DELTA SHOCK WAVE FORMATION IN THE CASE OF TRIANGULAR HYPERBOLIC SYSTEM OF CONSERVATION LAWS

V.G. DANILOV AND D. MITROVIC

Abstract. We describe δ-shock wave generation from continuous initial data in the case of triangular conservation law system arising from "generalized pressureless gas dynamics model". We use smooth approximations in the weak sense that are more general than small viscosity approximations.

In this paper we investigate formation of δ-shock wave in the case of triangular system of conservation laws:

$$u_t + (f(u))_x = 0, \tag{1}$$

$$v_t + (vg(u))_x = 0. \tag{2}$$

For the functions f and g we assume:

$$\begin{aligned} & f \in C^2([U_0, U_1]), \quad g \in C^1([U_0, U_1]), \\ & f'' > 0 \qquad \text{on } [U_0, U_1], \\ & g' - f'' \geq 0 \quad \text{on } [U_0, U_1], \\ & \exists \hat{U} \in (U_0, U_1) \text{ such that } g(\hat{U}) = f'(\hat{U}). \end{aligned} \tag{3}$$

As we will see in the next section, such conditions provide appearance of an admissible δ-shock wave as a solution to (1), (2) (of course, in certain sense; see Definition 3).

In a matter of fact, we propose a method for *constructing* explicit formulas which are *smooth in t* and represent global approximate solution to (1), (2), and whose weak limit contains Dirac δ distribution. The procedure that we use here we call the weak asymptotic method, because the above-mentioned approximate solution satisfy system (1), (2) up to terms that are small in the weak sense.

We stress that in previous works on the subject no method providing construction of explicit formulas representing an approximate solution to the problem was proposed. Another novelty is that, unlike usual and in principle simpler Riemann initial data, we are dealing with continuous initial data. More precisely, we shall show how (regularized) δ-shock wave naturally arises from continuous initial data for system (1), (2). This actually means that we are able to describe smoothly in

---

1991 *Mathematics Subject Classification.* 35L65, 35L67.

Key words and phrases. delta shock wave formation; hyperbolic system; non-classical solution; global approximate solution; weak asymptotic method.

The work of the first author is supported by RFFI grant 05-01-00912, DFG Project 436 RUS 113/895/0-1.

The work of the second author is supported in part by the Research Council of Norway.

Permanent address of the second author is University of Montenegro, Faculty of Mathematics, Cetinjski put bb, 81000 Podgorica, Montenegro.

$t \in \mathbf{R}^+$ passage from the classical to the weak solution concept which is interesting result in itself (compare with [17]). Further in the introduction, we will give more detailed overview of known facts concerning system (1), (2).

The construction is based on the construction of an approximate solution to equations (1), (2) with the following initial data:

$$u|_{t=0} = \hat{u}(x) = \begin{cases} U_1, & x < a_2 \\ u_0(x), & a_1 \leq x \leq a_2 \ , \\ U_0, & a_1 < x \end{cases} \tag{4}$$

$$v|_{t=0} = \hat{v}(x) = \begin{cases} V_1, & x < a_2 \\ v_0(x), & a_1 \leq x \leq a_2 \\ V_0, & a_1 < x \end{cases} \tag{5}$$

where u_0 and v_0 are continuous functions defined on $[a_2, a_1]$ such that v_0 is bounded and u_0 satisfies:

$$f'(u_0(x)) = -Kx + b, \quad x \in [a_2, a_1],$$

and K and b are constants determined from the continuity conditions:

$$f'(U_1) = -Ka_2 + b, \quad f'(U_0) = -Ka_1 + b$$

i.e.

$$K = \frac{f'(U_1) - f'(U_0)}{a_1 - a_2}, \quad b = \frac{f'(U_1)a_1 - f'(U_0)a_2}{a_1 - a_2}. \tag{6}$$

Initial data (4) correspond to the simplest version of shock wave formation for (1). Such initial data provide the formation of the jump with nonzero value at the instant of bifurcation. This phenomenon enables us to find approximate solution to the considered problem by using a variant of the method of characteristics. Actually, we find lines in the (x, t) plane along which the approximate solution u_ε of problem (1), (4) remains constant. We introduced them in [10] and named "new characteristics" (compare Figure 1 and Figure 2). But unlike ordinary characteristics, the "new characteristics" never intersect, and thus, they define the approximate solution along entire time axis. Knowing smooth global approximating solution to (1), (4), we can replace it in (2) and then solve obtained equation by using ordinary method of characteristics.

This approach can be extended to general initial data $(u_0(x), v_0(x))$ if we assume finite number of bifurcation points for each instant of time. We give rough description. Denote by $\{x_1^0, ..., x_k^0\}$ set of points which reach time of gradient catastrophe (bifurcation time) in the moment t^*. Then, instead of given initial data $(u_0(x), v_0(x))$ we put initial data $(u_{\varepsilon 0}(x), v_{\varepsilon 0}(x))$ which differs from $(u_0(x), v_0(x))$ only in the intervals $(x_i^0 - \varepsilon^\mu, x_i^0 + \varepsilon^\mu)$, $\mu \in (0, 1)$, $i = 1, ..., k$. In that intervals functions $u_{\varepsilon 0}$ and $v_{\varepsilon 0}$ have form (4) and (5), respectively. For system (1), (2) with initial data $(u_{\varepsilon 0}, v_{\varepsilon 0})$ we can find global approximate solution by combining method of standard characteristics (out of the intervals $(x_i^0 - \varepsilon^\mu, x_i^0 + \varepsilon^\mu)$, $i = 1, ..., k$) with the method to be presented here (in the intervals $(x_i^0 - \varepsilon^\mu, x_i^0 + \varepsilon^\mu)$, $i = 1, ..., k$). Obviously, the solution constructed in such manner will be an approximate solution to the original problem (the one with initial data $(u_0(x), v_0(x))$).

It is well known that if the solution to equation (1) has a jump then unknown function v contained in (2) can contain δ distribution (of course, in certain sense;

see Definitions 2, 3 or 5 below). Such form of the function v is natural from the viewpoint of applications.

It is clear that the function v is bounded (at least) Lipschitz continuous function until a jump of the function u appears. After that, v has the Dirac δ function as a summand propagating together with a jump (we repeat, in the sense of Definitions 2, 3 or 5, depending on the chosen concept of solution).

The main part of the text is dedicated to the construction of continuous approximation of solution to (1-(2)), ((4)-5) that contains both stages (regular one and one containing δ distribution) of the above dynamics. This approximation is understood in the following sense:

Definition 1. [6] By $O_{\mathcal{D}'}(\varepsilon^\alpha) \in \mathcal{D}'(\mathbf{R})$, $\alpha \in \mathbf{R}$, we denote the family of distributions depending on $\varepsilon \in (0,1)$ and $t \in \mathbf{R}^+$ such that for any test function $\eta(x) \in C_0^1(\mathbf{R})$, the estimate

$$\langle O_{\mathcal{D}'}(\varepsilon^\alpha), \eta(x)\rangle = O(\varepsilon^\alpha), \quad \varepsilon \to 0,$$

holds, where the estimate on the right-hand side is understood in the usual sense and locally uniformly in t, i.e., $|O(\varepsilon^\alpha)| \le C_T \varepsilon^\alpha$ for $t \in [0,T]$.

Now, we can give definition of our approximating solution:

Definition 2. The family of pairs of functions $(u_\varepsilon, v_\varepsilon) = (u_\varepsilon(x,t), v_\varepsilon(x,t))$, $\varepsilon > 0$, is called a weak asymptotic solution of problem (1-(2)), ((4)-5) if

$$\begin{aligned} u_{\varepsilon t} + (f(u_\varepsilon))_x &= \mathcal{O}_{\mathcal{D}'}(\varepsilon), \\ v_{\varepsilon t} + (v_\varepsilon g(u_\varepsilon))_x &= \mathcal{O}_{\mathcal{D}'}(\varepsilon), \\ u_\varepsilon\Big|_{t=0} - \hat{u} =& \mathcal{O}_{\mathcal{D}'}(\varepsilon), \quad v_\varepsilon\Big|_{t=0} - \hat{v} = \mathcal{O}_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to 0. \end{aligned} \tag{7}$$

In [9] it is proven that passing to the limit in (7) we obtain solution of (1-(2)), ((4)-5) in the following sense.

Suppose that $\Gamma = \{\gamma_i : \ i \in I\}$ is a graph in the upper half-plane $\{(x,t) : \ x \in \mathbf{R}, \ t \in \mathbf{R}^+\}$ containing smooth arcs γ_i, $i \in I$, where I is a finite set. By I_0 we denote a subset of I such that an arc γ_k for $k \in I_0$ starts from the points of the x-axis. The set $\Gamma_0 = \{x_k^0 : \ k \in I_0\}$ is the set of initial points of arcs γ_k, $k \in I_0$.

Consider δ-shock wave type initial data $(u_0(x), v_0(x))$, i.e. initial data of the form:

$$v_0(x) = V^0(x) + e^0\delta(\Gamma_0),$$

where $u_0, V^0 \in L^\infty(\mathbf{R})$, and $e^0\delta(\Gamma_0) := \sum_{k\in I_0} e_k^0 \delta(x - x_k^0)$ for constants e_k^0, $k \in I_0$. Notice that in our case we have $e_k^0 = 0$ for every $k \in I_0$.

Definition 3. [9] A pair of distributions (u, v) and the graph $\Gamma = \{\gamma_i : \ i \in I\}$, γ_i parametrized by $(t, x_i(t))$, $t \in \mathbf{R}^+$, where v is represented in the form of the sum

$$v(x,t) = V(x,t) + e(x,t)\delta(\Gamma),$$

where u, V are piecewise smooth functions, and $e(x,t)\delta(\Gamma) := \sum_{i\in I} e_i(x,t)\delta(\gamma_i)$, $e_i(x,t) \in C^1(\gamma_i)$, $i \in I$, is called a generalized δ-shock wave type solution of system

(1), (2) with the initial data $(u_0(x), v_0(x))$ if the integral identities:

$$\begin{aligned}
&\int_{\mathbf{R}^+}\int_{\mathbf{R}} (u\partial_t\varphi + f(u)\partial_x\varphi)\,dxdt + \int_{\mathbf{R}} u_0(x)\varphi(x,0)dx = 0,\\
&\int_{\mathbf{R}^+}\int_{\mathbf{R}} (V\partial_t\varphi + g(u)V\partial_x\varphi)\,dxdt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{d\varphi(x,t)}{dt}dt \\
&\qquad + \int_{\mathbf{R}} v_0(x)\varphi(x,0)dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) = 0,
\end{aligned} \tag{8}$$

hold for every test function $\varphi \in \mathcal{D}(\mathbf{R}\times\mathbf{R}^+)$, where $\frac{d}{dt} = \partial_t + \frac{dx_i}{dt}\partial_x$ is the tangential derivative on the graph Γ, and $\int_{\gamma_i}$ is a line integral over the arc γ_i.

As we shall see, in the case of problem (1-(2)), ((4)-5), the graph Γ will contain only one arc $\{(x,t):\ \ x = ct,\ \ t > t^*\}$ for a constants c and $t^* > 0$ (see (65) and (66)).

Also, there exists another method for deducing integral equalities (7) (more precisely, the ones corresponding to equation (2)). To motivate this approach, notice that the second equation of the system contains nonlinearity generally implying the problem of defining the product of δ distribution with Heaviside function.

Of course, this problem appears only if we directly substitute functions containing mentioned singularities into the equation. In that case, the product of δ and Heaviside function can be defined using measure theory [13, 14, 15, 16, 35]. For the completeness, we give definition of measure valued solution.

Definition 4. Let $BM(\mathbf{R})$ be the space of bounded Borel measures defined on $\mathbf{R}$. For a $\mu \in BM(\mathbf{R})$ by $L^p_\mu(\mathbf{R})$ we denote set of functions $f:\mathbf{R}\to\mathbf{R}$ such that

$$\left(\int_\Omega |f(x)|^p d\mu(x)\right)^{1/p} < \infty.$$

Definition 5. A pair (u,v) where $v(x,t) \in C(BM(\mathbf{R});\mathbf{R}^+)$ and $u(x,t) \in L^\infty(L^\infty_{v(\,\cdot\,,t)}(\mathbf{R});\mathbf{R}^+)$ for almost all $t \geq 0$ is said to be measure valued solution of Cauchy problem (1-(2)), ((4)-5) if the integral identities

$$\int_0^\infty\int_{\mathbf{R}} (u\partial_t\varphi + f(u)\partial_x\varphi)\,dxdt = 0, \qquad \int_0^\infty\int_{\mathbf{R}} (\partial_t\varphi + g(u)\partial_x\varphi)\,v(dx,t)dt = 0$$

hold for all $\varphi \in \mathcal{D}'(\mathbf{R}\times(0,\infty))$.

Within this framework, the following formulas representing the solution of system (1), (2) with Riemann initial data are derived:

$$(u(x,t),v(x,t)) = \begin{cases} (u_L, v_L), & x < \phi(t)\\ (u_\sigma, w(t)\delta(x-\phi(t)), & x = \phi(t),\\ (u_R, v_R), & x > \phi(t), \end{cases} \tag{9}$$

where δ is Dirac distribution and u_-, u_+ and u_σ are values of the function u before the discontinuity, after the discontinuity and on the discontinuity line, respectively. The function $\phi(t) = ct$, c is a constant, is the equation of the discontinuity line.

It is important to notice that *after computing integrals in Definition 5 for functions (9) we get known integral identities from Definition 3.* For instance, if we assume that the graph Γ contains only the arc $\{(x,t): \quad x = ct, \quad t > t^*\}$ then we will have $e(t) = w(t)$ and $u_\sigma = c$ (see also [9] for the pressureless gas dynamics).

Furthermore, notice that in [9], integral equalities from Definition 3 are derived without assumptions on the value of the function u on the discontinuity line. We consider this fact first, more corresponding to generally accepted concept of weak solutions as distributions over $\mathcal{D}(\mathbf{R}^+ \times \mathbf{R})$; and second, we consider unmotivated from the viewpoint of application to define value of a function on a set of zero Lebesgue measure.

Also, one can use regularization of δ and Heaviside distribution and define the product as the weak limit of the product of the approximations [21, 22, 29]. According to such concept, the solution to the system is a family of functions representing approximate solution to the system (see also [2, 27]). We follow this concept (see Definition 2), but we also prove that our approximating solution tends to a solution of the system in the sense of Definition 3 (see Theorem 14).

For other methods involving nonconservative products we address reader on [19, 20, 23, 28, 34].

According to all said above, we see that the problem of propagation of already formed δ-shocks has been explored rather thoroughly.

But, the problem of δ-shock formation is much less studied. Even if it is studied, it has been always done for Riemann problem and always using vanishing viscosity approach. One of the first result on finding global smooth approximation to the problem of type (1-(2)) with Riemann initial data can be found in [21] for the case $f(u) = u^2/2$ and $g(u) = u$. There, approximate solution u_ε to the first equation of the system is found in explicit form by using vanishing viscosity approximation and Hopf-Cole transformation. Then, substituting u_ε in the place of u in equation (2), the equation becomes linear equation in v and it is solved by the method of characteristics.

In [18] the situation with arbitrary f and g has been studied by using another version of the vanishing viscosity approach. In that paper, the vanishing term was of the form $\varepsilon t(u,v)_{xx}$. For the Riemann initial data author proves that system (1),(2) with the vanishing viscosity admits solutions converging to δ type distribution. Author obtains the result by using various relations which are satisfied by the family of the approximate solutions. Still, no explicit form of the approximate solution is given.

Another method for describing δ-shock wave formation in the case of similar systems one can find in [2, 27]. There, so called vanishing pressure approach is used (instead on the right hand side of a system as in vanishing viscosity approach, here one adds perturbation inside the flux).

According to Definition 2, an approximation constructed by the means of the vanishing viscosity or vanishing pressure is indeed weak asymptotic solution to equation (1). In the case of the quadratic nonlinearity (i.e. when $f(u) = u^2$) weak asymptotic solution is constructed in [4], and in the case when f is arbitrary convex function in [10]. In both of the latter papers Cauchy problems with special initial data of type (4) are considered.

The same problem as in [21] is solved in [26] using the weak asymptotic method. The approach used there is similar to one we will use here, i.e. approximate solution is constructed along some kind of "new characteristics". The "new characteristics" used in [26] are the same for both components of solution u and v (which is not the case in this paper). Also, approach from [26] can be applied only in the case of special initial data.

In the current paper, we consider more general initial conditions (continuous initial conditions), and also give explicit formula for approximate solution to the problem (see also [5]).

At the end of the Introduction, we expose the plan of the paper in more details.

In Section 1 we recall necessary conditions for appearance of admissible δ-shock wave for system (1), (2). Then, we quote result in the framework of the weak asymptotic method that we shall need.

In Section 2 we construct the weak asymptotic solution to problem (1), (4).

In Section 3 we construct the weak asymptotic solution to problem (2), (5).

Finally, in Section 4 we find weak limit of the constructed weak asymptotic solution to problem (1-(2)), ((4)-5) and prove that it satisfies Definition 3.

1. Conditions for δ-shock wave appearance and some weak asymptotic formulas

Consider system (1), (2) with Riemann initial data:

$$u|_{t=0} = \begin{cases} U_l, & x<0, \\ U_r, & x\geq 0, \end{cases} , \tag{10}$$

$$v|_{t=0} = \begin{cases} V_l, & x<0, \\ V_r, & x\geq 0. \end{cases} \tag{11}$$

Since the aim of the paper is to describe formation of δ-shock waves, we want to determine sufficient condition on f and g which provides δ-shock wave formation from initial data (10), (11). In other words, we want to determine conditions on f and g such that Riemann problem (1), (2), (10), (11) admits solution of the type:

$$u(x,t) = \begin{cases} U_l, & x<ct, \\ U_r, & x\geq ct, \end{cases} \tag{12}$$

$$v(x,t) = \begin{cases} V_l, & x<ct, \\ V_0, & x\geq ct \end{cases} + const.\cdot t\cdot \delta(x-ct). \tag{13}$$

The solution is understood in the sense of Definition 3.

As the admissibility conditions for δ-shocks we shall use overcompressivity conditions (as in [18, 22, 25, 33]):

$$\lambda_i(U_r,V_r)\leq c\leq \lambda_i(U_l,V_l), \quad i=1,2, \tag{14}$$

where λ_i, $i=1,2$, are eigenvalues of system (1),(2), i.e.

$$\lambda_1(u,v)=f'(u), \quad \lambda_2(u,v)=g(u).$$

From (14) and expressions for λ_i, $i=1,2$ we have:

$$\begin{aligned} f'(U_r)\leq c\leq f'(U_l) \\ g(U_r)\leq c\leq g(U_l). \end{aligned} \tag{15}$$

The following conditions were used in [18]:

$$g'>0, \quad f''>0, \quad f'<g.$$

Still, such conditions will not necessarily give δ-shock even if the classical solution u to (1), (4) blows up after certain time. Since in this paper we are interested only in the δ-shock appearance phenomenon, we shall need more restrictive conditions. The conditions which we shall derive below ensure δ-shock wave appearance if the classical solution to (1), (4) blows up. We stress that in the case of special initial data, the δ-shock wave can arise also in the case of less restrictive conditions on f and g.

We proceed with deriving the necessary conditions. Initial assumption is convexity of the function f, i.e. $f'' > 0$. We have to find conditions on g such that (15) is satisfied. The following condition obviously implies (15):

$$g(U_r) \leq f'(U_r) \leq c \leq f'(U_l) \leq g(U_l). \tag{16}$$

Since f' is increasing it is clear that it has to be $U_l > U_r$. If we assume that $F = g - f'$ is increasing in the interval $[U_r, U_l]$ and that F attains zero in that interval, obviously (16) will be satisfied (since F changes sign on $[U_r, U_l]$ from negative to positive). We can collect the previous considerations in the following theorem:

Theorem 6. *Assume that the functions $f, g \in C^2(\mathbf{R})$ satisfy conditions* (3).

Then Riemann problem (1),(2), (10),(11) admits δ-shock wave type solution of the form (12), (13) (in the sense of Definition 3).

Next, we give very important theorem in the framework of the weak asymptotic method (sometimes called nonlinear superposition law):

Theorem 7. [10] *Let $\omega_i \in C^\infty(\mathbf{R})$, $i = 1, 2$, where $\lim_{z\to+\infty} \omega_i(z) = 1$, $\lim_{z\to-\infty} \omega_i(z) = 0$ and $\frac{d\omega(z)}{dz} \in \mathcal{S}(\mathbf{R})$ where $\mathcal{S}(\mathbf{R})$ is the Schwartz space of rapidly decreasing functions. For the bounded functions a, b, c defined on $\mathbf{R}^+ \times \mathbf{R}$ and bounded functions φ_i, $i = 1, 2$, defined on $\mathbf{R}^+$, we have*

$$\begin{aligned} & f\left(a + b\omega_1(\frac{\varphi_1 - x}{\varepsilon}) + c\omega_2(\frac{\varphi_2 - x}{\varepsilon})\right) \\ =& f(a) + H(\varphi_1 - x)\left(f(a+b+c)B_1 + f(a+b)B_2 - f(a+c)B_1 - f(a)B_2\right) \\ & + H(\varphi_2 - x)\left(f(a+b+c)B_2 - f(a+b)B_2 + f(a+c)B_1 - f(a)B_1\right) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \end{aligned} \tag{17}$$

where H is the Heaviside function and $B_i = B_i(\frac{\varphi_2 - \varphi_1}{\varepsilon})$ is such that for every $\rho \in \mathbf{R}$ we have

$$B_1(\rho) = \int \dot{\omega}_1(z)\omega_2(z+\rho)dz \text{ and } B_2(\rho) = \int \dot{\omega}_2(z)\omega_1(z-\rho)dz, \tag{18}$$

and

$$B_1(\rho) + B_2(\rho) = 1.$$

Furthermore, we have:

$$\begin{aligned} B_1(\rho) = 1 - B_2(\rho) \to 1, &\quad \text{as} \quad \rho \to +\infty \\ B_1(\rho) = 1 - B_2(\rho) \to 0, &\quad \text{as} \quad \rho \to -\infty \end{aligned} \tag{19}$$

2. Weak asymptotic solution to (1), (4)

In this section we shall find the weak asymptotic solution to problem (1), (4). According to the concept of weak asymptotic method we replace the problem by the family of problems

$$u_{\varepsilon t} + (f(u_\varepsilon))_x = \mathcal{O}_{\mathcal{D}'}(\varepsilon), \tag{20}$$

$$u_\varepsilon|_{t=0} = \hat{u}(x) + \mathcal{O}_{\mathcal{D}'}(\varepsilon), \tag{21}$$

where $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ will be determined in (30).

Before we pass on solving the problem, we introduce the notation that we shall use (as usual $x \in \mathbf{R}$, $t \in \mathbf{R}^+$):

$$\begin{aligned}
&u_1 = u_1(x,t,\varepsilon), \quad B_i = B_i(\rho), \quad \varphi_i = \varphi_i(t,\varepsilon),\\
&H_i = H(\varphi_i - x), \quad \delta_i = \delta(\varphi_i - x), \quad i = 1,2,\\
&\tau = \frac{f'(U_1)t + a_2 - f'(U_0)t - a_1}{\varepsilon} = \frac{\psi_0(t)}{\varepsilon},\\
&t^* = \frac{a_1 - a_2}{f'(U_1) - f'(U_0)},\\
&x^* = f'(U_1)t^* + a_2 = f'(U_0)t^* + a_1 = \frac{f'(U_0)a_1 - f'(U_0)a_2}{f'(U_1) - f'(U_0)},
\end{aligned}$$

where H is the Heaviside function and δ Dirac distribution.

The function τ is so-called 'fast variable'. It is equal to difference of standard characteristics of equation (1) emanating from a_2 and a_1, respectively. Since $a_2 < a_1$, when we are in the domain of existence of classical solution to (1), (4) we have $\tau \to -\infty$ as $\varepsilon \to 0$, while when we are in the domain where solution to (1), (4) is discontinuous (i.e. in the form of the shock wave) we have $\tau \to \infty$ as $\varepsilon \to 0$.

The point (t^*, x^*) is the point of blow up of the classical solution to (1), (4).

We explain the procedure we shall use before we formulate the theorem.

It is well known problem (1), (4) will have classical solution up to the moment t^* given by (we introduced it at the beginning of the section but we find convenient to repeat since we shall use it often in the sequel):

$$t = t^* = \max_{x \in (a_2, a_1)} -\frac{1}{f''(u_0(x))u_0'(x)} = \frac{1}{K}, \tag{22}$$

where K is given by (6). The choice of our initial data is such that in the moment of blow up of the classical solution the shock wave will be formed and it will not change its shape for any $t > t^*$. This is because all the characteristics emanating from $[a_2, a_1]$ intersect in one point (t^*, x^*) (see Figure 1).

So, for $t > t^*$ we have to pass to the weak solution concept. In other words, in the moment $t = t^*$ we stop the time and solve Riemann problem for equation (1).

Our aim is to solve the problem globally in time without changing solution concept, i.e. to find globally defined approximate solution to (1), (4) which is at least continuous. To do this we have to avoid intersection of characteristics.

Natural idea is to smear the discontinuity line, i.e. to take ε neighborhood of the discontinuity line and to dispose characteristics in that neighborhood in a way that they do not intersect and as $\varepsilon \to 0$ all of them lump together into the discontinuity line. Of course, this will not be the standard characteristics for problem (1), (4).

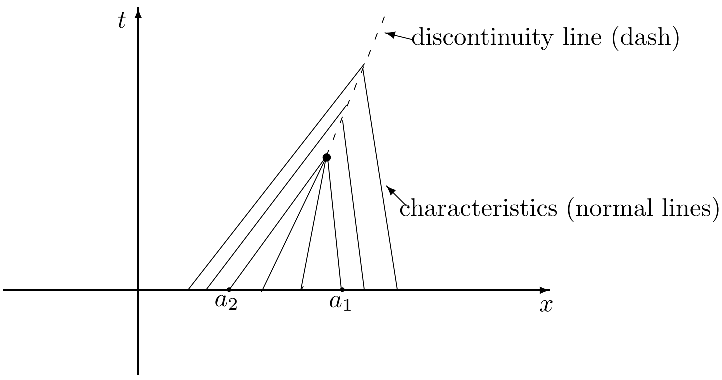

Figure 1. Standard characteristics for (1), (4). Dotted point in (t,x) plane is (t^*,x^*).

Nevertheless, along them approximate solution to our problem will remain constant. Such lines we call 'new characteristics'.

Another question that arises here is how to distribute 'new characteristics' in the ε neighborhood of the discontinuity line. The obvious way to accomplish this is to distribute the 'new characteristics' uniformly in the mentioned area, i.e. in a way that every of them is parallel to the discontinuity line.

Since all the characteristics emanating from the interval $[a_2,a_1]$ intersect in the same point, roughly speaking, it is enough to find the way to dispose 'new characteristics' emanating from a_2 and a_1 so that they do not intersect.

We use Theorem 7 and 'switch' functions B_i, $i=1,2$, appearing there.

Denote by φ_i, $i=1,2$, the new characteristics emanating from the points a_i, $i=1,2$, respectively. They are given by the following Cauchy problems:

$$\frac{d}{dt}\varphi_1(t,\varepsilon)=(B_2(\rho)-B_1(\rho))f'(U_0)+cB_1(\rho),\quad \varphi_1(0,\varepsilon)=a_1+A\varepsilon\frac{a_1-a_2}{2},\quad (23)$$

$$\frac{d}{dt}\varphi_2(t,\varepsilon)=(B_2(\rho)-B_1(\rho))f'(U_1)+cB_1(\rho),\quad \varphi_2(0,\varepsilon)=a_2-A\varepsilon\frac{a_1-a_2}{2},\quad (24)$$

for large enough constant A. As we shall see later, it will be necessary to extend a little bit the interval $[a_2,a_1]$ in order to prove that the 'new characteristics' do not mutually intersect. Therefore, we have $A\varepsilon\frac{a_1-a_2}{2}$ accompanying initial data in (23) and (24). Furthermore, notice that the latter Cauchy problems are simple and globally solvable since the unknown functions φ_i, $i=1,2$, are only on the left hand side. Namely, the function $\rho=\rho(\tau(t))$ appearing in (23) and (24) is defined as:

$$\rho=\frac{\varphi_2(t,\varepsilon)-\varphi_1(t,\varepsilon)}{\varepsilon},\quad (25)$$

and it is effectively given by Cauchy problem (35).

Characteristics given by (23) and (24) actually emanate from $a_1+A\varepsilon\frac{a_1-a_2}{2}$ and $a_2-A\varepsilon\frac{a_1-a_2}{2}$, respectively. Still, since this is perturbation of order $\mathcal{O}(\varepsilon)$ it does not affect our weak asymptotic solution.

According to what we said above, we expect that for every $t > t^*$ it should be (since new characteristics should be 'close' one to another for $t > t^*$):

$$\varphi_1(t,\varepsilon) - \varphi_2(t,\varepsilon) = \mathcal{O}(\varepsilon), \quad t > t^*,$$

and also:

$$\frac{d}{dt}\varphi_{i\varepsilon}(t,\varepsilon) = \frac{f(U_1) - f(U_0)}{U_1 - U_0} + \mathcal{O}(\varepsilon), \quad t > t^*, \tag{26}$$

since the new characteristics should be 'close' to the discontinuity line which is, according to Rankine-Hugoniot conditions, given by equation (26).

This hints us that if we put $c = 2\frac{f(U_1)-f(U_0)}{U_1-U_0}$ in (23) and (24), we can expect that for $t > t^*$ the expression $B_2(\rho) - B_1(\rho)$ is to zero thus eliminating nonlinearity f' appearing in the equations of new characteristics (23) and (24). Indeed, according to Theorem 7 we have $B_2(\rho) + B_1(\rho) = 1$ which together with expected $B_2(\rho) - B_1(\rho) \to 0$ means that as $\tau \to \infty$ the functions B_1 and B_2 are close to $1/2$. Due to our choice of c this implies that $B_1 c$ from (23) and (24) is close to $\frac{f(U_1)-f(U_0)}{U_1-U_0}$ (Rankine-Hugoniot conditions) which means that $\varphi_{i\varepsilon}$, $i = 1, 2$, satisfy (26) as expected.

Here we used the following simple observation.

Once the shock wave is formed, it continuous to move according to Rankine-Hugoniot conditions and it does not change its shape along entire time axis. Therefore, the linear equation:

$$\frac{\partial u}{\partial t} + \frac{c}{2}\frac{\partial u}{\partial x} = 0, \quad c = 2\frac{f(U_1) - f(U_0)}{U_1 - U_0}, \tag{27}$$

and equation (1) with the same initial condition:

$$u|_{t=0} = \begin{cases} U_1, & x < 0, \\ U_0, & x \geq 0, \end{cases}$$

will have the same solutions. Clearly, it is much easier to solve linear equation (27) then nonlinear equation (1). Still, the question is how to pass from nonlinear equation (1) to linear equation (27) in the domains where they give the same solution (in the case of our initial data it will be after the shock wave formation). We explain briefly how we do it.

Define the 'new characteristics' as the solutions to the following Cauchy problem:

$$\begin{aligned} &\dot{x} = (B_2 - B_1)f'(u_1) + cB_1, \quad \dot{u}_1 = 0, \\ &u_1(0) = u_0(x_0), \quad x(0) = x_0 + \varepsilon A(x_0 - \frac{a_1 + a_2}{2}), \quad x_0 \in [a_2, a_1]. \end{aligned} \tag{28}$$

Thus, $\varphi_i(t,\varepsilon) = x(a_i, t, \varepsilon)$, $i = 1, 2$, where x is the solution to (28). We will show later that it is possible to choose the constant A so that for $x_0 \in [a_2, a_1]$ every $t > 0$ we have

$$\frac{\partial x}{\partial x_0} > 0.$$

This means that 'new characteristics' do not mutually intersect which in turn means that there exists the solution $x_0 = x_0(x, t, \varepsilon)$ of the implicit equation:

$$x(x_0, t, \varepsilon) = x. \tag{29}$$

From here, it follows that $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ from (21) is given by

$$\mathcal{O}_{\mathcal{D}'}(\varepsilon) = \begin{cases} U_1, & x < a_2 \\ u_0(x + \varepsilon A(x - \frac{a_1 - a_2}{2})), & a_1 \leq x < a_2 \\ U_0, & a_2 \leq x. \end{cases} \tag{30}$$

Bearing in mind that $B_1 = 1 - B_2 \sim 0$ before the interaction and $B_1 \sim B_2$ after the interaction, we see that, using the new characteristics, we have smoothly passed from the characteristics of equation (1) to the characteristics of equation (28), i.e. from equation (1) to equation (27).

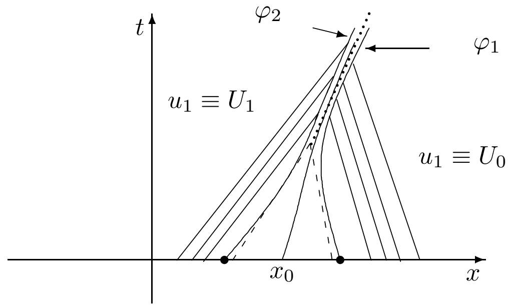

FIGURE 2. System of 'new characteristics' for u_ε. Dashed lines are standard characteristics emanating from a_1 and a_2. The points $a_1 + A\varepsilon\frac{a_1-a_2}{2}$ and $a_2 - A\varepsilon\frac{a_1-a_2}{2}$ are dotted on the x axis.

We formalize the previous considerations in Theorem 8. The theorem is analogue to the main result from [10]. The problem which we consider here, i.e. problem (1), (4) can be solved in more elegant manner (see Theorem 10 below). Still, approach used in Theorem 8 can be used on the case of arbitrary piecewise monotone initial data [11]. Also, Theorem 8 represents motivation for Theorem 10.

Theorem 8. *The weak asymptotic solution of problem (1), (4) has the form:*

$$u_\varepsilon(x,t) = U_0 + (u_1(x,t,\varepsilon) - U_0)\,\omega_1(\frac{\varphi_1(t,\varepsilon) - x}{\varepsilon}) \\ + (U_1 - u_1(x,t,\varepsilon))\,\omega_2(\frac{\varphi_2(t,\varepsilon) - x}{\varepsilon}), \tag{31}$$

where ω_i, $i = 1,2$, satisfy the conditions from Theorem 7.

The functions $\varphi_i(t,\varepsilon)$, $t \in \mathbf{R}^+$, $i = 1,2$, are given by (23) and (24), and the function ρ is given by (25).

The function $u_1(x,t,\varepsilon)$ is given by

$$u_1(x,t,\varepsilon) = u_0(x_0(x,t,\varepsilon))$$

where x_0 is the inverse function to the function $x = x(x_0,t,\varepsilon)$, $t > 0$, $\varepsilon > 0$, of 'new characteristics' defined through Cauchy problem (28).

Proof: We substitute ansatz (31) into (20):

$$\Big(U_0+(u_1(x,t,\varepsilon)-U_0)\,\omega_1(\frac{\varphi_1(t,\varepsilon)-x}{\varepsilon})+(U_1-u_1(x,t,\varepsilon))\,\omega_2(\frac{\varphi_2(t,\varepsilon)-x}{\varepsilon})\Big)_t$$
$$+\Big(f(U_0+(u_1(x,t,\varepsilon)-U_0)\,\omega_1(\frac{\varphi_1(t,\varepsilon)-x}{\varepsilon})+(U_1-u_1(x,t,\varepsilon))\,\omega_2(\frac{\varphi_2(t,\varepsilon)-x}{\varepsilon}))\Big)_x$$
$$=\mathcal{O}_{\mathcal{D}'}(\varepsilon)$$

After using (17) we get (we remind $u_1=u_1(x,t,\varepsilon)$ and $B_i=B_i(\rho)=B_i(\frac{\varphi_2-\varphi_1}{\varepsilon})$):

$$\Big(U_0+(u_1-U_0)\,\omega_1(\frac{\varphi_1(t,\varepsilon)-x}{\varepsilon})+(U_1-u_1)\,\omega_2(\frac{\varphi_2(t,\varepsilon)-x}{\varepsilon})\Big)_t+\Big(f(U_0)$$
$$+(f(U_1)B_1+f(u_1)B_2-f(U_0+U_1-u_1)B_1-f(U_0)B_2)\,H_1$$
$$+(f(U_1)B_2-f(u_1)B_2+f(U_0+U_1-u_1)B_1-f(U_0)B_1)\,H_2\Big)_x=\mathcal{O}_{\mathcal{D}'}(\varepsilon)$$

Notice that ($w-$ denotes distributional limit) $w-\lim\limits_{\varepsilon\to0}\omega_i(\frac{\varphi_i-x}{\varepsilon})=H_i=H(\varphi_i-x)$ and $w-\lim\limits_{\varepsilon\to0}\partial_x\omega_i(\frac{\varphi_i-x}{\varepsilon})=\delta_i=\delta(\varphi_i-x)$, $i=1,2$, for the Heaviside function H and the Dirac distribution δ. Taking this into account, we get from the previous expression upon differentiating and collecting terms multiplying H_i and $\delta_i=-\partial_xH_i$:

$$\Big[\frac{\partial u_1}{\partial t}+B_2f'(u_1)\frac{\partial u_1}{\partial x}+B_1f'(U_1+U_0-u_1)\frac{\partial u_1}{\partial x}\Big]\,H_1$$
$$+\Big[-\frac{\partial u_1}{\partial t}-B_2f'(u_1)\frac{\partial u_1}{\partial x}-B_1f'(U_1+U_0-u_1)\frac{\partial u_1}{\partial x}\Big]\,H_2$$
$$+((u_1-U_0)\varphi_{1t}-B_2\,(f(u_1)-f(U_0))-B_1\,(f(U_1)-f(U_1+U_0-u_1)))\,\delta_1$$
$$+((U_1-u_1)\varphi_{2t}-B_2\,(f(U_1)-f(u_1))-B_1\,(f(U_1+U_0-u_1)-f(U_0)))\,\delta_2$$
$$=\mathcal{O}_{\mathcal{D}'}(\varepsilon).$$

We rearrange this expression using the following simple formula $CH_1+DH_2=(C+D)H_2+C(H_1-H_2)$:

$$\Big(\frac{\partial u_1}{\partial t}+[(B_2-B_1)f'(u_1)]\frac{\partial u_1}{\partial x}\Big)\,(H_1-H_2)$$
$$+B_1[\frac{d}{dx}\,(f(U_1+U_0-u_1)+f(u_1))]\,(H_1-H_2)$$
$$+((u_1-U_0)\varphi_{1t}-B_2\,(f(u_1)-f(U_0))-B_1\,(f(U_1)-f(U_1+U_0-u_1)))\,\delta_1$$
$$+((U_1-u_1)\varphi_{2t}-B_2\,(f(U_1)-f(u_1))-B_1\,(f(U_1+U_0-u_1)-f(U_0)))\,\delta_2=\mathcal{O}_{\mathcal{D}'}(\varepsilon).$$

For an unknown constant c we add and subtract the term $cB_1\frac{\partial u_1}{\partial x}$ in the coefficient multiplying (H_1-H_2) and then we rewrite the last expression in the following form:

$$\Big(\frac{\partial u_1}{\partial t}+[(B_2-B_1)f'(u_1)+cB_1]\frac{\partial u_1}{\partial x}\Big)\,(H_1-H_2)$$
$$+B_1[\frac{d}{dx}\,(f(U_1+U_0-u_1)+f(u_1)-cu_1)]\,(H_1-H_2)$$
$$+((u_1-U_0)\varphi_{1t}-B_2\,(f(u_1)-f(U_0))-B_1\,(f(U_1)-f(U_1+U_0-u_1)))\,\delta_1$$
$$+((U_1-u_1)\varphi_{2t}-B_2\,(f(U_1)-f(u_1))-B_1\,(f(U_1+U_0-u_1)-f(U_0)))\,\delta_2=\mathcal{O}_{\mathcal{D}'}(\varepsilon). \tag{32}$$

We put

$$\frac{\partial u_1}{\partial t}+[(B_2-B_1)f'(u_1)+cB_1]\frac{\partial u_1}{\partial x}=0,\quad u_1(x,0,\varepsilon)=u_0(x),\quad x\in[a_2,a_1].$$

The system of characteristics for this problem reads:

$$\begin{aligned}\dot{x}=(B_2-B_1)f'(u_1)+cB_1,\quad \dot{u}_1=0,\\ u_1(0)=u_0(x_0),\quad x(0)=x_0\in[a_2,a_1].\end{aligned}\tag{33}$$

The aim is to prove that characteristics defined by the previous system do not intersect. It appears that it is much easier to accomplish this if we perturb initial data for x in the previous system for a parameter of order ε. More precisely, instead of (33) we shall consider system (28) (the same is done in [10]).

It is clear that such perturbation changes the solution of (28) for $\mathcal{O}_{\mathcal{D}'}(\varepsilon)$ since initial condition in (28) is continuous.

We pass to the proof that the characteristics given by (28) do not intersect. From the second equation in (28) it follows $u_1\equiv u_0(x_0)$. We substitute this into the first equation of (28) and use $f'(u_0(x_0))=-Kx_0+b$, $x_0\in[a_2,a_1]$. We have:

$$\dot{x}=(B_2-B_1)(-Kx_0+b)+cB_1,\quad x(0)=x_0+\varepsilon A\left(x_0-\frac{a_1+a_2}{2}\right).\tag{34}$$

Out of the segment $[a_2-\varepsilon A\frac{a_1-a_2}{2},a_1+\varepsilon A\frac{a_1-a_2}{2}]$ initial function is constant and we define the solution u_1 of problem (28) to be equal to U_1 on the left-hand side of the characteristic emanating from $a_2-A\frac{a_1-a_2}{2}$ and to be equal to U_0 on the right-hand side of the characteristic emanating from $a_1+A\frac{a_1-a_2}{2}$ (see Figure 1).

For the functions φ_1 and φ_2 as the characteristics emanating from $a_1+A\frac{a_1-a_2}{2}$ and $a_2-A\frac{a_1-a_2}{2}$ respectively, we have (23) and (24).

Now, we show how to effectively determine ρ given by (25). We apply standard procedure (see [4, 6, 10])). Subtracting (23) from (24) we get:

$$(\varphi_2-\varphi_1)_t=\varepsilon\left(\frac{\varphi_2-\varphi_1}{\varepsilon}\right)_t=\varepsilon\rho_t=(B_2-B_1)\psi_0(t).$$

Then, passing from the "slow" variable t to the "fast" variable τ we obtain (we also use $B_2+B_1=1$):

$$\rho_\tau=1-2B_1(\rho),\quad \frac{\rho}{\tau}\Big|_{\tau\to-\infty}=1.\tag{35}$$

We explain the condition $\lim\limits_{\tau\to-\infty}\frac{\rho}{\tau}=1$. We have from (23) and (24)

$$\frac{\rho}{\tau}=\frac{\int_0^t(f'(U_1)-f'(U_0))(B_2-B_1)dt'+a_2-a_1}{(f'(U_1)-f'(U_0))t+a_2-a_1}.$$

Putting $t=0$ in the previous relation we see that

$$\frac{\rho}{\tau}\Big|_{t=0}=1.\tag{36}$$

For $t=0$ we have $\tau\to-\infty$ as $\varepsilon\to0$. Therefore, from (36) it follows

$$\frac{\rho}{\tau}\Big|_{\tau\to-\infty}=1.\tag{37}$$

This relation practically means that new characteristics emanating from a_i, $i=1,2$, coincides at least in the initial moment with standard characteristics up to some small parameter ε. Still, since $\tau\to-\infty$ as $\varepsilon\to0$ for every $t<t^*$ (which means

$B_1 \to 0$; see (19) and (37)) we see from (23) and (24) that new characteristics coincides with standard ones for every $t < t^*$ up to some small parameter ε.

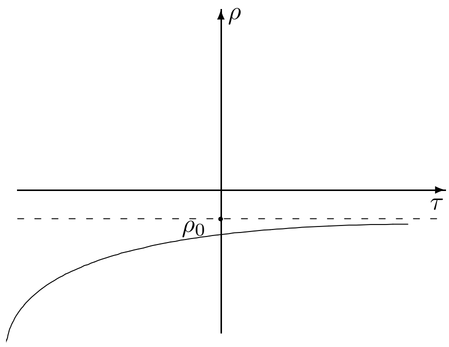

FIGURE 3. The curve represents solution of (35). Dot on the ρ axis, denoted by ρ_0, is the (smallest and in this case unique

Next, we analyze (35). From the standard theory of ODE we see that $\rho \to \rho_0$ as $\tau \to +\infty$ where ρ_0 is constant such that $B_1(\rho_0) = B_2(\rho_0) = 1/2$ (see Figure 3). That means that after the interaction, i.e. for $t > t^*$, we have

$$\rho = \frac{\varphi_1 - \varphi_2}{\varepsilon} = \rho_0 + \mathcal{O}(\varepsilon) \implies \varphi_1 = \varphi_2 + \mathcal{O}(\varepsilon), \quad \varepsilon \to 0,$$

or, after letting $\varepsilon \to 0$, for $t > t^*$ we have shock wave concentrated at (see text in front of theorem for notations):

$$\varphi(t) = \lim_{\varepsilon \to 0} \varphi_i(t, \varepsilon) = \frac{c}{2}(t - t^*) + x^*. \tag{38}$$

Now, we can prove global solvability of Cauchy problem (28).

Problem (28) is globally solvable if characteristics emanating from the interval $[a_2 - A\varepsilon \frac{a_1 - a_2}{2}, a_1 + A\varepsilon \frac{a_1 - a_2}{2}]$ do not intersect. To prove that we will use the inverse function theorem. We will prove that for every $t \in \mathbf{R}^+$ we have $\frac{\partial x}{\partial x_0} > 0$ which means that for every $x = x(x_0, t, \varepsilon)$, $x_0 \in [a_2, a_1]$, we have unique $x_0 = x_0(x, t, \varepsilon)$ and we can write $u_1(x(x_0, t, \varepsilon), t) = u_0(x_0(x, t, \varepsilon))$.

Differentiating (34) in x_0 and integrating from 0 to t we obtain (we remind $B_2 + B_1 = 1$):

$$\frac{\partial x}{\partial x_0} = 1 + \varepsilon A - K \int_0^t (B_2 - B_1) dt' = 1 + \varepsilon A - K \int_0^t (1 - 2B_1) dt'. \tag{39}$$

For $t \in [0, t^*]$ we have (notice that $1 - Kt^* = 0$):

$$\begin{aligned} \frac{\partial x}{\partial x_0} &= 1 + \varepsilon A - K \int_0^t dt + K \int_0^t 2B_1 dt \\ &\geq 1 + \varepsilon A - K \int_0^{t*} dt + K \int_0^t 2B_1 dt = \varepsilon A + K \int_0^t 2B_1 dt > 0. \end{aligned}$$

So, everything is correct for $t \leq t^*$.

To see what is happening for $t > t^*$, initially we estimate $1 - 2B_1(\rho)$ when $\tau \to \infty$. From equation (35) we have (we use Taylor expansion of B_1 around the point $\rho = \rho_0$):

$$\rho_\tau = 1 - 2B_1(\rho) = -2(\rho - \rho_0)B_1'(\tilde{\rho}) \Rightarrow \frac{d}{d\tau}\ln(\rho - \rho_0) = -2B_1'(\tilde{\rho}),$$

for some $\tilde{\rho}$ belonging to the interval with endpoints ρ and ρ_0. From here we see:

$$\rho - \rho_0 = (\rho(\tau_0) - \rho_0)\exp(\int_{\tau_0}^{\tau} -2B_1'(\tilde{\rho})d\tau') = (\rho(\tau_0) - \rho_0)\exp((\tau_0 - \tau)2B_1'(\tilde{\rho}_1))$$

for some fixed $\rho_0 \in \mathbf{R}$ and $\tilde{\rho}_1 \in (\rho(\tau_0), \rho(\tau)) \subset [\rho(\tau_0), \rho_0]$. We remind that $B_1'(\tilde{\rho}_1)) \geq \tilde{c} > 0$, for some constant $\tilde{c}$, since B_1 is increasing function and $\tilde{\rho}_1$ belongs to the compact interval $[\rho(\tau_0), \rho_0]$. Letting $\tau \to \infty$ we conclude that for any $N \in \mathbf{N}$

$$\rho - \rho_0 = \mathcal{O}(1/\tau^N), \quad \tau \to \infty.$$

From here we have $\rho_\tau = \mathcal{O}(1/\tau^N), \quad \tau \to \infty$, since:

$$\lim_{\tau\to\infty} \frac{\rho_\tau}{\rho - \rho_0} = \lim_{\tau\to\infty} \frac{1 - 2B_1(\rho)}{\rho - \rho_0} = \lim_{\tau\to\infty} -2B_1'(\rho) = -2B_1'(\rho_0) = \text{const.} < 0.$$

Accordingly, ρ_τ and $\rho - \rho_0$ have the same growth rate with respect to τ.

This, in turn, means that for every $N \in \mathbf{N}$ and $t > t^*$ we have

$$1 - 2B_1(\rho) = \rho_\tau = \mathcal{O}(\tau^{-N}) = \mathcal{O}(\varepsilon^N), \quad \varepsilon \to \infty, \tag{40}$$

since for fixed $t > t^*$ we have $\tau = \frac{\psi_0(t)}{\varepsilon} \to \infty$ as $\varepsilon \to 0$.

Now we can prove that $\frac{\partial x}{\partial x_0} > 0$ for $t > t^*$. We have

$$\begin{aligned} \frac{\partial x}{\partial x_0} &= 1 + \varepsilon A - 2K \int_0^t (1 - 2B_1)dt' \qquad (41) \\ &= 1 + \varepsilon A - 2K \int_0^{t^*} (1 - 2B_1)dt' - 2K \int_{t^*}^t (1 - 2B_1)dt' \\ &= \varepsilon A + 4K \int_0^{t^*} B_1 dt' - 2K \int_{t^*}^t (1 - 2B_1)dt' > \varepsilon A - 2K \int_{t^*}^t (1 - 2B_1)dt'. \end{aligned}$$

Recall that

$$B_1 = B_1(\rho(\tau)) = B_1(\rho(\frac{\psi_0(t)}{\varepsilon})).$$

Consider the last term in expression (41):

$$\begin{aligned} 2K \int_{t^*}^t (1 - 2B_1)dt' &= 2K \int_{t^*}^t (1 - 2B_1(\rho(\frac{\psi_0(t')}{\varepsilon})))dt' \\ &= \begin{pmatrix} \frac{\psi_0(t')}{\varepsilon} = z \implies & (f'(U_1) - f'(U_0))dt' = \varepsilon dz; \\ t^* < t' < t \implies & 0 < z < \frac{\psi_0(t)}{\varepsilon} \end{pmatrix} \\ &= \varepsilon \frac{2K}{f'(U_1) - f'(U_0)} \int_0^{\frac{\psi_0(t)}{\varepsilon}} (1 - 2B_1(\rho(z)))dz < \varepsilon 2KC, \end{aligned}$$

where

$$C = \frac{\int_0^\infty (1 - 2B_1(\rho(z)))dz}{f'(U_1) - f'(U_0)} < \infty,$$

since from (40) we know $1 - 2B_1(\rho(z)) = \mathcal{O}(z^{-N})$, $z \to \infty$ and $N \in \mathbf{N}$ arbitrary.

Therefore, from (41) it follows that for A large enough (more precisely for $A > C$) we have

$$\frac{\partial x}{\partial x_0} > \tilde{C}\varepsilon > 0$$

for a constant $\tilde{C} > 0$, what we wanted to prove.

Next step is to obtain the constant c. We multiply (32) by $\eta \in C_0^1(\mathbf{R})$, integrate over $\mathbf{R}$ with respect to x and use (28) (so, we remove the first term in (32)):

$$\begin{aligned}&\int B_1[\frac{d}{dx}\left(f(U_1+U_0-u_1)+f(u_1)-cu_1\right)]\left(H_1-H_2\right)\eta(x)dx\\&+\left((u_1-U_0)\varphi_{1t}-B_2\left(f(u_1)-f(U_0)\right)-B_1\left(f(U_1)-f(U_1+U_0-u_1)\right)\right)\delta_1\\&+\left((U_1-u_1)\varphi_{2t}+B_2\left(f(u_1)-f(U_1)\right)+B_1\left(f(U_0)-f(U_1+U_0-u_1)\right)\right)\delta_2=\mathcal{O}(\varepsilon).\end{aligned}$$

We apply partial integration on the first integral in the previous expression to obtain:

$$\begin{aligned}&\int B_1[f(U_1+U_0-u_1)+f(u_1)-cu_1]\left(H_1-H_2\right)\eta'(x)dx \qquad (42)\\&+\int\left((u_1-U_0)\varphi_{1t}-B_2\left(f(u_1)-f(U_0)\right)+B_1\left(f(u_1)+f(U_0)-cu_1\right)\right)\eta(x)\delta_1dx\\&+\int\left((U_1-u_1)\varphi_{2t}+B_2\left(f(u_1)-f(U_1)\right)-B_1\left(f(u_1)+f(U_1)-cu_1\right)\right)\eta(x)\delta_2dx=\mathcal{O}(\varepsilon).\end{aligned}$$

Bearing in mind that $\rho = \frac{\varphi_2-\varphi_1}{\varepsilon}$ and using definition of the Dirac δ distribution together with the fact that $u_1 \equiv U_1$ for $x \geq \varphi_1$ and $u_1 \equiv U_0$ for $x \leq \varphi_2$:

$$\begin{aligned}&\varepsilon\rho B_1\int[f(U_1+U_0-u_1)+f(u_1)-cu_1]\frac{H_1-H_2}{\varphi_2-\varphi_1}\eta'(x)dx \qquad (43)\\&+B_1\left(2f(U_0)-cU_0\right)\eta(\varphi_1)-B_1\left(-2f(U_1)+cU_1\right)\eta(\varphi_2)=\mathcal{O}(\varepsilon).\end{aligned}$$

To continue, notice that we have $|\rho B_1| < \infty$ for every $\tau \in \mathbf{R}$. Namely,

$$\begin{aligned}&|\rho B_1(\rho)|\to 0 \quad \text{as} \quad \tau\to-\infty \text{ since in that case } B_1(\rho(\tau))\sim B_1(\tau)\sim\frac{1}{\tau^N}\sim\frac{1}{\rho^N},\\&|\rho B_1(\rho)|\to\rho_0 B_1(\rho_0) \quad \text{as} \quad \tau\to\infty \text{ since in that case } \rho\to\rho_0.\end{aligned} \qquad (44)$$

This fact reduces expression (43) to:

$$B_1\left(2f(U_0)-cU_0\right)\eta(\varphi_1)-B_1\left(-2f(U_1)+cU_1\right)\eta(\varphi_2)=\mathcal{O}(\varepsilon). \qquad (45)$$

Rewrite this expression in the following manner:

$$\begin{aligned}&B_1\left(2(f(U_0)-f(U_1))-c(U_0-U_1)\right)\eta(\varphi_1)+B_1\left(-2f(U_1)+cU_1\right)\left(\eta(\varphi_2)-\eta(\varphi_1)\right)\\&=B_1\left(2(f(U_0)-f(U_1))-c(U_0-U_1)\right)\eta(\varphi_1)\\&+\varepsilon\rho B_1(\rho)\left(-2f(U_1)+cU_1\right)\frac{\eta(\varphi_2)-\eta(\varphi_1)}{\varphi_2-\varphi_1}=\\&=^{(44)} B_1\left(2(f(U_0)-f(U_1))-c(U_0-U_1)\right)\eta(\varphi_1)=\mathcal{O}(\varepsilon).\end{aligned}$$

From here, we see that the last relation is satisfied for

$$c = 2\frac{f(U_1)-f(U_0)}{U_1-U_0}. \qquad (46)$$

The theorem is proved. □

Remark 9. In the case such as our, when U_0 and U_1 are constants, is possible to replace formula (31) by

$$u_\varepsilon(x,t) = \hat{u}(x_0(x,t,\varepsilon)),$$

where, as before, the function x_0 is the solution to implicit equation (29) and $\hat{u}$ are initial data (4).

The proof of this fact obviously follows after comparing trajectories on Figure 2 and Figure 4. We give precise formulation in the next theorem. We leave it without proof since it is completely analogical to the proof of the previous theorem.

The difference between the previous and the next theorem is in the form of characteristics along which we solve our problem.

In the previous theorem, for fixed ε, the weak asymptotic solution u_ε to (1), (4) was generator of continuous semigroup of transformations (since characteristics intersect along $x = \varphi_i$) and in the following theorem the weak asymptotic solution u_ε to (1), (4) forms continuous group of transformation since appropriate characteristics do not intersect (compare Figure 1 and Figure 2). Still, approach from the next theorem can be used only in the case of special initial data.

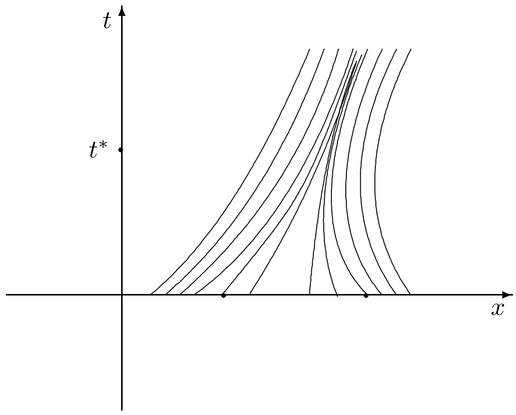

FIGURE 4. System of characteristics for u_ε defined in Theorem 10. The points $a_1 + \varepsilon A\frac{a_1-a_2}{2}$ and $a_2 - \varepsilon A\frac{a_1-a_2}{2}$ are dotted on the x axis.

Theorem 10. *The weak asymptotic solution u_ε, $\varepsilon > 0$, to Cauchy problem*

$$u_t + (f(u))_x = 0, \quad u|_{t=0} = \hat{u}(x), \tag{47}$$

is given by

$$u_\varepsilon(x,t) = \hat{u}(x_0(x,t,\varepsilon)), \tag{48}$$

where x_0 is inverse function to the function $x = x(x_0,t,\varepsilon)$, $t > 0$, $\varepsilon > 0$, of 'new characteristics' defined trough the Cauchy problem:

$$\begin{gathered}\dot{x} = f'(u_\varepsilon)(B_2(\rho) - B_1(\rho)) + cB_1(\rho), \quad \dot{u}_\varepsilon = 0,\\ x(0) = x_0 + \varepsilon A\left(x_0 - \frac{a_1+a_2}{\varepsilon}\right), \quad u_\varepsilon(0) = \hat{u}(x_0), \quad x_0 \in \mathbf{R}.\end{gathered} \tag{49}$$

where A is large enough so that

$$\frac{\partial x}{\partial x_0} > 0 \ \ \textit{for every} \ \ x_0 \in \mathbf{R} \ \ \text{and} \ \ t \in \mathbf{R}^+.$$

The functions B_1 and B_2 are defined in Theorem 7, constant c is given in (27) and $\rho = \rho(\psi_0(t)/\varepsilon)$ is the solution of Cauchy problem (25).

The following corollary is obvious. It claims that the weak asymptotic solution defined in arbitrary of the previous theorems tends to the shock wave with the states U_1 on the left and U_0 on the right (see (38)):

Corollary 11. *With the notations from the previous theorems, for $t > t^*$ the weak asymptotic solution u_ε to problem (1), (4) we have for every fixed $t > 0$:*

$$u_\varepsilon(x,t) \rightharpoonup \begin{cases} U_1, & x < \frac{c}{2}(t-t^*) + x^*, \\ U_0, & x > \frac{c}{2}(t-t^*) + x^*, \end{cases} \qquad \text{in} \ \ \mathcal{D}'(\mathbf{R}).$$

3. The weak asymptotic solution to (2), (5)

At the beginning of the section, we explain some general moments.

The plan is to substitute the weak asymptotic solution (u_ε) of problem (1), (4) into (2). Thus, we obtain the family of equations:

$$v_{\varepsilon t} + (v_\varepsilon g(u_\varepsilon))_x = 0, \quad \varepsilon > 0. \tag{50}$$

Augmented by initial data (5), this linear partial differential equation of the first order has global differentiable solution.

The function u_ε is given by (31) or by simpler version (48) (both formulas give the weak asymptotic solution to (1), (4)). For the simplicity we will substitute the function given by (48) in the place of u appearing in (2).

Weak asymptotic solution to Cauchy problem (2), (5) we will solve separately in five areas of (x,t) plane (see Figure 6).

In order to single out those domains we substitute (48) into (2) and use Leibnitz rule for derivative of product:

$$v_{\varepsilon t} + g(u_\varepsilon)v_{\varepsilon x} = -(g(u_\varepsilon))_x v_\varepsilon. \tag{51}$$

The system of characteristics corresponding to (51), (5) is:

$$\begin{aligned} \dot{X} &= g(u_\varepsilon), \quad X(0) = x_0, \\ \dot{v}_\varepsilon &= -\, v_\varepsilon (g(u_\varepsilon))_x, \quad v_\varepsilon(0) = \hat{v}(x_0). \end{aligned} \tag{52}$$

We prove global resoluteness of this ODE system for $x_0 \in \mathbf{R}$. According to the inverse function theorem it is enough to prove that along entire temporal axis we have

$$\frac{\partial X}{\partial x_0} > 0.$$

Denote by $J = \frac{\partial \tilde{x}}{\partial x_0}$ where $\tilde{x} = \tilde{x}(x_0, t, \varepsilon)$ is the solution of Cauchy problem (34). We have seen in Theorem 10 that $J > 0$ for every $t > 0$ and $x_0 \in \mathbf{R}$. Recall that

$$u_\varepsilon(x,t) = \hat{u}(\tilde{x}_0(x,t,\varepsilon)),$$

where $\tilde{x}_0$ is inverse function to the function $\tilde{x}(x_0,t,\varepsilon)$ with respect to x_0. From (52) we have (we write below $g'(\hat{u}) = g'(\hat{u}(\tilde{x}_0(X(x_0,t,\varepsilon),t,\varepsilon))))$

$$\frac{d}{dt}\frac{\partial X}{\partial x_0} = g'(\hat{u})\hat{u}'\frac{\partial \tilde{x}_0}{\partial X}\frac{\partial X}{\partial x_0} = g'(\hat{u})\hat{u}' J^{-1}\frac{\partial X}{\partial x_0}, \quad \frac{\partial X}{\partial x_0}\Big|_{t=0} = 1.$$

After integrating this differential equation with respect to the unknown function $\frac{\partial X}{\partial x_0}$ we obtain:

$$\frac{\partial X}{\partial x_0} = \exp(\int_0^t g'(\hat{u})\hat{u}' J^{-1} dt') > 0, \quad t > 0. \tag{53}$$

Furthermore, the inverse derivative of $\frac{\partial X}{\partial x_0}$ has the form:

$$\frac{\partial x_0}{\partial x} = \exp(-\int_0^t g'(\hat{u}(\tilde{x}_0(x,t)))\hat{u}'(\tilde{x}_0(x,t)) J^{-1} dt') > 0, \quad t > 0.$$

The subintegral function $g'(\hat{u}(\tilde{x}_0(x,t)))\hat{u}'(\tilde{x}_0(x,t))J^{-1}$ has discontinuity only along lines $(\varphi_i(t), t)$, $i = 1, 2$, since $\tilde{x}_0(\varphi_i(t), t) = a_i$ and $\hat{u}'(x_0)$ has discontinuities only for $x_0 = a_i$, $i = 1, 2$. But, since it it is integrated over $t' \in [0, t]$ we conclude that $\frac{\partial x_0}{\partial x}$ is continuous, and, thus, $\frac{\partial X}{\partial x_0}$ from (53) is continuous as well.

Therefore, the inverse function theorem implies existence of inverse function $x_0 = x_0(X, t, \varepsilon)$ along entire temporal axis, which, in turn, implies global resoluteness of problem (52).

Since for $t < t^*$ and $\varepsilon = 0$ the Jacobian J^{-1} is well defined for every $x_0 \in \mathbf{R}$ we see that for $t < t^*$ the function v is determined along characteristics. The characteristics are non-intersecting and their form is plotted on Figure 5.

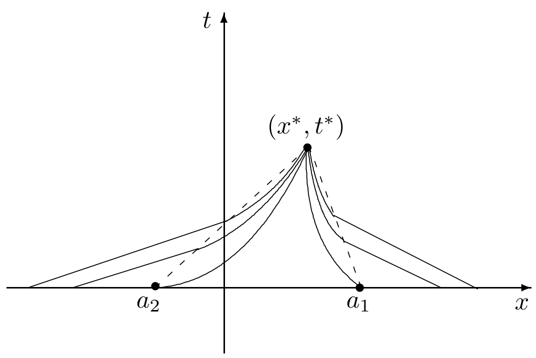

FIGURE 5. Standard characteristics for (2), (5) are plotted by normal lines. Dashed lines are characteristics for (1), (4) emanating from a_2 and a_1, respectively.

Denote by φ_i^*, $i = 1, 2$, solutions of the following Cauchy problems:

$$\begin{aligned} \dot{X} &= g(u_\varepsilon), \\ X(0) &= a_i, \quad i = 1, 2. \end{aligned}$$

Now, we can introduce domains in which we will separately solve Cauchy problem (50), (5).

We set

$$D_1 = \{(x,t)|\, x < \varphi_2\}, \quad D_2 = \{(x,t)|\, x > \varphi_1\},$$
$$D_3 = \{(x,t)|\, \varphi_2 < x < \varphi_2^*\}, \quad D_4 = \{(x,t)|\, \varphi_1^* < x < \varphi_1\},$$
$$D_5 = \{(x,t)|\, \varphi_2^* < x < \varphi_1^*\}$$

Domains are plotted in Figure 6. Since the function v_ε is everywhere continuous function (it is defined along nonintersecting characteristics; see (53)), it will be classical solution to problem (50), (4).

On the beginning, we prove that those domains are disjunct. Accordingly, we inspect relations between the functions φ_i and φ_i^*, $i = 1, 2$. We have to prove the following fact for every $t \in \mathbf{R}^+$:

$$\varphi_2(t,\varepsilon) \leq \varphi_2^*(t,\varepsilon) < \varphi_1^*(t,\varepsilon) \leq \varphi_1(t,\varepsilon), \tag{54}$$

First, we prove that

$$\varphi_2 \leq \varphi_2^*. \tag{55}$$

In the moment $t = 0$ we have

$$(\varphi_2)'_t = f(U_1)(B_2 - B_1) + cB_1 \text{ and } (\varphi_2^*)'_t = g(U_1), \tag{56}$$

and (see (16))

$$g(U_1) > f'(U_1) > f'(U_1)(B_2 - B_1) + cB_1,$$

since $f'(U_1) > c/2$. Using well known theorem from ODE-s (”who goes slower does not reach further”, [1]) from (56) we see that in some neighborhood of $t = 0$ we have $\varphi_2 < \varphi_2^*$. Assume now that t_0 is the smallest $t > 0$ such that $\varphi_2 = \varphi_2^*$. In this case we have the same situation as in the moment $t = 0$, i.e. there exists neighborhood $(t_0, t_0 + \delta)$ such that $\varphi_2 < \varphi_2^*$ in $(t_0, t_0 + \delta)$. Continuing like this we see that we indeed have (55).

In the completely same manner we prove that

$$\varphi_1^* \leq \varphi_1. \tag{57}$$

It is remained to prove that:

$$\varphi_2^* < \varphi_1^*. \tag{58}$$

This directly follows from the fact that characteristics of problem (52) do not intersect. That means that relation between two characteristics remains the same along entire time axis. Therefore,

$$\varphi_2^* = X(a_2, t, \varepsilon) < X(a_1, t, \varepsilon) = \varphi_1^*,$$

since $a_2 < a_1$. This proves (58).

Collecting (55), (57) and (58) we obtain (54). From (54) and the fact that for $t > t^*$

$$\lim_{\varepsilon \to 0} \varphi_i(t,\varepsilon) = \frac{c}{2}(t - t^*) + x^*, \quad i = 1, 2$$

it follows that for $t > t^*$ we have:

$$\lim_{\varepsilon \to 0} \varphi_i^*(t,\varepsilon) = \frac{c}{2}(t - t^*) + x^*, \quad i = 1, 2. \tag{59}$$

We remind that the constants t^* and x^* are introduced in front of Theorem 8.

Next, we solve problem (51), (5) separately in domains D_i, $i = 1, ..., 5$.

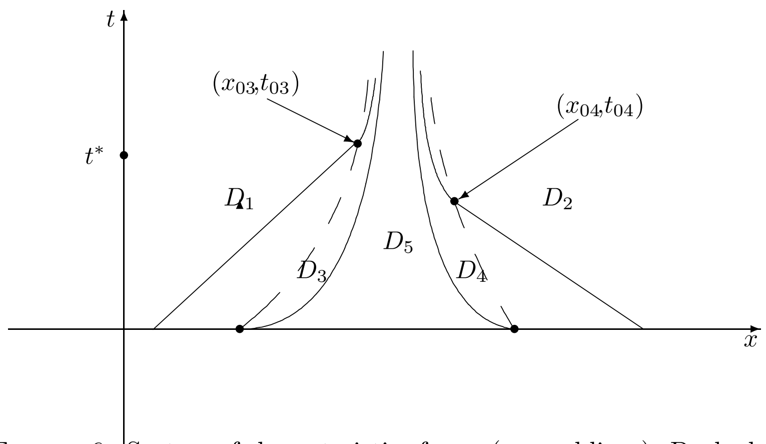

FIGURE 6. System of characteristics for v_ε (normal lines). Dashed lines are $x = \varphi_i(t)$, $i = 1, 2$, normal lines inside dashed ones are $x = \varphi_i^*$, $i = 1, 2$. The points $a_1 + \varepsilon A\frac{a_1 - a_2}{2}$ and $a_2 - \varepsilon A\frac{a_1-a_2}{2}$ are dotted on x axis.

In domains D_1 and D_2 we have $u_\varepsilon \equiv const.$ and therefore the characteristics corresponding to v_ε there are straight lines. More precisely, we have:

$$v_\varepsilon(x,t) \equiv V_1, \quad (x,t) \in D_1,$$
$$v_\varepsilon(x,t) \equiv V_0, \quad (x,t) \in D_2.$$

Another two domains are:

$$D_3 = \{(x,t) |\ \varphi_2 < x < \varphi_2^*\}, \quad D_4 = \{(x,t) |\ \varphi_1^* < x < \varphi_1\}.$$

In those domains we solve the following Cauchy problems:

$$\begin{aligned} v_{\varepsilon t} + g(u_\varepsilon)v_{\varepsilon x} &= -(g(u_\varepsilon)_x v_\varepsilon, \\ v_\varepsilon|_{x=\varphi_1} &= V_0 \text{ (initial data for the Cauchy problem in } D_3), \\ v_\varepsilon|_{x=\varphi_2} &= V_1 \text{ (initial data for the Cauchy problem in } D_4). \end{aligned}$$

We use standard method of characteristics. Note that in this case characteristics emanate from the lines $x = \varphi_i$, $i = 1, 2$, and not from x axis as usual (see Figure 6). The system of characteristics for ones emanating from the line φ_1 has the form:

$$\begin{aligned} \dot X =& g(u_\varepsilon), \\ \dot v_\varepsilon =& -v_\varepsilon(g(u_\varepsilon))_x, \\ X(t_0) =& \varphi_1(t_0) = x_0, \quad v_\varepsilon(t_0) = V_0. \end{aligned} \tag{60}$$

and for the characteristics emanating from the line φ_2 has the form:

$$\begin{aligned} \dot X =& g(u_\varepsilon), \\ \dot v_\varepsilon =& -v_\varepsilon(g(u_\varepsilon))_x, \\ X(t_0) =& \varphi_2(t_0) = x_0, \quad v_\varepsilon(t_0) = V_1. \end{aligned} \tag{61}$$

Global solvability of this system can be proved in the same way as the one for the system (52).

Next step is to solve the second equation from (60) (or analogically from (61)). We have for problem (60):

$$\begin{aligned}
&\dot v_\varepsilon=(-g(u_\varepsilon))_x v_\varepsilon \implies\\
&v_\varepsilon=V_0\exp(-\int_0^t (g(u_\varepsilon))_x dt') \implies\\
&v_\varepsilon=V_0\exp(-\int_0^t (\frac{dX}{dt'})_x dt') \implies\\
&v_\varepsilon=V_0\exp(-\int_0^t \frac{\partial}{\partial x_0}\frac{dX}{dt'}\cdot\frac{\partial \tilde x_0}{\partial X} dt') \implies\\
&v_\varepsilon=V_0\exp(-\int_0^t \frac{\frac{d}{dt'}\frac{\partial X}{\partial x_0}}{\frac{\partial X}{\partial x_0}} dt') \implies\\
&v_\varepsilon=\frac{V_0}{\frac{\partial X}{\partial x_0}},
\end{aligned} \tag{62}$$

and, similarly, for (61):

$$v_\varepsilon=\frac{V_1}{\frac{\partial X}{\partial x_0}}. \tag{63}$$

The previous implies:

$$\begin{aligned}
&v_\varepsilon(x,t)=V_1\frac{\partial x_{03}}{\partial x}(x,t,\varepsilon), \quad (x,t)\in D_3,\\
&v_\varepsilon(x,t)=V_0\frac{\partial x_{04}}{\partial x}(x,t,\varepsilon), \quad (x,t)\in D_4.
\end{aligned}$$

where $x_{03}=x_0(X,t,\varepsilon)=\varphi_1(t_{01})$ and $x_{04}=x_0(X,t,\varepsilon)=\varphi_2(t_{02})$ are inverse functions to the function X determined by (60) and (61), respectively (for appropriate t_{0i}, $i=1,2$, depending on (X,t); see Figure 6, domains D_3 and D_4).

Finally, we solve problem (51), (5) in the domain:

$$D_5=\{(x,t)|\ \varphi_2^*<x<\varphi_1^*\}.$$

We apply similar procedure as in the previous case. The solution in this domain is:

$$v_\varepsilon(x,t)=v_0(x_0(x,t,\varepsilon))\frac{\partial x_{05}}{\partial x}(x,t,\varepsilon),$$

where $x_{05}=x_0(X,t,\varepsilon)$ is inverse function to the function X determined by (52) (x_0 restricted on $[a_2-\varepsilon A\frac{a_1-a_2}{2}, a_1+\varepsilon A\frac{a_1-a_2}{2}]$).

So, we have proved the following theorem:

Theorem 12. *The function:*

$$v_\varepsilon(x,t)=\begin{cases}V_0, \quad ,(x,t)\in D_1,\\ V_0\frac{\partial x_{03}}{\partial x}(x,t,\varepsilon), \quad (x,t)\in D_3,\\ v_0(x_0(x,t,\varepsilon))\frac{\partial x_{05}}{\partial x}(x,t,\varepsilon), \quad (x,t)\in D_5,\\ V_1\frac{\partial x_{04}}{\partial x}(x,t,\varepsilon), \quad (x,t)\in D_4,\\ V_1, \quad (x,t)\in D_2.\end{cases} \tag{64}$$

represents classical solution to problem (51), (5) (and thus the weak asymptotic solution as well).

4. Weak limit of the solution

It remains to inspect the weak limit of the weak asymptotic solution $(u_\varepsilon, v_\varepsilon)$ of problem (2), (5) for $t > t^*$ (since for $t < t^*$ we have classical solution of the considered problem).

We have already known from Corollary 11 that for $t \geq t^*$ we have:

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) = \begin{cases} U_1, & x < \frac{c}{2}(t-t^*) + x^* \\ U_0, & x \geq \frac{c}{2}(t-t^*) + x^* \end{cases} \text{ in } \mathcal{D}'(\mathbf{R}). \tag{65}$$

So, we have to inspect weak limit of v_ε. ore precisely, in this section we will prove the following theorem:

Theorem 13. *For every fixed $t > t^*$ the function v_ε given by (64) satisfies as $\varepsilon \to 0$*

$$\begin{aligned} v_\varepsilon(x,t) \rightharpoonup v(x,t) = &\begin{cases} V_1, & x < \frac{c}{2}(t-t^*) + x^*, \\ V_0, & x \geq \frac{c}{2}(t-t^*) + x^* \end{cases} \\ &+ \Big[V_1(a_2 + g(U_1)t - \frac{c}{2}(t-t^*) - x^*) + V_0(\frac{c}{2}(t-t^*) + x^* - a_1 - g(U_0)t) \\ &+ \int_{a_2}^{a_1} v_0(x_0)dx_0\Big]\delta(x - \frac{c}{2}(t-t^*) - x^*) \text{ in } \mathcal{D}'(\mathbf{R}). \end{aligned} \tag{66}$$

Proof: To begin, note that we can write function v_ε from (64) in the following manner:

$$v_\varepsilon(x,t) = \hat{v}(x_0(x,t,\varepsilon))\frac{\partial x_0}{\partial x}(x,t,\varepsilon),$$

where (see Figure 6):

$$x_0(x,t,\varepsilon) = \begin{cases} x - g(U_1)t, \quad (x,t) \in \bar{D}_1, \\ x_{03}^{-1}(x,t,\varepsilon) - g(U_1)\varphi_2^{-1}(x_{03}^{-1}(x,t,\varepsilon)), \quad (x,t) \in D_3 \\ \text{ (here first we go by } x_{03}^{-1} \text{ to the line } \varphi_2 \text{ so that } x_{03}^{-1}(x,t,\varepsilon) = \varphi_2(t_{03}) \\ \text{ and then proceed to the line } t = 0 \text{ along the straight line } x - g(U_1)t), \\ x_{05}^{-1}(x,t,\varepsilon), \quad (x,t) \in \bar{D}_5, \\ x_{04}^{-1}(x,t,\varepsilon) - g(U_0)\varphi_1^{-1}(x_{04}^{-1}(x,t,\varepsilon)), \quad (x,t) \in D_4, \\ \text{ (here first we go by } x_{04}^{-1} \text{ to the line } \varphi_1 \text{ so that } x_{04}^{-1}(x,t,\varepsilon) = \varphi_1(t_{04}) \\ \text{ and then proceed to the line } t = 0 \text{ along the straight line } x - g(U_0)t), \\ x - g(U_0)t, \quad (x,t) \in \bar{D}_2, \end{cases} \tag{67}$$

and

$$\frac{\partial x_0}{\partial x}(x,t,\varepsilon) = \begin{cases} 1, & (x,t) \in \bar{D}_1, \\ \frac{\partial x_{03}}{\partial x}, & (x,t) \in D_3, \\ \frac{\partial x_{05}}{\partial x}, & (x,t) \in \bar{D}_5, \\ \frac{\partial x_{04}}{\partial x}, & (x,t) \in D_4, \\ 1, & (x,t) \in \bar{D}_2. \end{cases}$$

We take $\eta \in C_0^1(\mathbf{R})$ and write using (64):

$$\int v_\varepsilon(x,t)\eta(x)dx = \int_{-\infty}^{\varphi_2-\varepsilon} v_\varepsilon(x,t)\eta(x)dx + \int_{\varphi_2-\varepsilon}^{\varphi_2^*} v_\varepsilon(x,t)\eta(x)dx$$
$$+ \int_{\varphi_2^*}^{\varphi_1^*} v_\varepsilon(x,t)\eta(x)dx + \int_{\varphi_1^*}^{\varphi_1+\varepsilon} v_\varepsilon(x,t)\eta(x)dx + \int_{\varphi_1+\varepsilon}^{\infty} v_\varepsilon(x,t)\eta(x)dx$$
$$= \int_{-\infty}^{\varphi_2-\varepsilon} V_1\eta(x)dx + \int_{\varphi_2-\varepsilon}^{\varphi_2^*} V_1\frac{\partial x_{04}}{\partial x}\eta(x)dx + \int_{\varphi_2^*}^{\varphi_1^*} v_0(x_0(x,t,\varepsilon))\frac{\partial x_{05}}{\partial x}\eta(x)dx$$
$$+ \int_{\varphi_1^*}^{\varphi_1+\varepsilon} V_0\frac{\partial x_{03}}{\partial x}\eta dx + \int_{\varphi_1+\varepsilon}^{\infty} V_0\eta(x)dx.$$

Here, we have written $\varphi_i \pm \varepsilon$ in order to avoid complications due to possible $\varphi_i = \varphi_i^*$.

Then, we use the change of variables $x = X(x_0,t,\varepsilon)$ where X is inverse function of the function $x_0 = x_0(X,t,\varepsilon)$ given by (67). We have:

$$\int v_\varepsilon(x,t)\eta(x)dx = \int_{-\infty}^{\varphi_2-\varepsilon} v_\varepsilon(x,t)\eta(x)dx \tag{68}$$
$$+ \int_{x_0(\varphi_2-\varepsilon,t,\varepsilon)}^{a_2} V_1\eta(x(x_0,t,\varepsilon))dx_0 + \int_{a_2}^{a_1} v_0(x_0)\eta(X(x_0,t,\varepsilon))dx_0$$
$$+ \int_{a_1}^{x_0(\varphi_1+\varepsilon,t,\varepsilon)} V_0\eta(x(x_0,t,\varepsilon))dx_0 + \int_{\varphi_1+\varepsilon}^{\infty} V_0\eta(x)dx,$$

and we remind that:

$$x_0(\varphi_1+\varepsilon,t,\varepsilon) = \varphi_1+\varepsilon-g(U_0)t, \quad x_0(\varphi_2-\varepsilon,t,\varepsilon) = \varphi_2-\varepsilon-g(U_1)t.$$

Furthermore, for $t > t^*$ we have (see (23) and (24)):

$$x_0(\varphi_1+\varepsilon,t,\varepsilon) \to \frac{c}{2}(t-t^*) + x^* - g(U_0)t, \quad \varepsilon \to 0,$$
$$x_0(\varphi_2-\varepsilon,t,\varepsilon) \to \frac{c}{2}(t-t^*) + x^* - g(U_1)t, \quad \varepsilon \to 0.$$

Accordingly, for $t > t^*$ after letting $\varepsilon \to 0$ we have from (68) exactly (66).

This concludes the theorem. □

It remains to give a comment concerning admissibility of the singularities appearing in u and v, and to inspect weather u and v defined by (65) and (66), respectively, represent solution of the system in the sense of Definition 3. We have the following theorem:

Theorem 14. *Delta shock wave appearing in the function v defined by (66) for $t > t^*$ is overcompressive with respect to system (1),(2). Furthermore, the functions u and v defined by*

$$v = w - \lim_{\varepsilon\to 0} v_\varepsilon$$
$$u = w - \lim_{\varepsilon\to 0} u_\varepsilon,$$

are solutions to Cauchy problem (1-(2)*), (*(4)*-5) in the sense of Definition 3.*

Proof: We recall that δ-shock is overcompresive with respect to system (1),(2) if (15) holds. This follows directly from assumptions on f' and g quoted in Theorem 6 providing:

$$g(U_1) < f'(U_1), \quad f'(U_0) < g(U_0). \tag{69}$$

Admissibility of the shock wave appearing in the solution to problem (1), (4) implies

$$f'(U_1) < c/2 < f'(U_0)$$

which together with (69) implies:

$$g(U_1) < c/2 < g(U_0),$$

which proves overcompressivity of the shock and δ-shock wave appearing in (65) and (66).

We pass to the prove of the other statement of the theorem.

Recall that the functions u and v are defined along the 'new characteristics' (see (28) and (67)), which, for $t < t^*$ converge to standard characteristics as $\varepsilon \to 0$. Therefore, for $t < t^*$ the pair (u, v) is solution to problem (1-(2)), ((4)-5) along characteristics. Taking this into account and substituting $V = v$ and u in the second equation of (8) we have for an arbitrary $\varphi \in C_0^1(\mathbf{R}^+ \times \mathbf{R})$:

$$\begin{aligned}
&\int_{\mathbf{R}^+}\int_{\mathbf{R}} (v\partial_t\varphi + g(u)v\partial_x\varphi)\, dxdt \\
&+ \int_{\{x=\frac{c}{2}(t-t^*)+x^*,\ t>t^*\}} e(t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}dl + \int_{\mathbf{R}} \hat{v}(x)\varphi(x,0)dx \\
&= \int_{t^*}^{\infty}\int_{\mathbf{R}} (v\partial_t\varphi + g(u)v\partial_x\varphi)\, dxdt \\
&+ \int_{\{x=\frac{c}{2}(t-t^*)+x^*,\ t>t^*\}} e(t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}dl + \int_{\mathbf{R}} \hat{v}(x)\varphi(x,0)dx = 0,
\end{aligned} \tag{70}$$

where (see (66))

$$\begin{aligned}
e(t) = \Big[&V_1(a_2 + g(U_1)t - \frac{c}{2}(t-t^*) - x^*) + V_0(\frac{c}{2}(t-t^*) + x^* - a_1 - g(U_0)t) + \\
&\int_{a_2}^{a_1} v_0(x_0)dx_0\Big]\delta(x - \frac{c}{2}(t-t^*) - x^*)
\end{aligned}$$

Then, notice that for $t > t^*$ the functions u and v are given by (65) and (66), respectively. Therefore, we have from (70) after parametrizing $\{x = \frac{c}{2}(t-t^*) + x^*,\ t > t^*\}$:

$$\begin{aligned}
&\int_{t^*}^{\infty}\int_{\mathbf{R}} (v\partial_t\varphi + g(u)v\partial_x\varphi)\, dxdt \\
&+ \int_{\{x=\frac{c}{2}(t-t^*)+x^*,\ t>t^*\}} e(t)\left(\frac{\partial\varphi(x,t)}{\partial t} + \frac{\partial\varphi(x,t)}{\partial x}\frac{c}{2}\right)dt + \int_{\mathbf{R}} \hat{v}(x)\varphi(x,0)dx \\
&= \int_{t^*}^{\infty}\int_{-\infty}^{\frac{c}{2}(t-t^*)+x^*} (V_0\partial_t\varphi + g(U_0)V_0\partial_x\varphi)\, dxdt \\
&+ \int_{t^*}^{\infty}\int_{\frac{c}{2}(t-t^*)+x^*}^{\infty} (V_1\partial_t\varphi + g(U_1)V_1\partial_x\varphi)\, dxdt \\
&+ \int_{t^*}^{\infty} e(t)\partial_t(\varphi(t, \frac{c}{2}(t-t^*) + x^*))dt \\
&= \int_{t^*}^{\infty}\left(-e'(t) + ([vg(u)] - [v]\frac{f(u)}{[u]})\right)\varphi(t, \frac{c}{2}(t-t^*) + x^*)dt = 0
\end{aligned}$$

implying *Rankine-Hugoniot conditions for δ-shock* on the line $x = \frac{c}{2}(t - t^*) + x^*$:

$$e'(t) = \left([vg(u)] - [v]\frac{f(u)}{[u]} \right) \Big|_{x=\frac{c}{2}(t-t^*)+x^*}, \tag{71}$$

By direct substitution, it is trivial to check that (71) is satisfied.
□

Bibliography

[1] V.I.Arnold, *Obiknovenie differencial'nie uravneniya*, Izdatel'stvo "NAUKA", Moskva 1971. (in Russian)

[2] G-Q. Chen, H. Liu, Formation of δ-shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids, SIAM J. Math. Anal. **34** (2003), no. 4, 925–938.

[3] C. M. Dafermos *Hyperbolic Conservation Laws in Continuum Physics*, Berlin; Heidelberg; New York; Barcelona; Hong Kong; London; Milan; Paris; Singapore; Tokyo: Springer, 2000.

[4] V. G. Danilov, *Generalized Solution Describing Singularity Interaction*, International Journal of Mathematics and Mathematical Sciences, Volume 29, No. 22. February 2002, pp. 481-494.

[5] V. G. Danilov, *On singularities of conservation equation solution*, Nonlinear Analysis(2007), doi:10.1016/j.na.2006.12.044

[6] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of nonlinear waves to quasilinear equations*, in: *Proceedings of Eight International Conference on Hyperbolic Problems. Theory-Numerics-Applications*, Univ. Magdeburg, Magdeburg, 2000, pp. 326–328.

[7] V. G. Danilov, V. M. Shelkovich, *Propagation and Interaction of shock waves of quasilinear equations*, Nonlinear Stud. 8 (1) (2001) 135-169.

[8] V.G. Danilov, V.M.Shelkovich, *Dynamics of propagation and interaction of δ-shock waves in conservation law system*, J. Differential Equations 211 (2005) 333-381.

[9] V.G. Danilov, V.M.Shelkovich, *Delta-shock wave type solution of hyperbolic systems of conservation laws*, Quart. Appl. Math. 63 (2005), 401-427

[10] V. G. Danilov, D. Mitrovic, *Weak asymptotic of shock wave formation process*, Journal of Nonlinear Analysis - Theory, Methods and Applications, 61(2005) 613-635.

[11] V. G. Danilov, D. Mitrovic, *Smooth approximations of global in time solutions to scalar conservation law*, preprint

[12] V. G. Danilov, G. A. Omelianov, V. M. Shelkovich, *Weak Asymptotic Method and Interaction of Nonlinear Waves* in: M.Karasev (Ed.), Asymptotic Methods for Wave and Quantum Problems, American Mathematical Society Translation Series, vol. 208, 2003, pp. 33-165.

[13] X. Ding, Z. Wang, *Existence and Uniqueness of Discontinuous Solution defined by Lebesgue-Stieltjes integral*, Sci. China Ser. A, 39 (1996), no.8., 807-819

[14] F. Huang, *Existence and Uniqueness of Discontinuous Solutions for a Class of Non-strictly Hyperbolic System*, Advances in nonlinear partial differential equations and related areas (Beijing, 1997), 187-208, World Sci. Publ., River Edge, NJ, 1998.

[15] F. Huang, *Weak solution to pressureless type system*, Comm. Partial Differential Equations **30** (2005), no. 1-3, 283–304.

[16] F. Huang, *Existence and uniqueness of discontinuous solutions for a hyperbolic system* Proc.Roy.Soc.Edinburgh Sect. A **127** (1997), no. 6, 1193-1205.

[17] A. M. Ilin, *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*, Nauka, Moscow, 1989; English transl., American Mathematical Society, RI, 1992.

[18] G. Ercole, *Delta-shock waves as self-similar viscosity limits*, Quart. Appl. Math. LVIII (1) (2000) 177-199.

[19] A. Forestier, P. G. LeFloch, *Multivalued solutions to some nonlinear and nonstrictly hyperbolic systems*, Japan J. Indus. Appl. Math. 9 (1992), 1–23.

[20] B. Hayes, P. G. LeFloch, *Measure-solutions to a strictly hyperbolic system of conservation laws*, Nonlinearity 9 (1996), 1547–1563.

[21] K. T. Joseph, *A Riemann problem whose viscosity solution contains δ measures*, Asymptotic Analysis 7 (1993) 105-120.

[22] B. L. Keyfitz, H. C. Krantzer, *Spaces of weighted measures for conservation laws with singular shock solutions*, J. Differential Equations 118 (1995) 420-451.

[23] P. G. LeFloch, *An existence and uniqueness result for two nonstrictly hyperbolic systems*, IMA Volumes in Math. and its Appl., "Nonlinear evolution equations that change type", ed. B.L. Keyfitz and M. Shearer, Springer Verlag, Vol. 27, 1990, pp. 126–138.

[24] P. Le Floch, *An existence and uniqueness result for two nonstrictly hyperbolic systems* in: B. Keyfitz, M. Shearer (Eds.), Nonlinear Evolution Equations that Change Type, Springer, Berlin, 1990, pp. 126-138.

[25] Y.-P. Liu, Z. Xin, *Overcompressive shock waves*, in: B.Keyfitz, M.Shearer (Eds.) Nonlinear Evolution Equations that Change Type, Springer, Berlin, 1990, pp. 149-145.

[26] D. Mitrovic, J. Susic, *Global in time solution to Hopf equation and application to a non-strictly hyperbolic system of conservation laws*, Electronic J. of Differential Equations, Vol. 2007(2007), No. 114, pp. 1-22.

[27] D. Mitrovic, M. Nedeljkov, *Delta shock waves as a limit of shock waves*, J. of Hyperbolic Differential Equations, Vol. 4, No. 4 (2007), 629-653.

[28] M. Nedeljkov, *Unbounded solutions to some systems of conservation laws - split delta shock waves*, Matematički Vesnik 54 (2002), 145-149.

[29] M. Nedeljkov, *Delta and singular delta locus for one-dimensional systems of conservation laws*, Math. Meth. Appl. Sci. **27** (2004), 931–955.

[30] E. Yu. Panov, V. M. Shelkovich, *δ'-shock waves as a new type of solutions to systems of conservation laws*, J.Differential Equations 228 (2006), no. 1, 49-86.

[31] V. M. Shelkovich, *The Riemann problem admitting δ-, δ'-shocks, and vacuum states (the vanishing viscosity approach)*, J. Differential Equations 231 (2006), no. 2, 459-500.

[32] W. Sheng, T. Zhang, *The Riemann problem for transportation equations in gas dynamics*, Mem. Amer. Math. Soc. 137 (645) (1999) 1-77.

[33] D. Tan, T. Zhang, Y. Zheng, *Delta shock waves as a limits of vanishing viscosity for a system of conservation laws*, J. Differential Equations 112 (1994) 1-32.

[34] A. I. Volpert, *The space BV and quasilinear equations*, Math. USSR Sb. 2 (1967) 225-267.

[35] H. Yang, *Riemann problems for class of coupled hyperbolic system of conservation laws*, Journal of Differential Equations, 159(1999) 447-484.

Vladimir G. Danilov, Moscow Technical University of Communication and Informatics, Aviamotornaya 8a, 111024 Moscow, Russia
E-mail address: danilov@miem.edu.ru

Darko Mitrovic, Norwegian University of Science and Technology, Department of Mathematical Sciences, Alfred Getz vei 1, NO-7491 Trondheim, Norway
E-mail address: matematika@cg.yu