A Model Problem for the Motion of a Compressible, Viscous Fluid Flow with the No-Slip Boundary Condition

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Abstract

We consider the system of Navier-Stokes equations as a model of the motion of compressible, viscous, "pressureless" fluids in the domain $\Omega = \mathbb{R}^3_+$ with the no-slip boundary condition. We construct a global in time regular weak solution, provided that the initial density, ρ_0, is bounded and the magnitude of the initial velocity $\mathbf{u}_0$ is suitably restricted in the semi-norm $\|\sqrt{\rho_0(\cdot)}\mathbf{u}_0(\cdot)\|_{L^2(\Omega)} + \|\nabla \mathbf{u}_0(\cdot)\|_{L^2(\Omega)}$.

1 Introduction

Consider the system of Navier-Stokes equations that describe the motion of compressible, viscous fluids in the isentropic approximation.

$$\frac{\partial}{\partial t}\rho + \mathrm{div}\ (\rho \mathbf{u}) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\rho \mathbf{u} + \mathrm{div}\ (\rho \mathbf{u} \otimes \mathbf{u}) - (\lambda + \mu)\mathrm{div}\, \mathbf{u} - \mu \Delta \mathbf{u} + \nabla P = 0, \tag{2}$$

$$\mu > 0,\ 2\lambda + 3\mu > 0, \tag{3}$$

$$P = A\rho^\gamma,\ \gamma \geq 1, \quad (t,\, x) \in \mathbb{R}_+ \times \Omega,\ \Omega \subset \mathbb{R}^3.$$

Here, ρ is the density of a fluid, $\mathbf{u}$ is the velocity. The values of unknown functions, ρ, $\mathbf{u}$, are assumed to be given at the time $t = 0$:

$$\rho = \rho_0,\ \mathbf{u} = \mathbf{u}_0. \tag{4}$$

The problem is supplemented by the various boundary conditions, for example, (a) the no-slip boundary condition; $\mathbf{u} = 0$ on $\partial\Omega$, (b) periodic flows; functions are assumed to be spatially periodic, (c) Cauchy problem; $\Omega = \mathbb{R}^3$.

There is an extensive literature devoted to the study of these problems. When the data of the problem are smooth, for example from the space $\left(W^{3,2}\right)$, the initial-boundary value problem is known to be well-posed, meaning that there is time interval on which the solution exists and retains its initial smoothness, see [10]. All types of boundary problems, mentioned above can be treated the methods presented there.

On the other hand there is a well-developed theory of weak solutions, see [9], [4]. Solutions exist globally in time but enjoy minimal regularity properties, namely, the total energy $\int_\Omega \rho(t,\cdot)|\mathbf{u}(t,\cdot)|^2/2 + P(\rho(t,\cdot))/(\gamma-1)$ and the total dissipated energy $\iint_{(0,t)\times\Omega} \mu|\nabla\mathbf{u}|^2$, $t>0$, are finite.

The theory of weak solutions is not complete as the question remains open either the weak solution constructed in [9] and [4] can be considered reasonable from the physical point of view. In particular, one would like to know that a weak solution does not allow the spontaneous formation of vacuum. The dimension $N=1$ is special in this respect as non-formation of vacuum is a generic property of weak solutions, see [7]. On the other hand, this problem can also be resolved in a multidimensional case provided that the initial velocity does not deviate to much from the zero state when measured by the energy norm. Specifically, the following theorem holds.

Theorem (D.Hoff, [5], [6])**.** *Let $N=2,3$. Let $\hat{\rho}>0$ and $L>0$ be given. There is a positive number $c=c(N)$ and a pair of positive numbers A, C depending on $(\mu,\lambda,\hat{\rho},L,N,c)$, with the property that if*

$$\lambda+\mu\le c\mu \tag{5}$$

and the initial data $(\rho_0,\mathbf{u}_0)$ satisfy bounds

$$0\le\rho_0\le\hat{\rho},\qquad a.e.\ \mathbb{R}^3_+,$$

$$\int_{\mathbb{R}^N}|\mathbf{u}_0(y)|^2+(\rho_0(y)-\hat{\rho})^2\,dy\le A$$

and

$$|\mathbf{u}_0|_{L^{2^N}(\mathbb{R}^N)^N}\le L,$$

then, there is a global weak solution $(\rho,\mathbf{u})$ of the problem (1)–(4) *for which*

$$\|\rho\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^N)}\le C\hat{\rho},$$

$$\mathbf{u}\in L^\infty\left(\{t\,:\,t>\tau\}\times\mathbb{R}\right)^N,\quad\forall\tau>0.$$

(We refer the reader to [6] for the complete statement of the Theorem.)

The analogous result is obtained for flows in domains with boundaries, with the Navier slip boundary condition at the boundary, i.e. the tangential velocity at the boundary is proportional to the tangential component of the stress, see [6].

The natural question then is the possibility of extending the theory to flows in which the velocity adheres to the boundary of the domain without a slip (the no-slip boundary condition). This condition stems from the postulate of continuity of the velocity field throughout the flow domain, see [1], and is confirmed by many experimental data.

In this paper we consider a simplified model, that consist of equation (1) and the system of equations (2) without the pressure term ∇P. Precisely, the model we study consists of equations

$$\frac{\partial}{\partial t}\rho + \operatorname{div}\,(\rho\mathbf{u})\,=0, \tag{6}$$

$$\frac{\partial}{\partial t}\rho\mathbf{u} + \operatorname{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) - (\lambda+\mu)\operatorname{div}\mathbf{u} - \mu\Delta\mathbf{u} = 0, \tag{7}$$

$$\mu>0,\; 2\lambda+3\mu>0,$$

that hold in the domain

$$(t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+,$$

the no-slip boundary condition

$$\mathbf{u}(t,x)=0, \quad x\in\partial\mathbb{R}^3_+, \tag{8}$$

and the initial conditions

$$\rho=\rho_0,\; \mathbf{u}=\mathbf{u}_0, \quad t=0. \tag{9}$$

For the problem (6)–(9) we prove global existence of a small energy weak solution with the density being a L^∞ function throughout the domain. We prove the following theorem.

Theorem 1. *For any $\hat\rho>0$ there is a $c_0=c_0(\hat\rho,\,\lambda,\,\mu)>0$ such that if a pair $(\rho_0,\,\mathbf{u}_0)$ of measurable functions verifies the bounds*

$$0\le\rho_0<\hat\rho, \qquad a.e.\;\mathbb{R}^3_+, \tag{10}$$

and

$$\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} + \|\nabla_x\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} \le c_0, \tag{11}$$

then, there exists a global weak solution of (6)–(9)*, see Definition 1 for the definition of a weak solution. Moreover,*

$$\left.\begin{aligned} &\mathbf{u}\in L^2\left(\mathbb{R}_+\,:\,L^6\left(\mathbb{R}^3_+\right)\right),\\ &D^2_x\mathbf{u}\in L^2\left(\mathbb{R}_+\,:\,L^2\left(\mathbb{R}^3_+\right)\right), \end{aligned}\right\} \tag{12}$$

and there is $c>0$ that verifies the following estimates.

$$\left.\begin{aligned} &0\le\rho(t,\cdot)<c\hat\rho, \quad a.e.\;\mathbb{R}^3_+,\\ &\|\nabla_x\mathbf{u}(t,\cdot)\|_{L^2(\mathbb{R}^3_+)}\le c\|\nabla_x\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)}, \qquad a.e.\;in\;\mathbb{R}_+. \end{aligned}\right\} \tag{13}$$

Remark 1. *If one imposes a structural condition on the relative size of viscosity coefficients* (5)*, then one can show that a weak solution with uniformly bounded density exists if the condition* (11) *is substituted with weaker conditions*

$$\mathbf{u}_0 \in L^p\left(\mathbb{R}^3_+\right),$$

for some $p > 6$ *and*

$$\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} \leq c_0,$$

for suitable $c_0 = c_0(\hat{\rho}, \lambda, \mu, \|\mathbf{u}_0\|_{L^p})$*. The proof of this statement can be derived from results of [6].*

Remark 2. *If additionally to the condition* (5) *the initial density* $\rho_0 > \check{\rho} > 0$*, a.e. in* $\mathbb{R}^3_+$ *then a weak solution exists which is bounded away from vacuum, i.e.* $\rho(t,x) > c > 0$*, a.e.*(t,x).

The framework of the analysis was established in [5], [6]. The analysis divides into the study of point-wise bounds on the $\operatorname{div}\mathbf{u}$ and several energy-type estimates. A point-wise bound on the divergence are obtained from the equations (2) considered as a system of Lamé equations:

$$(\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \mu\Delta\mathbf{u} = (\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P$$

with the zero boundary condition for $\mathbf{u}$. In the case of the Cauchy problem or Navier boundary conditions this system can be reduced to two Laplace's equations for the $\operatorname{curl}\mathbf{u}$ and $(\lambda+2\mu)\operatorname{div}\mathbf{u} - P$ − viscous flux, with favorable boundary conditions, see [6]. The study of these equations reveals that the viscous flux is more regular than the pressure or $\operatorname{div}\mathbf{u}$, the crucial fact in obtaining a priori estimates. On the other hand, the argument fails when the no-slip condition is prescribed at the boundary.

We approach this problem by analyzing an integral representation for $\mathbf{u}$, utilizing the Green's matrix for the Lamé equations, see [12]. From this representation one can see that the structure of the pressure near the boundary significantly affects the smoothness of the viscous flux. In particular, the L^∞ bound of the pressure does not imply L^∞ bound on the viscous flux. For this reason, in this paper, we concentrate on the study of pressureless model, being the first step in understanding the full system of Navier-Stokes equations. The integral representation for $\mathbf{u}$ is used to show that $\operatorname{div}\mathbf{u}$ is bounded provided that u is C^α in the spacial variable. The later fact is established by energy estimates. The analysis of the continuity equation, then, provides uniform bounds on density. At the end, a weak solution is constructed as a limit of classical solutions, which exists globally in a view of a priori estimates we obtain in the first part of the paper.

1.1 Functional setting

By $B(r,x)$, $r>0$, $x\in\mathbb{R}^3$, we denote a ball with radius r, centered at $x\in\mathbb{R}^3_+$. We use symbol ∇ to denote spacial gradient of a function and D^2 the set of all spacial second derivatives. Let L^p, $1\le p\le+\infty$, be the Lebesgue space of functions from $\mathbb{R}^3_+$ to $\mathbb{R}$, integrable with exponent p (essentially bounded when $p=+\infty$). We use the standard notation $W^{k,p}\left(\mathbb{R}^3_+\right)$, $k\in\mathbb{N}$, $1\le p<+\infty$ for the space of weakly differentiable, up to the order k, functions, with derivatives from $L^p\left(\mathbb{R}^3_+\right)$ space. In this paper we will abbreviate $L^p\left(\mathbb{R}^3_+\right)$ to L^p and use the same notation for norms of scalar and vector functions. Denote by

$$[\mathbf{u}]_{C^\alpha}=\sup_{x,y\in\mathbb{R}^3_+,\,x\neq y}\frac{|\mathbf{u}(x)-\mathbf{u}(y)|}{|x-y|^\alpha},\ 0<\alpha<1,$$

– Hölder semi-norm. The following estimates are well-known, see [3](Theorem 7.10, Theorem 7.17)

Lemma 1. *Let u be a locally integrable function such that $\nabla u\in L^2\left(\mathbb{R}^3_+\right)$ with zero trace on the boundary $\partial\mathbb{R}^3_+$. Then, $u\in L^6\left(\mathbb{R}^3_+\right)$ and there is $c>0$, independent of u, such that*

$$\|u\|_{L^6}\le c\|\nabla u\|_{L^2}.$$

Lemma 2. *Let u be a locally integrable function with $\nabla u\in L^p\left(\mathbb{R}^3_+\right)$, $p>3$. Then, there is $c=c(p)$ such that for a.e. $x,y\in\mathbb{R}^3_+$ it holds*

$$|u(x)-u(y)|\le c|x-y|^\alpha\|\nabla u\|_{L^p},\ \alpha=1-3p^{-1}.$$

Definition 1. *A pair of functions*

$$(\rho,\,\mathbf{u})=(\rho(t,x),\,u_1(t,x),\,u_2(t,x),\,u_3(t,x))$$

is called a weak solution of (6)-(9) *if*

$$\rho,\,\rho u_i,\,\nabla u_i\in L^1_{loc}\left(\mathbb{R}_+\times\mathbb{R}^3_+\right),\ i=1,2,3,$$

$$\rho u_k\otimes u_l\in L^1_{loc}\left(\mathbb{R}_+\times\mathbb{R}^3_+\right),\ i,k,l=1..3,$$

and for all test functions ϕ, $\psi_k\in C^\infty\left([t,\,T]\,:\,C_0^\infty\left(\mathbb{R}^3_+\right)\right)$, $k=1..3$, and $0\le t<T<+\infty$ it holds (summation over the repeated indexes is assumed)

$$\int\int_{\mathbb{R}_+\times\mathbb{R}^3_+}\rho\partial_t\phi+\rho\mathbf{u}\cdot\nabla\phi-\int_{\mathbb{R}^3_+}\rho(\tau,\cdot)\phi(\tau,\cdot)\Big|_t^T=0,$$

$$\int\int_{\mathbb{R}_+\times\mathbb{R}^3_+} \rho u_k \partial_t \psi_k + \rho u_k u_j \partial_k \psi_j$$
$$- \int\int_{\mathbb{R}_+\times\mathbb{R}^3_+} (\lambda+\mu)\mathrm{div}\,\mathbf{u}\mathrm{div}\,\psi + \mu \partial_k u_l \partial_k \psi_l$$
$$- \int_{\mathbb{R}^3_+} \rho(\tau,\cdot)u_k(\tau,\cdot)\psi_k(\tau,\cdot)\Big|_t^T = 0.$$

1.2 Lamé equations

The principal part of (7) is an elliptic system of Lamé equations (14). Consider the problem

$$\left.\begin{aligned} -(\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u} - \mu\Delta\mathbf{u} = \mathbf{F}, \quad & \mathbb{R}^3_+, \\ u = 0, \quad & \partial\mathbb{R}^3_+, \end{aligned}\right\} \tag{14}$$

with the conditions $\mu > 0$, $\lambda + \mu > 0$. Here, $\mathbf{F} = (F_1(x),\, F_2(x),\, F_3(x))$. The system is $\left(W_0^{1,2}\right)^3$ – elliptic, see Chap. 3, sec. 7 of [11], meaning that the bilinear form

$$a(\mathbf{u},\mathbf{v}) = \int_{\mathbb{R}^3_+} (\lambda+\mu)\mathrm{div}\,\mathbf{u}\mathrm{div}\,\mathbf{v} + \mu\nabla\mathbf{u} : \nabla\mathbf{v},$$

is coercive, i.e.

$$a(\mathbf{u},\mathbf{u}) \geq \mu\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^3_+)}.$$

This condition is sufficient to imply the existence of the strong solution, Theorem 2.1 in [11], Chap. 3.

Lemma 3. *Let $\mathbf{F} \in L^2\left(\mathbb{R}^3_+\right)$. Then, there is a unique strong solution of* (14)*, such that*

$$\|D^2\mathbf{u}\|_{L^2(\mathbb{R}^3_+)} \leq c\|\mathbf{F}\|_{L^2(\mathbb{R}^3_+)} + c\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^3_+)},$$
$$\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^3_+)} \leq c\|\mathbf{F}\|_{L^{\frac{6}{5}}(\mathbb{R}^3_+)}.$$

The simple structure of the boundary $\partial\mathbb{R}^3_+$ allows us to write down the Green's matrix for the problem (14), see [12] for the derivation of the formula. Let

$$A = \frac{\lambda+\mu}{2\mu(\lambda+2\mu)},\ B = \frac{\lambda+3\mu}{\lambda+\mu}$$

and δ_{ik} be the Kronecker symbol. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ let $x^* = (-x_1, x_2, x_3)$. The solution can be written in the integral form

$$u_i = \int_{\mathbb{R}^3_+} G_i^k(x,y)F_k(y)\,dy, \tag{15}$$

where

$$G_i^k(x,y) = A\left[\left(B\delta_{ik} + (x_i - y_i)\frac{\partial}{\partial y_k}\right)\left(\frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-y^*|}\right)\right] \\ + x_1\left(\delta_{ik} - B^{-1}y_1\frac{\partial}{\partial y_k}\right)\frac{1}{2\pi}\frac{\partial}{\partial x_i}\frac{1}{|x-y^*|}, \quad i,k = 1,2,3, \quad (16)$$

and we assume the summation over repeated indexes. We also have

$$\operatorname{div}\mathbf{u}(x) = \int_{\mathbb{R}^3_+} \theta^k(x,y)F_k(y)\,dy, \quad (17)$$

where

$$-(\lambda+2\mu)\theta^k(x,y) = \frac{\partial}{\partial y_k}\left(\frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-y^*|}\right) \\ + \left(\delta_{1k} - B^{-1}y_1\frac{\partial}{\partial y_k}\right)\frac{1}{2\pi}\frac{\partial}{\partial y_1}\frac{1}{|x-y^*|}, \quad k = 1,2,3. \quad (18)$$

2 The proof of Theorem 1

Let $(\rho, \mathbf{u})$ be a unique classical solution of (6)-(9) that corresponds to the smooth, C^∞, initial datum $(\rho_0, \mathbf{u}_0)$, where $0 < \check{\rho} < \rho_0(x) < \hat{\rho}$ The solution exists locally in time on some interval $(0,T)$, $T > 0$. This was proved for the full system of Navier-Stokes equations in [10]. In fact, the absence of the pressure term only simplifies the problem, when the problem is considered locally in time. For such solution we now derive some estimates in strong norms, independent of the interval of the existence of smooth solution, that imply that the solution can be continued for all times $t > 0$, see [10].

2.1 An uniform estimate on $\operatorname{div}\mathbf{u}$.

Motivated by the original problem (7) we take $\mathbf{F} = -\frac{\partial}{\partial t}\rho\mathbf{u} - \operatorname{div}\rho\mathbf{u}\otimes\mathbf{u}$ in (14). Then we can formally write ($(-\Delta_L)^{-1}[\cdot]$ – a solution operator of (14))

$$\operatorname{div}\mathbf{u} = -\frac{d}{dt}\operatorname{div}(-\Delta_L)^{-1}[\rho\mathbf{u}] \\ + (\mathbf{u}\cdot\nabla)\operatorname{div}(-\Delta_L)^{-1}[\rho\mathbf{u}] - \operatorname{div}(-\Delta_L)^{-1}[\operatorname{div}\rho\mathbf{u}\otimes\mathbf{u}] \\ = \frac{d}{dt}I_1 + I_2. \quad (19)$$

Suppressing the dependence of functions in consideration on t we can write for $x \in \mathbb{R}^3_+$ (summation over repeated indexes is assumed):

$$
\begin{aligned}
-(\lambda+2\mu)I_2 = & -(\lambda+2\mu)u_i(x)\partial_i^x \int_{\mathbb{R}^3_+} \theta^k(x,y)\rho(y)u_k(y) \\
& -(\lambda+2\mu)p.v. \int_{\mathbb{R}^3_+} \partial_i^y\left(\theta(x,y)\right)\rho(y)u_i(y)u_k(y) \\
= & \; p.v. \int_{\mathbb{R}^3_+} \partial_i^y \partial_k^y \left(\frac{1}{4\pi|x-y|}\right)\rho(y)u_k(y)(u_i(y)-u_i(x)) \\
& + \int_{\mathbb{R}^3_+} \partial_1^y \partial_k^y \left(\frac{1}{4\pi|x-y^*|}\right)\rho(y)u_k(y)(u_1(y)+u_1(x)) \\
& - \sum_{i=2,3} \int_{\mathbb{R}^3_+} \partial_i^y \partial_k^y \left(\frac{1}{4\pi|x-y^*|}\right)\rho(y)u_k(y)(u_i(y)-u_i(x)) \\
+ \frac{\delta_{1k}}{2\pi} & \left[\int_{\mathbb{R}^3_+} \partial_i^x \partial_1^y \frac{1}{|x-y^*|}\rho(y)u_k(y)u_i(x) + \int_{\mathbb{R}^3_+} \partial_i^y \partial_1^y \frac{1}{|x-y^*|}\rho(y)u_k(y)u_i(y)\right] \\
& + \frac{1}{2\pi}\left[\int_{\mathbb{R}^3_+} B^{-1}y_1\partial_i^x\partial_k^y\partial_1^y \frac{1}{|x-y^*|}\rho(y)u_k(y)u_i(x)\right. \\
& \left. + \int_{\mathbb{R}^3_+} B^{-1}y_1\partial_i^y\partial_k^y\partial_1^y \frac{1}{|x-y^*|}\rho(y)u_k(y)u_i(y)\right] \\
& + \frac{1}{2\pi}\int_{\mathbb{R}^3_+} \delta_{1i}B^{-1}\partial_k^y\partial_1^y \frac{1}{|x-y^*|}\rho(y)u_k(y)u_i(y) \triangleq \sum_{k=1..6} I_2^k. \qquad (20)
\end{aligned}
$$

We estimate ($\alpha \in (0,1)$):

$$
\begin{aligned}
|I_2^1(x)| \le C[\mathbf{u}]_{C^\alpha(\mathbb{R}^3_+)} & \left|\int_{\mathbb{R}^3_+} \frac{1}{|x-y|^{3-\alpha}}\rho(y)\mathbf{u}(y)\right| \\
& \le C[\mathbf{u}]_{C^\alpha(\mathbb{R}^3_+)}\left(\sup_{x\in\mathbb{R}^3_+}\|\rho\mathbf{u}\|_{L^p(B(1,x))} + \|\rho\mathbf{u}\|_{L^q(\mathbb{R}^3_+)}\right), \qquad (21)
\end{aligned}
$$

where $1 < q < \frac{3}{\alpha} < p$. Term I_2^3 is estimated in the same way, we just have to notice that $|x-y| \le |x-y^*|$. Then,

$$
|I_2^2(x)| \le C\int_{\mathbb{R}^3_+} \frac{1}{|x-y^*|^3}|\rho(y)\mathbf{u}(y)|(|\mathbf{u}(y)|+|\mathbf{u}(x)|).
$$

Let $\zeta = \zeta(x,y) \in \partial\mathbb{R}^3_+$, $x \ne y$, be the point of intersection of the line containing x and y^* and the hyperplane $\partial\mathbb{R}^3_+$. We have $|y-\zeta| \le |x-y^*|$ and $|x-\zeta| \le |x-y^*|$. And, consequently,

$$
\left\{
\begin{array}{lcl}
|\mathbf{u}(y)| = |\mathbf{u}(y)-\mathbf{u}(\zeta)| & \le & C[\mathbf{u}]_{C^\alpha(\mathbb{R}^3_+)}|x-y^*|, \\
|\mathbf{u}(x)| = |\mathbf{u}(x)-\mathbf{u}(\zeta)| & \le & C[\mathbf{u}]_{C^\alpha(\mathbb{R}^3_+)}|x-y^*|,
\end{array}
\right. \qquad (22)
$$

where $C > 0$ is a constant of the embedding theorem. Thus, we obtain

$$|I_2^2(x)| \leq C[\mathbf{u}]_{C^\alpha\left(\mathbb{R}^3_+\right)} \int_{\mathbb{R}^3_+} \frac{1}{|x-y^*|^{3-\alpha}} |\rho(y)\mathbf{u}(y)|. \tag{23}$$

The expression on the right is bounded by the term appearing on the right hand side in (21), with possibly different choice of C. By the same arguments, the rest of the terms, $|I_2^4|$, $|I_2^5|$ and $|I_2^6|$ are also bounded by (21) (note, that $y_1 \leq |x-y^*|$, $x, y \in \mathbb{R}^3_+$). We summarize this analysis by the next estimate.

$$|I_2(x)| \leq C[\mathbf{u}]_{C^\alpha\left(\mathbb{R}^3_+\right)} \left(\sup_{x\in\mathbb{R}^3_+} \|\rho\mathbf{u}\|_{L^p(B(1,x))} + \|\rho\mathbf{u}\|_{L^q\left(\mathbb{R}^3_+\right)} \right), \tag{24}$$

where

$$1 < q < \frac{3}{\alpha} < p. \tag{25}$$

Terms $|I_1|$ and $|I_3|$ in (19) estimated using the following inequality. For $g(\cdot) \in L^{p_1}_{loc} \cap L^{q_1}\left(\mathbb{R}^3_+\right)$, $1 < q_1 < 3 < p_1$ and $x \in \mathbb{R}^3_+$ it holds:

$$\int_{\mathbb{R}^3_+} \frac{1}{|x-y|^2} |g(y)|\, dy \leq C \left(\sup_{x\in\mathbb{R}^3_+} \|g\|_{L^{p_1}(B(1,x))} + \|g\|_{L^{q_1}\left(\mathbb{R}^3_+\right)} \right).$$

Then, carrying out the differentiation in (18), we see that $|\theta^k(x,y)| \leq C|x-y|^{-2}$, $x, y \in \mathbb{R}^3_+$. By taking $g(\cdot) = |\rho(t,\cdot)\mathbf{u}(t,\cdot)|$ and $g(\cdot) = |\mathbf{f}(t,\cdot)|$ in the above inequality we obtain (for $1 < q_1 < 3 < p_1$):

$$|I_1(t,x)| \leq C \sup_{x\in\mathbb{R}^3_+} \|\rho(t,\cdot)\mathbf{u}(t,\cdot)\|_{L^{p_1}(B(1,x))} + C\|\rho(t,\cdot)\mathbf{u}(t,\cdot)\|_{L^{q_1}\left(\mathbb{R}^3_+\right)}, \tag{26}$$

with C depending on $\sup_{(o,\infty)\times\Omega} |\rho|$.

2.2 Energy estimates

Let us assume that there is $M > 0$, such that

$$\rho(t,x) < M, \quad t > 0,\, x \in \mathbb{R}^3_+. \tag{27}$$

In what follows by C we mean a generic function of the parameters of the model and/or M. Multiplying equations (7) by $\mathbf{u}$, using (6) and integrating over $\mathbb{R}^3_+$ we obtain:

$$\sup_{t\in[0,T)} \int_{\mathbb{R}^3_+} \rho(t,\cdot)\frac{\|\mathbf{u}(t,\cdot)\|^2_{L^2}}{2} + \int_0^T \int_{\mathbb{R}^3_+} (\lambda+\mu)|\operatorname{div}\mathbf{u}|^2 + \mu|\nabla\mathbf{u}|^2 \leq \int_{\mathbb{R}^3_+} \rho_0(\cdot)\frac{\|\mathbf{u}_0(\cdot)\|^2_{L^2}}{2}. \tag{28}$$

Let us consider equations (7), divide them by ρ and take operators div and curl of the result. We get:

$$\frac{d}{dt}\operatorname{div}\mathbf{u} + \operatorname{div}\,[(\mathbf{u}\cdot\nabla)\mathbf{u}] - (\mathbf{u}\cdot\nabla)\operatorname{div}\mathbf{u} \\ - \operatorname{div}\,\left[\rho^{-1}(\lambda+2\mu)\nabla\operatorname{div}\mathbf{u} - \rho^{-1}\mu\operatorname{curl}\operatorname{curl}\mathbf{u}\right] = 0, \quad (29)$$

$$\frac{d}{dt}\operatorname{curl}\mathbf{u} + \operatorname{curl}\,[(\mathbf{u}\cdot\nabla)\mathbf{u}] - (\mathbf{u}\cdot\nabla)\operatorname{curl}\mathbf{u} \\ - \operatorname{curl}\,\left[\rho^{-1}(\lambda+2\mu)\nabla\operatorname{div}\mathbf{u} - \rho^{-1}\mu\operatorname{curl}\operatorname{curl}\mathbf{u}\right] = 0. \quad (30)$$

We multiply the first equation by $\operatorname{div}\mathbf{u}$, the second by $\operatorname{curl}\mathbf{u}$, add them together and integrate over the $\mathbb{R}^3_+$. After carrying out the integration by parts on the principal part we obtain (for notational convenience we suppress the dependence of functions on t)

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^3_+} |\operatorname{div}\mathbf{u}|^2 + |\operatorname{curl}\mathbf{u}|^2 + \int_{\mathbb{R}^3_+} \rho^{-1}\,|(\lambda+2\mu)\nabla\operatorname{div}\mathbf{u} + \mu\operatorname{curl}\operatorname{curl}\mathbf{u}|^2 \\ = \frac{1}{2}\int_{\mathbb{R}^3_+} (|\operatorname{div}\mathbf{u}|^2 + |\operatorname{curl}\mathbf{u}|^2)\operatorname{div}\mathbf{u} + \left\{\int_{\mathbb{R}^3_+} (\operatorname{div}((\mathbf{u}\cdot\nabla)\mathbf{u}) - (\mathbf{u}\cdot\nabla)\operatorname{div}\mathbf{u})\operatorname{div}\mathbf{u} \right. \\ \left. + \int_{\mathbb{R}^3_+} (\operatorname{curl}((\mathbf{u}\cdot\nabla)\mathbf{u}) - (\mathbf{u}\cdot\nabla)\operatorname{curl}\mathbf{u})\cdot\operatorname{curl}\mathbf{u}\right\} \triangleq J_1 + J_2. \quad (31)$$

Both terms, $|J_1|$ and $|J_2|$ are bounded by $C\int_{\mathbb{R}^3_+}|\nabla\mathbf{u}|^3$ for suitable $C>0$. On the other hand, by Lemma 1, we have:

$$\|\nabla\mathbf{u}\|^3_{L^3} \le \|\nabla\mathbf{u}\|^{\frac{3}{2}}_{L^2}\|\nabla\mathbf{u}\|^{\frac{3}{2}}_{L^6} \le C\|\nabla\mathbf{u}\|^{\frac{3}{2}}_{L^2}\|D^2\mathbf{u}\|^{\frac{3}{2}}_{L^2},$$

where $D^2\mathbf{u}$ denotes the vector of all second derivatives of $\mathbf{u}$. By using Lemma 3 and Young inequality we obtain

$$\|\nabla\mathbf{u}\|^3_{L^3} \le C\|\nabla\mathbf{u}\|^{\frac{3}{2}}_{L^2}\left(\|(\lambda+2\mu)\nabla\operatorname{div}\mathbf{u} + \mu\operatorname{curl}\operatorname{curl}\mathbf{u}\|^{\frac{3}{2}}_{L^2} + \|\nabla\mathbf{u}\|^{\frac{3}{2}}_{L^2}\right).$$

By utilizing the well-known elliptic estimate,

$$\|\nabla\mathbf{u}\|^2_{L^2} \le C\|\operatorname{div}\mathbf{u}\|^2_{L^2} + C\|\operatorname{curl}\mathbf{u}\|^2_{L^2}, \quad (32)$$

the above inequality implies that

$$\|\nabla\mathbf{u}\|^3_{L^3} \le C\|\nabla\mathbf{u}\|^6_{L^2} + \frac{1}{4}\int_{\mathbb{R}^3_+} \rho^{-1}\,|(\lambda+2\mu)\nabla\operatorname{div}\mathbf{u} + \mu\operatorname{curl}\operatorname{curl}\mathbf{u}|^2.$$

Combining estimates on $|J_i|$, $i=1,2$, we conclude that

$$\frac{d}{dt}\int_{\mathbb{R}^3_+} |\operatorname{div}\mathbf{u}|^2 + |\operatorname{curl}\mathbf{u}|^2 + \int_{\mathbb{R}^3_+} \rho^{-1}\,|(\lambda+2\mu)\nabla\operatorname{div}\mathbf{u} + \mu\operatorname{curl}\operatorname{curl}\mathbf{u}|^2 \\ \le C\|\nabla\mathbf{u}\|^6_{L^2}, \quad (33)$$

for some $C > 1$. Postulating smallness of initial data

$$2C(1 + \|\nabla \mathbf{u}_0\|_{L^2}^2)\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2 < 2^{-1}, \tag{34}$$

the following estimates are easily deduced from inequality (33) and the inequality (32)

$$\sup_{t\in(0,T)} \|\nabla \mathbf{u}(t,\cdot)\|_{L^2} \leq C\|\nabla \mathbf{u}_0\|_{L^2}, \tag{35}$$

$$\int_0^T \|D^2\mathbf{u}(t,\cdot)\|^2 \leq C(1 + \|\nabla \mathbf{u}_0\|_{L^2}^4)\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}^2, \tag{36}$$

with some C independent of $T > 0$.

2.3 An estimate on $\rho(t,x)$

Using the representation formula (19), equation (6) can be written as

$$\frac{d}{dt}(\log\rho - \log\hat{\rho}) = -\operatorname{div}\mathbf{u} = -\frac{d}{dt}I_1 - I_2. \tag{37}$$

Integrating this equation along a particle trajectory and taking maximum over all trajectories we deduce that

$$\sup_{\{\rho(T,\cdot)>\hat{\rho}\}} \log\left[\frac{\rho(T,\cdot)}{\hat{\rho}}\right] \leq 2 \sup_{(0,T)\times\mathbb{R}^3_+} |I_1| + \int_0^T \|I_2(t,\cdot)\|_{L^\infty}\,dt. \tag{38}$$

We estimate terms appearing on the right-hand side of (38). Let us take $q_1 = 2$ and $p_1 = 6$ in (26). Using Lemma 1 and estimate (28) we derive:

$$|I_1(t,x)| \leq C\|\nabla \mathbf{u}(t,\cdot)\|_{L^2} + C\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2}. \tag{39}$$

Consider the bound (24). Take $q = 6$, $\alpha = 1/4$ and $p = 13 > 3\alpha^{-1}$. Then,

$$\sup_{x\in\mathbb{R}^3_+} \|\rho\mathbf{u}\|_{L^{13}(B(1,x))} \leq C\|\nabla\mathbf{u}\|_{L^{\frac{39}{16}}} \leq \|\nabla\mathbf{u}\|_{L^2}^{\theta}\|\nabla\mathbf{u}\|_{L^6}^{1-\theta} \leq C\|\nabla\mathbf{u}\|_{L^2}^{\theta}\|D^2\mathbf{u}\|_{L^2}^{1-\theta},$$

where $16/39 = \theta/2 + (1-\theta)/6$. Also, by Lemma 2 and Lemma 3, we have:

$$[\mathbf{u}]_{C^\alpha} \leq C\|\nabla\mathbf{u}\|_{L^4} \leq C\|\nabla\mathbf{u}\|_{L^2}^{\theta_1}\|\nabla\mathbf{u}\|_{L^6}^{1-\theta_1} \leq C\|\nabla\mathbf{u}\|_{L^2}^{\theta_1}\|D^2\mathbf{u}\|_{L^2}^{1-\theta_1},$$

where $1/4 = \theta_1/2 + (1-\theta_1)/6$. Thus, combining last two estimates and using once again Lemma 2 we obtain that

$$[\mathbf{u}]_{C^\alpha} \sup_{x\in\mathbb{R}^3_+} \|\rho\mathbf{u}\|_{L^{13}(B(1,x))} \leq C\|\nabla\mathbf{u}\|_{L^2}^{\theta+\theta_1}\|D^2\mathbf{u}\|_{L^2}^{2-\theta-\theta_1} \leq C\|\nabla\mathbf{u}\|_{L^2}^2 + C\|D^2\mathbf{u}\|_{L^2}^2.$$

Similarly,

$$[\mathbf{u}]_{C^\alpha}\|\rho\mathbf{u}\|_{L^6} \leq c(M)\|\nabla\mathbf{u}\|_{L^2}^{\theta_1}\|D^2\mathbf{u}\|_{L^2}^{2-\theta_1} \leq C\|\nabla\mathbf{u}\|_{L^2}^2 + C\|D^2\mathbf{u}\|_{L^2}^2.$$

It follows that

$$|I_2(t,x)| \leq C\|\nabla \mathbf{u}\|_{L^2}^2 + C\|D^2\mathbf{u}\|_{L^2}^2. \tag{40}$$

Combining estimates (39), (40), (35), (36) in (38) we obtain the following inequality true for any $t > 0$.

$$\sup_{\{\rho(t,\cdot)>\hat{\rho}\}} \log\left[\frac{\rho(t,\cdot)}{\hat{\rho}}\right] \leq C\|\nabla \mathbf{u}_0\|_{L^2} + C\|\sqrt{\rho_0}\mathbf{u}_0\|_{L^2} + C\|\mathbf{u}_0\|_{L^2}^2\left(1 + \|\nabla \mathbf{u}_0\|_{L^2}^4\right).$$

We recall that C depends on the upper bound for $\rho(t,x)$. Clearly, by suitably restricting the initial data $\mathbf{u}_0$, the last inequality provides an a priori estimate for the density:

$$\rho(t,x) \leq C(\lambda, \mu, \hat{\rho}, \mathbf{u}_0), \quad t > 0,\, x \in \mathbb{R}^3_+. \tag{41}$$

Remark 3. *If in addition to the upper bound $\rho \leq \hat{\rho}$ we had a lower bound*

$$\rho_0 \geq \check{\rho} > 0,$$

then, the argument similar to the one that lead us to (41) *would imply that the solution is lower bounded,*

$$\rho(t,x) \geq c(\lambda, \mu, \hat{\rho}, \mathbf{u}_0).$$

2.4 Proof of the existence

Consider now a sequence of initial data

$$\rho_0^n,\, \mathbf{u}_0^n \in C^\infty\left(\mathbb{R}^3_+\right) \times C_0^\infty\left(\mathbb{R}^3_+\right),$$

which approximates the given initial data in the space $L^6_{loc}(\mathbb{R}^3_+) \times L^6\left(\mathbb{R}^3_+\right)^3$. We require that $M > \rho_0^n > m(n)$, and $\|\nabla \mathbf{u}_0\|_{L^2}$ small as required by analysis of the previous sections. Such a sequence, clearly exists. We can take $\rho_0^n(x) = (\rho_0(x) + n^{-1}) * \omega_{n^{-1}}(x)$, $\mathbf{u}_0^n = (\mathbf{u}_0(x)) * \omega_{n^{-1}}(x)$, where ω_ϵ is the standard mollifier. Accordingly, let $(\rho^n, \mathbf{u}^n)$ be the sequence of smooth solutions of the problem with $(\rho_0^n, \mathbf{u}_0^n)$ as the initial data. As we mentioned before, the existence of such solution is implied by the result of [10] and a priori estimates we just obtained. In particular, we established that the following norms are bounded with bounds independent of n.

$$\{\rho^n\} \quad \text{bounded in} \quad L^\infty\left(\mathbb{R}_+ \times \mathbb{R}^3_+\right), \tag{42}$$

$$\{\sqrt{\rho^n}\mathbf{u}^n\} \quad \text{bounded in} \quad L^\infty\left(\mathbb{R}_+ : L^2(\mathbb{R}^3_+)\right), \tag{43}$$

$$\{\nabla \mathbf{u}^n\} \quad \text{bounded in} \quad L^2\left(\mathbb{R}_+ \times \mathbb{R}^3_+\right), \tag{44}$$

$$\{D^2\mathbf{u}^n\} \quad \text{bounded in} \quad L^2\left(\mathbb{R}_+ \times \mathbb{R}^3_+\right). \tag{45}$$

By the weak stability result of P.-L. Lions, see Theorem 5.1 of [9], bounds (42)–(44) imply the existence of an accumulation point $(\rho, \mathbf{u})$ of the sequence $\{\rho^n,\, \mathbf{u}^n\}$ in the weak topology of $L^6_{\text{loc}}(\mathbb{R}^3_+) \times L^6(\mathbb{R}^3_+)$, which is a weak solution of (6), (7). Moreover, the bounds in the spaces from (42)–(45) hold for this (ρ, $\mathbf{u}$).

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