)\big]\nu_1+\big[v\big]\nu_2-\frac{\partial e(x,t)}{\partial\mathbf{l}}\Big)\varphi(x,t)\,dl=0$$

for all $\varphi(x,t)\in\mathcal{D}(\Omega)$. Thus the second relation (2.2) holds.

We obtain the proof of the first relation (2.2) using formula (2.4).

If $\Gamma=\{(x,t):x=\phi(t)\}$ then $\mathbf{n}=(\nu_1,\nu_2)=\frac{1}{\sqrt{1+(\dot\phi(t))^2}}\big(1,-\dot\phi(t)\big)$ and

$$\begin{aligned}\frac{\partial\varphi(x,t)\big|_\Gamma}{\partial\mathbf{l}}&=\frac{1}{\sqrt{1+(\dot\phi(t))^2}}\Big(\frac{\partial\varphi(\phi(t),t)}{\partial t}+\dot\phi(t)\frac{\partial\varphi(\phi(t),t)}{\partial x}\Big)\\&=\frac{1}{\sqrt{1+(\dot\phi(t))^2}}\frac{d\varphi(\phi(t),t)}{dt}.\end{aligned}\tag{2.6}$$

In view of (2.6), relations (2.2) imply (2.3). □

The first equation (2.2) (or (2.3)) is the *standard* Rankine–Hugoniot condition. The left-hand side of the second equation (2.2) (or (2.3)) is called the *Rankine–Hugoniot deficit*.

Now we introduce a definition of a *δ-shock wave type solution* for system (1.2). This definition for the case of "zero-pressure gas dynamics system" was first presented in [13]. Suppose that arcs of the graph $\Gamma=\{\gamma_i:i\in I\}$ have the form $\gamma_i=\{(x,t):x=\phi_i(t)\}$, $i\in I$.

Definition 2.2. A pair of distributions $\big(u(x,t),v(x,t)\big)$ and graph Γ from Definition 2.1 is called a *generalized δ-shock wave type solution* of system (1.2) with the initial data $\big(u^0(x),v^0(x);\dot\phi_k(0),k\in I_0\big)$ if the integral identities

$$\begin{aligned}\int_0^\infty\!\!\int\Big(V\varphi_t+G(u,V)\varphi_x\Big)\,dx\,dt+\sum_{i\in I}\int_{\gamma_i}e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl&\\+\int V^0(x)\varphi(x,0)\,dx+\sum_{k\in I_0}e_k^0\varphi(x_k^0,0)&=0,\\\int_0^\infty\!\!\int\Big(uV\varphi_t+H(u,V)\varphi_x\Big)\,dx\,dt&\\+\sum_{i\in I}\int_{\gamma_i}e_i(x,t)\dot\phi_i(t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl&\\+\int u^0(x)V^0(x)\varphi(x,0)\,dx+\sum_{k\in I_0}e_k^0\dot\phi_k(0)\varphi(x_k^0,0)&=0,\end{aligned}\tag{2.7}$$

hold for all $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\infty))$.

Theorem 2.2. *Let us assume that $\Omega\subset\mathbb{R}\times(0,\ \infty)$ is some region cut by a smooth curve Γ into a left- and right-hand parts $\Omega_\mp$, $(u(x,t),v(x,t))$ and Γ is a generalized*

δ-shock wave type solution of system (1.2) *and* $(u(x,t), v(x,t))$ *is smooth in* $\Omega_{\pm}$. *Then the Rankine–Hugoniot conditions for δ-shocks*

$$\begin{aligned} \dot{e}(t) &= \Big([G(u,v)] - [v]\dot{\phi}(t)\Big)\Big|_{x=\phi(t)}, \\ \frac{d\big(e(t)\dot{\phi}(t)\big)}{dt} &= \Big([H(u,v)] - [uv]\dot{\phi}(t)\Big)\Big|_{x=\phi(t)}, \end{aligned} \tag{2.8}$$

hold along Γ.

Theorem 2.2 is proved similarly to Theorem 2.1.

Remark 2.1. As was first pointed out in [13], the Cauchy problem for system (1.2) is well-posed if in addition to the initial data $(u^0(x), v^0(x))$ *we add the initial velocities* $\dot{\phi}_k(0)$, $k \in I_0$. This modification is a direct consequence of the fact that in this case, according to (2.8), the trajectory of a singularity $x = \phi(t)$ and the coefficient of the δ-function $e(t)$ are determined by a system of *second-order equations*. For one-dimensional system of "zero-pressure gas dynamics" (1.9) this problem was considered in detail in [13]. In particular, the results [13, Theorem 4.4, Corollary 4.5.] related to system (1.9) coincide with the analogous statement from [7], [32], [55] if we *identify* the velocity on the discontinuity line $x = \phi(t)$ in formula (1.12) with the phase velocity of nonlinear wave:

$$u_\delta(t) = \dot{\phi}(t).$$

Let us note that the Rankine–Hugoniot conditions (2.8) are analogous to the Rankine–Hugoniot conditions [60, (3.7)].

The systems of *δ-shocks* integral identities (2.1) and (2.7) are *natural generalization* of the *usual* system of integral identities (1.3). The integral identities (2.1) differ from integral identities (1.3) by an additional term

$$\int_\Gamma e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl = \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl$$

in the second identity. This term appears due to the *Rankine–Hugoniot deficit*. The integral identities (2.7) differ from integral identities (1.3) by additional terms

$$\int_\Gamma e(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl, \quad \int_\Gamma e(x,t)\dot{\phi}(t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl = \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\dot{\phi}_i(t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\,dl.$$

2.2. **Geometrical and physical sense of δ-shock Rankine–Hugoniot conditions.** It is well known that if a pair of functions $\big(u(x,t), v(x,t)\big) \in L^\infty\big(\mathbb{R} \times (0,\infty); \mathbb{R}^2\big)$ compactly supported with respect to x is a generalized solution of system (1.1) then integrals of the solution on the whole space

$$\int u(x,t)\,dx = \int u^0(x)\,dx, \qquad \int v(x,t)\,dx = \int v^0(x)\,dx, \qquad t \ge 0 \tag{2.9}$$

(that is, the total area, mass, momentum, energy, etc.) are independent of time, where $(u^0(x), v^0(x))$ is initial data (see Fig. 1.).

For a δ-shock wave type solution this fact *does not hold*. However, there is a "generalized" analog of conservation laws (2.9).

Denote by

$$\begin{array}{rcl} S_u(t) & = & \int_{-\infty}^{\phi(t)} u(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)\,dx, \\ S_v(t) & = & \int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx, \\ S_{uv}(t) & = & \int_{-\infty}^{\phi(t)} u(x,t)v(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)v(x,t)\,dx, \\ S_u(0) & = & \int_{-\infty}^{0} u^0(x)\,dx + \int_{0}^{+\infty} u^0(x)\,dx, \\ S_v(0) & = & \int_{-\infty}^{0} v^0(x)\,dx + \int_{0}^{+\infty} v^0(x)\,dx, \\ S_{uv}(0) & = & \int_{-\infty}^{0} u^0(x)v^0(x)\,dx + \int_{0}^{+\infty} u^0(x)v^0(x)\,dx, \end{array} \tag{2.10}$$

the areas under the graphs $y = u(x,t)$, $y = V(x,t)$, $y = u(x,t)V(x,t)$, and $y = u^0(x)$, $y = V^0(x)$, $y = u^0(x)V^0(x)$, respectively, where $x = \phi(t)$ is a line in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\}$ issued from $\phi(0) = 0$.

Theorem 2.3. ([54]) *Let the pair of distributions $\big(u(x,t), v(x,t)\big)$ be a generalized δ-shock wave type solution of the Cauchy problem* (1.1) *with δ-shock wave type initial data, where $v(x,t) = V(x,t) + e(t)\delta(\Gamma)$, $\Gamma = \{(x,t) : x = \phi(t)\}$ is the discontinuity line, and $u(x,t)$, $V(x,t)$ are compactly supported functions with respect to x. Then the following balance relations hold:*

$$\dot{S}_u(t) = 0, \qquad \dot{S}_v(t) = -\dot{e}(t), \tag{2.11}$$

where

$$\dot{e}(t) = \Big([G(u,v)] - [v]\frac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)}$$

is the Rankine–Hugoniot deficit. Thus,

$$\begin{aligned} &\int_{-\infty}^{\phi(t)} u(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)\,dx \\ &\qquad = \int_{-\infty}^{0} u^0(x)\,dx + \int_{0}^{+\infty} u^0(x)\,dx, \\ &\int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx + e(t) \\ &\qquad = \int_{-\infty}^{0} v^0(x)\,dx + \int_{0}^{+\infty} v^0(x)\,dx + e^0, \end{aligned} \tag{2.12}$$

where e^0 is an initial amplitude of the δ-function.

Proof of Theorem 2.3. Let us prove the second relation (2.11). We denote $v_{\pm} = \lim_{x\to\phi(t)\pm 0} v(x,t)$. Differentiating the second relation (2.10) and using the second equation of system (1.1), we obtain

$$\begin{aligned} \dot{S}_v(t) &= v_-\dot{\phi}(t) - v_+\dot{\phi}(t) + \int_{-\infty}^{\phi(t)} v_t(x,t)\,dx + \int_{\phi(t)}^{+\infty} v_t(x,t)\,dx \\ &= [v]\Big|_{x=\phi(t)}\dot{\phi}(t) - \int_{-\infty}^{\phi(t)} \big(G(u,v)\big)_x\,dx - \int_{\phi(t)}^{+\infty} \big(G(u,v)\big)_x\,dx \\ &= [v]\Big|_{x=\phi(t)}\dot{\phi}(t) - [G(u,v)]\Big|_{x=\phi(t)} \\ &\qquad + G\big(u(-\infty,t), v(-\infty,t)\big) - G\big(u(+\infty,t), v(+\infty,t)\big). \end{aligned}$$

Taking into account that

$$G\big(u(-\infty,t), v(-\infty,t)\big) = G\big(u(+\infty,t), v(+\infty,t)\big) = G(0,0) = 0$$

and using the Rankine–Hugoniot conditions (2.3), we obtain

$$\dot{S}_v(t) = \left([v] \frac{[F(u,v)]}{[u]}\Big|_{x=\phi(t)} - [G(u,v)] \right)\Big|_{x=\phi(t)}.$$

The first relation (2.11) is the well-known relation for $\in L^\infty$-generalized solutions of conservation laws. The proof of this relation is carried out in the same way. Integrating expressions (2.11), we obtain (2.12). □

From the second relation (2.12), we can see that the sense of amplitude $e(t)$ of δ function is the *"area" of the discontinuity line*. Moreover, the *"total area"* $S_v(t)+e(t)$ *is independent of time*.

Consider the geometric aspect of δ-shock formation from sufficiently smooth compactly supported initial data (u^0, v^0) for system (1.1).

It is well known that the solution u and v must become *multivalued* at finite time. Any *multivalued* part of the wave profile must be replaced by an appropriate discontinuity. Construction for the position of shock in a breaking wave was considered in [59, 2.8.]. Construction for the position of δ-shock in a breaking wave will be given below. Let $A_u(t)$, $A_v(t)$ be the areas of the lobes to the left of discontinuity, and $B_u(t)$, $B_v(t)$ be the areas of the lobes to the right of discontinuity.

Let $t = t^*$ be the time of δ-shock formation. Then, according to (2.12) (for $t = t^*$) the *correct* initial positions for δ-shock discontinuities in u and v are such that these discontinuities must cut off lobes of equal area, as on Fig. 1.. If $t > t^*$, according to (2.12), the *correct* positions for δ-shock discontinuities in u and v are such that the discontinuity in u must cut off lobes of equal area $B_u(t) = A_u(t)$ (see Fig. 1.), while the discontinuity in v must cut off lobes whose areas satisfy the relation $B_v(t) = A_v(t) + e(t)$ (see Fig. 2.).

It remains to note that at the time $t = t^*$ of δ-shock wave formation the area $S_v(t)$ is a *continuous* function with respect to t but its derivative has a *jump*.

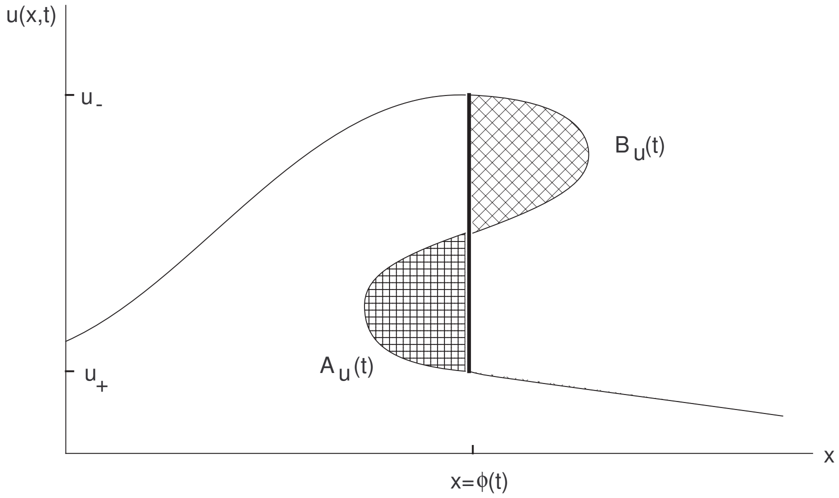

FIGURE 1. Equal area construction for the position of δ-shock in breaking wave $u(x,t)$.

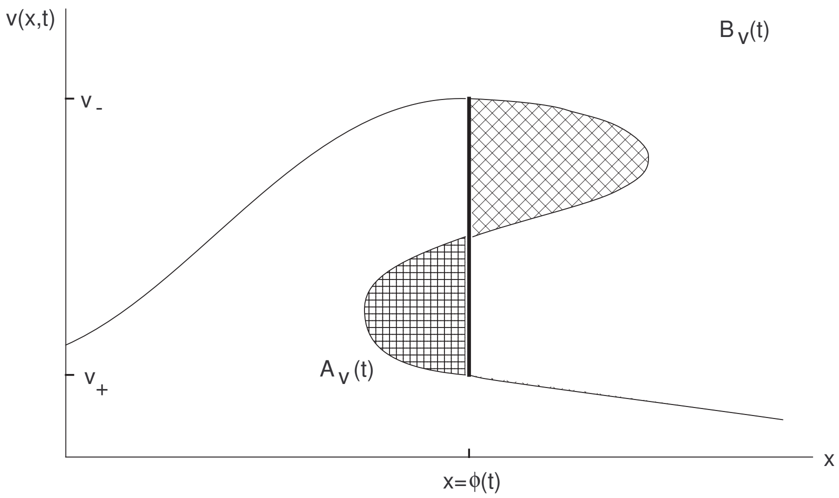

FIGURE 2. Nonequal area construction for the position of δ-shock in breaking wave $v(x,t)$.

Repeating the proof of Theorem 2.3 almost word for word, we obtain the following assertion.

Theorem 2.4. ([54]) *Let the pair of distributions* $\big(u(x,t), v(x,t)\big)$ *be a generalized* δ*-shock wave type solution of the Cauchy problem* (1.2) *with* δ*-shock wave type initial data, where* $v(x,t) = V(x,t) + e(t)\delta(\Gamma)$, $\Gamma = \{(x,t) : x = \phi(t)\}$ *is the discontinuity line, and* $u(x,t)$, $V(x,t)$ *are compactly supported functions with respect to* x*. Then the following balance relations hold:*

$$\dot{S}_v(t) = -\dot{e}(t), \qquad \dot{S}_{uv}(t) = -\frac{d\big(e(t)\dot{\phi}(t)\big)}{dt}, \tag{2.13}$$

where $\dot{e}(t)$, $\frac{d\left(e(t)\dot{\phi}(t)\right)}{dt}$ *are defined by system* (2.8). *Thus,*

$$
\begin{aligned}
&\int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx + e(t) \\
&\qquad\qquad = \int_{-\infty}^{0} v^0(x)\,dx + \int_{0}^{+\infty} v^0(x)\,dx + e^0, \\
&\int_{-\infty}^{\phi(t)} u(x,t)v(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)v(x,t)v(x,t)\,dx + e(t)\dot{\phi}(t) \\
&\qquad = \int_{-\infty}^{0} u^0(x)v^0(x)\,dx + \int_{0}^{+\infty} u^0(x)v^0(x)\,dx + e^0\dot{\phi}(0),
\end{aligned}
\tag{2.14}
$$

where e^0 *is the initial amplitude of* δ*-function,* $\dot{\phi}(0)$ *is the initial velocity of* δ*-shock.*

According to Theorem 2.4, the *"total areas"* $S_v(t)+e(t)$ and $S_{uv}(t)+e(t)\dot{\phi}(t)$ are independent of time.

The geometric aspect of δ-shock wave formation for system (1.2) can be considered in the same way as that for system (1.1) above.

Consider the case of "zero-pressure gas dynamics" system (1.9). This system is a particular case of system (1.2), where $G(u,v)=uv$, $H(u,v)=vu^2$. In this case $v(x,t)\ge 0$ is density, and $u(x,t)$ is velocity and hence, the area $S_v(t)=M(t)$ is mass, and the area $S_{uv}(t)=p(t)$ is momentum.

As has already been pointed in [54],

$$\dot{e}(t) > 0. \tag{2.15}$$

Indeed, according to (2.8), the Rankine–Hugoniot conditions have the following form

$$
\begin{aligned}
\dot{e}(t) &= [uv] - [v]\dot{\phi}(t)\Big|_{x=\phi(t)}, \\
\frac{d\left(e(t)\dot{\phi}(t)\right)}{dt} &= [u^2v] - [uv]\dot{\phi}(t)\Big|_{x=\phi(t)}.
\end{aligned}
\tag{2.16}
$$

System (1.9) has a double eigenvalue $\lambda_1(u)=\lambda_2(u)=u$, and in this case the entropy "overcompression" condition (1.17) is

$$u_+ \le \dot{\phi}(t) \le u_-. \tag{2.17}$$

Taking (2.15) and (2.16) into account, we see that

$$\dot{e}(t) = v_-\big(u_- - \dot{\phi}(t)\big) + v_+\big(\dot{\phi}(t) - u_+\big),$$

i.e., the inequality (2.15) holds.

According to Theorem 2.4, if (u,v) is compactly supported generalized δ-shock wave type solution of "zero-pressure gas dynamics" system, we have the following mass and momentum balance relations

$$\dot{e}(t) = -\dot{M}(t), \qquad \frac{d\left(e(t)\dot{\phi}(t)\right)}{dt} = -\dot{p}(t), \tag{2.18}$$

and

$$
\begin{aligned}
M(t) + e(t) &= M(0) + e^0, \\
p(t) + e(t)\dot{\phi}(t) &= p(0) + e^0\phi(0),
\end{aligned}
\tag{2.19}
$$

where $M(0)=S_v(0)$, $p(0)=S_{uv}(0)$ are initial mass and momentum, respectively. In addition, equations (2.18) and (2.15) imply

$$\dot{M}(t) < 0, \tag{2.20}$$

i.e., the mass under the graph $y=V(x,t)$ is a monotonically decreasing function.

From formulas (2.19), we can see that the sense of amplitude $e(t)$ of δ function is the *"mass" of discontinuity line*, and the sense of the term $e(t)\dot{\phi}(t)$ is the *"momentum" of discontinuity line*. Moreover, the *"total mass"* $M(t)+e(t)$ and the *"total momentum"* $p(t)+e(t)\dot{\phi}(t)$ are independent of time.

In the special case of the initial data $M(0)=-e^0$, $p(0)=-e^0\phi(0)$, from (2.19) we can readily see that the discontinuity point $x=\phi(t)$ moves at the velocity

$$\dot{\phi}(t)=\frac{p(t)}{M(t)}, \tag{2.21}$$

i.e., in such a way as if the total mass were concentrated at the point $x=\phi(t)$. Thus the point $x=\phi(t)$ can be in a sense considered as the system barycenter.

In view of (2.20) and (2.19), it is clear that in the finite time interval $\tilde{t}$ the whole mass $M(0)+e^0$ will be concentrated at the point $x=\phi(\tilde{t})$ of the discontinuity line $x=\phi(t)$. After that we have vacuum states $v_-=v_+=0$ everywhere except for the discontinuity line, and according to (2.16), the above mentioned point of the mass $e(\tilde{t})=M(0)+e^0$ will move with the velocity $\dot{\phi}(\tilde{t})$ along the straight line

$$x=\phi(t)=\dot{\phi}(\tilde{t})\big(t-\tilde{t}\big)+\phi(\tilde{t}).$$

The model of "zero-pressure gas dynamics" can be described at a discrete level by a finite collection of particles. In view of (2.15) and (2.20), the *mass transportation* from area $S_v(t)$ to the discontinuity curve is going on. Thus, the particles stick more and more as the time increases, i.e., the *concentration process* on the discontinuity curve $x=\phi(t)$ is going on. Thus at collision the colliding particles get stuck together and form a new massive particle.

2.3. **Weak asymptotic solutions.** Now we introduce a definition of a *weak asymptotic solution*, which is one of the most important notions in the *weak asymptotics method*.

Denote by $O_{\mathcal{D}'}(\varepsilon^\alpha)$ the collection of distributions $f(x,t,\varepsilon)\in\mathcal{D}'(\mathbb{R}_x)$ such that

$$\langle f(x,t,\varepsilon),\ \psi(x)\rangle=O(\varepsilon^\alpha),$$

for any test function $\psi(x)\in\mathcal{D}(\mathbb{R}_x)$. Moreover, $\langle f(x,t,\varepsilon),\ \psi(x)\rangle$ is a continuous function in t, where the estimate $O(\varepsilon^\alpha)$ is understood in the standard sense and is uniform with respect to t. The relation $o_{\mathcal{D}'}(\varepsilon^\alpha)$ is understood in a corresponding way.

Definition 2.3. ([13], [14]) A pair of functions $\big(u(x,t,\varepsilon),v(x,t,\varepsilon)\big)$ smooth as $\varepsilon>0$ is called a *weak asymptotic solution* of systems (1.1) or (1.2) with the initial data $(u^0(x),v^0(x))$ if

$$\begin{array}{rcl}
\int L_1[u(x,t,\varepsilon),v(x,t,\varepsilon)]\psi(x)\,dx &=& o(1),\\
\int L_2[u(x,t,\varepsilon),v(x,t,\varepsilon)]\psi(x)\,dx &=& o(1),\\
\int \Big(u(x,0,\varepsilon)-u^0(x)\Big)\psi(x)\,dx &=& o(1),\\
\int \Big(v(x,0,\varepsilon)-v^0(x)\Big)\psi(x)\,dx &=& o(1),\quad \varepsilon\to+0,
\end{array}$$

for all $\psi(x)\in\mathcal{D}(\mathbb{R}_x)$, i.e.,

$$\begin{array}{rcl}
L_1[u(x,t,\varepsilon),v(x,t,\varepsilon)] &=& o_{\mathcal{D}'}(1),\\
L_2[u(x,t,\varepsilon),v(x,t,\varepsilon)] &=& o_{\mathcal{D}'}(1),\\
u(x,0,\varepsilon) &=& u^0(x)+o_{\mathcal{D}'}(1),\\
v(x,0,\varepsilon) &=& v^0(x)+o_{\mathcal{D}'}(1),\quad \varepsilon\to+0,
\end{array} \tag{2.22}$$

where the first two estimates are uniform in t.

Within the framework of the *weak asymptotics method*, we find the *generalized δ-shock wave type solution* $(u(x,t), v(x,t))$ to the Cauchy problem as the limit

$$u(x,t) = \lim_{\varepsilon\to+0} u(x,t,\varepsilon), \qquad v(x,t) = \lim_{\varepsilon\to+0} v(x,t,\varepsilon), \tag{2.23}$$

of the *weak asymptotic solution* $(u(x,t,\varepsilon), v(x,t,\varepsilon))$ to this problem, where limits are understood in the weak sense (in the sense of the space of distributions $\mathcal{D}'(\mathbb{R}\times[0,\,\infty))$). Constructing the *weak asymptotic solution* and multiplying the first two relations (2.22) by a test function $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,\infty))$, integrating these relations by parts and then passing to the limit as $\varepsilon\to+0$, we see that the pair of distributions (2.23) satisfy integral identities (2.1) or (2.7). Thus, we will prove that the left-hand sides of the following relations

$$\begin{aligned}
\lim_{\varepsilon\to+0}\int_0^\infty\int L_1[u(x,t,\varepsilon), v(x,t,\varepsilon)]\varphi(x,t)\,dx\,dt &= 0,\\
\lim_{\varepsilon\to+0}\int_0^\infty\int L_2[u(x,t,\varepsilon), v(x,t,\varepsilon)]\varphi(x,t)\,dx\,dt &= 0,
\end{aligned}$$

coincide with the left-hand sides of relations (2.1) or (2.7) for all test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,\infty))$.

3. Propagation of δ-shocks in system (1.14)

3.1. **Choosing corrections.** Let us consider the propagation of a single δ-shock wave of system (1.14), i.e., consider the Cauchy problem (1.14), (1.15). In this case the graph Γ contains only one arc. Suppose this arc has the form $\Gamma=\{(x,t): x=\phi(t)\}$, and hence $e(x,t)\big|_\Gamma = e(t)$.

The eigenvalues of the characteristic matrix of system (1.14) are

$$\lambda_{1,2}(u) = \frac{1}{2}\Big(f'(u)\pm\sqrt{\big(f'(u)\big)^2-4g'(u)}\Big), \qquad \big(f'(u)\big)^2\ge 4g'(u).$$

We assume that the "overcompression" condition (1.17) is satisfied.

In the framework of our approach, we will seek a *δ-shock wave type solution* in the form of the *singular ansatz* (1.16) and construct a *weak asymptotic solution* in the form of the *smooth ansatz* (1.20).

Since a *generalized δ-shock wave type solution* is defined as a weak limit (2.23) of (1.20), in view of the estimates (1.18), the corrections $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$ *do not make a contribution* to the generalized solution of the problem. However, according to (3.9), (3.10), these terms *make a contribution* to the weak asymptotics of the superposition $f\big(u(x,t,\varepsilon)\big)-v(x,t,\varepsilon)$ and $g\big(u(x,t,\varepsilon)\big)$, and hence play an *essential* role in the construction of the generalized solution to the problem. Without introducing these terms, we cannot solve the Cauchy problem with arbitrary initial data (see Remark 3.1 below).

Here we choose the *corrections* in the special form

$$\begin{aligned}
R_u(x,t,\varepsilon) &= P(t)\frac{1}{\varepsilon^{1/n}}\Omega_P\Big(\frac{-x+\phi(t)}{\varepsilon}\Big)\\
&\qquad\qquad +Q(t)\frac{1}{\varepsilon^{1/(n+1)}}\Omega_Q\Big(\frac{-x+\phi(t)}{\varepsilon}\Big),\\
R_v(x,t,\varepsilon) &= 0,
\end{aligned} \tag{3.1}$$

where $P(t)$, $Q(t)$ are the desired functions, $\frac{1}{\varepsilon}\Omega_P^n\big(x/\varepsilon\big)$, $\frac{1}{\varepsilon}\Omega_Q^{n+1}\big(x/\varepsilon\big)$ are regularizations (1.21) of the delta function, mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ have properties (a)–(c). Consequently, the estimates (1.18) hold.

It is clear that we can construct the *weak asymptotic solution*, using the *corrections* of a different structure.

In addition to (3.1), we can choose mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ such that

$$\begin{aligned} \int \Omega_P^k(\eta)\Omega_Q^{n+1-k}(\eta)\,d\eta = 0, \qquad & k = 1, 2, \dots n+1, \\ \int \Omega_Q^{n+1}(\eta)\,d\eta \neq 0, \qquad & \int \Omega_P^n(\eta)\,d\eta \neq 0. \end{aligned} \tag{3.2}$$

In particular, for Keyfitz–Kranzer system (1.8) $f(u) = u^2$, $g(u) = \frac{1}{3}u^3 - u$ and relations (3.2) have the form

$$\int \Omega_P^3(\eta)\,d\eta = 0, \quad \int \Omega_P^2(\eta)\Omega_Q(\eta)\,d\eta = 0, \quad \int \Omega_P(\eta)\Omega_Q^2(\eta)\,d\eta = 0.$$

In this case, for example, we can choose $\Omega_P(\eta) = \eta e^{-\eta^2}$, $\Omega_Q(\eta) = \big(1 - 2\eta^2\big)e^{-\eta^2}$. This example was proposed by V. I. Polischook.

3.2. **A weak asymptotic solution.** The first step of our approach is to find a *weak asymptotic solution* of the Cauchy problem (1.14), (1.15).

Theorem 3.1. ([54]) *Let*

$$\lambda_+(u_+^0(0)) \le \frac{[f(u^0)] - [v^0]}{[u^0]}\bigg|_{x=0} \le \lambda_-(u_-^0(0)), \tag{3.3}$$

($u_+^0 = u_0^0$, $u_-^0 = u_0^0 + u_1^0$) *then there exists* $T > 0$ *such that, for* $t \in [0, T)$, *the Cauchy problem* (1.14), (1.15) *has a weak asymptotic solution* (1.20), (3.1), (3.2) *if and only if*

$$\begin{array}{rcl} L_{11}[u_+, v_+] &=& 0, \quad x > \phi(t), \\ L_{11}[u_-, v_-] &=& 0, \quad x < \phi(t), \\ L_{12}[u_+, v_+] &=& 0, \quad x > \phi(t), \\ L_{12}[u_-, v_-] &=& 0, \quad x < \phi(t), \\ \dot\phi(t) &=& \frac{[f(u)]-[v]}{[u]}\Big|_{x=\phi(t)}, \\ \dot e(t) &=& \Big([g(u)] - [v]\frac{[f(u)]-[v]}{[u]}\Big)\Big|_{x=\phi(t)}, \end{array} \tag{3.4}$$

$$\begin{array}{rcl} P(t) &=& \Big(\dfrac{e(t)}{aA_n}\Big)^{1/n}, \\ Q(t) &=& \bigg\{\dfrac{e(t)}{cB_{n+1}}\Big(\dfrac{[f(u)] - [v]}{[u]} - \dfrac{1}{A_n}\Big(B_n \\ && +\Big(\big(1 - \dfrac{b}{a}\big)u_+ + \dfrac{b}{a}u_-\Big)(n+1)B_{n+1}\Big)\Big)\bigg|_{x=\phi(t)}\bigg\}^{1/(n+1)}, \end{array} \tag{3.5}$$

where $u_+ = u_0$, $v_+ = v_0$, $u_- = u_0 + u_1$, $v_- = v_0 + v_1$,

$$\begin{array}{rcl} a &=& \displaystyle\int \Omega_P^n(\eta)\,d\eta > 0, \\ b &=& \displaystyle\int \omega_{0u}(\eta)\Omega_P^n(\eta)\,d\eta, \\ c &=& \displaystyle\int \Omega_Q^{n+1}(\eta)\,d\eta \neq 0. \end{array} \tag{3.6}$$

The initial data for system (3.4), (3.5) *are defined from* (1.15), *and*

$$
\begin{array}{rcl}
e(0) & = & e^0, \qquad \phi(0)=0, \\
P(0) & = & \Big(\dfrac{e^0}{aA_n}\Big)^{1/n}, \\
Q(0) & = & \Big\{\dfrac{e^0}{cB_{n+1}}\Big(\dfrac{[f(u^0)]-[v^0]}{[u^0]}-\dfrac{1}{A_n}\Big(B_n \\
 & & +\Big((1-\dfrac{b}{a})u^0_+ + \dfrac{b}{a}u^0_-\Big)(n+1)B_{n+1}\Big)\Big)\Big\}^{1/(n+1)}\Big|_{x=0}.
\end{array}
$$

Proof of Theorem 3.1. With the help of (3.2), (3.6) and the fifth and sixth relations (6.36) from Lemma 6.3, we find the following weak asymptotics

$$
\begin{array}{rcl}
R^k(x,t,\varepsilon) & = & o_{\mathcal{D}'}(1), \quad k\le n-1, \\
R^n(x,t,\varepsilon) & = & aP^n(t)\delta(-x+\phi(t))+o_{\mathcal{D}'}(1), \\
R^{n+1}(x,t,\varepsilon) & = & cQ^{n+1}(t)\delta(-x+\phi(t))+o_{\mathcal{D}'}(1), \\
H(-x+\phi(t),\varepsilon)R^n(x,t,\varepsilon) & = & bP^n(t)\delta(-x+\phi(t))+o_{\mathcal{D}'}(1),
\end{array} \tag{3.7}
$$

where a, b, c are defined by (3.6).

Using the first, fifth and sixth relations (6.36) from Lemma 6.3, one can calculate

$$
\begin{array}{rcl}
\big(u(x,t,\varepsilon)\big)^k & = & u_0^k+\big((u_0+u_1)^k-u_0^k\big)H(-x+\phi(t)) \\
 & & \qquad\qquad +o_{\mathcal{D}'}(1), \quad k\le n-1, \\
\big(u(x,t,\varepsilon)\big)^n & = & u_0^n+\big((u_0+u_1)^n-u_0^n\big)H(-x+\phi(t)) \\
 & & \qquad\qquad +R^n(x,t,\varepsilon)+o_{\mathcal{D}'}(1), \\
\big(u(x,t,\varepsilon)\big)^{n+1} & = & u_0^{n+1} \\
 & & +\big((u_0+u_1)^{n+1}-u_0^{n+1}\big)H(-x+\phi(t)) \\
 & & \quad +(n+1)\big(u_0+u_1H(-x+\phi(t),\varepsilon)\big) \\
 & & \times R^n(x,t,\varepsilon)+R^{n+1}(x,t,\varepsilon)+o_{\mathcal{D}'}(1).
\end{array} \tag{3.8}
$$

In particular, we have

$$
\begin{array}{rcl}
\big(u(x,t,\varepsilon)\big)^2 & = & u_0^2+\Big((u_0+u_1)^2-u_0^2\Big)H(-x+\phi(t)) \\
 & & \qquad +aP^2(t)\delta(-x+\phi(t))+o_{\mathcal{D}'}(1), \\
\big(u(x,t,\varepsilon)\big)^3 & = & u_0^3+\Big((u_0+u_1)^3-u_0^3\Big)H(-x+\phi(t)) \\
 & & +\Big(3(au_0+bu_1)P^2(t)+cQ^3(t)\Big)\delta(-x+\phi(t)) \\
 & & \qquad\qquad +o_{\mathcal{D}'}(1), \quad \varepsilon\to+0.
\end{array}
$$

Taking into account relations (3.7), (3.8), we obtain the following weak asymptotics

$$
\begin{array}{c}
f\big(u(x,t,\varepsilon)\big)=f(u_0)+\Big(f(u_0+u_1)-f(u_0)\Big)H(-x+\phi(t)) \\
\qquad +aA_nP^n(t)\delta(-x+\phi(t))+o_{\mathcal{D}'}(1),
\end{array} \tag{3.9}
$$

$$
\begin{array}{l}
g\big(u(x,t,\varepsilon)\big)=g(u_0)+\Big(g(u_0+u_1)-g(u_0)\Big)H(-x+\phi(t)) \\
\qquad +\Big\{aB_nP^n(t)+(n+1)\big(au_0+bu_1\big)B_{n+1}P^n(t) \\
\qquad +cB_{n+1}Q^{n+1}(t)\Big\}\delta(-x+\phi(t))+o_{\mathcal{D}'}(1), \quad \varepsilon\to+0.
\end{array} \tag{3.10}
$$

Substituting the smooth ansatz (1.20) and relations (3.9), (3.10) into the left-hand side of system (1.14), we obtain, up to $o_{\mathcal{D}'}(1)$, the following relations

$$
\begin{array}{l}
L_{11}[u(x,t,\varepsilon),v(x,t,\varepsilon)] \\
\qquad =L_{11}[u_+,v_+]+\Big\{\dfrac{\partial[u]}{\partial t}+\dfrac{\partial}{\partial x}\big[f(u)-v\big]\Big\}H(-x+\phi(t))
\end{array}
$$

$$+\Big\{[u]\dot{\phi}(t) - \big[f(u) - v\big]\Big\}\delta(-x+\phi(t))$$

$$+\Big\{e(t) - aA_nP^n(t)\Big\}\delta'(-x+\phi(t)) + o_{\mathcal{D}'}(1), \tag{3.11}$$

$$L_{12}[u(x,t,\varepsilon), v(x,t,\varepsilon)]$$

$$= L_{22}[u_+, v_+] + \Big\{\frac{\partial[v]}{\partial t} + \frac{\partial}{\partial x}\Big[g(u)\Big]\Big\}H(-x+\phi(t))$$

$$= \Big\{[v]\dot{\phi}(t) + \dot{e}(t) - \Big[g(u)\Big]\Big\}\delta(-x+\phi(t))$$

$$+\Big\{e(t)\dot{\phi}(t) - aB_nP^n(t) - (n+1)\big(au_+ + b[u]\big)B_{n+1}P^n(t)$$

$$-cB_{n+1}Q^{n+1}(t)\Big\}\delta'(-x+\phi(t)) + o_{\mathcal{D}'}(1), \quad \varepsilon \to +0. \tag{3.12}$$

Here we take into account estimates (1.18).

Setting the left-hand side of (3.11), (3.12) equal to zero, we obtain the necessary and sufficient conditions for the first two equalities (2.22), i.e., systems (3.4), (3.5).

Consider the Cauchy problem

$$\begin{array}{rcl} L_{11}[u,V] &=& 0, \quad u(x,0) = u^0(x), \\ L_{12}[u,V] &=& 0, \quad V(x,0) = V^0(x) = v_0^0(x) + v_1^0(x)H(-x), \end{array} \tag{3.13}$$

assuming that condition (3.3) holds. The last condition means that $(u^0(x), V^0(x))$ is entropy initial data.

According to [38, Ch.4.2.], we extend a pair of functions

$$\begin{array}{rr} \big(u_+^0(x) = u_0^0(x),\ V_+^0(x) = v_0^0(x)\big), & x \le 0, \\ \big(u_-^0(x) = u_0^0(x) + u_1^0(x),\ V_-^0(x) = v_0^0(x) + v_1^0(x)\big), & x \ge 0, \end{array}$$

in a bounded C^1 fashion and continue to denote the extended pair of functions by $\big(u_\pm^0(x), V_\pm^0(x)\big)$. By $\big(u_\pm(x,t), V_\pm(x,t)\big)$ we denote the C^1 solutions of the problems

$$\begin{array}{rclcrcl} L_{11}[u,V] &=& 0, & & u_\pm(x,0) &=& u_\pm^0(x), \\ L_{12}[u,V] &=& 0, & & V_\pm(x,0) &=& V_\pm^0(x), \end{array}$$

which, according to [38, Ch.2.1.], [44, Ch.I,§8.], exist for small enough time interval $[0,\ T_1]$. The pair $\big(u_\pm(x,t), V_\pm(x,t)\big)$ determines a two-sheeted covering of the plane (x,t). Next, we define the function $x = \phi(t)$ as a solution of the problem

$$\dot{\phi}(t) = \frac{f(u_-(x,t)) - f(u_+(x,t)) - V_-(x,t)) + V_+(x,t)}{u_-(x,t) - u_+(x,t)}\Bigg|_{x=\phi(t)},$$

$\phi(0) = 0$. It is clear that there exists a unique function $\phi(t)$ for sufficiently short times $[0,\ T_2]$. Therefore, for $T = \min(T_1, T_2)$ we define the shock solution by

$$(u(x,t), V(x,t)) = \left\{\begin{array}{lcl} (u_+(x,t), V_+(x,t)), & x &>& \phi(t), \\ (u_-(x,t), V_-(x,t)), & x &<& \phi(t). \end{array}\right.$$

Thus the first five equations of system (3.4) define a unique solution of the Cauchy problem (3.13) for $t \in [0,\ T)$. Solving this problem, we obtain $u(x,t)$, $V(x,t)$, $\phi(t)$.

Then, substituting these functions into (3.4), (3.5), we obtain $e(t)$, $v(x,t) = V(x,t) + e(t)\delta(-x+\phi(t))$, and $P(t)$, $Q(t)$. It is clear that mollifiers $\Omega_P(\eta)$, $\Omega_Q(\eta)$ can be chosen to satisfy relations (3.2). $\square$

3.3. **A generalized solution.** At the second step, using the *weak asymptotic solution* constructed by Theorem 3.1, we obtain a generalized solution of the Cauchy problem (1.14), (1.15).

Theorem 3.2. *There exists $T > 0$ given by Theorem 3.1 such that the Cauchy problem (1.14), (1.15), (3.3) for $t \in [0, T)$ has a unique generalized solution*

$$\begin{array}{rcl} u(x,t) & = & u_0(x,t) + u_1(x,t)H(-x+\phi(t)), \\ v(x,t) & = & v_0(x,t) + v_1(x,t)H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{array}$$

which satisfies the integral identities (2.1):

$$\begin{array}{rcl} \displaystyle\int_0^T \int \Big(u\varphi_t + \big(f(u) - V\big)\varphi_x\Big)\,dx\,dt & & \\ \displaystyle + \int u^0(x)\varphi(x,0)\,dx & = & 0, \\ \displaystyle\int_0^T \int \Big(V\varphi_t + g(u)\varphi_x\Big)\,dx\,dt + \int V^0(x)\varphi(x,0)\,dx & & \\ \displaystyle + \int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl + e^0\varphi(0,0) & = & 0, \end{array} \qquad (3.14)$$

where $\Gamma = \{(x,t) : x = \phi(t), \quad t \in [0, T)\}$,

$$\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl = \int_0^T e(t)\frac{d\varphi(\phi(t),t)}{dt}\,dt,$$

$V(x,t) = v_0(x,t) + v_1(x,t)H(-x+\phi(t))$, functions $u_k(x,t)$, $v_k(x,t)$, $\phi(t)$, $e(t)$ are defined by system (3.4), and (see (2.6)) $\frac{d\varphi(\phi(t),t)}{dt} = \varphi_t(\phi(t),t) + \dot\phi(t)\varphi_x(\phi(t),t)$.

Proof of Theorem 3.2. According to Eqs. (3.9), (3.10), (1.20), (3.1),

$$\begin{array}{l} f\big(u(x,t,\varepsilon)\big) - v(x,t,\varepsilon) = f(u_0) - v_0 + \Big[f(u) - v\Big]H(-x+\phi(t)) \\ \qquad + \Big\{aA_nP^n(t) - e(t)\Big\}\delta(-x+\phi(t)) + o_{\mathcal{D}'}(1), \end{array} \qquad (3.15)$$

$$\begin{array}{l} g\big(u(x,t,\varepsilon)\big) = g(u_0) + \Big[g(u)\Big]H(-x+\phi(t)) \\ \qquad + \Big\{aB_nP^n(t) + (n+1)\big(au_0 + bu_1\big)B_{n+1}P^n(t) \\ + cB_{n+1}Q^{n+1}(t)\Big\}\delta(-x+\phi(t)) + o_{\mathcal{D}'}(1), \quad \varepsilon \to +0. \end{array} \qquad (3.16)$$

where the correction functions $P(t)$, $Q(t)$ are given by (3.5). Substituting $P(t)$, $Q(t)$ into expressions (3.15), (3.16) we have

$$\begin{array}{l} f\big(u(x,t,\varepsilon)\big) - v(x,t,\varepsilon) \\ \qquad = f(u_0) - v_0 + \Big[f(u) - v\Big]H(-x+\phi(t)) + o_{\mathcal{D}'}(1), \end{array} \qquad (3.17)$$

$$\begin{array}{l} g\big(u(x,t,\varepsilon)\big) = g(u_0) + \Big[g(u)\Big]H(-x+\phi(t)) \\ \qquad + e(t)\dfrac{[f(u)]}{[u]}\delta(-x+\phi(t)) + o_{\mathcal{D}'}(1), \quad \varepsilon \to +0. \end{array} \qquad (3.18)$$

By Theorem 3.1 we have the following estimates:

$$L_{11}[u(x,t,\varepsilon)v(x,t,\varepsilon)] = o_{\mathcal{D}'}(\varepsilon), \quad L_{12}[u(x,t,\varepsilon), v(x,t,\varepsilon)] = o_{\mathcal{D}'}(\varepsilon).$$

Let us apply the left-hand and right-hand sides of these relations to an arbitrary test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0, T))$. Since for $\varepsilon > 0$ the functions $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ are smooth, we obtain, integrating by parts,

$$\int_0^T \int \Big(u(x,t,\varepsilon)\varphi_t(x,t) + \Big(f(u(x,t,\varepsilon)) - v(x,t,\varepsilon) \Big) \varphi_x(x,t) \Big) \, dxdt \\ + \int u(x,0,\varepsilon)\varphi(x,0)\, dx = o(1),$$

$$\int_0^T \int \Big(v(x,t,\varepsilon)\varphi_t(x,t) + g(u(x,t,\varepsilon))\varphi_x(x,t) \Big) \, dxdt \\ + \int v(x,0,\varepsilon)\varphi(x,0)\, dx = o(1), \quad \varepsilon \to +0.$$

Passing to the limit as $\varepsilon \to +0$, and taking into account (1.20), (3.1), (3.17), (3.18), and the fact that

$$\lim_{\varepsilon \to +0} \int_0^T \int_{-\infty}^{\infty} e(t)\delta\big(-x+\phi(t),\varepsilon\big)\varphi(x,t)\, dxdt = \int_0^T e(t)\varphi(\phi(t),t)\, dt,$$

$$\lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} e(0)\delta\big(-x,\varepsilon\big)\varphi(x,0)\, dx = e(0)\varphi(0,0),$$

we obtain the integral identities (3.14).

In view of the above remark, system (3.4) has a unique solution. □

The fifth and sixth equations of systems (3.4) are the Rankine–Hugoniot conditions of δ-shocks, and the right-hand side of the sixth equation is the *Rankine–Hugoniot deficit*.

If $A_n > 0$, $e^0 \geq 0$, according to (3.5), the amplitude $e(t)$ of δ-function is positive.

Corollary 3.1. ([53]) *For* $t \in [0, \infty)$, *the Cauchy problem* (1.14), (1.15), (3.3), *with piecewise constant initial data* $u_0^0 = u_0$, $u_1^0 = u_1$, $v_0^0 = v_0$, $v_1^0 = v_1$ *has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0 + u_1 H(-x+\phi(t)), \\ v(x,t) &= v_0 + v_1 H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

where

$$\begin{aligned} \phi(t) &= \tfrac{[f(u)]-[v]}{[u]} t, \\ e(t) &= e^0 + \Big([g(u)] - [v]\tfrac{[f(u)]-[v]}{[u]} \Big) t. \end{aligned}$$

Applying Theorem 3.2 to Keyfitz–Kranzer system (1.8), we have the following statement.

Theorem 3.3. ([54]) *There exists* $T > 0$ *given by Theorem* 3.1 *such that the Cauchy problem* (1.8), (1.15),

$$u_0^0(0) + 1 \leq \frac{[(u^0)^2] - [v^0]}{[u^0]}\bigg|_{x=0} \leq u_0^0(0) + u_1^0(0) - 1, \tag{3.19}$$

for $t \in [0, T)$ *has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) H(-x+\phi(t)), \\ v(x,t) &= v_0(x,t) + v_1(x,t) H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

which satisfies the integral identities (3.14), *where* $f(u) = u^2$, $g(u) = \frac{1}{3}u^3 - u$, *and functions* $u_k(x,t)$, $v_k(x,t)$, $\phi(t)$, $e(t)$ *are defined by system* (3.4).

Corollary 3.2. ([53], [54]) *For* $t \in [0, \infty)$*, the Cauchy problem* (1.8)*,* (1.15)*,* (3.19)*, with piecewise constant initial data* $u_0^0 = u_0$*,* $u_1^0 = u_1$*,* $v_0^0 = v_0$*,* $v_1^0 = v_1$ *has a unique generalized solution*

$$\begin{aligned} u(x,t) &= u_0 + u_1 H(-x+\phi(t)), \\ v(x,t) &= v_0 + v_1 H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \end{aligned}$$

where

$$\begin{aligned} \phi(t) &= \tfrac{[u^2]-[v]}{[u]} t, \\ e(t) &= e^0 + \Big(\tfrac{[u^3]}{3} - [u] - [v]\tfrac{[u^2]-[v]}{[u]}\Big) t. \end{aligned}$$

Moreover, if $e^0 = 0$*, the Rankine–Hugoniot deficit is positive:*

$$\dot e(t) = \frac{[u^3]}{3} - [u] - [v]\frac{[u^2]-[v]}{[u]} > 0$$

(as in [26]).

Here $\dot e(t) > 0$, according to the seventh equation (3.5).

Remark 3.1. To find a generalized solution of the Cauchy problem (1.14), (1.15) and (1.8), (1.15) we construct a weak asymptotic solution of problem (1.20), where the functions $u_k(x,t)$, $v_k(x,t)$, $\phi(t)$, $e(t)$, $k = 0, 1$ are determined by relations (3.4), and the functions $\omega_{0u}(\eta)$, $\Omega_P(\eta)$, $\Omega_Q(\eta)$, $P(t)$, $Q(t)$ are determined by relations (3.2), (3.5), (3.6).

In view of estimate (1.18) (see also formulas (3.17), (3.18)), the generalized solution (1.16) of the Cauchy problem *does not depend* on correction functions $P(t)$, $Q(t)$. However, the correction term

$$P(t)\frac{1}{\varepsilon^{1/n}}\Omega_P\Big(\frac{-x+\phi(t)}{\varepsilon}\Big) + Q(t)\frac{1}{\varepsilon^{1/(n+1)}}\Omega_Q\Big(\frac{-x+\phi(t)}{\varepsilon}\Big),$$

plays an important role in constructing δ-shock solution.

According to (3.5), we can see that if we introduce only the first term, we can construct a weak asymptotic solution of the Cauchy problem *only* if the following the relation

$$\frac{[f(u)]-[v]}{[u]}\Big|_{x=\phi(t)} = \frac{1}{A_n}\Big(B_n + \Big(u_+\big(1-\frac{b}{a}\big) + \frac{b}{a}u_-\Big)\Big|_{x=\phi(t)}(n+1)B_{n+1}\Big), \quad (3.20)$$

holds, where the constants a, b are defined by (3.6). In the general case, relation (3.6) makes the Cauchy problem (1.14), (1.15) overdetermined.

In [26], in the framework of the Colombeau theory, an *approximate solution* of the Cauchy problem for system (1.8) with piecewise constant initial data (1.15) was constructed. It is a particular case of a weak asymptotic solution (1.20), where only a term of the type

$$P(t)\frac{1}{\sqrt{\varepsilon}}\Omega_P\Big(\frac{-x+\phi(t)}{\varepsilon}\Big)$$

is used. In this case relation (3.20) has the following form

$$\frac{u_0 + u_1 - \frac{v_1}{u_1}}{u_1} = \frac{b}{a},$$

where $a = \int \Omega_P^2(\eta)\,d\eta$, $b = \int \omega_{0u}(\eta)\Omega_P^2(\eta)\,d\eta$. This relation can be rewritten as

$$\frac{u_0 - \frac{v_1}{u_1}}{u_1} = \frac{\dot\phi(t) - u_-}{u_1} = \frac{b-a}{a}, \quad (3.21)$$

where $u_- = u_0 + u_1$. In [26] the parameter $a = \int \Omega_P^2(\eta)\,d\eta$ was set to be 1. Hence (see (3.6))

$$-1 < \frac{b-a}{a} = \int \Big(\omega_{0u}(\eta) - 1\Big)\Omega_P^2(\eta)\,d\eta < 0.$$

Here relation (3.21) coincides with the second relation of [26, Proposition 2] and the last inequality coincides with the statement of Lemma 1 from [26]. However, according to [26, Proposition 2], in this case relation (3.21) still leaves one degree of freedom, to connect $u_- = u_0 + u_1$ and $u_+ = u_0$.

If we set in (3.5) $P(t) = 0$ then $e(t) = 0$, i.e., the Cauchy problem (1.14), (1.15) cannot have δ-shock solutions. Moreover, in this case the Cauchy problem (1.14), (1.15) can be solved only if the relation

$$\Big([g(u)] - [v]\frac{[f(u)] - [v]}{[u]}\Big)\Big|_{x=\phi(t)} = 0$$

holds. Thus, without introduction δ-shocks, we have the *overdetermined* Cauchy problem.

3.4. **On singular superpositions (products) of distributions.** Let $F(u,v)$ be a smooth function, and let $u(x,t)$, $v(x,t)$ be distributions. It seems natural to introduce a *singular superposition (product) of distributions* $F(u(x,t), v(x,t))$ as the weak limit of $F(u(x,t,\varepsilon), v(x,t,\varepsilon))$, as $\varepsilon \to 0$, where $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ are regularizations of distributions $u(x,t)$, $v(x,t)$. If we construct *singular superpositions* $f\big(u(x,t)\big) - v(x,t)$, $g\big(u(x,t)\big)$ by using relations (3.15), (3.16), these superpositions *depend* on the regularizations of the Heaviside function, delta function, and the correction functions $P(t)$, $Q(t)$.

Substituting $P(t)$, $Q(t)$ from (3.5) into (3.15), (3.16), we obtain relations (3.17), (3.18). Now the weak limits of (3.17), (3.18) *do not depend* on the regularizations of the Heaviside function, or delta function, or the correction functions $P(t)$, $Q(t)$. Using formulas (3.17), (3.18), we can introduce the *"right" singular superpositions* by the following definition:

$$\begin{aligned} f\big(u(x,t)\big) - v(x,t) &\stackrel{def}{=} \lim_{\varepsilon\to+0}\Big(f\big(u(x,t,\varepsilon)\big) - v(x,t,\varepsilon)\Big) \\ &= f(u_0) - v_0 + \Big[f(u) - v\Big]H(-x+\phi(t)), \end{aligned} \tag{3.22}$$

$$\begin{aligned} g\big(u(x,t)\big) &\stackrel{def}{=} \lim_{\varepsilon\to+0}\Big(g\big(u(x,t,\varepsilon)\big)\Big) \\ &= g(u_0) + \Big[g(u)\Big]H(-x+\phi(t)) + e(t)\frac{\big[f(u)\big]}{[u]}\delta(-x+\phi(t)), \end{aligned} \tag{3.23}$$

where distributions $u(x,t)$, $v(x,t)$ are defined in (1.16) and the limits are understood in the weak sense.

Note that in (3.15), (3.16) the pair $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ is understood in the sense of *regularizations of distributions* (1.16), while in (3.17), (3.18) and (3.22), (3.23) this pair is understood in the sense of the *weak asymptotic solution* of the Cauchy problem (1.14), (1.15). It is clear that the *unique "right" singular superpositions* (3.22), (3.23) can be obtained *only by the construction of a weak asymptotic solution* of the Cauchy problem (1.14), (1.15).

As was already mentioned above, systems (1.14) and (1.8) have a *specific* property. We stress that, in contrast to systems (1.6), (1.9), in the case of systems (1.14), (1.8) we *do not define (!)* the *product of the Heaviside function and the δ-function*. Moreover, although (according to (1.16)), $u(x,t)$ *does not depend (!)* on the term $e(t)\delta(-x+\phi(t))$, the *"right" singular superposition* $g\big(u(x,t)\big)$ determined by (3.23), *does depend (!)* on this term. Thus one can say that the term $e(t)\delta(-x+\phi(t))$ "appears from nothing", and the *"right" singular superposition* $g\big(u(x,t)\big)$ is determined *in the context of solving* the Cauchy problem.

It remains to note that, since according to (3.22), (3.23), in the "specific" systems (1.14), (1.8) there are no terms of the type of (1.7), it is *impossible* to construct a *δ-shock wave type solution* for them by using the *nonconservative product* [30], [31], [42].

4. δ-Shocks in multidimensional "zero-pressure gas dynamics" system

Let us consider the propagation of a single δ-shock wave of system (1.23), i.e., consider the Cauchy problem (1.23), (1.24), (1.24).

The system (1.23) can be represented as

$$L[W]=\frac{\partial W}{\partial t}+\sum_{j=1}^{n}A_j(W)\frac{\partial W}{\partial x_j}=0, \tag{4.1}$$

where $W=(\rho,U)^T$,

$$A_1(W)=\begin{pmatrix} u_1 & \rho & 0 & \cdots & 0\\ 0 & u_1 & 0 & \cdots & 0\\ 0 & 0 & u_1 & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdots & u_1 \end{pmatrix},\dots,A_n(W)=\begin{pmatrix} u_n & 0 & 0 & \cdots & \rho\\ 0 & u_n & 0 & \cdots & 0\\ 0 & 0 & u_n & \cdots & 0\\ \cdot & \cdot & \cdot & \cdots & \cdot\\ \cdot & \cdot & \cdot & \cdots & u_n \end{pmatrix}.$$

Since in the direction $\nu=(\nu_1,\dots,\nu_n)$ the characteristic equation of $L[W]$ is

$$\begin{vmatrix} \sum_{j=1}^n\nu_ju_j-\lambda & \nu_1\rho & \cdots & \nu_n\rho\\ 0 & \sum_{j=1}^n\nu_ju_j-\lambda & \cdots & 0\\ \cdot & \cdot & \cdots & \cdot\\ 0 & 0 & \cdots & \sum_{j=1}^n\nu_ju_j-\lambda \end{vmatrix}=0,$$

this system is an *extremely degenerate* hyperbolic system with repeated eigenvalues $\lambda=\sum_{j=1}^n\nu_ju_j$. The right eigenvectors of $L[W]$ corresponding to the eigenvalue λ are

$$\begin{array}{rclllllll} r_1 & = & (1, & 0, & 0, & \dots, & 0, & 0, & 0)^T,\\ r_2 & = & (0, & \nu_n, & 0, & \dots, & 0, & 0, & -\nu_1)^T,\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot,\\ r_n & = & (0, & 0, & 0, & \dots, & 0, & \nu_n, & -\nu_{n-1})^T, \end{array}$$

and

$$\nabla_W\lambda\cdot r_j=0,\quad j=1,\dots,n.$$

Thus system (4.1) is linear degenerate.

Since (1.23) is extremely degenerate and has non-conservative form, the Cauchy problem for this system is a *nonclassical* problem. It is known that the solutions to systems (1.23), (1.13) are not always bounded for bounded and smooth initial condition $(\rho(x,0),u(x,0))$. Namely, there are two kinds of blowup mechanisms. The density ρ itself and the gradient of the velocity ∇U may become singular measures [2]. Hence, even if $\widehat{e}^{\,0}(x)=0$, in order to solve the Cauchy problem (1.23), (1.24), (1.25), it is natural to introduce a delta function into the density ρ. Physically, such solutions can be interpreted as trajectories of concentrated particles (galaxies in the universe, in the above application), the surface of which carries discontinuities.

As in the one-dimensional case (see Subsec. 2.3.), by $f(x,t,\varepsilon)=O_{\mathcal{D}'}(\varepsilon^{\alpha})$ we denote the collection of distributions $f(x,t,\varepsilon)\in\mathcal{D}'(\mathbb{R}^n)$ (depending on t and ε as on parameters) such that for any test function $\psi\in\mathcal{D}(\mathbb{R}^n)$ satisfies the relation

$$\langle f(\cdot,t,\varepsilon),\ \psi(\cdot)\rangle=O(\varepsilon^{\alpha}),$$

where the estimation $O(\varepsilon^{\alpha})$ is treated in the usual sense and is uniform with respect to t. The relation $o_{\mathcal{D}'}(\varepsilon^{\alpha})$ is understood in a corresponding way.

Just as in Sec. 3, we will seek a *δ-shock wave type solution* of the Cauchy problem (1.23), (1.24) in the form of the *singular ansatz* (1.26).

We assume that $\nabla S(x,t)\big|_{S=0} \neq 0$, $\nabla S(x,t) = (S_{x_1}, \dots, S_{x_n})$ for all $t \in [0,T)$, i.e., the δ-shock wave front $\Gamma_t = \big\{x : S(x,t) = 0\big\}$ is a smooth surface of codimension 1 in the space $\mathbb{R}^n \times \mathbb{R}$. That is, in a neighborhood of any point of the surface Γ_t, one can introduce local coordinates $(\tau, \tilde{\tau}, t)$, where $\tau = S(x,t)$, while the other coordinates $\tilde{\tau} = (\tilde{\tau}_2, \dots, \tilde{\tau}_n)$ can be chosen so that the formulas relating (x,t) and $(\tau, \tilde{\tau}, t)$ are determined by infinitely differentiable functions with positive Jacobian. It is clear that in local coordinates the function $\widehat{e}(x,t)$ depends *only* on $(\tilde{\tau}, t)$.

We shall seek a *δ-shock wave type solution* of the Cauchy problem (1.23), (1.24), (1.25) in the form (1.26), and a *weak asymptotic solution* in the form 1.27).

In contrast to the case of system (1.14), we shall construct a *weak asymptotic solution* (1.27) of systems (1.23) *without introducing corrections*.

Definition 4.1. The smooth ansatz (1.27) $(\rho(x,t,\varepsilon), U(x,t,\varepsilon))$, $\varepsilon > 0$, is called a *weak asymptotic solution* of the Cauchy problem (1.23), (1.24) in the domain $\Omega \times [0,\ T) \subset \mathbb{R}^n \times \mathbb{R}$ if

$$
\begin{aligned}
\int_{\Omega} \mathcal{L}_1[\rho(x,t,\varepsilon), U(x,t,\varepsilon)]\psi(x)\,dx &= o(\varepsilon),\\
\int_{\Omega} \mathcal{L}_2[U(x,t,\varepsilon)]\psi(x)\,dx &= o(\varepsilon),\\
\int_{\Omega} \big(\rho(x,0,\varepsilon) - \rho^0(x)\big)\psi(x)\,dx &= o(\varepsilon),\\
\int_{\Omega} \big(U(x,0,\varepsilon) - U^0(x)\big)\psi(x)\,dx &= o(\varepsilon), \quad \varepsilon \to +0,
\end{aligned}
$$

for all $\psi \in \mathcal{D}(\Omega)$, where the first two estimates are uniform with respect to $t \in [0,\ T)$.

The last relations can be rewritten as

$$
\begin{aligned}
\mathcal{L}_1[\rho(x,t,\varepsilon), U(x,t,\varepsilon)] &= o_{\mathcal{D}'}(1)(\varepsilon),\\
\mathcal{L}_2[U(x,t,\varepsilon)] &= o_{\mathcal{D}'}(1)(\varepsilon),\\
\rho(x,0,\varepsilon) - \rho^0(x) &= o_{\mathcal{D}'}(1)(\varepsilon),\\
U(x,0,\varepsilon) - U^0(x) &= o_{\mathcal{D}'}(1)(\varepsilon), \quad \varepsilon \to +0,
\end{aligned}
\tag{4.2}
$$

Since system (1.23) has a non-conservative form, the definition of a *generalized solution* is not given in the form of integral identities. We introduce the *generalized solution* $(\rho(x,t), U(x,t))$ of the Cauchy problem as a weak limit (in $\mathcal{D}'(\mathbb{R}^{n+1})$) of a weak asymptotic solution $(\rho(x,t,\varepsilon), U(x,t,\varepsilon))$ as $\varepsilon \to +0$.

Definition 4.2. Let $(\rho(x,t,\varepsilon), U(x,t,\varepsilon))$ be a weak asymptotic solution of the Cauchy problem (1.23), (1.24). A pair of distributions $\big(\rho(x,t), U(x,t)\big)$ is called a *generalized δ-shock wave type solution* of the Cauchy problem in the domain $\Omega \times [0,\ T)$ if

$$
\begin{aligned}
\int_0^T \int_{\Omega} \rho(x,t)\varphi(x,t)\,dxdt &= \lim_{\varepsilon\to+0} \int_0^T \int_{\Omega} \rho(x,t,\varepsilon)\varphi(x,t)\,dxdt,\\
\int_0^T \int_{\Omega} U(x,t)\varphi(x,t)\,dxdt &= \lim_{\varepsilon\to+0} \int_0^T \int_{\Omega} U(x,t,\varepsilon)\varphi(x,t)\,dxdt
\end{aligned}
$$

for all $\varphi \in \mathcal{D}(\Omega \times [0,\ T))$.

Definitions 4.1, 4.2 are similar to those introduced in [10]– [12] for the case $n = 1$.

Remark 4.1. In this paper we use Definition 4.2, but a *δ-shock wave type solution* can be introduced by the following definition in form of integral identities.

Let Ω be a bounded domain in $\mathbb{R}^n$. A singular ansatz (1.26) and the hypersurface $\Gamma_t = \big\{x : S(x,t) = 0\big\} \subset \Omega \times [0,\ T)$ is called a *generalized δ-shock wave type solution* of the Cauchy problem (1.23), (1.24) in the domain $\Omega \times [0,\ T)$ if the integral identities hold:

$$\begin{array}{rcl} \displaystyle\int_{\Omega\times[0,\ T)\setminus\Gamma_t} \Big(\rho\varphi_t + \rho U\cdot\nabla\varphi\Big)\,dx\,dt + \int_{\Gamma_t} \widehat{e}\,\frac{D\varphi}{Dt}\sigma(x,t) &=& 0, \\ \displaystyle\int_{\Omega\times[0,\ T)\setminus\Gamma_t} \Big(U_t + (U\cdot\nabla)U\Big)\varphi\,dx\,dt - \int_{\Gamma_t} [U]\frac{DS}{Dt}\varphi\sigma(x,t) &=& 0, \end{array} \tag{4.3}$$

for all $\varphi \in \mathcal{D}(\Omega \times (0,\ T))$. Here $\sigma(x,t) = \frac{dS(x,t)}{|\nabla_{(x,t)}S|}$ is the Leray measure with respect to the spacetime variables $x_1,\dots,x_n,t$, $\frac{D}{Dt} = \frac{\partial}{\partial t} + U_\delta\cdot\nabla$ is the operator of differentiation with respect to t (so-called Lagrangian derivative), where U_δ is the velocity of a moving surface Γ_t.

As we will see in Sec. 6, $U_\delta = \frac{U^- + U^+}{2}$, where $U^\pm$ is the velocity behind the δ-shock wave front and ahead of it, respectively.

5. Propagation of δ-shocks in multidimensional "zero-pressure gas dynamics" system

5.1. **A weak asymptotic solution.** Let $\Omega_t^- = \{(x,t) : S(x,t) < 0\}$ and $\Omega_t^+ = \{(x,t) : S(x,t) > 0\}$ denote the domains behind the δ-shock wave front and ahead of it, and let $\rho^\pm$, $U^\pm$ be the states of pressureless gas in the domains $\Omega_t^\pm$. Obviously, $\rho^- = \rho_0 + \rho_1$, $U^- = U_0 + U_1$ for $(x,t) \in \Omega_t^-$ and $\rho^+ = \rho_0$, $U^+ = U_0$ for $(x,t) \in \Omega_t^+$.

The *stability condition* for the δ-shock front is

$$U^+(x,t)\cdot\nu\big|_{\Gamma_t} < U_\delta(x,t)\cdot\nu\big|_{\Gamma_t} < U^-(x,t)\cdot\nu\big|_{\Gamma_t}, \tag{5.1}$$

where U_δ is the velocity of motion of the δ-shock front, ν is the unit space normal to the surface Γ_t pointing from Ω_t^- to Ω_t^+. This condition means that all characteristics on two sides of a δ-shock wave front Γ_t are incoming. The inequality (5.1) implies

$$[U]\cdot\nabla S\big|_{\Gamma_t} > 0. \tag{5.2}$$

The direction of the vector $\frac{\nabla S}{|\nabla S|}$ coincides with the direction in which the function S increases, i.e., inward the domain Ω_t^+. The above reasoning allows us to choose the normal as $\nu = \frac{\nabla S}{|\nabla S|}$.

Denote by $\mathbf{N} = \frac{(\nu,-G)}{\sqrt{1+G^2}} = \frac{\nabla_{(x,t)}S}{|\nabla_{(x,t)}S|}$ the unit spacetime normal to the surface Γ_t, where $G = -\frac{S_t}{|\nabla S|}$, $\nabla_{(x,t)} = \big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial t}\big)$ (see Subsec. 6.2.).

Now, we will construct a *weak asymptotic solution* of the problem (1.23), (1.24), (1.25).

Theorem 5.1. *Let condition* (1.25) *be satisfied. Then there exists a sufficiently short-time $T > 0$ and a compact $K \subset \mathbb{R}^n$ such that, for $(x,t) \in K \times [0,T)$, there exists a weak asymptotic solution* (1.27) *of the Cauchy problem* (1.23), (1.24), (1.25) (*in the sense of Definition* 4.1) *if and only if the vector-functions $U_0 = U^+$, $U_1 = U^- - U^+$, and the functions $\rho_0 = \rho^+$, $\rho_1 = \rho^- - \rho^+$, $\widehat{e}$, S, are satisfy the following systems*

$$\begin{array}{rcll} \rho_t^- + \nabla\cdot(\rho U^-) &=& 0, & \\ U_t^- + (U^-\cdot\nabla)U^- &=& 0, & (x,t) \in \Omega_t^-, \end{array} \tag{5.3}$$

$$\begin{array}{rcll} \rho_t^+ + \nabla\cdot(\rho U^+) &=& 0, & \\ U_t^+ + (U^+\cdot\nabla)U^+ &=& 0, & (x,t) \in \Omega_t^+, \end{array} \tag{5.4}$$

$$\Big\{S_t + U_\delta \cdot \nabla S\Big\}\Big|_{\Gamma_t} = 0, \tag{5.5}$$

$$\frac{\delta \widehat{e}}{\delta t} + \mathrm{div}_{\Gamma_t}(\widehat{e}U_\delta) = \big([\rho U] - [\rho]U_\delta\big)\cdot \nabla S\Big|_{\Gamma_t}, \tag{5.6}$$

where

$$U_\delta = \frac{U^- + U^+}{2} = \Big(\frac{[(u_1)^2/2]}{[u_1]}, \dots, \frac{[(u_n)^2/2]}{[u_n]}\Big) \tag{5.7}$$

is the velocity of motion of the δ-shock front,

$$[f(\rho, U)]\big|_{\Gamma_t} = \big(f(\rho^-, U^-) - f(\rho^+, U^+)\big)\big|_{\Gamma_t}$$

is, as usual, a jump in the quantity $f(\rho, U)$ across the δ-shock front $\Gamma_t = \big\{x : S(x,t) = 0\big\}$, div_{Γ_t} is a surface (tangent) divergence (6.7), *δ-derivatives are defined by* (6.5), (6.6). *Here mollifiers ω_j, ω_δ are such that*

$$\int \omega_{0r}(\eta)\omega_j(\eta)\,d\eta = \int \omega_{0j}(\eta)\omega_\delta(\eta)\,d\eta = \frac{1}{2}, \qquad r, j = 1, \dots, n. \tag{5.8}$$

The initial data for above systems are defined from (1.24), *and $S(x,0) = S^0(x)$.*

Proof of Theorem 5.1. 1. With the help of Lemma 6.3, we find the following asymptotics for $\varepsilon \to +0$:

$$\begin{aligned}\rho(x,t,\varepsilon)u_j(x,t,\varepsilon) &= \rho_0 u_{0j} + \big[\rho u_j\big]H(-S)\\ &\quad + \widehat{e}\big(u_{0j} + a_j u_{1j}\big)\delta(S) + O_{\mathcal{D}'}(\varepsilon),\end{aligned} \tag{5.9}$$

$$\Big(u_j(x,t,\varepsilon)\Big)^2 = \big(u_{0j}\big)^2 + \big[(u_j)^2\big]H(-S) + O_{\mathcal{D}'}(\varepsilon), \tag{5.10}$$

$$\begin{aligned}&u_r(x,t,\varepsilon)\frac{\partial u_r(x,t,\varepsilon)}{\partial x_r}\\ &= \Big(u_{0r} + u_{1r}H_r(-S,\varepsilon)\Big)\Big(\frac{\partial u_{0r}}{\partial x_r} + \frac{\partial u_{1r}}{\partial x_r}H_r(-S,\varepsilon) + u_{1r}\frac{dH_r(-S,\varepsilon)}{dS}(-S_{x_r})\Big)\\ &= u_{0r}\frac{\partial u_{0r}}{\partial x_r} + \Big[u_r\frac{\partial u_r}{\partial x_r}\Big]H(-S) - u_{1r}\Big(u_{0r} + \frac{1}{2}u_{1r}\Big)S_{x_r}\delta(S) + O_{\mathcal{D}'}(\varepsilon),\end{aligned} \tag{5.11}$$

$$\begin{aligned}&u_r(x,t,\varepsilon)\frac{\partial u_j(x,t,\varepsilon)}{\partial x_r}\\ &= \Big(u_{0r} + u_{1r}H_r(-S,\varepsilon)\Big)\Big(\frac{\partial u_{0j}}{\partial x_r} + \frac{\partial u_{1j}}{\partial x_r}H_j(-S,\varepsilon) + u_{1j}\frac{dH_j(-S,\varepsilon)}{dS}(-S_{x_r})\Big)\\ &\quad = u_{0r}\frac{\partial u_{0j}}{\partial x_r} + \Big[u_r\frac{\partial u_j}{\partial x_r}\Big]H(-S)\\ &\qquad - u_{1j}\Big(u_{0r} + c_{jr}u_{1r}\Big)S_{x_r}\delta(S) + O_{\mathcal{D}'}(\varepsilon), \qquad \varepsilon \to +0,\end{aligned} \tag{5.12}$$

where, according to Lemma 6.3, we have

$$a_j = \int \omega_{0j}(\eta)\omega_\delta(\eta)\,d\eta, \quad c_{jr} = \int \omega_{0r}(\eta)\omega_j(\eta)\,d\eta = 1 - c_{rj}, \quad r \neq j, \tag{5.13}$$

where $c_{jr} \in (0,\ 1]$, $r, j = 1, \dots, n$.

Taking into account formulas (6.30), (6.31), we can readily calculate the derivatives of the relations (1.27), (5.9) as follows:

$$\frac{\partial u_j(x,t,\varepsilon)}{\partial t} = \frac{\partial u_{0j}}{\partial t} + \frac{\partial [u_j]}{\partial t}H(-S) - \big[u_j\big]S_t\delta(S) + O_{\mathcal{D}'}(\varepsilon), \tag{5.14}$$

$$
\begin{aligned}
\frac{\partial\rho(x,t,\varepsilon)}{\partial t} &= \frac{\partial\rho_0}{\partial t} + \frac{\partial[\rho]}{\partial t}H(-S) - \big[\rho\big]S_t\delta(S) \\
&\qquad\qquad + \frac{\delta\widehat{e}}{\delta t}\delta(S) + \widehat{e}\frac{\partial}{\partial t}\delta(S) + O_{\mathcal{D}'}(\varepsilon) \\
&= \frac{\partial\rho_0}{\partial t} + \frac{\partial[\rho]}{\partial t}H(-S) + \Big\{ - \big[\rho\big]S_t + \frac{\delta\widehat{e}}{\delta t} + 2\mathcal{H}\frac{\widehat{e}S_t}{|\nabla S|}\Big\}\delta(S) \\
&\qquad\qquad + \frac{\widehat{e}S_t}{|\nabla S|}d_\nu\widehat{\delta}(S) + O_{\mathcal{D}'}(\varepsilon),
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
&\frac{\partial\rho(x,t,\varepsilon)u_j(x,t,\varepsilon)}{\partial x_j} = \frac{\partial(\rho_0u_{0j})}{\partial x_j} + \frac{\partial\big[\rho u_j\big]}{\partial x_j}H(-S) - \big[\rho u_j\big]S_{x_j}\delta(S) \\
&\quad + \frac{\delta}{\delta x_j}\Big(\widehat{e}\big(u_{0j} + a_ju_{1j}\big)\Big)\delta(S) + \widehat{e}\big(u_{0j} + a_ju_{1j}\big)\Big|_{\Gamma_t}\frac{\partial}{\partial x_j}\delta(S) + O_{\mathcal{D}'}(\varepsilon) \\
&= \frac{\partial(\rho_0u_{0j})}{\partial x_j} + \frac{\partial\big[\rho u_j\big]}{\partial x_j}H(-S) + \Big\{ - \big[\rho u_j\big]S_{x_j} + \frac{\delta}{\delta x_j}\Big(\widehat{e}\big(u_{0j} + a_ju_{1j}\big)\Big) \\
&\quad + 2\mathcal{H}\frac{\widehat{e}\big(u_{0j} + a_ju_{1j}\big)S_{x_j}}{|\nabla S|}\Big\}\delta(S) + \frac{\widehat{e}\big(u_{0j} + a_ju_{1j}\big)S_{x_j}}{|\nabla S|}d_\nu\delta(S) + O_{\mathcal{D}'}(\varepsilon),
\end{aligned}
\tag{5.16}
$$

where $\mathcal{H}$ is the mean curvature (6.10) of the surface Γ_t.

Substituting expressions (5.10)–(5.12) and (5.14)– (5.16) into the left-hand side of system (1.23), we obtain

$$
\begin{aligned}
\mathcal{L}_1[\rho(x,t,\varepsilon),U(x,t,\varepsilon)] &= \frac{\partial\rho(x,t,\varepsilon)}{\partial t} + \sum_{r=1}^{n}\frac{\partial\big(\rho(x,t,\varepsilon)u_r(x,t,\varepsilon)\big)}{\partial x_r} \\
&= \frac{\partial\rho_0}{\partial t} + \sum_{r=1}^{n}\frac{\partial(\rho_0u_{0r})}{\partial x_r} + \Big\{\frac{\partial[\rho]}{\partial t} + \sum_{r=1}^{n}\frac{\partial[\rho u_r]}{\partial x_r}\Big\}H(-S) \\
&\quad + \Big\{ - \big[\rho\big]S_t + \frac{\delta\widehat{e}}{\delta t} + 2\mathcal{H}\frac{\widehat{e}S_t}{|\nabla S|} + \sum_{r=1}^{n}\Big(- \big[\rho u_r\big]S_{x_r} \\
&\quad + \frac{\delta}{\delta x_r}\Big(\widehat{e}\big(u_{0r} + a_ru_{1r}\big)\Big) + 2\mathcal{H}\frac{\widehat{e}\big(u_{0r} + a_ru_{1r}\big)S_{x_r}}{|\nabla S|}\Big)\Big\}\delta(S) \\
&\quad + \frac{\widehat{e}}{|\nabla S|}\Big\{S_t + \sum_{r=1}^{n}\big(u_{0r} + a_ru_{1r}\big)S_{x_r}\Big\}d_\nu\delta(S) + O_{\mathcal{D}'}(\varepsilon),
\end{aligned}
\tag{5.17}
$$

$$
\begin{aligned}
\mathcal{L}_2[u_j(x,t,\varepsilon)] &= \frac{\partial u_j(x,t,\varepsilon)}{\partial t} + \sum_{r=1}^{n}u_r(x,t,\varepsilon)\frac{\partial u_j(x,t,\varepsilon)}{\partial x_r} \\
&= \frac{\partial u_{0j}}{\partial t} + \sum_{r=1}^{n}u_{0r}\frac{\partial u_{0j}}{\partial x_r} + \Big\{\frac{\partial[u_j]}{\partial t} + \sum_{r=1}^{n}\Big[u_r\frac{\partial u_j}{\partial x_r}\Big]\Big\}H(-S) \\
&\quad - u_{1j}\Big\{S_t + \sum_{r=1}^{n}\big(u_{0r} + c_{jr}u_{1r}\big)S_{x_r}\Big\}\delta(S) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0,
\end{aligned}
\tag{5.18}
$$

where the coefficients c_{jr} for $r \neq j$ are defined by formula (5.13), and $c_{jj} = \frac{1}{2}$, $r,j = 1,\dots,n$.

Next, using Lemma 6.2, and setting equal to zero the smooth coefficients of the distributions $H(S)$, $\widehat{\delta}(S)$, $d_\nu\widehat{\delta}(S)$ of the left-hand sides of relations (5.17), (5.18), we find the following system of necessary and sufficient conditions for

$$
\mathcal{L}_1[\rho(x,t,\varepsilon),U(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon), \qquad \mathcal{L}_2[U(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon).
$$

$$\begin{aligned} \frac{\partial \rho_0}{\partial t} + \sum_{r=1}^{n} \frac{\partial(\rho_0 u_{0r})}{\partial x_r} &= 0, \\ \frac{\partial u_{0j}}{\partial t} + \sum_{r=1}^{n} u_{0r} \frac{\partial u_{0j}}{\partial x_r} &= 0, \quad S > 0, \end{aligned} \tag{5.19}$$

$$\begin{aligned} \frac{\partial(\rho_0+\rho_1)}{\partial t} + \sum_{r=1}^{n} \frac{\partial\big((\rho_0+\rho_1)(u_{0r}+u_{1r})\big)}{\partial x_r} &= 0, \\ \frac{\partial(u_{0j}+u_{1j})}{\partial t} + \sum_{r=1}^{n} (u_{0r}+u_{1r}) \frac{\partial(u_{0j}+u_{1j})}{\partial x_r} &= 0, \quad S < 0, \end{aligned} \tag{5.20}$$

$$\Big\{ S_t + \sum_{r=1}^{n} \big(u_{0r} + c_{jr} u_{1r}\big) S_{x_r} \Big\}\Big|_{\Gamma_t} = 0, \tag{5.21}$$

$$\Big\{ S_t + \sum_{r=1}^{n} \big(u_{0r} + a_r u_{1r}\big) S_{x_r} \Big\}\Big|_{\Gamma_t} = 0, \tag{5.22}$$

$$\begin{aligned} \Big\{ \frac{\delta \widehat{e}}{\delta t} &+ \sum_{r=1}^{n} \frac{\delta}{\delta x_r} \Big(\widehat{e}\big(u_{0r} + a_r u_{1r}\big) \Big) \\ &+ \frac{2\mathcal{H}\widehat{e}}{|\nabla S|} \Big(S_t + \sum_{r=1}^{n} \big(u_{0r} + a_r u_{1r}\big) S_{x_r} \Big) \\ &\qquad - \Big(\big[\rho\big] S_t + \sum_{r=1}^{n} \big[\rho u_r\big] S_{x_r} \Big) \Big\}\Big|_{\Gamma_t} = 0, \end{aligned} \tag{5.23}$$

where $j = 1, \dots, n$.

2. According to (5.19)–(5.23), two smooth functions U^- and U^+, defined on respective domains Ω_t^- and Ω_t^+ on either side of hypersurface Γ_t so that

$$\mathcal{L}_2[U^{\pm}] = 0, \qquad (x,t) \in \Omega_t^{\pm}. \tag{5.24}$$

The boundary values of U^- and U^+ restricted to the hypersurface Γ_t satisfy the following *overdetermined* system of $n+1$ Rankine–Hugoniot type conditions (5.21), (5.22):

$$\begin{aligned} \Big\{ S_t + \sum_{r=1}^{n} \big(u_{0r} + a_r u_{1r}\big) S_{x_r} \Big\}\Big|_{\Gamma_t} &= 0, \\ \Big\{ S_t + \sum_{r=1}^{n} \Big(u_{0r} + c_{jr} u_{1r}\Big) S_{x_r} \Big\}\Big|_{\Gamma_t} &= 0, \quad j = 1, \dots, n. \end{aligned} \tag{5.25}$$

The initial data of problem (5.24), (5.25) are

$$U^0(x) = U_0^0(x) + U_1^0(x) H\big(-S^0(x)\big), \qquad S(x,0) = S^0(x), \tag{5.26}$$

defined from (1.24) under assumption (1.25).

Note that we construct the asymptotic solution (1.27) of the Cauchy problem (1.23), (1.24), (1.25) which is *suitable* for any entropy initial data. To this end, the solution of problem (5.24), (5.25), (5.26) must be *suitable for any entropy initial data* (5.26). Now, let us consider the case of the piecewise constant initial data (5.26), where

$$u_{1k}^0 \neq 0, \quad u_{1j}^0 = 0, \quad j \neq k, \qquad S^0(x) = \nu^0 \cdot x, \quad |\nu^0| = 1, \quad \nu_k^0 \neq 0,$$

and k is a certain integer, $(k = 1, \dots, n)$, $x \in \mathbb{R}^n$. In this case, the solution of problem (5.24), (5.25), (5.26) is given by system (5.25), where U_1^0 is described above. Subtracting one of the Rankine–Hugoniot conditions (5.25) from the other, we obtain

$$(c_{jk} - c_{ik})u^0_{1k}S_{x_k} = 0, \quad i \neq j, \quad i,j = 1,\dots,n.$$

Since $u^0_{1k} \neq 0$, if $c_{jk} - c_{ik} \neq 0$ then $S_{x_k} = 0$. Taking $\nu^0_k \neq 0$ into account, we *cannot* solve the Cauchy problem with an *arbitrary* initial surface $S^0(x)$. Thus we must set $c_{jk} = c_{ik}$, $i \neq j$, $i,j = 1,\dots,n$. Since $c_{rr} = \frac{1}{2}$, for all $r = 1,\dots,n$, we obtain $c_{jr} = a_r = \frac{1}{2}$, for all $j,r = 1,\dots,n$.

Thus, instead of system (5.25), we obtain the Rankine–Hugoniot condition

$$S_t + \sum_{r=1}^{n} \big(u_{0r} + \frac{1}{2}u_{1r}\big)S_{x_r}, \quad (x,t) \in \Gamma_t = 0,$$

i.e., (5.5).

Using (5.22), (5.5), we can rewrite relation (5.23) in the form

$$\frac{\delta \widehat{e}}{\delta t} + \mathrm{div}_{\Gamma_t}(\widehat{e}U_\delta) = \big[\rho\big]S_t + \sum_{r=1}^{n}\big[\rho u_r\big]S_{x_r}, \quad (x,t) \in \Gamma_t = 0. \tag{5.27}$$

Using the Rankine–Hugoniot condition (5.5), we readily see that relation (5.27) can be reduced to relation (5.6).

3. It remains to study the problem of existence of entropy weak solution of system (5.3)–(5.6).

Since $U_\delta = \frac{1}{2}\big(U^- + U^+\big)$, then in view of the fact that the initial data satisfy the entropy condition (1.25), relation (5.1) holds for $t = 0$.

The pair of vector-functions U^-, U^+ and the space-time hypersurface $\Gamma_t = \{S(x,t) = 0\}$ form the solution of the classical shock-front problem [38, 4.2.]

$$\begin{array}{rcl} \mathcal{L}_2[U^\pm] &=& 0, \qquad (x,t) \in \Omega^\pm_t, \\ S_t\Big|_{\Gamma_t} &=& -\sum_{j=1}^{n} \frac{[(u_j)^2/2]}{[u_j]} S_{x_j}\Big|_{\Gamma_t}, \end{array} \tag{5.28}$$

with the initial data (5.26), (1.25).

The boundary values U^- and U^+ restricted to the hypersurface Γ_t are not arbitrary but satisfy the Rankine–Hugoniot condition of shocks (5.5), i.e., the second equation in (5.28). Thus system (5.28) describes a classical "free boundary" problem for a nonlinear hyperbolic system of equations. A classical idea in free boundary problems is to introduce the equation of the *front as one of the unknowns* and use a change of variables to reduce the problem to a *fixed* domain [36]– [38], [43].

Using the well-known works [36]– [38], [43] we conclude that under assumption (1.25) for the piecewise-smooth initial data (5.26), in view of the *theorem of the existence of shock fronts* [37, p.8], for a sufficiently short-time $T > 0$ and for some compact K there is a shock front solution of the problem (5.28), (5.26), (1.25), where $(x,t) \in K \times [0,T)$.

Solving problem (5.28), we find $U^\pm$, S. Then substituting $U^\pm$, S into the system

$$\begin{array}{rcl} \mathcal{L}_1[\rho^-, U^-] &=& 0, \quad S \ < \ 0, \\ \mathcal{L}_1[\rho^+, U^+] &=& 0, \quad S \ > \ 0, \end{array}$$

we obtain $\rho^\pm$. Next, from (5.6) we find $\widehat{e}$.

Thus, after finding the functions ρ_k, U_k, $\widehat{e}$, S, $k = 1,2$, we construct a *weak asymptotic solution* (1.27) of the Cauchy problem (1.23), (1.24), (1.25). □

5.2. **A generalized solution.** Using the *weak asymptotic* solution constructed in Theorem 5.1, we obtain a *generalized solution* in the sense of Definition 4.2.

Theorem 5.2. *There is a compact* K *and* $T > 0$*, such that the Cauchy problem* (1.23)*,* (1.24)*,* (1.25) *for* $(x,t) \in K \times [0,T)$ *has a unique generalized solution* (1.26) (*in the sense of Definition* 4.2)

$$\begin{aligned} \rho(x,t) &= \rho_0(x,t) + \rho_1(x,t)H\big(-S(x,t)\big) + \widehat{e}(x,t)\delta\big(S(x,t)\big), \\ U(x,t) &= U_0(x,t) + U_1(x,t)H\big(-S(x,t)\big), \end{aligned}$$

where

$$\begin{aligned} \rho_t^{\pm} + \nabla\cdot(\rho U^{\pm}) &= 0, \qquad (x,t) \in \Omega_t^{\pm}, \\ U_t^{\pm} + (U^{\pm}\cdot\nabla)U^{\pm} &= 0, \qquad (x,t) \in \Omega_t^{\pm}, \end{aligned} \tag{5.29}$$

$$\begin{aligned} S_t\big|_{\Gamma_t} &= -U_\delta\cdot\nabla S\big|_{\Gamma_t}, \\ \frac{\delta\widehat{e}}{\delta t} + \mathrm{div}_{\Gamma_t}(\widehat{e}U_\delta) &= \big([\rho U] - [\rho]U_\delta\big)\cdot\nabla S\big|_{\Gamma_t}. \end{aligned} \tag{5.30}$$

Here K*,* $T > 0$ *are given by Theorem* 5.1*. The initial data for above systems are defined from* (1.24)*, and* $S(x,0) = S^0(x)$*.*

Moreover,

$$\frac{\delta\widehat{e}}{\delta t} + \mathrm{div}_{\Gamma_t}(\widehat{e}U_\delta) > 0, \tag{5.31}$$

i.e., the mass flows into the surface Γ_t *and, therefore, the concentration process on the surface is going on.*

Proof of Theorem 5.2. In view of the above remarks in the proof of Theorem 5.1, system (5.29)–(5.30) has a unique solution and, consequently, the Cauchy problem (1.23), (1.24), (1.25) has a unique generalized solution (1.26).

A *δ-shock wave type solution* of the Cauchy problem is constructed as a weak limit of the *weak asymptotic solution* (1.27), where mollifiers ω_j, ω_δ satisfy relations (5.8). But the solution itself, i.e., the weak limit of (1.27), *does not depend* on these relations.

The *pair* of equations (5.30) of system (5.29)–(5.30) is the *Rankine–Hugoniot conditions for δ-shocks*, where the first equation in (5.30) is the *standard* Rankine–Hugoniot condition. The right-hand side of the second equation in (5.30) is the *Rankine–Hugoniot deficit*. The Rankine–Hugoniot conditions can be rewritten as

$$\begin{aligned} S_t\Big|_{\Gamma_t} &= -\overline{U}\cdot\nabla S\Big|_{\Gamma_t}, \\ \frac{\delta\widehat{e}}{\delta t} + \mathrm{div}_{\Gamma_t}(\widehat{e}U_\delta) &= \overline{\rho}\,[U]\cdot\nabla S\Big|_{\Gamma_t}, \end{aligned} \tag{5.32}$$

where $\overline{\rho} = \frac{1}{2}\big(\rho^- + \rho^+\big)$ and $\overline{U} = \frac{1}{2}\big(U^- + U^+\big)$ represent the average density and the δ-shock velocity on the two sides of the δ-shock wave front Γ_t, respectively. Thus the δ-shock velocity and the average velocity on both sides of Γ_t *coincide*. Since $\rho^{\pm} > 0$ and $\nu = \frac{\nabla S}{|\nabla S|}$ is the space normal pointing from Ω_t^- to Ω_t^+, it follows from (5.2) that inequality (5.31) is satisfied. $\Box$

Taking into account formulas (6.5), (6.6), the second condition (5.30) readily implies the equation

$$\frac{\delta\widehat{e}}{\delta t} + \mathrm{div}_{\Gamma_t}(\widehat{e}U_\delta) = |\nabla_{(x,t)}S|\big([\rho U],\ [\rho]\big)\cdot\mathbf{N}, \tag{5.33}$$

where $\mathbf{N}$ is the unit spacetime normal to the surface Γ_t.

If points x of a moving surface Γ_t are indexed smoothly by the time t as $x = x(t)$ then the velocity of motion of the δ-shock front is $\frac{dx}{dt} = U_\delta$.

Thus we calculate

$$\frac{D\widehat{e}}{Dt} = \frac{\partial \widehat{e}}{\partial t} + \sum_{j=1}^{n} \frac{\partial \widehat{e}}{\partial x_j}\frac{dx_j}{dt} = \frac{\partial \widehat{e}}{\partial t} + U_\delta \cdot \nabla \widehat{e}, \tag{5.34}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + U_\delta \cdot \nabla$ is the operator of differentiation with respect to t (Lagrangian derivative). By using (6.5), (6.6), and the last relation, the second condition (5.32) is represented in the form

$$\frac{D\widehat{e}}{Dt} + \widehat{e}\,\mathrm{div}_{\Gamma_t} U_\delta = \overline{\rho}\,[U] \cdot \nabla S\big|_{\Gamma_t}. \tag{5.35}$$

Remark 5.1. Taking into account the fact that $\mathcal{L}_1[\rho(x,t,\varepsilon), U(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon)$ and $\mathcal{L}_2[U(x,t,\varepsilon)] = O_{\mathcal{D}'}(\varepsilon)$, where $\big(\rho(x,t,\varepsilon), U(x,t,\varepsilon)\big)$ is the *weak asymptotic solution* (1.27) constructed in Theorem 5.1, and repeating the constructions of the proof of Theorem 3.2), we can prove that (1.26) is a generalized solution of the Cauchy problem (1.23), (1.24), (1.25) in the sense of Definition (4.3).

Namely, applying the left-hand and right-hand sides of relation (5.17) to an arbitrary test function $\varphi(x,t) \in \mathcal{D}(\Omega \times [0, T))$, integrating by parts with the help of Lemma 6.1, and then passing to the limit as $\varepsilon \to +0$, we obtain the first integral identity of Definition (4.3). Applying the left-hand and right-hand sides of relation (5.18) to an arbitrary test function $\varphi(x,t) \in \mathcal{D}(\Omega \times [0, T))$, and then passing to the limit as $\varepsilon \to +0$, we obtain the second integral identity of Definition (4.3).

Remark 5.2. Setting $\widehat{e}^{\,0}(x) = \widehat{e}(x,t) = 0$, from the second relation in (5.32) we have

$$[U] \cdot \nabla S\big|_{\Gamma_t} = 0. \tag{5.36}$$

Comparing these two formulas, as well as (5.2) and (5.36), we readily see that the Cauchy problem (1.23), (1.24), (1.25) in the case of piecewise constant initial data *has not been solved*.

However, in this case δ-shock Rankine–Hugoniot conditions (5.30) formally imply the relation

$$[\rho(U_\delta - U)] \cdot \nu\big|_{\Gamma_t} = 0,$$

which coincides with the first shock Rankine–Hugoniot condition for gas dynamics system in conservative form [9, (4.4)].

Let us construct a planar δ-shock in system (1.23). To this end, consider the case of piecewise constant initial data (1.24), where $U_k^0 = U_k$, $\rho_k^0 = \rho_k$ are constants, $k = 0,1$, $\Gamma_0 = \big\{x : S^0(x) = 0\big\}$, $S^0(x) = \nu^0 \cdot x$, $|\nu^0| = 1$, $x \in \mathbb{R}^n$.

Corollary 5.1. *The Cauchy problem* (1.23), (1.24), (1.25) *with the piecewise constant initial data for* $(x,t) \in \mathbb{R}^n \times [0,\infty)$ *has a unique generalized solution* (1.26)*:*

$$\begin{aligned} \rho(x,t) &= \rho_0^0 + \rho_1^0 H\big(-\nu^0 \cdot (x - U_\delta t)\big) + \widehat{e}(t)\delta\big(\nu^0 \cdot (x - U_\delta t)\big), \\ U(x,t) &= U_0^0 + U_1^0 H\big(-\nu^0 \cdot (x - U_\delta t)\big), \end{aligned}$$

where $U_\delta = U_0^0 + \frac{1}{2}U_1^0 = \frac{1}{2}(U^- + U^+)$*, and*

$$\begin{aligned} t &= \tfrac{\nu^0 \cdot x}{\nu^0 \cdot U_\delta}, \\ \widehat{e}(t) &= \widehat{e}^{\,0} + \nu^0 \cdot \Big([\rho U] - [\rho] U_\delta\Big) t. \end{aligned} \tag{5.37}$$

In this case system (5.29), (5.30) is reduced to the system of PDEs

$$\begin{aligned} S_t\big|_{\Gamma_t} &= -U_\delta \cdot \nabla S\big|_{\Gamma_t}, \\ \frac{D\widehat{e}}{Dt} &= \overline{\rho}\,[U] \cdot \nabla S\big|_{\Gamma_t}, \end{aligned}$$

with constant coefficients. By solving the first linear PDE of this system, we obtain the first relation in (5.37). Since in this case U_δ and ∇S are constants, integrating the second equation of this system, we obtain the second relation in (5.37).

Note that in this case the Rankine–Hugoniot condition (5.33) coincides with the second Rankine–Hugoniot condition [34, (2.11)], [33, (16a)].

5.3. **Geometrical and physical sense of δ-shock Rankine–Hugoniot conditions.** Let us assume that a moving surface Γ_t permanently separates $\mathbb{R}^n_x$ into two parts $V_t^- = \{x \in \mathbb{R}^n : S(x,t) < 0\}$ and $V_t^+ = \{x \in \mathbb{R}^n : S(x,t) > 0\}$. Let (U,ρ) be compactly supported with respect to x. Denote by

$$\begin{array}{rcl} M(t) &=& \int_{V_t^-} \rho(x,t)\,dx + \int_{V_t^+} \rho(x,t)\,dx, \\ M(0) &=& \int_{V_0^-} \rho^0(x)\,dx + \int_{V_0^+} \rho^0(x)\,dx, \\ m(t) &=& \int_{\Gamma_t} \widehat{e}(x,t)\sigma(x) = \int_{\Gamma_t} \widehat{e}(x,t)\frac{dS(x)}{|\nabla S|}, \\ m(0) &=& \int_{\Gamma_0} \widehat{e}^{\,0}(x)\sigma(x) = \int_{\Gamma_0} \widehat{e}^{\,0}(x)\frac{dS(x)}{|\nabla S|} \end{array} \tag{5.38}$$

the masses of the domains $V_t^- \cup V_t^+$, $V_0^- \cup V_0^+$ and the "masses" of the surfaces Γ_t, Γ_0, respectively, where $\sigma(x)$ is the Leray measure defined in Subsec. 6.2..

Theorem 5.3. *Let (U,ρ) be a generalized δ-shock wave type solution* (1.26) *of the Cauchy problem* (1.23), (1.24), (1.25), *compactly supported with respect to x. Then the following balance relation holds:*

$$\dot{M}(t) = -\dot{m}(t), \qquad \dot{m}(t) > 0, \tag{5.39}$$

i.e., the mass transportation process from the volume into the discontinuity surface Γ_t is going on. Thus,

$$\begin{aligned} &\int_{V_t^-} \rho(x,t)\,dx + \int_{V_t^+} \rho(x,t)\,dx + \int_{\Gamma_t} \widehat{e}(x,t)\frac{dS(x)}{|\nabla S|} \\ &\qquad = \int_{V_0^-} \rho^0(x)\,dx + \int_{V_0^+} \rho(x,t)\,dx + \int_{\Gamma_0} \widehat{e}^{\,0}(x)\frac{dS(x)}{|\nabla S|}. \end{aligned} \tag{5.40}$$

Proof of Theorem 5.3. Let us assume that the support of $U(x,t)$ and $\rho(x,t)$ with respect to x is a compact $K \in \mathbb{R}^n_x$ bounded by ∂K. Let $K_t^\pm = V_t^\pm \cap K$. By ν we denote the space normal to Γ_t pointing from V_t^- to V_t^+. Differentiating the first relation (5.38) and using the volume transport Theorem 6.1, we obtain

$$\begin{aligned} \dot{M}(t) = \int_{K_t^-} \frac{\partial \rho^-}{\partial t}\,dx &+ \int_{K_t^+} \frac{\partial \rho^+}{\partial t}\,dx \\ &+ \int_{\partial K_t^-} G\rho^-\,dS(x) + \int_{\partial K_t^+} G\rho^+\,dS(x). \end{aligned}$$

Next, taking into account the first equation of system (5.29), the first Rankine–Hugoniot condition (5.30), and applying Gauss's divergence theorem, we transform the last relation to the form

$$\begin{aligned} \dot{M}(t) &= -\int_{K_t^-} \operatorname{div}(\rho^- U^-)\,dx - \int_{K_t^+} \operatorname{div}(\rho^+ U^+)\,dx + \int_{\Gamma_t} G[\rho]\,dS(x) \\ &= -\int_{\Gamma_t} \rho^- U^- \cdot \nu\,dS(x) + \int_{\Gamma_t} \rho^+ U^+ \cdot \nu\ \,dS(x) + \int_{\Gamma_t} G[\rho]\,dS(x) \\ &= -\int_{\Gamma_t} \big([\rho U] - [\rho]U_\delta\big) \cdot \nu\,dS(x), \end{aligned} \tag{5.41}$$

where $G = U_\delta \cdot \nu$ on Γ_t. Here we use the fact that the vector $U^\pm$ and function $\rho^\pm$ are equal to zero on the surface $\partial K_t^\pm$ except Γ_t.

According to the second Rankine–Hugoniot condition (5.30), relation (5.41) can be rewritten as

$$\dot{M}(t) = -\int_{\Gamma_t} \Big(\frac{\delta \widehat{e}}{\delta t} + \sum_{r=1}^{n} \frac{\delta(\widehat{e} u_{\delta r})}{\delta x_r}\Big) \frac{dS(x)}{|\nabla S|}. \tag{5.42}$$

Using the surface transport Theorem 6.2, we see that the right-hand side of (5.42) coincides with $-\dot{m}(t)$. The inequality (5.31) implies $\dot{m}(t) > 0$. To complete the proof of the theorem, it remains to integrate (5.39) with respect to t. $\Box$

According to Theorem 5.3, the rate at which the mass in $V_t^- \cup V_t^+$ decreases must be equal to the rate at which the mass flows into the surface Γ_t. But we see that the *total mass* $M(t) + m(t)$ *is independent of time*.

The quantity $\widehat{e}(x,t)$ is the *surface density of the mass*. Now, taking into account the geometrical sense of the Leray measure [19, ch.III,§1.2.] (see also Subsec. 6.2.), we can also explain the sense of the quantity $\widehat{e}(x,t)\sigma(x)$. Suppose that in a neighborhood of the point of the surface $S(x,t) = 0$ we have $S_{x_j} \neq 0$. Let us pass to coordinates $\tau_1 = x_1, \dots, \tau_j = S, \dots, \tau_n = x_n$. Consider the surfaces $S(x,t) = 0$, $S(x,t) = h$, where $h = dS$ is small. Let us take a small area $d\pi$ on the surface $S(x,t) = 0$ and transfer this area onto the surface $S(x,t) = h$ by the coordinate lines $\tau_1 = x_1, \dots, \tau_{j-1}, \tau_{j+1}, \dots, \tau_n = x_n$. Thus we obtain a cylindrical volume Δv with the mass $\Delta M = \rho \Delta v$. In view of formula (6.13), we see that the quantity $\widehat{e}(x,t)\sigma(x)$ is the *velocity of changing elementary mass with respect to changing the quantity* S, i.e.,

$$\widehat{e}(x,t)\sigma(x) = \frac{dM}{dS} = \lim_{h\to 0} \frac{\Delta M}{h}.$$

5.4. **On multiplication of distributions.** The problem of multiplication of distributions is very important to define *δ-shock type solutions*. This problem arises due to the fact that system (1.23) has a non-conservative form (see [30], [31]). In the background of relations (5.10)–(5.12) is the *construction of multiplication of distributions*. Namely, we can define *singular compositions of the Heaviside function and the delta function* by the following definitions:

$$\text{(a) } \rho(x,t)U(x,t) \stackrel{def}{=} \lim_{\varepsilon\to+0} \rho(x,t,\varepsilon)U(x,t,\varepsilon)$$

$$= \rho^+ U^+ + [\rho U] H(-S) + \widehat{e}\,\frac{U^- + U^+}{2}\,\delta(S), \tag{5.43}$$

$$\text{(b) } \big(u_j(x,t)\big)^2 \stackrel{def}{=} \lim_{\varepsilon\to+0} \big(u_j(x,t,\varepsilon)\big)^2 = \big(u_j^+\big)^2 + [u_j^2] H(-S), \tag{5.44}$$

$$\text{(c) } u_r(x,t)\frac{\partial u_j(x,t)}{\partial x_r} \stackrel{def}{=} \lim_{\varepsilon\to+0} u_r(x,t,\varepsilon)\frac{\partial u_j(x,t,\varepsilon)}{\partial x_r}$$

$$= u_r^+ \frac{\partial u_j^+}{\partial x_r} + \Big[u_r \frac{\partial u_j}{\partial x_r}\Big] H(-S) + \big(u_j^- - u_j^+\big)\frac{u_r^- + u_r^+}{2} S_{x_r}\delta(S), \tag{5.45}$$

where $\rho^{\pm}$, $U^{\pm}$, $\widehat{e}$, S are given by (5.29), (5.30), and the limits are understood in the weak sense, $r, j = 1, \dots, n$.

6. Some auxiliary formulas

6.1. **Moving surfaces of discontinuity.** Now we present some results concerning moving surfaces from [24, 5.2.], [3], [4]. Let Γ_t be a moving smooth surface of codimension 1 in the space $\mathbb{R}^n$. Such a surface can be represented locally either in the form $\Gamma_t = \big\{x \in \mathbb{R}^n : S(x,t) = 0\big\}$, or in terms of the curvilinear Gaussian coordinates $s = (s_1, \dots, s_{n-1})$ on the surface:

$$x_j = x_j(s_1, \dots, s_{n-1}, t), \qquad s \in \mathbb{R}^{n-1}.$$

It should be remarked that Γ_t could be considered as a submanifold of the spacetime $\mathbb{R}^n \times \mathbb{R}$. We shall assume that $\nabla S(x,t)\big|_{\Gamma_t} \neq 0$ for all fixed values of t, where $\nabla = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$. Let ν be the unit space normal to the surface Γ_t pointing in the positive direction such that $\frac{\partial S}{\partial x_j} = |\nabla S| \nu_j$, $j = 1, \dots, n$.

Let $f(x,t)$ be a function defined on the surface Γ_t, and let $\frac{\delta f}{\delta t}$ to denote the derivative with respect to time as it would be computed by an observer moving with the surface. This derivative has the following geometrical interpretation. Let M_0 be a point on the surface at time $t = t_0$. Construct the normal line to the surface at M_0. At time $t = t_0 + \Delta t$, Δt an infinitesimal, this normal meets the surface $\Gamma_{t+\Delta t}$ at the point $M = M(t + \Delta t)$. Then the δ-derivative is defined as

$$\frac{\delta f(M_0, t_0)}{\delta t} = \lim_{\Delta t \to 0} \frac{f(M) - f(M_0)}{\Delta t}. \tag{6.1}$$

If Δs is the distance between M_0 and M, then

$$G = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \tag{6.2}$$

is the velocity (normal velocity $U_\delta \cdot \nu$) of the moving surface Γ_t and

$$\frac{\delta x_j}{\delta t} = \lim_{\Delta t \to 0} \frac{\Delta x_j}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \frac{\Delta x_j}{\Delta s} = G \nu_j, \quad j = 1, \dots, n, \tag{6.3}$$

where U_δ is the velocity of Γ_t.

Since the essential feature of δ-derivative is that it is computed on the surface, and S remains constant on the surface then $\frac{\delta S}{\delta t} = 0$. Thus we have

$$0 = \frac{\delta S}{\delta t} = \frac{\partial S}{\partial t} + \sum_{j=1}^{n} \frac{\delta S}{\delta x_j} \frac{\delta x_j}{\delta t} = \frac{\partial S}{\partial t} + \sum_{j=1}^{n} G |\nabla S| \nu_j^2,$$

i.e.,

$$S_t = -G |\nabla S|. \tag{6.4}$$

From this formula we can see that $-G = \frac{S_t}{|\nabla S|}$ can be interpreted as the component of the normal vector in the time direction.

Let $f(x,t)$ be a function defined only on Γ_t. Then the first order derivatives of f with respect to the space and time variables are defined by the following formulas [24, 5.2.(15),(16)]:

$$\frac{\delta f}{\delta t} = \frac{\partial \widetilde{f}}{\partial t} + G \frac{d \widetilde{f}}{d\nu}, \tag{6.5}$$

$$\frac{\delta f}{\delta x_j} = \frac{\partial \widetilde{f}}{\partial x_j} - \nu_j \frac{d \widetilde{f}}{d\nu}, \tag{6.6}$$

where $\widetilde{f}$ is any smooth extension of f to a neighborhood of Γ_t in $\mathbb{R}^n \times \mathbb{R}$, $j = 1, \dots, n$, and

$$\frac{d \widetilde{f}}{d\nu} = \nu \cdot \nabla \widetilde{f} = \sum_{j=1}^{n} \frac{\partial \widetilde{f}}{\partial x_j} \nu_j = \frac{\partial \widetilde{f}}{\partial S} |\nabla S|$$

is a normal derivative. In the sequel we shall drop tilde from f. Thus the gradient along a direction normal to the surface and the gradient tangent to the surface are defined as

$$\nabla_\nu = \nu \big(\nu \cdot \nabla\big), \qquad \nabla_{\Gamma_t} = \nabla - \nabla_\nu = \Big(\frac{\delta}{\delta x_1}, \dots, \frac{\delta}{\delta x_n}\Big),$$

respectively. For a vector $A(x,t) = (A_1(x,t), \dots, A_n(x,t))$ defined only on Γ_t, we introduce a *surface (tangent) divergence* by the following formula

$$\mathrm{div}_{\Gamma_t} A = \nabla_{\Gamma_t} \cdot A = \sum_{j=1}^{n} \frac{\delta A_j}{\delta x_j}. \tag{6.7}$$

Note that $\frac{\delta f}{\delta x_j}$ and $\frac{\delta f}{\delta t}$ depend only on the value of f on Γ_t, i.e., if $f=0$ on Γ_t then $\frac{\delta f}{\delta x_j}$ and $\frac{\delta f}{\delta t}$ on Γ_t, $j=1,\dots,n$. Indeed, let $(x_0,t_0) \in \Gamma_t$. If $\nabla_{(x,t)} f(x_0,t_0) = 0$ then $\nabla_{\Gamma_t} f(x_0,t_0) = 0$ and $\frac{\delta f}{\delta t}(x_0,t_0)$. If $\nabla f(x_0,t_0) \neq 0$ then in a neighborhood of the point (x_0,t_0) the surface Γ_t has the unit space normal $\nu = \frac{\nabla f}{|\nabla f|}$ and $G = -\frac{\frac{\partial f}{\partial t}}{|\nabla f|}$. Consequently, $\nabla_{\Gamma_t} f(x_0,t_0) = 0$ and $\frac{\delta f}{\delta t}(x_0,t_0) = 0$.

The quantities

$$\mu_{ij} = \frac{\delta \nu_i}{\delta x_j}, \qquad i,j = 1,\dots,n \tag{6.8}$$

are components of the *second fundamental form* of the surface Γ_t. Observe that μ_{ij} is a symmetric surface tensor, that is

$$\mu_{ij} = \mu_{ji}, \qquad \sum_{j=1}^{n} \mu_{ij}\nu_j = 0, \qquad i,j = 1,\dots,n. \tag{6.9}$$

The trace of the matrix μ_{ij} is equal to $-2\mathcal{H}$, where

$$\mathcal{H} = -\frac{1}{2}\,\mathrm{div}_{\Gamma_t}(\nu) = -\frac{1}{2}\sum_{j=1}^{n} \frac{\delta \nu_j}{\delta x_j} \tag{6.10}$$

is called the *mean curvature* of the surface Γ_t.

Let $A(x,t) = (A_1(x,t), \dots, A_n(x,t))$ be a smooth vector defined only on a surface Γ_t. Using (6.6), (6.7), (6.10), it is easy to calculate

$$\begin{aligned} \mathrm{div}_{\Gamma_t}(A) &= -2\mathcal{H} A \cdot \nu + \mathrm{div}(A) - \sum_{j,k=1}^{n} \nu_j \frac{\partial (A_j \nu_k)}{\partial x_k} \\ &= -2\mathcal{H} A \cdot \nu + \sum_{j,k=1}^{n} \nu_j \Big(\frac{\partial (A_k \nu_j)}{\partial x_k} - \frac{\partial (A_j \nu_k)}{\partial x_k} \Big). \end{aligned} \tag{6.11}$$

If $n=3$ then $\mathrm{div}_{\Gamma_t}(A) = -2\mathcal{H} A \cdot \nu + \nu \cdot \mathrm{curl}(\nu \times A)$.

6.2. **Distributions related to a moving surface.** Consider some statements about distributions related to a moving surface [24, 5.2.], [3], [4].

Let us suppose for a moment that time t is absent from our analysis. According to [19, ch.III,§1.], we introduce the distributions related to the surface $\Gamma = \big\{x : P(x) = 0\big\}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. We assume that $P \in C^\infty(\mathbb{R}^n)$, $\nabla P(x)\big|_{P=0} \neq 0$ for $x \in \Omega \subset \mathbb{R}^n$, i.e., Γ is a smooth surface of codimension 1 in the space $\mathbb{R}^n$, $x \in \Omega$.

The Heaviside function $H(P)$ is introduced by the following definition:

$$\big\langle H(P),\ \psi \big\rangle = \int_{P \geq 0} \psi(x)\,dx,$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R}^n)$.

The delta function on the surface Γ is defined as a functional acting by the rule [19, ch.III,§1.3.]:

$$\big\langle \delta(P),\ \psi \big\rangle = \int_{P=0} \psi(x)\sigma = \int_{\Gamma} \frac{\psi(x)}{|\nabla P|}\,dP, \tag{6.12}$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R}^n)$, where the 1-form $\sigma = \sigma(x)$ is the Leray measure with respect to the space variables $x_1, \dots, x_n$. This form is a solution of the equation $dP \wedge \sigma = dx_1 \wedge \cdots \wedge dx_n$, where $\wedge$ is the exterior product. According to [19, ch.III,§1.2.], the Leray measure is the velocity of changing elementary volume with respect to changing the quantity P, i.e.,

$$\sigma = \frac{dv}{dP}. \tag{6.13}$$

The distribution $\delta'(P)$ is defined by the relation [19, ch.III,§1.5.]:

$$\big\langle \delta'(P),\ \psi \big\rangle = -\int_{P=0} \sigma_1(\psi(x)) \tag{6.14}$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R}^n)$, where the form $\sigma_1(\psi)$ is a solution of the equation $dP \wedge \sigma_1(\psi) = d\sigma_0(\psi),\ \ \sigma_0(\psi) = \psi \wedge \sigma$.

In particular, if in a neighborhood of the point x we have $P_{x_j} \neq 0$ then, according to [19, ch.III,§1.2.,§1.5.], we can pass to coordinates $\tau_1 = x_1, \dots, \tau_{j-1} = x_{j-1}, \tau_j = P, \tau_{j+1} = x_{j+1}, \dots, \tau_n = x_n$. Since the Jacobian can be rewritten as

$$\frac{\partial(x)}{\partial(\tau)} = \frac{\partial(x_1, \dots, x_n)}{\partial(\tau_1, \dots, \tau_n)} = \frac{1}{\frac{\partial(\tau)}{\partial(x)}} = \frac{1}{\frac{\partial P}{\partial x_j}}, \tag{6.15}$$

we have [19, ch.III,§1.2.]:

$$\sigma = (-1)^{j-1} \frac{d\tau_1 \wedge \cdots \widehat{d\tau_j} \cdots \wedge d\tau_n}{\frac{\partial P}{\partial x_j}}, \tag{6.16}$$

and [19, ch.III,§1.5.]:

$$\sigma_1(\psi) = (-1)^{j-1} \frac{\partial}{\partial \tau_j} \Big(\widetilde{\psi} \frac{\partial(x)}{\partial(\tau)} \Big) d\tau_1 \wedge \cdots \widehat{d\tau_j} \cdots \wedge d\tau_n, \tag{6.17}$$

where $\widetilde{\psi}(\tau) = \psi(x)$. The hat $\widehat{\ }$ over $d\tau_j$ denotes deletion of that factor from the product $d\tau_1 \wedge \cdots \wedge d\tau_n$.

The relations $\delta(P) = H'(P)$ and $\delta'(P) = \big(\delta(P)\big)'$ are understood in the sense that

$$\frac{\partial H(P)}{\partial x_j} = P_{x_j} \delta(P), \quad \frac{\partial \delta(P)}{\partial x_j} = P_{x_j} \delta'(P), \qquad j = 1, \dots, n. \tag{6.18}$$

According to [24, 5.3.(1)], we now introduce the delta function $\widehat{\delta}(S)$ on the surface $\Gamma_t = \big\{x : S(x,t) = 0\big\}$, whose action on a test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$ is given by

$$\big\langle \widehat{\delta}(S),\ \varphi \big\rangle = \int_{-\infty}^{\infty} \int_{\Gamma_t} \varphi(x,t)\, dS(x)\, dt, \tag{6.19}$$

where $dS(x)$ is the surface measure on Γ_t. Here the integration with respect to the space variables is surface integration while that with respect to time is ordinary integration. If time t is treated as an ordinary variable, we shall introduce a different delta function $\widetilde{\delta}(S)$, which is related to $\widehat{\delta}(S)$ by the following formula:

$$\widetilde{\delta}(S) = \sqrt{1 + G^2}\ \widehat{\delta}(S), \tag{6.20}$$

because in this case the spacetime unit normal to the surface Γ_t is given by $N = \frac{(\nu, -G)}{\sqrt{1+G^2}}$, where $\sqrt{1+G^2} = \frac{|\nabla_{(x,t)} S|}{|\nabla S|}$, $\nabla_{(x,t)} = \big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\big)$. Consequently, (6.19) can be rewritten as

$$\big\langle \widehat{\delta}(S),\ \varphi \big\rangle = \int_{\Gamma_t} \varphi(x,t) \frac{|\nabla S|}{|\nabla_{(x,t)} S|}\, dS(x,t), \quad \varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}). \tag{6.21}$$

The relation between $\delta(S)$ defined by (6.12) and $\widehat{\delta}(S)$ is

$$\delta(S)=\frac{\widehat{\delta}(S)}{|\nabla S|}. \tag{6.22}$$

Now we introduce the normal derivative of the delta function [24, 5.3.(7)], [57, 2.,§6.5] $d_\nu\delta(S)$ whose action is given by the formula

$$\big\langle d_\nu\delta(S),\ \varphi\big\rangle=-\Big\langle\delta(S),\ \frac{d\varphi}{d\nu}\Big\rangle=-\int_{\Gamma_t}\frac{d\varphi(x,t)}{d\nu}\frac{dS(x,t)}{|\nabla_{(x,t)}S|} \tag{6.23}$$

for any $\varphi\in\mathcal{D}(\mathbb{R}^n\times\mathbb{R})$, where $\frac{d\varphi}{d\nu}=\nu\cdot\nabla\varphi$ is the normal derivative of φ. Let $f(x,t)$ be a continuous function defined on Γ_t. Then the distribution $d_\nu(f\delta(S))$ (the so-called *double layer*) is a functional acting by the rule

$$\big\langle d_\nu\big(f\delta(S)\big),\ \varphi\big\rangle=-\Big\langle\delta(S),\ f\frac{d\varphi}{d\nu}\Big\rangle=-\int_{\Gamma_t}f\frac{d\varphi(x,t)}{d\nu}\frac{dS(x,t)}{|\nabla_{(x,t)}S|} \tag{6.24}$$

for any $\varphi\in\mathcal{D}(\mathbb{R}^n\times\mathbb{R})$.

Owing to (6.18), we have [24, 5.5.(11)]

$$\begin{array}{rcccl} S_{x_i}\dfrac{\partial}{\partial x_j}\delta(S) &=& S_{x_i}S_{x_j}\delta'(S) &=& S_{x_j}\dfrac{\partial}{\partial x_i}\delta(S),\\ S_t\dfrac{\partial}{\partial x_j}\delta(S) &=& S_tS_{x_j}\delta'(S) &=& S_{x_j}\dfrac{\partial}{\partial t}\delta(S). \end{array} \tag{6.25}$$

Multiplying (6.25) by ν_i and summing over i, we obtain relations [24, 5.5.(8),(8)]

$$\begin{array}{rcccl} \dfrac{\partial}{\partial x_j}\delta(S) &=& S_{x_j}\delta'(S) &=& \nu_j\displaystyle\sum_{i=1}^n\nu_i\frac{\partial}{\partial x_i}\delta(S),\\ \dfrac{\partial}{\partial t}\delta(S) &=& S_t\delta'(S) &=& -G\displaystyle\sum_{i=1}^n\nu_i\frac{\partial}{\partial x_i}\delta(S), \end{array} \tag{6.26}$$

and

$$\delta'(S)=\frac{1}{|\nabla S|}\sum_{i=1}^n\nu_i\frac{\partial}{\partial x_i}\delta(S). \tag{6.27}$$

Note that $|\nabla S|\delta'(S)$ is not a normal derivative operator.

Now we readily calculate [24, 5.3.(9); 5.5.(8),(9)]

$$|\nabla S|\delta'(S)=\sum_{i=1}^n\nu_i\frac{\partial}{\partial x_i}\delta(S)=2\mathcal{H}\delta(S)+d_\nu\delta(S), \tag{6.28}$$

where the mean curvature is given by (6.10). Indeed,

$$\Big\langle\sum_{i=1}^n\nu_i\frac{\partial}{\partial x_i}\delta(S),\ \varphi\Big\rangle=-\sum_{i=1}^n\Big\langle\delta(S),\ \frac{\partial}{\partial x_i}\big(\nu_i\varphi\big)\Big\rangle$$

$$=-\sum_{i=1}^n\Big\langle\delta(S),\ \varphi\frac{\partial\nu_i}{\partial x_i}+\nu_i\frac{\partial\varphi}{\partial x_i}\Big\rangle=-\Big\langle\delta(S),\ -2\mathcal{H}\varphi+\frac{d\varphi}{d\nu}\Big\rangle$$

$$=\big\langle 2\mathcal{H}\delta(S)+d_\nu\delta(S),\ \varphi\big\rangle.$$

Thus, as it follows from (6.26), (6.28),

$$\begin{array}{rcl} \dfrac{\partial}{\partial x_j}\delta(S) &=& \nu_j\big(2\mathcal{H}\delta(S)+d_\nu\delta(S)\big),\\ \dfrac{\partial}{\partial t}\delta(S) &=& -G\big(2\mathcal{H}\delta(S)+d_\nu\delta(S)\big). \end{array} \tag{6.29}$$

With the help of relations (6.5), (6.6), (6.29), for a differentiable function $f(x,t)$ defined on Γ_t we have [24, 12.6.(15),(16)]

$$\frac{\partial}{\partial x_j}\big(f\delta(S)\big) = \frac{\delta f}{\delta x_j}\delta(S) + S_{x_j} f\delta'(S)$$
$$= \Big(\frac{\partial f}{\partial x_j} - \nu_j \frac{df}{d\nu} + 2\mathcal{H}\nu_j f\Big)\delta(S) + \nu_j f d_\nu \delta(S), \tag{6.30}$$

where $j = 1, \dots, n$,

$$\frac{\partial}{\partial t}\big(f\delta(S)\big) = \frac{\delta f}{\delta t}\delta(S) + S_t f\delta'(S)$$
$$= \Big(\frac{\partial f}{\partial t} + G\frac{df}{d\nu} - 2\mathcal{H}Gf\Big)\delta(S) - Gf d_\nu \delta(S). \tag{6.31}$$

It is clear that the following relation holds:

$$d_\nu(\nu_j\delta(S)) = \nu_j d_\nu(\delta(S)) + \sum_{k=1}^n \frac{\delta\nu_j}{\delta x_k}\nu_k\delta(S) = \nu_j d_\nu(\delta(S)). \tag{6.32}$$

Indeed, according to (6.23), we have

$$\big\langle d_\nu(\nu_j\delta(S)),\ \varphi\big\rangle = -\Big\langle \nu_j\delta(S),\ \frac{d\varphi}{dn}\Big\rangle$$
$$= -\Big\langle \nu_j\delta(S),\ \sum_{k=1}^n \frac{\partial\varphi}{\partial x_k}\nu_k\Big\rangle = \sum_{k=1}^n \Big\langle \frac{\partial}{\partial x_k}\big(\nu_j\nu_k\delta(S)\big),\ \varphi\Big\rangle$$

for any $\varphi(x,t) \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R})$. By formulas (6.29), (6.30), we readily obtain

$$\Big\langle d_\nu(\nu_j\delta(S)),\ \varphi\Big\rangle = \Big\langle \nu_j \sum_{k=1}^n \frac{\partial\delta(S)}{\partial x_k}\nu_k + \frac{\delta}{\delta x_k}\big(\nu_j\nu_k\big)\delta(S),\ \varphi\Big\rangle$$
$$= \Big\langle \nu_j\big(2\mathcal{H}\delta(S) + d_\nu\delta(S)\big) + \sum_{k=1}^n \Big(\frac{\delta\nu_j}{\delta x_k}\nu_k + \nu_j\frac{\delta\nu_k}{\delta x_k}\Big)\delta(S),\ \varphi\Big\rangle.$$

Using (6.9), (6.10), we calculate

$$\big\langle d_\nu(\nu_j\delta(S)),\ \varphi\big\rangle = \big\langle \nu_j\big(2\mathcal{H}\delta(S) + d_\nu\delta(S)\big) - 2\mathcal{H}\nu_j\delta(S),\ \varphi\big\rangle$$
$$= \big\langle \nu_j d_\nu\delta(S),\ \varphi\big\rangle.$$

Lemma 6.1. (*see* [24, 5.2.(25),(26)]) *Let $f(x,t)$, $g(x,t)$ be compactly supported smooth functions defined only on a surface Γ_t. Then the formulas for integration by parts hold:*

$$\begin{aligned} \int_{\Gamma_t} f\frac{\delta g}{\delta x_j}\sigma(x,t) &= -\int_{\Gamma_t} \frac{\delta^* f}{\delta x_j} g\sigma(x,t), \\ \int_{\Gamma_t} f\frac{\delta g}{\delta t}\sigma(x,t) &= -\int_{\Gamma_t} \frac{\delta^* f}{\delta t} g\sigma(x,t), \end{aligned} \tag{6.33}$$

where $\frac{\delta^*}{\delta x_j}$ and $\frac{\delta^*}{\delta t}$ are the adjoint operators defined as $\frac{\delta^* f}{\delta x_j} = \frac{\delta f}{\delta x_j} + 2\mathcal{H}\nu_j f$ and $\frac{\delta^* f}{\delta t} = \frac{\delta f}{\delta t} - 2\mathcal{H}Gf$, respectively, and $\sigma(x,t)$ is the Leray measure with respect to the spacetime variables x, t.

Proof of Lemma 6.1. According to (6.6), (6.12), (6.23), (6.24), (6.29), (6.32), we have

$$\int_{\Gamma_t} f\frac{\delta g}{\delta x_j}\sigma(x,t) = \int_{\Gamma_t} f\frac{\delta g}{\delta x_j}\frac{dS(x,t)}{|\nabla_{(x,t)}S|} = \Big\langle f\delta(S),\ \frac{\delta g}{\delta x_j}\Big\rangle$$
$$= \Big\langle f\delta(S),\ \frac{\partial g}{\partial x_j} - \nu_j\frac{dg}{dn}\Big\rangle = -\Big\langle \frac{\partial}{\partial x_j}\big(f\delta(S)\big),\ g\Big\rangle + \Big\langle d_\nu\big(\nu_j f\delta(S)\big),\ g\Big\rangle$$

$$= -\Big\langle \frac{\delta f}{\delta x_j}\delta(S) + f\nu_j\big(2\mathcal{H}\delta(S) + d_\nu\delta(S)\big),\ g\Big\rangle$$
$$+\Big\langle \nu_j\delta(S)\sum_{k=1}^{n}\frac{\delta f}{\delta x_k}\nu_k + f\nu_j d_\nu\delta(S),\ g\Big\rangle.$$

To complete the proof of the first relation (6.33), note that $\sum_{k=1}^{n}\frac{\delta f}{\delta x_k}\nu_k = 0$.

In order to prove the second relation (6.33), it is necessary to repeat the proof of the first relation (6.33) almost word for word. Elementary calculations show that

$$\int_{\Gamma_t} f\frac{\delta g}{\delta t}\sigma(x,t) = \Big\langle f\delta(S),\ \frac{\delta g}{\delta t}\Big\rangle = \Big\langle f\delta(S),\ \frac{\partial g}{\partial t} + G\frac{dg}{dn}\Big\rangle$$
$$= -\Big\langle \frac{\partial}{\partial t}\big(f\delta(S)\big),\ g\Big\rangle - \Big\langle d_\nu\big(Gf\delta(S)\big),\ g\Big\rangle$$
$$= -\Big\langle \frac{\delta f}{\delta t}\delta(S) - fG\big(2\mathcal{H}\delta(S) + d_\nu\delta(S)\big),\ g\Big\rangle$$
$$-\Big\langle \delta(S)\sum_{k=1}^{n}\frac{\delta(Gf)}{\delta x_k}\nu_k + fGd_\nu\delta(S),\ g\Big\rangle = -\Big\langle \frac{\delta f}{\delta t}\delta(S) - 2\mathcal{H}Gf\delta(S),\ g\Big\rangle.$$

Here we take into account the relation $\sum_{k=1}^{n}\frac{\delta(Gf)}{\delta x_k}\nu_k = 0$. □

By formulas (6.30), (6.31), it is easy to prove the following almost obvious statement.

Lemma 6.2. *Let $A(x,t)$, $B(x,t)$, $C(x,t)$, $D(x,t)$, $E(x,t)$, and $S(x,t)$ be smooth functions, $\nabla S\big|_{\Gamma_t} \neq 0$. Then*

$$A(x,t) + B(x,t)H(S) + C(x,t)\delta(S) + D(x,t)\Big|_{\Gamma_t}\frac{\partial\delta(S)}{\partial x_j} + E(x,t)\Big|_{\Gamma_t}\frac{\partial\delta(S)}{\partial t} = 0$$

if and only if

$$\begin{array}{rclcrcl} A &=& 0, & \textit{for} & S &<& 0,\\ A+B &=& 0, & \textit{for} & S &>& 0,\\ C &=& 0, & \textit{for} & S &=& 0,\\ DS_{x_j} + ES_t &=& 0, & \textit{for} & S &=& 0. \end{array}$$

6.3. **Transport theorems.** Here we give the following *transport theorems*.

Theorem 6.1. ([24, 12.8.(3)], [3], [5], [6]) *Let $f(x,t)$ be a sufficiently smooth function defined in a moving solid $V(t)$ and let a moving hypersurface $\partial V(t)$ be its boundary. Let ν be the outward unit space normal to the surface $\partial V(t)$ and $W(x,t)$ be the velocity of a point x in $V(t)$. Then the volume transport theorem holds:*

$$\frac{d}{dt}\int_{V(t)} f(x,t)\,dx = \int_{V(t)}\frac{\partial f}{\partial t}\,dx + \int_{\partial V(t)} fW\cdot\nu\,dS(x)$$
$$= \int_{V(t)}\Big(\frac{\partial f}{\partial t} + \operatorname{div}(fW)\Big)\,dx. \tag{6.34}$$

Using (6.11) and transport theorems from [24, 12.8.(9)], [5], [6], it is easy to prove the following assertion.

Theorem 6.2. *If $f(x,t)$ is a quantity defined only on the moving surface Γ_t then the surface transport theorem holds:*

$$\frac{d}{dt}\int_{\Gamma_t} f(x,t)\frac{dS(x)}{|\nabla S|} = \int_{\Gamma_t}\Big(\frac{\delta f}{\delta t} + \operatorname{div}_{\Gamma_t}(fU_\delta)\Big)\frac{dS(x)}{|\nabla S|}, \tag{6.35}$$

where U_δ is the velocity of Γ_t.

6.4. **Some weak asymptotic expansions.** To construct *δ-shock type solutions* to systems (1.14) and (1.23) we need to calculate the weak asymptotics of some products of regularizations of distributions (1.21), (1.22).

In order to find a *weak asymptotic solution* of the Cauchy problem (1.14), (1.15) we need to construct the weak asymptotics of the products of regularizations of distributions given by the following formulas.

Lemma 6.3. *Let*

$$\delta(x,\varepsilon)=\frac{1}{\varepsilon}\omega_\delta\Big(\frac{x}{\varepsilon}\Big), \qquad \frac{1}{\varepsilon}\omega\Big(\frac{x}{\varepsilon}\Big)$$

be regularizations (1.21) *of the delta function, and*

$$H_j(\xi,\varepsilon)=\omega_{0j}\Big(\frac{\xi}{\varepsilon}\Big)=\int_{-\infty}^{\frac{x}{\varepsilon}}\omega_j(\eta)\,d\eta, \quad j=1,2$$

be a regularization (1.22) *of the Heaviside function $H(x)$, $x\in\mathbb{R}$. Then we have the following weak asymptotic expansions:*

$$\begin{array}{rcl} \big(H_j(\xi,\varepsilon)\big)^r &=& H(\xi)+O_{\mathcal{D}'}(\varepsilon),\\ H_1(\xi,\varepsilon)H_2(\xi,\varepsilon) &=& H(\xi)+O_{\mathcal{D}'}(\varepsilon),\\ H_j(\xi,\varepsilon)\dfrac{dH_j(\xi,\varepsilon)}{d\xi} &=& \frac{1}{2}\delta(\xi)+O_{\mathcal{D}'}(\varepsilon),\\ H_j(\xi,\varepsilon)\dfrac{dH_k(\xi,\varepsilon)}{d\xi} &=& c_{kj}\delta(\xi)+O_{\mathcal{D}'}(\varepsilon), \quad j\neq k,\\ \Big(H_j(x,\varepsilon)\Big)^r\delta(x,\varepsilon) &=& B_{j,r}\delta(x)+O_{\mathcal{D}'}(\varepsilon),\\ \delta(x,\varepsilon)\Big(\omega\Big(\dfrac{x}{\varepsilon}\Big)\Big)^r &=& A_r\delta(x)+O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon\to+0, \end{array} \tag{6.36}$$

where

$$c_{kj}=\int\omega_{0j}(\eta)\omega_k(\eta)\,d\eta\in(0,\ 1], \qquad c_{kj}=1-c_{jk},$$

$$B_{j,r}=\int\omega_{0j}^r(\eta)\omega_\delta(\eta)\,d\eta, \qquad A_r=\int\omega_\delta(\eta)\omega^r(\eta)\,d\eta,$$

$r=1,2,\dots,\ \ j,k=1,2.$

Proof of Lemma 6.3. From (1.22), we obviously have the first two relations in (6.36).

Consider the asymptotics of the product $H_j(\xi,\varepsilon)\frac{dH_j(\xi,\varepsilon)}{d\xi}$. Making the change of variables $\xi=\varepsilon\eta$, we have

$$\Big\langle H_j(\cdot,\varepsilon)\frac{dH_j(\cdot,\varepsilon)}{d\xi},\psi(\cdot)\Big\rangle=\int\omega_{0j}(\eta)\omega_j(\eta)\psi(\varepsilon\eta)\,d\eta,$$

where $\omega_j(\eta)=\omega_{0j}'(\eta)$. Since $\omega_j(\eta)$ decreases sufficiently rapidly as $|\eta|\to\infty$ and $\lim_{\eta\to+\infty}\omega_{0j}(\eta)=1$, $\lim_{\eta\to-\infty}\omega_{0j}(\eta)=0$, we obtain

$$\Big\langle H_j(\cdot,\varepsilon)\frac{dH_j(\cdot,\varepsilon)}{d\xi},\psi(\cdot)\Big\rangle=\frac{1}{2}\psi(0)+O(\varepsilon), \quad \varepsilon\to+0,$$

for all $\psi\in\mathcal{D}(\mathbb{R})$, $j=1,2$.

Analogously, making the change of variables $\xi=\varepsilon\eta$, we obtain the fourth relation

$$\Big\langle H_j(\cdot,\varepsilon)\frac{dH_k(\cdot,\varepsilon)}{d\xi},\psi(\cdot)\Big\rangle=\int\omega_{0j}(\eta)\omega_k(\eta)\psi(\varepsilon\eta)\,d\eta=c_{kj}\psi(0)+O(\varepsilon), \quad \varepsilon\to+0,$$

$j\neq k$. Since $\omega_j=\omega_{0j}'$, the integration by parts gives

$$c_{kj}=\int\omega_{0j}(\eta)\omega_k(\eta)\,d\eta=\omega_{0j}(\eta)\omega_{0k}(\eta)\Big|_{-\infty}^{\infty}-\int\omega_{0k}(\eta)\omega_j(\eta)\,d\eta=1-c_{jk}.$$

Making the change of variables $x = \varepsilon\eta$, we obtain

$$\Big\langle \frac{1}{\varepsilon}\omega_\delta\big(\frac{x}{\varepsilon}\big)\Big(\omega_{0j}\big(\frac{x}{\varepsilon}\big)\Big)^r, \psi(x)\Big\rangle = \int \omega_{0j}^r(\eta)\omega_\delta(\eta)\psi(\varepsilon\eta)\,d\eta = B_{j,r}\psi(0) + O(\varepsilon), \quad \varepsilon \to +0,$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$, i.e., the fifth relation is proved.

Since $\omega_\delta(\eta)\omega^r(\eta)$ decreases sufficiently rapidly as $|\eta| \to \infty$, then following the same reasoning, we obtain

$$\Big\langle \frac{1}{\varepsilon}\omega_\delta\big(\frac{x}{\varepsilon}\big)\Big(\omega\big(\frac{x}{\varepsilon}\big)\Big)^r, \psi(x)\Big\rangle = \int \omega_\delta(\eta)\omega^r(\eta)\psi(\varepsilon\eta)\,d\eta = A_r\psi(0) + O(\varepsilon), \quad \varepsilon \to +0,$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$, $r = 1, 2, \dots$. □

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