# *The Navier-Stokes equations for the motion of compressible, viscous fluid flows with the no-slip boundary condition*

Mikhail Perepelitsa

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**Abstract**

The Navier-Stokes equations for the motion of compressible, viscous fluids in the half-space $\mathbb{R}^3_+$ with the no-slip boundary condition are studied. Given a constant equilibrium state $(\bar{\rho},\, \mathbf{0})$, we construct a global in time, regular weak solution, provided that the initial data $\rho_0, \mathbf{u}_0$ are close to the equilibrium state when measured by the norm

$$|\rho_0 - \bar{\rho}|_{L^\infty} + |\mathbf{u}_0|_{H^1}$$

and discontinuities of $\rho_0$ decay near the boundary of $\mathbb{R}^3_+$.

## 0.1 Introduction

We consider a model for the motion of a compressible, isothermal, viscous flow based on the Navier-Stokes equations. With $\rho(t,x)$ and $\mathbf{u}(t,x)$ being the density and the velocity of the fluid, the model consist of the equations:

$$\frac{\partial}{\partial t}\rho + \operatorname{div}\,(\rho\mathbf{u})\, = 0, \tag{1}$$

$$\frac{\partial}{\partial t}\,(\rho\mathbf{u})\, + \operatorname{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) - (\lambda+\mu)\operatorname{div}\mathbf{u} - \mu\Delta\mathbf{u} + \nabla P = \rho\mathbf{f}, \tag{2}$$

$$3\lambda + 2\mu \geq 0, \quad \mu > 0,$$

$$(t,\, x) \in \mathbb{R}_+ \times \Omega, \quad P(\rho) = A\rho,\, A > 0,$$

and a set of initial and boundary conditions:

$$(\rho(0,x),\, \mathbf{u}(0,x)) = (\rho_0(x),\, \mathbf{u}_0(x))\,, \quad x \in \Omega, \tag{3}$$

$$\mathbf{u}(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega. \tag{4}$$

There is an extensive literature concerning different aspects of the problem (1) – (4). For the detailed discussion of the results we refer the reader to the recent monograph [13]. We shortly mention some of them. It is known that if the initial data of the problem are smooth then the problem is well-posed. Moreover, a unique, global solution exists if the initial data are close to a static equilibrium state, measured in strong norms, for example in $H^3\left(\mathbb{R}^3_+\right)$, see [11, 15]. On the other hand, there is a well-developed theory of weak solutions of the problem (1) – (4) and other related problems, see [8, 3]. A typical result is contained in the following theorem.

**Theorem** (P.-L. Lions, [8])**.** *Suppose that $\gamma \geq \frac{9}{5}$ and $\Omega \in C^{2+\theta}$, $\theta > 0$. Suppose that the initial data $(\rho_0, m_0)$ satisfy $\rho \in L^\gamma(\Omega)$, $|m_0|^2/\rho_0 \in L^1(\Omega)$, where we agree that $m_0 = 0$ on $\{\rho_0(\cdot) = 0\}$. Then there is a global weak solution of the problem (1)–(4), $(\rho, \mathbf{u})$, such that $\rho(0,\cdot) = \rho_0(\cdot)$ and $\rho(0,\cdot)\mathbf{u}(t,\cdot) = m_0$ Moreover, for any $t > 0$ the energy inequality holds.*

$$\int_\Omega \left(\frac{1}{2}\rho(t,\cdot)|\mathbf{u}(t,\cdot)|^2 + \frac{A\rho(t,\cdot)^\gamma}{\gamma-1}\right) + \int_0^t \int_\Omega \mu|\nabla \mathbf{u}|^2 + (\lambda+\mu)|\operatorname{div}\mathbf{u}|^2 \leq \int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}_0|^2 + \frac{A\rho_0^\gamma}{\gamma-1}\right).$$

Solutions constructed in the above theorem have somewhat limited regularity properties: $\rho \in L^\infty(\mathbb{R}_+ : L^\gamma(\Omega))$ and $\mathbf{u} \in L^2\left(\mathbb{R}_+ : W^{1,2}(\Omega)\right)^3$, and thus, may incorporate some non-physical phenomena.

For the Cauchy problem, i.e. when the flow occupies the whole space $\mathbb{R}^3$, global existence of weak solutions that remain near a static equilibrium sate, $(\bar{\rho}, \mathbf{0})$, was proved in [4], see also [5] for related results. In contrast with the result of [11], solutions built in [4] are *essentially* weak; the density is an element of $L^\infty$. On the other hand, solutions possess many favorable properties, such as, impossibility of spontaneous formation of vacuum, the fact which is being implicitly assumed when equations (1) – (2) are used to model a motion of real fluids.

**Theorem** (D. Hoff, [4, 5])**.** *Let $\Omega = \mathbb{R}^N$, $N = 2, 3$. Let $\bar{\rho} > 0$ and $L > 0$ be given. There is a positive number $c = c(N)$ and a pair of positive numbers $A$, $C$ depending on $(\mu, \lambda, \bar{\rho}, L, N, c)$, with the property that if*

$$\lambda + \mu \leq c\mu \tag{5}$$

*and the initial data $(\rho_0, \mathbf{u}_0)$ satisfy bounds*

$$0 \leq \rho_0 \leq \bar{\rho}, \qquad a.e.\ \mathbb{R}^3_+,$$

$$\int_{\mathbb{R}^N} |\mathbf{u}_0(y)|^2 + (\rho_0(y) - \bar{\rho})^2\, dy \leq A$$

*and*

$$|\mathbf{u}_0|_{L^{2N}\left(\mathbb{R}^N\right)^N} \leq L,$$

*then, a global weak solution $(\rho, \mathbf{u})$ of the problem (1)–(3) exists for which*

$$|\rho|_{L^\infty\left(\mathbb{R}_+\times\mathbb{R}^N\right)} \leq C\bar{\rho},$$

$$\mathbf{u} \in L^\infty\left(\{t\,:\,t > \tau\} \times \mathbb{R}\right)^N, \quad \forall \tau > 0.$$

*(We refer the reader to [5] for the complete statement of the Theorem.)*

The analogous result was obtained for flows in domains with boundaries under the Navier boundary condition, i.e. the condition that tangential velocity at the boundary is proportional to the tangential component of the stress, see [5].

In this work we present a development of the existence theory of the near equilibrium weak solutions initiated in [4] to the problems with the no-slip boundary condition (4). The density component of the weak solution that we construct is $L^\infty$ away form the boundaries and such that discontinuities in $\rho(t,\cdot)$ decay near the boundary. Specifically, we measure $\rho(t,\cdot)$ by the norm

$$\langle \rho(t,\cdot)\rangle_\alpha + |\rho(t,\cdot)|_{L^\infty\cap L^2}, \quad \alpha \in ]0,1[,$$

where $\langle\cdot\rangle_\alpha$ is defined in (13). The localization of discontinuities in $\rho(t,\cdot)$ inside the domain corresponds to a physical situation when motion of a fluid results from disturbances that occur in the interior of the domain. At the level of technical description of the proof, the introduction of the above functional to measure the density is dictated by the fact that the $L^\infty$ norm along is not suitable to control sound waves reflected from the boundary. Moreover, for weak solutions to remain near the equilibrium state we impose a certain structural restriction on the model, i.e. on the relative size of $\lambda$ and $\mu$, given by (6), which guarantees that sound waves reflected from the boundary are, in fact, *weaker* than of incident waves. More detailed descussion of the result is given in the next section.

## 0.2 Statement of the result

We prove the following theorem.

**Theorem A.** *For any $\bar\rho > 0$ and $\alpha \in ]0,\frac{1}{4}[$ there are $c_0 = c_0(\alpha) > 0$, $c_i = c_i(\bar\rho,\lambda,\mu,\alpha,A)$, $i = 1,2$, and a continuous, non-increasing function $\alpha(t) > 0$, $\alpha(0) = \alpha$ such that if*

$$\frac{c(\alpha)\mu}{\lambda + 3\mu} < 1, \tag{6}$$

*and a pair of measurable functions $(\rho_0,\, \mathbf{u}_0)$ verifies a smallness assumption:*

$$|\rho_0 - \bar\rho|_{L^2(\mathbb{R}^3_+)} + |\rho_0 - \bar\rho|_{L^\infty(\mathbb{R}^3_+)} + \langle\rho_0\rangle_\alpha + |\mathbf{u}_0|_{H^1(\mathbb{R}^3_+)} \le c_1, \tag{7}$$

*then, there exists a weak solution, $(\rho,\mathbf{u})$, of the problem (1) – (4), defined for all times $t > 0$, see Definition 1 for the definition of a weak solution. Moreover, for a.e. $t$ in $\mathbb{R}_+$, the following estimates hold.*

$$\left.\begin{aligned}
&\operatorname{osc}\rho(t,\cdot) + \langle\rho(t,\cdot)\rangle_{\alpha(t)} < c_2(\operatorname{osc}\rho(t,\cdot) + \langle\rho(t,\cdot)\rangle_\alpha + |\mathbf{u}_0|_{H^1}),\\
&|\mathbf{u}(t,\cdot)|_{H^1(\mathbb{R}^3_+)} + |\rho(t,\cdot) - \bar\rho|_{L^2(\mathbb{R}^3_+)} \le c_2(|\mathbf{u}_0|_{H^1(\mathbb{R}^3_+)} + |\rho_0 - \bar\rho|_{L^2(\mathbb{R}^3_+)}),\\
&\nabla_x \mathbf{u} \in L^2(\mathbb{R}_+ \,:\, L^6\left(\mathbb{R}^3_+\right)).
\end{aligned}\right\} \tag{8}$$

**Remark 1.** *Since the solution, constructed in the Theorem, is such that the oscillations in density are small, there is no loss of generality in assuming the pressure law $P = A\rho$ instead of the "isentropic" $\gamma$–law, $P = A\rho^\gamma$, $\gamma \ge 1$. Indeed, the derivation of a priori estimates in case $\gamma > 1$ is identical to case $\gamma = 1$. Moreover, the strong convergence of the sequence of the approximate, classical solutions for $\gamma > 1$ is established by the Lions-Feireisl theory, see FEIREISL[3].*

The framework of the analysis was established in works [4, 8]. We shortly describe the new issues appearing in the problem with the no-slip boundary conditions. First, we notice that unlike the situation for the Cauchy problem, $L^\infty$ norm alone is not well-suited to measure the density of a solution. Indeed, the Navier-Stokes equations (2) can be written as a problem

$$\left.\begin{array}{rcl} (\lambda+\mu)\nabla \operatorname{div}\mathbf{u} + \mu\Delta\mathbf{u} &=& \mathbf{a} + \nabla(\rho-\bar\rho), \\ \mathbf{u} &=& 0, \quad \partial\mathbb{R}^3_+, \end{array}\right\} \tag{9}$$

where $\mathbf{a} = \rho D_t\mathbf{u}$ – the acceleration. Using a classical method of Lichtestein and assuming for a moment that $\mathbf{a} = 0$, the divergence of $\mathbf{u}$ can be represented as

$$\begin{aligned}(\lambda+2\mu)\operatorname{div}\mathbf{u}(t,x) = & -\frac{\lambda+\mu}{\lambda+3\mu}(\rho(t,x)-\bar\rho) \\ & + \frac{2\mu}{\lambda+2\mu}\int_{\mathbb{R}^3_+}\nabla_x G(x,y)\cdot\nabla_y(\rho(t,y)-\bar\rho)dy, \end{aligned} \tag{10}$$

where $G(x,y)$ is the Green's function for the Laplace's equation in $\mathbb{R}^3_+$, see section 0.6 for details. It can be seen from the above formula that $\operatorname{div}\mathbf{u}$ is not bounded in sup-norm if $\rho-\bar\rho$ is a generic function in $L^2\cap L^\infty\left(\mathbb{R}^3_+\right)$. The divergence, $\operatorname{div}\mathbf{u}$, may blow-up at boundary points. Moreover, the singularities caused by $\nabla(\rho-\bar\rho)$ are unlikely to be balanced by the acceleration, $\mathbf{a}$, because the later vanishes at the boundary of the domain. On the other hand, $\operatorname{div}\mathbf{u}$ is the rate of the production of mass which, when unbounded, may cause the blow-up of the density in $L^\infty$ norm. A better substitute is a norm

$$|\rho|_{X_0} := \langle\rho\rangle_\alpha + |\rho-\bar\rho|_2 + \operatorname{osc}\rho. \tag{11}$$

where $\langle\cdot\rangle_\alpha$ is defined by (13). It can be shown that

$$|\operatorname{div}\mathbf{u}|_{X_0} \le c|\rho|_{X_0},$$

see Lemma 9. The question to ask then, is either the motion caused by the reflection of "waves" from the boundary can destabilize a solution which is close to the static equilibrium $(\bar\rho,\mathbf{0})$ initially. We consider the reflection in the context of the linear elliptic problem (9). We say that *sound waves reflected from the boundary of* $\partial\mathbb{R}^3_+$ *are weaker then incident waves when measured by* $|\cdot|_X$ *if*

$$|(\lambda+2\mu)\operatorname{div}\mathbf{u} - (\rho-\bar\rho)|_X \le c|\rho-\bar\rho|_X, \quad \text{and} \quad 0<c<1. \tag{12}$$

For example, the representation formula (10) can be used to show that the above property holds with $X = L^2\left(\mathbb{R}^3_+\right)$, if $\lambda+\mu>0$, $\mu>0$. For $X_0$ introduced in (11) we were able to show that the above estimate holds with

$$c = \frac{c_0(\alpha)\mu}{\lambda+2\mu},$$

where the function $c_0(\alpha)$ is of the order $\alpha^{-1}$. We do not know if this number is smaller than 1 for the full range of $\{(\lambda,\mu) : \mu>0,\, 3\lambda+2\mu>0\}$ and this is the reason for including a structural condition (6) in the Theorem. On the other hand, the constant $c$ is less than 1 if the ratio $\frac{\mu}{\lambda+\frac{2}{3}\mu}$ is small. In Hydrodynamics the quantity $\lambda+\frac{2}{3}\mu$ is called a second viscosity coefficient or bulk viscosity. Under certain conditions such as propagation of high frequency sound waves, the bulk viscosity shows a dispersion relation on the frequency and the values of $\mu/(\lambda+\frac{2}{3}\mu)$ are in fact small, see Section 81 of [6]. For that reason our result may be thought of as a model for such type of motion. Note also that the space that we are using for the density supports oscillations of arbitrary high frequencies. We use the estimate (12) with $X = X_0$ in the equation (1) to show that the oscillations of the density in reflected sound waves are damped by pressure.

Another issue is also connected with the stability of the flow in $\langle\rho\rangle_a$ norm. Because of the hyperbolicity of the equation (1) the norm $\langle\rho(t,\cdot)\rangle_\alpha$ critically depends on the regularity of the flow $X^t$ generated by the velocity field $\mathbf{u}$. Using the integral representation of $\mathbf{u}$ as a solution of (9) we derive a system of ODE's (56) for the flow trajectories, from which, based on the energy estimates, we are able to show that the flow is Hölder continuous with the constant of Hölder continuity being generally a decreasing function of time, see Lemma 13. The later property causes the degradation of regularity of the density. In Theorem A this fact is represented by a non-increasing function $\alpha(t)$ in the estimate (8). From the estimates of Lemma 13 one can also see that flow trajectories approach the boundary of the domain at the rate proportional to $\langle\rho(t,\cdot)\rangle_\alpha$. This, potentially singular phenomenon is counterbalanced by the damping effect produced by the pressure on the density. The intensity of the damping is measured by $\inf\rho(t,\cdot)$, which is required to be positive, see Lemma 14.

A main part of the paper is devoted to derivation of *a priori* estimates for a generic classical solution of (1)–(2). A weak solution is constructed as a limit of global smooth solutions with appropriately smoothed initial data using Lions-Feireisl theory, [8, 3].

## 0.3 Functional setting

By $B(r,x)$, $r>0$, $x\in\mathbb{R}^3$, we denote a ball with radius $r$, centered at $x\in\mathbb{R}^3_+$. We use symbol $\nabla$ to denote the spacial gradient of a function and $D^2$ the set of all spacial second derivatives. Let $L^p$, $1\le p\le+\infty$, be the Lebesgue space of functions from $\mathbb{R}^3_+$ to $\mathbb{R}$, integrable with exponent $p$ (essentially bounded when $p=+\infty$). We use the standard notation $W^{k,p}\left(\mathbb{R}^3_+\right)$, $k\in\mathbb{N}$, $1\le p<+\infty$ for the space of weakly differentiable, up to the order $k$, functions, with derivatives from $L^p\left(\mathbb{R}^3_+\right)$ space. We use notation $H^1:=W^{1,2}$. In this paper we will abbreviate $L^p\left(\mathbb{R}^3_+\right)$ to $L^p$ and use the same notation for norms of scalar and vector functions. Denote by

$$[\mathbf{u}]_\alpha = \sup_{x,y\in\mathbb{R}^3_+,\, x\neq y} \frac{|\mathbf{u}(x)-\mathbf{u}(y)|}{|x-y|^\alpha},\ \alpha\in]0,1[,$$

$$\langle\mathbf{u}\rangle_\alpha = \sup_{x\in\partial\mathbb{R}^3_+, y\in\mathbb{R}^3_+,\, 0<|x-y|<1} \frac{|\mathbf{u}(x)-\mathbf{u}(y)|}{|x-y|^\alpha},\ \alpha\in]0,1[, \tag{13}$$

various Hölder semi-norms. The following estimates are well-known, see [2](Theorem 7.10, Theorem 7.17)

**Lemma 1.** *Let $u$ be a locally integrable function such that $\nabla u\in L^2\left(\mathbb{R}^3_+\right)$ and having zero trace on the boundary $\partial\mathbb{R}^3_+$. Then, $u\in L^6\left(\mathbb{R}^3_+\right)$ and there is $c>0$, independent of $u$, such that*

$$\|u\|_{L^6}\le c\|\nabla u\|_{L^2}.$$

**Lemma 2.** *Let $u$ be a locally integrable function with $\nabla u \in L^p\left(\mathbb{R}^3_+\right)$, $p > 3$. Then, there is $c = c(p)$ such that for a.e. $x, y \in \mathbb{R}^3_+$ it holds*

$$|u(x) - u(y)| \leq c|x-y|^{\alpha}\|\nabla u\|_{L^p},\ \alpha = 1 - \frac{3}{p}.$$

**Definition 1.** *A pair of functions*

$$(\rho,\, \mathbf{u}) = (\rho(t,x),\, u_1(t,x),\, u_2(t,x),\, u_3(t,x))$$

*is called a weak solution of* (1)-(4) *if*

$$\rho,\, \rho u_i,\, \nabla u_i \in L^1_{loc}\left(\mathbb{R}_+ \times \mathbb{R}^3_+\right),\ i = 1,2,3,$$

$$\rho u_k \otimes u_l \in L^1_{loc}\left(\mathbb{R}_+ \times \mathbb{R}^3_+\right),\ i,k,l = 1..3,$$

$$\left.\begin{array}{rcl} \nabla \mathbf{u} & \in & L^2\left(\mathbb{R}_+ \times \mathbb{R}^3_+\right), \\ \mathbf{u} & = & 0, \ \ on\ \partial\mathbb{R}^3_+, \end{array}\right\}$$

*and for all test functions $\phi,\, \psi_i \in C^\infty\left([t,\, T]\, :\, C_0^\infty\left(\mathbb{R}^3_+\right)\right)$, $i = 1,2,3$, with $0 \leq t < T < +\infty$ it holds (summation over the repeated indexes is assumed)*

$$\int\int_{\mathbb{R}_+\times\mathbb{R}^3_+} \rho\partial_t\phi + \rho\mathbf{u}\cdot\nabla\phi - \int_{\mathbb{R}^3_+} \rho(\tau,\cdot)\phi(\tau,\cdot)\Big|_t^T = 0,$$

$$\int\int_{\mathbb{R}_+\times\mathbb{R}^3_+} \rho u_k\partial_t\psi_k + \rho u_k u_j \partial_k\psi_j$$
$$- \int\int_{\mathbb{R}_+\times\mathbb{R}^3_+} (\lambda+\mu)\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\psi + \mu\partial_k u_l\partial_k\psi_l + (P - \bar{P})\partial_k\psi_k$$
$$- \int_{\mathbb{R}^3_+} \rho(\tau,\cdot)u_k(\tau,\cdot)\psi(\tau,\cdot)\Big|_t^T = 0.$$

To simplify the presentation we assume that the constant $A = 1$ in (2). It is always possible to reduced to this case through the substitution $(t, x, \rho, \mathbf{u}) \to (a^2 t,\, ax,\, \rho,\, au)$, $a = A^{-\frac{1}{2}}$, without changing the viscosity coefficients.

## 0.4 The Lamé equations

In this section we recall some elliptic estimates. The principal part of (2) is an elliptic system of Lamé equations (14). Consider the problem

$$\left.\begin{array}{rr} (\lambda+\mu)\nabla\,\mathrm{div}\,\mathbf{u} + \mu\Delta\mathbf{u} = \mathbf{F}, & \mathbb{R}^3_+, \\ u = 0, & \partial\mathbb{R}^3_+, \end{array}\right\} \tag{14}$$

with the conditions $\mu > 0$, $\lambda + \mu > 0$. Here, $\mathbf{F} = (F_1(x),\, F_2(x),\, F_3(x))$. The system is $\left(W_0^{1,2}\right)^3$ – elliptic, see Chap. 3, sec. 7 of [12], meaning that the bilinear form

$$a(\mathbf{u},\mathbf{v}) = \int_{\mathbb{R}^3_+} (\lambda+\mu)\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v} + \mu\nabla\mathbf{u} : \nabla\mathbf{v},$$

is coercive, i.e.

$$a(\mathbf{u},\mathbf{u}) \geq \mu |\nabla \mathbf{u}|^2_{L^2(\mathbb{R}^3_+)}.$$

This condition is sufficient to imply the existence of the strong solution, Theorem 2.1 in [12], Chap. 3.

**Lemma 3.** *Let $\mathbf{F} \in L^2\left(\mathbb{R}^3_+\right)$. Then, there is a unique strong solution of (14), such that*

$$|D^2\mathbf{u}|_{L^2(\mathbb{R}^3_+)} \leq c|\mathbf{F}|_{L^2(\mathbb{R}^3_+)} + c|\nabla \mathbf{u}|_{L^2(\mathbb{R}^3_+)},$$

$$|\nabla \mathbf{u}|_{L^2(\mathbb{R}^3_+)} \leq c|\mathbf{F}|_{L^{\frac{6}{5}}(\mathbb{R}^3_+)}.$$

The following Lemma is a well-known fact, whose prove is based on the Calderon-Zygmund estimates on singular integrals.

**Lemma 4.** *Suppose that in the problem (14) $\mathbf{F} = \nabla P$, for some $P \in L^p$, $p \in ]1,\infty[$. Then, the problem has a unique weak solution $\mathbf{u} \in L^{\frac{3p}{3-p}}$, $p < 3$ or $\mathbf{u} \in L^\infty_{\rm loc}$, $p > 3$, such that*

$$|\nabla \mathbf{u}|_{L^p} \leq c|P|_{L^p},$$

*for some $c = c(p)$.*

**Corollary 1.** *In the problem (14), let $\mathbf{F} = \mathbf{F}_1 + \nabla P$, where $\mathbf{F}_1 \in L^2(\mathbb{R}^3_+)$ and $P \in L^6(\mathbb{R}^3_+)$. Then, a unique solution of the problem exists and there is $c > 0$ such that*

$$|\nabla \mathbf{u}|_6 \leq c\left(|\nabla \mathbf{u}|_2 + |\mathbf{F}_1|_2 + |P|_6\right). \tag{15}$$

## 0.5 Energy estimates

In all subsequent estimate we assume

**Hypothesis $\mathbb{H}_0$.** *For all $(t,x)$,*

$$\rho(t,x) < 10\bar{\rho} := M.$$

**Lemma 5.** *Let*

$$\Phi(\rho) = \rho \int_{\bar{\rho}}^{\rho} s^{-2}(s-\bar{\rho})\,ds, \quad \rho \geq 0$$

*and*

$$E(t) = \int_{\mathbb{R}^3_+} \rho(t,\cdot)|\mathbf{u}(t,\cdot)|^2/2 + \Phi(\rho(t,\cdot)).$$

*Then, for any smooth solution $(\rho, \mathbf{u})$ of the problem (1)–(4) the following equality holds.*

$$E(t) + \int_0^t \int_{\mathbb{R}^3_+} (\lambda+2\mu)|\operatorname{div}\mathbf{u}(t,\cdot)|^2 + \mu|\operatorname{curl}\mathbf{u}(t,\cdot)|^2 = E(0). \tag{16}$$

The proof of this Lemma is well-known and can be found, for example, in [4].

**Lemma 6.** *With the notation $F = (\lambda+2\mu)\operatorname{div}\mathbf{u} - (\rho - \bar{\rho})$,*

$$\mathbf{F}_1 = \nabla F + \mu \operatorname{curl} \circ \operatorname{curl}\mathbf{u},$$

$$Y(t) = |\rho(t,\cdot) - \bar{\rho}|_2^2 + |F(t,\cdot)|_2^2 + |\operatorname{curl}\mathbf{u}(t,\cdot)|_2^2$$

*and in the conditions of the previous lemma, there are* $c_i = c_i(\lambda,\, \mu,\, M, |\nabla \mathbf{u}_0|_2)$, $i = 1, 2$, *such that for* $t > 0$ *it holds:*

$$Y(t) + \int_0^t |\mathbf{F}_1(\tau, \cdot)|_2^2 + |\rho(\tau, \cdot) - \bar{\rho}|_2^2 \, d\tau \leq c_2 \left(E(0) + Y(0)\right), \tag{17}$$

*provided that*

$$E(0) \leq c_1.$$

*Proof.* We divide equations (2) by $\rho$ and take operators $\operatorname{div}$ and $\operatorname{curl}$ of the result. We get:

$$\frac{d}{dt} \operatorname{div} \mathbf{u} + \operatorname{div} ((\mathbf{u} \cdot \nabla)\mathbf{u}) - (\mathbf{u} \cdot \nabla) \operatorname{div} \mathbf{u} - \operatorname{div} \left[\rho^{-1} \nabla F - \rho^{-1} \mu \operatorname{curl} \operatorname{curl} \mathbf{u}\right] = 0,$$

$$\frac{d}{dt} \operatorname{curl} \mathbf{u} + \operatorname{curl} ((\mathbf{u} \cdot \nabla)\mathbf{u}) - (\mathbf{u} \cdot \nabla) \operatorname{curl} \mathbf{u} - \operatorname{curl} \left[\rho^{-1} \nabla F - \rho^{-1} \mu \operatorname{curl} \operatorname{curl} \mathbf{u}\right] = 0.$$

Then, we multiply the first equation by $F$, second by $\mu \operatorname{curl} \mathbf{u}$, add them and integrate over $\mathbb{R}^3_+$. After carrying out the integration by parts on the principal part we obtain (dependence on $t$ is not explicitely written):

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int \frac{F^2}{\lambda + 2\mu} + \mu |\operatorname{curl} \mathbf{u}|^2 + \int \rho^{-1} |\mathbf{F}_1|^2 &= \frac{1}{2} \int \left(\frac{|F|^2}{\lambda + 2\mu} + \mu |\operatorname{curl} \mathbf{u}|^2\right) \operatorname{div} \mathbf{u} + \left\{ \int ( \operatorname{div} ((\mathbf{u} \cdot \nabla)\mathbf{u}) - (\mathbf{u} \cdot \nabla) \operatorname{div} \mathbf{u}) F \right. \\ &\quad \left. + \int ( \operatorname{curl} ((\mathbf{u} \cdot \nabla)\mathbf{u}) - (\mathbf{u} \cdot \nabla) \operatorname{curl} \mathbf{u}) \cdot \mu \operatorname{curl} \mathbf{u} \right\} \\ &\quad + \int \frac{\rho \operatorname{div} \mathbf{u} F}{\lambda + 2\mu} = J_1 + J_2 + J_3, \end{aligned} \tag{18}$$

Both terms,

$$|J_1|,\, |J_2| \leq c_\delta |\nabla \mathbf{u}|_3^3 + \delta |\rho - \bar{\rho}|_2^2, \tag{19}$$

for any $\delta > 0$. On the other hand,

$$\begin{aligned} J_3 = \int_{\mathbb{R}^3_+} \frac{(\rho - \bar{\rho}) \operatorname{div} \mathbf{u}((\lambda + 2\mu) \operatorname{div} \mathbf{u} - (\rho - \bar{\rho}))}{\lambda + 2\mu} \\ + \bar{\rho} \int_{\mathbb{R}^3_+} \frac{\operatorname{div} \mathbf{u}((\lambda + 2\mu) \operatorname{div} \mathbf{u} - (\rho - \bar{\rho}))}{\lambda + 2\mu} \leq c |\nabla \mathbf{u}|_2^2 - \frac{1}{2(\lambda + 2\mu)} |\rho - \bar{\rho}|_2^2. \end{aligned} \tag{20}$$

Let us estimate $|\nabla \mathbf{u}|_3$. We have $|\nabla \mathbf{u}|_3^3 \leq c |\nabla \mathbf{u}|_2^{\frac{3}{2}} |\nabla \mathbf{u}|_6^{\frac{3}{2}}$ and thus by Corollary 1 we get (for any $\varepsilon,\, \delta > 0$):

$$|\nabla \mathbf{u}|_3^3 \leq \varepsilon |\mathbf{F}_1|_2^2 + c_{\varepsilon,\delta} |\nabla \mathbf{u}|_2^2 (|\nabla \mathbf{u}|_2^4 + 1) + \delta |\rho - \bar{\rho}|_2^2. \tag{21}$$

A simple energy estimate of the Lamé equations (14) leads to the following estimate.

$$|\nabla \mathbf{u}|_2^2 \leq c |F|_2^2 + c |\operatorname{curl} \mathbf{u}|_2^2 + c |\rho - \bar{\rho}|_2^2.$$

Using this estimate in (21) and combining estimates (19)–(21) in (18) we obtain

$$\frac{d}{dt}\left(|F|_2^2 + |\operatorname{curl}\mathbf{u}|_2^2\right) + c|\mathbf{F}_1|_2^2 + c|\rho-\bar\rho|_2^2 \\ \leq c^{-1}|\nabla\mathbf{u}|_2^2((|F|_2^2 + |\operatorname{curl}\mathbf{u}|_2^2)^4 + |\rho-\bar\rho|_2^2 + 1).$$

Finally, from the equation (1) it follows that

$$\frac{d}{dt}|\rho-\bar\rho|_2^2 \leq \delta|\rho-\bar\rho|_2^2 + c_\delta|\nabla\mathbf{u}|_2^2.$$

Note, that we allow $c$ depend on $\sup\rho$. We conclude from the last two inequalities that

$$Y'(t) + |\mathbf{F}_1(t,\cdot)|_2^2 + |\rho(t,\cdot)-\bar\rho|_2^2 \leq c|\nabla\mathbf{u}(t,\cdot)|_2^2(Y(t)^4+1).$$

The conclusion of the lemma follows from the last inequality and Lemma 5. □

## 0.6 Representation formulas for the solution of (14)

Let us recall a classical method of Lichtenstein for the reduction of a boundary value problem for the elliptic system of Lamé equations to a boundary integral equation for the $\operatorname{div}\mathbf{u}$. The later can be explicitly solved in the half-space $\mathbb{R}^3_+$. The exposition of this method can be found for example in [10].

For $x=(x_1,x_2,x_3)\in\mathbb{R}^3_+$, let $x^*=(-x_1,x_2,x_3)$. Let

$$H(x,y) = \frac{1}{4\pi|x-y|},$$

and denote by

$$G(x,y) = -H(x,y) + H(x,y^*) \tag{22}$$

the Green's function for the Laplace's equation in $\mathbb{R}^3_+$. We look at the system of equations (2) as elliptic problem (14) where we set

$$\mathbf{F} = \mathbf{a} + \nabla(\rho-\bar\rho),$$

with

$$\mathbf{a} = (a^1,\, a^2,\, a^3) = (\rho\mathbf{u})_t + \operatorname{div}\rho\mathbf{u}\otimes\mathbf{u}$$

– the inertia force. Let

$$F = (\lambda+2\mu)\operatorname{div}\mathbf{u} - (\rho-\bar\rho), \tag{23}$$

be the notation for the viscous flux. Applying div to (14) we derive:

$$\Delta F = \operatorname{div}\mathbf{a}, \tag{24}$$

and the following integral representation holds (the dependence of functions on $t$ is not written for notational convenience).

$$F(x) = \int_{\partial\mathbb{R}^3_+} \partial_{n_y}G(x,\cdot)F(\cdot) + \int_{\mathbb{R}^3_+} G(x,\cdot)\operatorname{div}\mathbf{a}(\cdot). \tag{25}$$

Using (24), equations (14) can be written in the following form.

$$\Delta\left[\frac{\lambda+\mu}{2(\lambda+2\mu)}Fx+\mu\mathbf{u}\right]=\mathbf{a}+\frac{\lambda+\mu}{2(\lambda+2\mu)}\,\mathrm{div}\,(\mathbf{a})x+\frac{\mu}{\lambda+2\mu}\nabla(\rho-\bar\rho)$$

and so,

$$\begin{aligned}\mu\mathbf{u}(x)+\frac{\lambda+\mu}{2(\lambda+2\mu)}F(x)x&=\frac{\lambda+\mu}{2(\lambda+2\mu)}\int_{\partial\mathbb{R}^3_+}\partial_{n_y}G(x,y)F(y)y\,dS_y\\&+\int_{\mathbb{R}^3_+}G(x,y)\left[\mathbf{a}(y)+\frac{\lambda+\mu}{2(\lambda+2\mu)}\,\mathrm{div}\,\mathbf{a}(y)y+\frac{\mu}{\lambda+2\mu}\nabla_y(\rho(y)-\bar\rho)\right]dy.\end{aligned}\tag{26}$$

We set

$$\alpha_1=\frac{\lambda+\mu}{2(\lambda+2\mu)},\ \alpha_2=\frac{\mu}{\lambda+2\mu},\ \alpha_3=\frac{\lambda+3\mu}{2(\lambda+2\mu)}.\tag{27}$$

We take div of the last equations and use integral representation (25) for $F$ to get the following equation (here and below the summation over repeated indexes is assumed).

$$\begin{aligned}(2\alpha_1+\alpha_3)F(x)=&-\alpha_2(\rho(x)-\bar\rho)\\&+\alpha_1\int_{\partial\mathbb{R}^3_+}\partial_{n_y}\partial_{x_i}G(x,y)(y_i-x_i)F(y)\,dS_y+\alpha_1\int_{\mathbb{R}^3_+}\nabla_xG(x,y)\cdot(y-x)\,\mathrm{div}\,\mathbf{a}(y)\,dy\\&+\int_{\mathbb{R}^3_+}\nabla_xG(x,y)\cdot\mathbf{a}(y)\,dy+\alpha_2\int_{\mathbb{R}^3_+}\nabla_xG(x,y)\cdot\nabla_y(\rho(y)-\bar\rho)\,dy.\end{aligned}$$

One can easily verify that

$$\partial_{n_y}\partial_{x_i}G(x,y)(y_i-x_i)=2\partial_{n_y}G(x,y),\quad y\in\partial\mathbb{R}^3_+.$$

We use this identity in the last equation together with (25) to obtain the following representation formula for $F$.

$$\begin{aligned}\alpha_3F(x)=&-\alpha_2(\rho(x)-\bar\rho)\\&-2\alpha_1\int_{\mathbb{R}^3_+}G(x,y)\,\mathrm{div}\,\mathbf{a}\,dy+\alpha_1\int_{\mathbb{R}^3_+}\nabla_xG(x,y)\cdot(y-x)\,\mathrm{div}\,\mathbf{a}(y)\,dy\\&+\int_{\mathbb{R}^3_+}\nabla_xG(x,y)\cdot\mathbf{a}(y)\,dy+\alpha_2\int_{\mathbb{R}^3_+}\nabla_xG(x,y)\cdot\nabla_y(\rho(y)-\bar\rho)\,dy.\end{aligned}\tag{28}$$

Now, we work on the representation for the right-hand side of the above equation that we will need for the derivation of favorable estimates on $F$. First, notice that

$$\int_{\mathbb{R}^3_+}\partial_{x_i}G(x,y)\partial_{y_i}(\rho(y)-\bar\rho)\,dy=-(\rho(x)-\bar\rho)+\int_{\mathbb{R}^3_+}\partial_{y_i}\partial_{x_i}H(x,y^*)(\rho(y)-\bar\rho)\,dy.$$

Let

$$\begin{aligned}I(x)=2\alpha_1\int_{\mathbb{R}^3_+}\nabla_yG(x,y)\cdot\mathbf{a}(y)\,dy+\alpha_1\int_{\mathbb{R}^3_+}\nabla_xG(x,y)\cdot(y-x)\,\mathrm{div}\,\mathbf{a}(y)\,dy\\+\int_{\mathbb{R}^3_+}\nabla_xG(x,y)\cdot\mathbf{a}(y)\,dy.\end{aligned}$$

A direct computation shows that

$$\nabla_x G(x,y)\cdot(y-x) = G(x,y) + \tilde{H}(x,y^*),$$

where

$$\tilde{H}(x,y) = \frac{y_1(x_1-y_1)}{2\pi|x-y|^3}$$

and

$$\nabla_x G(x,y) = \, -\nabla_y G(x,y) + 2\partial_{y_1} H(x,y^*)(1,\,0,\,0).$$

Thus, we can write

$$I(t,x) = (\alpha_1+1)\int_{\mathbb{R}^3_+} \nabla_y G(x,y)\cdot \mathbf{a}(t,y)\,dy - \int_{\mathbb{R}^3_+} \Big[\alpha_1 \nabla_y \tilde{H}(x,y^*) \\ + \; 2\partial_{y_1}(H(x,y^*),0,0)\big]\cdot \mathbf{a}(t,y)\,dy. \quad (29)$$

As the next step, we recall the definition of $\mathbf{a}$ and perform the following operations. Note that $D_t$ stands for the material derivative.

$$\int_{\mathbb{R}^3_+} \nabla_y G(x,y)\cdot \mathbf{a}(t,y)\,dy = D_t \int_{\mathbb{R}^3_+} \nabla_y G(x,y)\cdot \rho\mathbf{u}\big|_{(t,y)}\,dy \\ - \mathbf{u}(t,x)\cdot\nabla_x \int_{\mathbb{R}^3_+} \nabla_y G(x,y)\rho\mathbf{u}\big|_{(t,y)}\,dy - \int_{\mathbb{R}^3_+} \nabla_y\otimes\nabla_y G(x,y) : \rho\mathbf{u}\otimes\mathbf{u}\big|_{(t,y)}\,dy \\ = D_t \int_{\mathbb{R}^3_+} \nabla_y G(x,y)\cdot\rho\mathbf{u}\big|_{(t,y)}\,dy + \int_{\mathbb{R}^3_+} \nabla_x\otimes\nabla_y H(x,y) : \rho\mathbf{u}\big|_{(t,y)}\otimes(\mathbf{u}(t,y)-\mathbf{u}(t,x))\,dy \\ - \int_{\mathbb{R}^3_+} \nabla_x\otimes\nabla_y H(x,y^*) : \rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,x)\,dy \\ - \int_{\mathbb{R}^3_+} \nabla_y\otimes\nabla_y H(x,y^*) : \rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,y)\,dy. \quad (30)$$

Similarly, we compute

$$\int_{\mathbb{R}^3_+} \nabla_y \tilde{H}(x,y^*)\cdot\mathbf{a}(t,y)\,dy = D_t\int_{\mathbb{R}^3_+} \nabla_y\tilde{H}(x,y^*)\cdot\rho\mathbf{u}\big|_{(t,y)}\,dy \\ - \int_{\mathbb{R}^3_+} \nabla_x\otimes\nabla_y\tilde{H}(x,y^*) : \rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,x)\,dy \\ - \int_{\mathbb{R}^3_+} \nabla_y\otimes\nabla_y\tilde{H}(x,y^*) : \rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,y)\,dy \quad (31)$$

and

$$\int_{\mathbb{R}^3_+} 2\partial_{y_1}(H(x,y^*),0,0)\cdot\mathbf{a}(t,y)\,dy = D_t\int_{\mathbb{R}^3_+} 2\partial_{y_1}(H(x,y^*),0,0)\cdot\rho\mathbf{u}\big|_{(t,y)}\,dy \\ - \int_{\mathbb{R}^3_+} \nabla_x\otimes(2\partial_{y_1},0,0)H(x,y^*) : \rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,x)\,dy \\ - \int_{\mathbb{R}^3_+} \nabla_y\otimes(2\partial_{y_1},0,0)H(x,y^*) : \rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,y)\,dy \quad (32)$$

Collecting (30)–(32) in (29) we can write the following representation formula for $I(t,x)$.

$$\begin{aligned}
I(t,x) = D_t \int_{\mathbb{R}^3_+} & L(x,y)\cdot\rho(t,y)\mathbf{u}(t,y)\,dy \\
&+ \int_{\mathbb{R}^3_+} \mathbb{L}_1(x,y) : \rho(t,y)\mathbf{u}(t,y)\otimes(\mathbf{u}(t,y)-\mathbf{u}(t,x))\,dy \\
&+ \int_{\mathbb{R}^3_+} \mathbb{L}_2^*(x,y) : \rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,x)\,dy \\
&+ \int_{\mathbb{R}^3_+} \mathbb{L}_3^*(x,y) : \rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,y)\,dy, \qquad (33)
\end{aligned}$$

where

$$\left[\begin{aligned}
L(x,y) &= (\alpha_1+1)\nabla_y G(x,y) - \alpha_1\nabla_y H(x,y^*) - (2\partial_{y_1},0,0)H(x,y^*), \\
\mathbb{L}_1(x,y) &= (\alpha_1+1)\nabla_x\otimes\nabla_y H(x,y), \\
\mathbb{L}_2^*(x,y) &= -(\alpha_1+1)\nabla_x\otimes\nabla_y H(x,y^*) - \alpha_1\nabla_x\otimes\nabla_y\tilde{H}(x,y^*) \\
&\quad - \nabla_x\otimes(2\partial_{y_1},0,0)H(x,y^*), \\
\mathbb{L}_3^*(x,y) &= -(\alpha_1+1)\nabla_y\otimes\nabla_y H(x,y^*) - \alpha_1\nabla_y\otimes\nabla_y\tilde{H}(x,y^*) \\
&\quad - \nabla_y\otimes(2\partial_{y_1},0,0)H(x,y^*).
\end{aligned}\right] \qquad (34)$$

Let us also set

$$K^*(x,y) = \partial_{x_i}\partial_{y_i}H(x,y^*). \qquad (35)$$

Then, using (33) in (28) we deduce the following representation for $F$.

$$\begin{aligned}
\alpha_3 F(x) = D_t \int_{\mathbb{R}^3_+} & L(x,y)\cdot\rho(t,y)\mathbf{u}(t,y)\,dy \\
&+ \int_{\mathbb{R}^3_+} \mathbb{L}_1(x,y) : \rho(t,y)\mathbf{u}(t,y)\otimes(\mathbf{u}(t,y)-\mathbf{u}(t,x))\,dy \\
&+ \int_{\mathbb{R}^3_+} \mathbb{L}_2^*(x,y) : \rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,x)\,dy \\
&+ \int_{\mathbb{R}^3_+} \mathbb{L}_3^*(x,y) : \rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,y)\,dy \\
&+ \left\{\alpha_2(\rho(x)-\bar{\rho}) + \alpha_2\int_{\mathbb{R}^3_+} \operatorname{div}_y\nabla G(x,y)(\rho(y)-\bar{\rho})\,dy\right\}. \qquad (36)
\end{aligned}$$

For notational convenience we abbreviate the above formula as

$$F(t,x) = D_t J_1(t,x) + \sum_{i=2}^{4} J_i(t,x) + \frac{\alpha_2}{\alpha_3}P_1(t,x). \qquad (37)$$

Now, we derive a representation formula for $\mathbf{u}$. Let $\mathbb{G}$ be a Green's matrix for the problem (14), i.e.

$$\mathbf{u}(x) = \int_{\mathbb{R}^3_+} \mathbb{G}(x,y)\mathbf{F}(y)\,dy.$$

The explicit expression for the Green's matrix can be found, for example, in [14]. Let

$$A = \frac{\lambda+\mu}{2\mu(\lambda+2\mu)},\ B = \frac{\lambda+3\mu}{\lambda+\mu}$$

and $\delta_{ik}$ be the Kronecker symbol. Then,

$$\mathbb{G}_i^k(x,y) = A\left[\left(B\delta_{ik} + (x_i - y_i)\frac{\partial}{\partial y_k}\right)\left(\frac{1}{4\pi|x-y|} - \frac{1}{4\pi|x-y^*|}\right)\right] + x_1\left(\delta_{ik} - B^{-1}y_1\frac{\partial}{\partial y_k}\right)\frac{1}{2\pi}\frac{\partial}{\partial x_i}\frac{1}{|x-y^*|}, \quad i,k = 1,2,3. \tag{38}$$

We split $\mathbf{u}$ according to the following formula

$$\mathbf{u}(t,x) = \int_{\mathbb{R}^3_+} \mathbb{G}(x,y)((\rho\mathbf{u})_t + \operatorname{div}\mathbf{u}\otimes\mathbf{u})\Big|_{(t,y)}\,dy + \int_{\mathbb{R}^3_+} \mathbb{G}(x,y)\nabla(\rho(t,y) - \bar{\rho})\,dy := \mathbf{w}(t,x) + \int_{\mathbb{R}^3_+} \mathbb{G}(x,y)\nabla(\rho(t,y) - \bar{\rho})\,dy \tag{39}$$

## 0.7 Some potential estimates

To deal with rather lengthy representation formulas from the last subsection we introduce the following classes of functions.

**Property $\mathbb{S}$.** *We say that a function $K(x,y)$ is of the class $\mathbb{S}$ if there is a constant $c > 0$ such that*

*1.*

$$|K(x,y)| \le c|x-y|^3, \quad \forall x \ne y,$$

*2.* $\forall x \ne z$, $y \in \mathbb{R}^3_+$ *such, that* $|(x+z)/2 - y| > 2|z-x|$ *it holds*

$$|K(x,y) - K(z,y)| \le c|z-x||(x+z)/2 - y|^{-4}.$$

**Property $\mathbb{S}^*$.** *We say that a function $K(x,y)$ is of the class $\mathbb{S}^*$ if there is a constant $c > 0$ such that*

*1.*

$$|K(x,y)| \le c|x'-y|^3, \quad \forall x \ne y,$$

*where $x'$ is the projection of the point $x \in \mathbb{R}^3_+$ onto $\partial\mathbb{R}^3_+$,*

*2.* $\forall x \ne z$, $y \in \mathbb{R}^3_+$ *such, that* $|(x+z)/2 - y| > 2|z-x|$ *it holds*

$$|K(x,y) - K(z,y)| \le c|z-x||\xi - y|^{-4},$$

*where $\xi$ – the projection of the point $(x+z)/2$ onto $\partial\mathbb{R}^3_+$.*

The following two Lemmas are verified by direct computations.

**Lemma 7.** *The elements of vector $L$ and matrix $\mathbb{L}_1$, from* (34)*, satisfy property $\mathbb{S}$.*

**Lemma 8.** *The function $K^*$ from* (35) *and the elements of matrices $\mathbb{L}_2^*$, $\mathbb{L}_3^*$, from* (34)*, satisfy property $\mathbb{S}^*$.*

Let

$$P_1(x) = (\rho(x) - \bar{\rho}) + \int_{\mathbb{R}^3_+} \nabla_x G(x,y)(\rho(y) - \bar{\rho}).$$

Then, the following Lemma holds.

**Lemma 9.** *For any* $\alpha \in ]0,1[$ *and* $\delta > 0$ *there are* $c > 0$, $c_\delta > 0$, *independent of* $(\lambda, \mu, \bar\rho, t)$, *such, that*

$$\langle P_1 \rangle_\alpha \leq c\left(|\rho(\cdot) - \bar\rho|_2 + \langle \rho \rangle_\alpha\right), \tag{40}$$

*and*

$$|P_1(x)| \leq \delta \langle \rho \rangle_\alpha + c_\delta |\rho(\cdot) - \bar\rho|_2. \tag{41}$$

*Proof.* We proof only the first part of the Lemma. The proof for the second part goes along the same line of arguments. Let $x_1,\, x_2 \in \mathbb{R}^3_+$, $|x_1 - x_2| < \delta$ and set $B_2 = B(x_1, 2)$, $B_1 = B(x_1, 4\delta)$, $B = B(x_1, 2|x_1 - x_2|)$,

$$S_2 = \partial\left\{\mathbb{R}^3_+ \cap B_2\right\} \setminus \partial \mathbb{R}^3_+,$$

and

$$S = \partial\left\{\mathbb{R}^3_+ \cap B\right\} \setminus \partial \mathbb{R}^3_+.$$

We can write the following representation for

$$\begin{aligned}
P_1(x_1) - P_1(x_2) &= (\rho(x_1) - \rho(x_2)) \\
&\quad - \int_{\mathbb{R}^3_+ \setminus B_2} \{\operatorname{div}_y \nabla_x G(x_1, y) - \operatorname{div}_y \nabla_x G(x_2, y)\}\,(\rho(y) - \bar\rho) \\
&\quad + \int_{S_2} \{\nabla_x G(x_1, y) - \nabla_x G(x_2, y)\} \cdot \mathbf{n}_y (\rho(y) - \bar\rho)\, dS_y \\
&\quad - \int_{B_2 \setminus B_1} \{\operatorname{div}_y \nabla_x G(x_1, y) - \operatorname{div}_y \nabla_x G(x_2, y)\}\,(\rho(y) - \rho(x_1)) \\
&\quad - \int_{B_1} \{\operatorname{div}_y \nabla_x G(x_1, y) - \operatorname{div}_y \nabla_x G(x_2, y)\}\,(\rho(y) - \rho(x_1)) \\
&\quad + \int_{S_2} \{\nabla_x G(x_1, y) - \nabla_x G(x_2, y)\} \cdot (-\mathbf{n}_y)(\rho(y) - \rho(x_1))\, dS_y.
\end{aligned} \tag{42}$$

We set $x_2'$ to be the projection of point $x_2$ onto $\partial\mathbb{R}^3_+$. and consequently,

$$\begin{aligned}
P_1(x_1) - P_1(x_2) &= (\rho(x_1) - \rho(x_2)) \\
&\quad - \int_{\mathbb{R}^3_+ \setminus B_2} \{\operatorname{div}_y \nabla_x G(x_1, y) - \operatorname{div}_y \nabla_x G(x_2, y)\}\,(\rho(y) - \bar\rho) \\
&\quad + \int_{S_2} \{\nabla_x G(x_1, y) - \nabla_x G(x_2, y)\} \cdot \mathbf{n}_y (\rho(x_1) - \bar\rho)\, dS_y \\
&\quad - \int_{B_2 \setminus B_1} \{\operatorname{div}_y \nabla_x G(x_1, y) - \operatorname{div}_y \nabla_x G(x_2, y)\}\,(\rho(y) - \rho(x_1)) \\
&\quad - \int_{B_1 \setminus B} \{\operatorname{div}_y \nabla_x G(x_1, y) - \operatorname{div}_y \nabla_x G(x_2, y)\}\,(\rho(y) - \rho(x_1)) \\
&\quad - \int_{B} \operatorname{div}_y \nabla_x G(x_1, y)(\rho(y) - \rho(x_1)) \\
&\quad + \int_{B} \operatorname{div}_y \nabla_x G(x_2, y)(\rho(y) - \rho(x_2')) \\
&\quad + (\rho(x_2') - \rho(x_1)) \int_S \operatorname{div}_y \nabla_x G(x_2, y) \cdot \mathbf{n}_y\, dS_y.
\end{aligned} \tag{43}$$

Finally, we split $G = H + H^*$ as in (22) and use the fact that $H$ is a fundamental solution of the Laplace's equation. We obtain, see (35) for definition of $K^*$,

$$\begin{aligned} P_1(x_1) - P_1(x_2) = (\rho(x_1) - \rho(x_2')) \\ - \int_{\mathbb{R}^3_+\setminus B_2} \{K^*(x_1,y) - K^*(x_2,y)\}(\rho(y) - \bar\rho) \\ + \int_{S_2} \{\nabla_x G(x_1,y) - \nabla_x G(x_2,y)\}\cdot \mathbf{n}_y(\rho(x_1) - \bar\rho)\, dS_y \\ - \int_{B_2\setminus B_1} \{K^*(x_1,y) - K^*(x_2,y)\}(\rho(y) - \rho(x_1)) \\ - \int_{B_1\setminus B} \{K^*(x_1,y) - K^*(x_2,y)\}(\rho(y) - \rho(x_1)) \\ - \int_B K^*(x_1,y)(\rho(y) - \rho(x_1)) \\ + \int_B K^*(x_2,y)(\rho(y) - \rho(x_2')) \\ + (\rho(x_2') - \rho(x_1)) \int_S \nabla_x G(x_2,y)\cdot \mathbf{n}_y\, dS_y. \end{aligned} \quad (44)$$

In the above representation formula we take $x_1 \in \partial\mathbb{R}^3_+$ and $x_2 \in \mathbb{R}^3_+$. The first term on the right is bounded by $\langle\rho\rangle_{\alpha,\delta}|x_1 - x_2|^\alpha$. The second is bounded by $|\rho(\cdot) - \bar\rho|_2|x_1 - x_2|$. The third – by $\operatorname{osc}\rho(\cdot)|x_1 - x_2|$. The fourth – by

$$\operatorname{osc}\rho(\cdot)\delta^{-\alpha}\alpha^{-1}|x_1 - x_2|^\alpha$$

and the rest of the terms are bounded by

$$\alpha^{-1}\langle\rho\rangle_{\alpha,\delta}|x_1 - x_2|^\alpha,$$

because $K^*$ possesses property $\mathbb{S}^*$. The same bound holds for the last term on the right, since

$$|\int_S \nabla_x G(x_2,y)\cdot \mathbf{n}_y\, dS_y| \le c,$$

with $c$ independent of $x_1$, $x_2$. Setting $\delta = 1$ we obtain the required estimate. □

We prove the following lemma.

**Lemma 10.** *There are $c_i = c_i(p,q)$, $i = 0,1$ and $c_2 = c_2(\beta,p)$ such that*

$$|J_1|_\infty + [J_1]_\gamma \le c_0\left(|\rho\mathbf{u}|_p + |\rho\mathbf{u}|_q\right), \quad \gamma = 1 - 3p^{-1},\ p > 3,\ q \in ]1,3[ \quad (45)$$

*and for $i = 2..4$*

$$|J_i|_\infty \le c_1 M[\mathbf{u}]_\alpha \left(\sup_{x\in\mathbb{R}^3_+} |\mathbf{u}|_{p,B(1,x)} + |\mathbf{u}|_q\right), \quad 1 < q < 3/\alpha < p, \quad (46)$$

$$\langle J_i\rangle_\beta \le c_2 M\left([\mathbf{u}]^2_\alpha + [\mathbf{u}]_\alpha|\mathbf{u}|_p\right), \quad \beta \in ]0,\alpha[,\ p > 1. \quad (47)$$

*Proof.* We only prove (47). The proofs of (45) and (46) are similar. Lets consider $J_2(x)$. Let $x_1 \in \partial\mathbb{R}^3_+$, $x_2 \in \mathbb{R}^3_+$ and $|x_1 - x_2| \le 1$. Let $x_0 = \frac{x_1+x_2}{2}$ and $B = B(x_0, 2|x_1 - x_2|) \cap \mathbb{R}^3_+$ and $B_1 = B(x_0, 2) \cap \mathbb{R}^3_+$. We can write

$$
\begin{aligned}
J_2(x_1) - J_2(x_2) &= \int_B \mathbb{L}_{1,ij}(x_1, y)\,[\rho(y)u_i(y)(u_j(x_1) - u_j(y))] \\
&\quad - \int_B \mathbb{L}_{1,ij}(x_2, y)\,[\rho(y)u_i(y)(u_j(x_2) - u_j(y))] \\
&\quad + \int_{B_1\setminus B} \{\mathbb{L}_{1,ij}(x_1, y) - \mathbb{L}_{1,ij}(x_2, y)\}\,\rho(y)u_i(y)(u_j(x_0) - u_j(y)) \\
&\quad + \int_{\mathbb{R}^3_+\setminus B_1} \{\mathbb{L}_{1,ij}(x_1, y)(u_j(x_1) - u_j(y)) - \mathbb{L}_{1,ij}(x_2, y)(u_j(x_1) - u_j(y))\}\,\rho(y)u_i(y) \\
&\quad + (u_j(x_1) - u_j(x_0)) \int_{B_1\setminus B} \mathbb{L}_{1,ij}(x_1, y)\rho(y)u_i(y) \\
&\quad - (u_j(x_2) - u_j(x_0)) \int_{B_1\setminus B} \mathbb{L}_{1,ij}(x_2, y)\rho(y)u_i(y) \triangleq \sum_1^6 J_2^i. \qquad (48)
\end{aligned}
$$

Note, that $\mathbf{u} = 0$ on $\partial\mathbb{R}^3_+$. Since functions $\mathbb{L}_{1,ij}$ verify property $\mathbb{S}$, it is easy to see that $|J_2^1|$, $|J_2^2|$ and $|J_2^3|$ are bounded by $c|\rho|_\infty[\mathbf{u}]^2_\alpha|x_1 - x_2|^\alpha$, for suitable $c$. For $J_2^5$ (and $J_2^6$) we have the following estimate.

$$|J_2^5| \le c|\rho|_\infty[\mathbf{u}]^2_\alpha|x_1 - x_2|^\alpha \log|x_1 - x_2|^{-1}.$$

On the other hand, due to the property $\mathbb{S}$

$$|J_2^4| \le c|\rho|_\infty[\mathbf{u}]_\alpha|\mathbf{u}|_p|x_1 - x_2|^\alpha, \quad p > 1.$$

We proved estimate (47) for $J_2$. Lets consider $J_3$ ($J_4$ is estimated in the same way). Let $x_i$, $i = 0, 1, 2$ be chosen as above.

$$
\begin{aligned}
J_3(x_1) - J_3(x_2) &= \int_{\mathbb{R}^3_+\setminus B_1} \left\{\mathbb{L}^*_{2,ij}(x_1, y) - \mathbb{L}^*_{2,ij}(x_2, y)\right\}\rho(y)u_i(y)u_j(y) \\
&\quad + \int_{B_1\setminus B} \left\{\mathbb{L}^*_{2,ij}(x_1, y) - \mathbb{L}^*_{2,ij}(x_2, y)\right\}\rho(y)u_i(y)u_j(y) \\
&\quad + \int_B \mathbb{L}^*_{2,ij}(x_1, y)\rho(y)u_i(y)u_j(y) \\
&\quad + \int_B \mathbb{L}^*_{2,ij}(x_1, y)\rho(y)u_i(y)u_j(y). \qquad (49)
\end{aligned}
$$

Now, since $\mathbb{L}^*_{2,ij}$ has the property $\mathbb{S}*$, the first term on the right is bounded by

$$|\rho|_\infty|\mathbf{u}|^2_p|x_1 - x_2|,\ p > 1,$$

the second by (note that $\mathbf{u} = 0$ on $\partial\mathbb{R}^3_+$)

$$|\rho|_\infty[\mathbf{u}]^2_\alpha|x_1 - x_2|^\alpha.$$

The last bound is also true for the third and forth terms. □

The purpose of the following corollary is to combine the results of the previously obtained Lemmas with the energy estimates of the section 0.5.

**Corollary 2.** *There is $c > 0$ such that*

$$\left.\begin{array}{rcl} |J_1(t,\cdot)|_\infty + [J_1(t,\cdot)]_{\frac{1}{2}} & \leq & c\,(E(0)+Y(0))^{\frac{1}{2}}\,, \\ \int_0^T |J_i(t,\cdot)|_\infty + \langle J_i(t,\cdot)\rangle_{\frac{1}{4}}\,dt & \leq & c\,(E(0)+Y(0))\,, \end{array}\right\} T>0, \tag{50}$$

$i = 2..4$.

*Proof.* In the estimate (45) we set $q = 2$, $p = 6$ and $\gamma = 1/2$. Then, by the Lemma 1 we can write

$$|J_1|_\infty + [J_1]^{\frac{1}{2}} \leq c\,(|\sqrt{\rho}\mathbf{u}|_2 + |\nabla\mathbf{u}|_2)\,,$$

which by the energy estimates of Lemma 5 and Lemma 6 is not greater than $(E(0)+Y(0))^{\frac{1}{2}}$. To estimate $J_i$, $i = 2..4$, we first notice that by the Sobolev's Lemma 1 and elliptic estimate (15) we know that

$$|\mathbf{u}|_6 \leq c|\nabla\mathbf{u}|_2, \quad |\nabla\mathbf{u}|_6 \leq c\,(|\mathbf{F}_1|_2 + |\rho-\bar{\rho}|_6)\,.$$

Thus, with an appropriate choice of $\theta \in ]0,1[$, we derive that

$$|\mathbf{u}|_{13} \leq |\nabla|_{\frac{39}{16}} \leq |\nabla\mathbf{u}|_2^\theta \|\nabla\mathbf{u}|_6^{1-\theta} \leq c|\nabla\mathbf{u}|_2^\theta(|\mathbf{F}_1|_2 + |\rho-\bar{\rho}|_6)^{1-\theta}.$$

Also,

$$[\mathbf{u}]_{\frac{1}{4}} \leq c|\nabla\mathbf{u}|_4 \leq c|\nabla\mathbf{u}|_2^{\theta_1}|\nabla\mathbf{u}|_6^{1-\theta_1} \leq c|\nabla\mathbf{u}|_2^{\theta_1}(|\mathbf{F}_1|_2 + |\rho-\bar{\rho}|_6)^{1-\theta_1}.$$

It follows then, that (Lemma 6 and Lemma 6)

$$\int_0^t |\mathbf{u}|_{13}[\mathbf{u}]_{\frac{1}{4}} \leq c(M)\int_0^t (|\nabla\mathbf{u}|_2^2 + |\mathbf{F}_1|_2^2 + |\rho-\bar{\rho}|_2^2) \leq c(E(0)+Y(0)).$$

The above quantity is a bound for $\int_0^t |J_1|_\infty$ as can be seen from the estimate (46) where we set $q = 6$, $p = 13$ and $\alpha = 1/2$. The estimate on $\int_0^t \langle J_1\rangle_{\frac{1}{4}}$ is obtained in exactly the same way from the estimate (47) with $p = 13$, $\alpha = 1/2$ and $\beta = 1/4$. □

## 0.8 Hölder continuity of flow trajectories

In this section we consider the regularity of the flow generated by the velocity $\mathbf{u}$.

Let $X(t,x;T)$ denote the trajectory of flow, i.e.

$$\frac{d}{dt}X(t,x;T) = \mathbf{u}(T, X(t,x;T)), \quad X(T,x;T) = x,\, x \in \mathbb{R}^3_+. \tag{51}$$

We choose two points $x_1 \in \mathbb{R}^3_+$ and $x_2 \in \partial\mathbb{R}^3_+$ and consider trajectories $X(t,x_1;T)$ and $X(t,x_2;T)$ which we abbreviate to $X_1^t$ and $X_2^t$. Let $\mathbf{w}(t,x)$ be the vector field from the representation (39). We can write that (note, that $X_2^t = x_2$ and $\mathbf{w}(t,x_2)=0$, $\forall t>0$)

$$\begin{aligned}
\mathbf{w}(t,X_1^t)-\mathbf{w}(t,X_2^t) = D_t \int_{\mathbb{R}^3_+} &\left\{\mathbb{G}(X_1^t,y)-\mathbb{G}(X_2^t,y)\right\}\rho(t,y)\mathbf{u}(t,y) \\
&- \int_{\mathbb{R}^3_+} (\mathbf{u}(t,X_1^t)\cdot\nabla_x)\mathbb{G}(X_1^t,y)\rho(t,y)\mathbf{u}(t,y) \\
&+ \int_{\mathbb{R}^3_+} (\mathbf{u}(t,X_2^t)\cdot\nabla_x)\mathbb{G}(X_2^t,y)\rho(t,y)\mathbf{u}(t,y) \\
&\quad + \int_{\mathbb{R}^3_+} \left\{\mathbb{G}(X_1^t,y)-\mathbb{G}(X_2^t,y)\right\} \operatorname{div}\left(\rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,y)\right). \qquad (52)
\end{aligned}$$

We introduce a point

$$\xi_\lambda^t = \lambda X_1^t + (1-\lambda)X_2^t$$

and develop the above formula into another one (summation over repeated indexes is assumed).

$$\begin{aligned}
\mathbf{w}(t,X_1^t)-\mathbf{w}(t,X_2^t) &= D_t \left\{\int_{\mathbb{R}^3_+}\int_0^1 \partial_{x_i}\mathbb{G}(\xi_\lambda^t,y)\rho(t,y)\mathbf{u}(t,y)\,d\lambda\,dy\right\}(X_1^t-X_2^t)_i \\
&+ \int_{\mathbb{R}^3_+}\int_0^1 \partial_{x_i}\mathbb{G}(\xi_\lambda^t,y)\rho(t,y)\mathbf{u}(t,y)\,d\lambda\,dy(u_i(t,X_1^t)-u_i(t,X_2^t)) \\
&- \int_{\mathbb{R}^3_+} (\mathbf{u}(t,X_1^t)\cdot\nabla_x)\mathbb{G}(X_1^t,y)\rho(t,y)\mathbf{u}(t,y) \\
&+ \int_{\mathbb{R}^3_+} (\mathbf{u}(t,X_2^t)\cdot\nabla_x)\mathbb{G}(X_2^t,y)\rho(t,y)\mathbf{u}(t,y) \\
&- \int_{\mathbb{R}^3_+} \nabla_y\left\{\mathbb{G}(X_1^t,y)-\mathbb{G}(X_2^t,y)\right\} : \left(\rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,y)\right) \\
&= D_t \left\{\int_{\mathbb{R}^3_+}\int_0^1 \partial_{x_i}\mathbb{G}(\xi_\lambda^t,y)\rho(t,y)\mathbf{u}(t,y)\,d\lambda\,dy\right\}(X_1^t-X_2^t)_i \\
&- u_i(t,X_1^t)\int_{\mathbb{R}^3_+}\int_0^1 \left\{\partial_{x_i}\mathbb{G}(X_1^t,y)-\partial_{x_i}\mathbb{G}(\xi_\lambda^t,y)\right\}\rho(t,y)\mathbf{u}(t,y) \\
&+ u_i(t,X_2^t)\int_{\mathbb{R}^3_+}\int_0^1 \left\{\partial_{x_i}\mathbb{G}(X_2^t,y)-\partial_{x_i}\mathbb{G}(\xi_\lambda^t,y)\right\}\rho(t,y)\mathbf{u}(t,y) \\
&- \int_{\mathbb{R}^3_+} \nabla_y\left\{\mathbb{G}(X_1^t,y)-\mathbb{G}(X_2^t,y)\right\}\left(\rho(t,y)\mathbf{u}(t,y)\otimes\mathbf{u}(t,y)\right). \qquad (53)
\end{aligned}$$

Let us introduce a matrix $\mathbb{A}$ with elements

$$\mathbb{A}_{ij}(t) = \int_{\mathbb{R}^3_+}\int_0^1 \partial_{x_i}\mathbb{G}_{kj}(\xi_\lambda^t,y)\rho(t,y)u_k(t,y)\,d\lambda\,dy, \qquad (54)$$

and vector

$$\mathbf{B}(t) = -u_i(t, X_1^t) \int_{\mathbb{R}^3_+} \int_0^1 \left\{ \partial_{x_i} \mathbb{G}(X_1^t, y) - \partial_{x_i} \mathbb{G}(\xi_\lambda^t, y) \right\} \rho(t,y) \mathbf{u}(t,y)$$
$$+ u_i(t, X_2^t) \int_{\mathbb{R}^3_+} \int_0^1 \left\{ \partial_{x_i} \mathbb{G}(X_2^t, y) - \partial_{x_i} \mathbb{G}(\xi_\lambda^t, y) \right\} \rho(t,y) \mathbf{u}(t,y)$$
$$- \int_{\mathbb{R}^3_+} \nabla_y \left\{ \mathbb{G}(X_1^t, y) - \mathbb{G}(X_2^t, y) \right\} (\rho(t,y) \mathbf{u}(t,y) \otimes \mathbf{u}(t,y))$$
$$- \int_O (\nabla_y \mathbb{G}(X_1^t, y) - \nabla_y \mathbb{G}(X_2^t, y))(\rho(t,y) - \bar{\rho}). \quad (55)$$

We use formulas (53), (39) with notation (54), (55) in the equation (51) to write it as

$$D_t \left( X_1^t - X_2^t \right) = D_t \mathbb{A} \left( X_1^t - X_2^t \right) + \mathbf{B}. \quad (56)$$

In the next lemma we estimate $\mathbf{B}$ in terms of the difference $X_1^t - X_2^t$.

**Lemma 11.** *Let*

$$\ln^+ r = \max\{1, -\ln r\},\ r > 0.$$

*For any* $p > 1$, $\alpha \in ]0\,1[$, *there is a number* $c = c(p, \alpha) > 0$ *such that*

$$|\mathbf{B}| \le c a_1(t) |X_1^t - X_2^t| \ln^+ |X_1^t - X_2^t| + c a_2(t) |X_1^t - X_2^t|,$$

*where*

$$a_1 = M[\mathbf{u}(t,\cdot)]_{\frac{1}{2}}^2 + M[\mathbf{u}(t,\cdot)]_{\frac{1}{2}} |\mathbf{u}(t,\cdot)|_p + M|\mathbf{u}(t,\cdot)|_p^2$$

*and*

$$a_2 = |\rho(t,\cdot) - \bar{\rho}|_2 + \langle \rho(t,\cdot) \rangle_\alpha.$$

*Proof.* It is a well-known fact that the vector field

$$\int \nabla_y \frac{1}{|x-y|} \omega(y)\, dy$$

is $log - Lipschitz$ if $\omega(y)$ is bounded, see, for example, Lemma 8.1 of [9] for the precise statement. All terms, except the last one, in the representation of $\mathbf{B}$, (55), can be treated similarly to this term. The last term is treated exactly as in the Lemma 9. The details of the proof are left to the reader. □

We also state the following lemma, which proof is similar to the proof of (45) of Lemma 10.

**Lemma 12.** *For any* $p > 3$ *and* $q \in ]1, 3[$ *there is* $c > 0$ *such that*

$$|\mathbb{A}_{ij}(t)| \le c \left( \sup_{x \in \mathbb{R}^3_+} |\rho(t,\cdot)\mathbf{u}(t,\cdot)|_{p, B(x,2)} + |\rho(t,\cdot)\mathbf{u}(t,\cdot)|_q \right).$$

In the next corollary, we combine the estimates of the last two Lemmas with the energy estimates from the section 0.5.

**Corollary 3.** *There is a constant $c = c(\lambda, \mu, M) > 0$ such that*

$$\left.\begin{array}{lcl} |\mathbb{A}_{ij}(t)| & \leq & c\,(E(t) + Y(t))^{\frac{1}{2}}\,, \\ |\mathbf{B}(t)| & \leq & b_1(t)|X_1^t - X_2^t| \ln^+ |X_1^t - X_2^t| + b_2(t)|X_1^t - X_2^t|, \end{array}\right\} \tag{57}$$

*where*

$$b_1(t) = c\left(|\nabla \mathbf{u}(t,\cdot)|_2^2 + |\mathbf{F}_1(t,\cdot)|_2^2 + |\rho(t,\cdot) - \bar{\rho}|_2^2\right)$$

*and*

$$b_2 = c\,(|\rho(t,\cdot) - \bar{\rho})|_2 + \langle \rho(t,\cdot)\rangle_\alpha)\,.$$

Now, from the system of ODE's (56) we derive the next Lemma.

**Lemma 13.** *There is $c_1 > 0$ such, that if*

$$E(0) + Y(0) \leq c_1,$$

*then*

$$|X_1^t - X_2^t| \leq [2|x_1 - x_2|]^{e^{-2\int_t^T b_1(s)\,ds}} e^{2\int_t^T (b_1(s)+b_2(s))\,ds}, \quad t \in\, ]0,\, T[.$$

We introduce a non-increasing function of time

$$\alpha(t) = \alpha_0 e^{-2\int_0^t b_1(s)\,ds}. \tag{58}$$

Note, that by Corollary 3 and the Lemma 6 there is an $\breve{\alpha} > 0$, independent of time such, that

$$\alpha(t) > \breve{\alpha} > 0, \quad t > 0.$$

It can be deduced from the last Lemma that

$$\frac{|X_1^t - X_2^t|^{\alpha(t)}}{|x_1 - x_2|^{\alpha(T)}} \leq c e^{2\alpha_0 \int_t^T b_2(s)\,ds}, \quad t \in\, ]0,\, T[, \tag{59}$$

with an appropriate choice of $c$.

*Proof.* For the proof we only note that ODE's (56) can be written as a system of integral equations

$$X_1^t - X_2^t = e^{\mathbb{A}(t) - \mathbb{A}(T)}(x_1 - x_2) - e^{\mathbb{A}(t)} \int_t^T e^{-\mathbb{A}(s)} \mathbf{B}(s)\,ds,$$

for $t \in\, ]0,\, T[$. Then, by Corollary 3 and energy estimates from Lemmas (5) and (6) the initial data can be restricted in the norm $E(0) + Y(0)$ in such a way that

$$|e^{\mathbb{A}(t)}|^2 \leq 2, \quad t \in\, ]0,\, T[.$$

The Lemma is concluded by a Grönwall-type estimate. □

## 0.9 Uniform estimates on density

In this section we assume

**Hypothesis $\mathbb{H}_1$.** *For all* $(t,x)$,

$$\rho(t,x) > 0.1\bar{\rho} := m.$$

The final *a priori* estimate is contained in the

**Lemma 14.** *Assuming the conditions of Lemma 5 and Lemma 6, with* $\alpha(t)$ *defined in* (58), *there exist* $c_i = c_i(\lambda,\mu,\bar{\rho},\alpha_0)$, $i=1,2,$ *and* $c_0 = c_0(\alpha_0)$, $\alpha_0 \in \,]0,1/4[$, *such that*

$$\operatorname{osc}\rho(t,\cdot) + \langle\rho(t,\cdot)\rangle_{\alpha(t)} \le c_2(\operatorname{osc}\rho_0 + c\langle\rho_0\rangle_{\alpha_0}) + (E(0)+Y(0))^{\frac{1}{2}}, \tag{60}$$

*provided that*

$$E(0) + Y(0) + \operatorname{osc}\rho_0 + \langle\rho_0\rangle_{\alpha_0} < c_1$$

*and*

$$\frac{c_0\mu}{\lambda+3\mu} < 1,$$

*Proof.* In the estimates below we allow a generic constant $c$ depend on both $m$ and $M$ (lower and upper bound on $\rho$). Take $x_1 \in \mathbb{R}^3_+$, $x_2 \in \mathbb{R}^3_+$ and consider two flow trajectories $X(s,x_1,t)$, $X(s,x_2,t)$ with the initial data

$$X(t,x_1,t) = x_1, \quad X(t,x_2,t) = x_2.$$

We abbreviate them as $X_1^s$ and $X_2^s$. Let $\rho^i(\tau)$, $F^i$, $i=1,2$, be the restriction of $\rho(t,x)$ and $F(t,x)$ to the first and second trajectories. Set

$$\Delta\log\rho = \log\rho^1 - \log\rho^2, \quad \Delta\rho = \rho^1 - \rho^2$$

and

$$\Delta F = F^1 - F^2.$$

Generally, by $\Delta f$ we will refer to the difference of the values of the function $f(t,x)$ on these trajectories. One readily notices that

$$\Delta\rho = \tilde{\rho}(t)\Delta\log\rho$$

where $\tilde{\rho} \in Int[\rho^1,\rho^2]$. We set

$$\omega(t) = \int_0^t \frac{\tilde{\rho}}{\lambda+2\mu}\,dt.$$

It holds that

$$\omega(s) - \omega(t) \le \frac{m}{\lambda+2\mu}(s-t), \quad s<t.$$

Consider equation (1). It can be written in the following form,

$$D_t\Delta\log\rho + \frac{\tilde{\rho}}{\lambda+2\mu}\Delta\log\rho$$
$$= -\frac{1}{\lambda+2\mu}\Delta F = -\frac{1}{\lambda+2\mu}\left\{D_t\Delta J_1 + \sum_2^4 \Delta J_i + \frac{\alpha_2}{\alpha_3}\Delta P_1\right\}.$$

This equation is integrated to obtain an inequality

$$|\Delta\rho(t)| \le Me^{-\omega(t)}|\Delta\rho(0)| \\ + \frac{M}{\lambda+2\mu}\left|\int_0^t e^{\omega(s)-\omega(t)}\left\{D_s\Delta J_1 + \sum_2^4 \Delta J_i + \frac{\alpha_2}{\alpha_3}\Delta P_1\right\} ds\right|. \quad (61)$$

First, we obtain an estimate on $\operatorname{osc}\rho(t)$. Using Lemma 10 and Corollary 2 we can write

$$\left|\int_0^t e^{\omega(s)-\omega(t)} D_s\Delta J_1\, ds\right| \le c\sup_{[0,t]}|J_1(s,\cdot)|_\infty \le c(E(0)+Y(0))^{\frac{1}{2}}.$$

Also, by Lemma 10 and Corollary 2,

$$\left|\int_0^t e^{\omega(s)-\omega(t)}\Delta J_i\, ds\right| \le \int_0^t |J_i(s,\cdot)|_\infty\, ds \le c(E(0)+Y(0)).$$

Finally, by the estimate 41

$$\left|\int_0^t e^{\omega(s)-\omega(t)}\Delta P_1\, ds\right| \le \int_0^t e^{\omega(s)-\omega(t)}(c_\delta|\rho(s,\cdot)-\bar\rho|_2 + \delta\langle\rho(s,\cdot)\rangle_\alpha)\, ds \\ \le c_\delta Y(0)^{\frac{1}{2}} + \delta\int_0^t e^{\omega(s)-\omega(t)}\langle\rho(s,\cdot)\rangle_\alpha\, ds.$$

Using the last estimates in (61) and taking the maximum over all points $x_1$, $x_2$ we obtain that

$$\operatorname{osc}\rho(t,\cdot) \le c\operatorname{osc}\rho_0(\cdot) + c_\delta(E(0)+Y(0))^{\frac{1}{2}} + c(E(0)+Y(0)) \\ + \delta\int_0^t e^{-\frac{m}{\lambda+2\mu}(t-s)}\langle\rho(s,\cdot)\rangle_\alpha\, ds. \quad (62)$$

Now, we concentrate on derive the estimate on $\langle\rho\rangle_\alpha$. For that in the equation (61) we restrict $x_1 \in \partial\mathbb{R}^3_+$, $|x_1-x_2|<1$ and divide the equation by $|x_1-x_2|^{\alpha(t)}$, where $\alpha(t)$ was introduced in (58). Note, that

$$\alpha(t) \in ]\breve\alpha, \alpha_0[,\ \alpha_0 < 1/4.$$

We obtain

$$M^{-1}\frac{|\Delta\rho|}{|x_1-x_2|^{\alpha(t)}} \le e^{-\omega(t)}\frac{|\Delta\rho_0|}{|X_1^0-X_2^0|^{\alpha_0}}\frac{|X_1^0-X_2^0|^{\alpha_0}}{|x_1-x_2|^{\alpha(t)}} \\ + \frac{1}{\lambda+2\mu}\left|\int_0^t e^{-\omega(t)+\omega(s)}D_s\frac{\Delta J_1(s)}{|x_1-x_2|^{\alpha(t)}}\, ds \right. \\ \left. + \frac{1}{\lambda+2\mu}\int_0^t e^{-\omega(t)+\omega(s)}\frac{\Delta J_i(s)+\frac{\alpha_2}{\alpha_3}\Delta P_1(s)}{|X_1^s-X_2^s|^{\alpha(s)}}\frac{|X_1^s-X_2^s|^{\alpha(s)}}{|x_1-x_2|^{\alpha(t)}}\, ds\right|. \quad (63)$$

Some simplifications are in order. The ratio $\frac{|X_1^s-X_2^s|^{\alpha(s)}}{|x_1-x_2|^{\alpha(t)}}$ was estimated in LEMMA (13) by

$$e^{c\int_s^t b_1(s)+b_2(s)\, ds},$$

where $b_i(s)$ are given in Corollary 3. We also obtain from Corollary 3 and Lemma 6 that

$$-\omega(t)+\omega(s)+c\int_s^t b_1(s)+b_2(s)\, ds < -\frac{m}{\lambda+2\mu}(t-s) + c\int_s^t \langle\rho(z,\cdot)\rangle_{\alpha(z)} + \epsilon\, dz + c_\epsilon Y(0).$$

We make the following

**Hypothesis** $\mathbb{H}_2$**.** *Function* $\langle\rho(s,\cdot)\rangle_{\alpha(s)}$ *and number* $Y(0)$ *verify the next bounds*

$$\begin{array}{rcl} c(\sup_{s\in[0,t]}\langle\rho(s,\cdot)\rangle_{\alpha(s)}+\epsilon) & < & \frac{m}{2(\lambda+2\mu)}, \\ c_\epsilon Y(0) & \leq & 1, \end{array}$$

*for some value of* $\epsilon > 0$*. Here,* $c$ *and* $c_\epsilon$ *are numbers from the previous estimate.*

Thus we have estimated

$$e^{-\omega(t)+\omega(s)}\frac{|X_1^s - X_2^s|^{\alpha(s)}}{|x_1-x_2|^{\alpha(t)}} \leq Ce^{-\frac{m}{2(\lambda+2\mu)}(t-s)}, \quad s<t,$$

where $C$ *does not depend on* $\lambda,\mu,m$, $M$, $\bar\rho$ or $t$. It follows then, that the first term on the right-hand side in (63) is bounded by

$$\langle\rho_0\rangle_{\alpha_0} + \operatorname{osc}\rho_0.$$

The second, by (Corollary 2),

$$c(E(0)+Y(0))^{\frac{1}{2}}$$

The third, by (Corollary (2) and Lemma (9) and formulas for $\alpha_i$ from (27)),

$$\begin{aligned} c(E(0)+Y(0)) + \frac{C}{(\lambda+2\mu)}\frac{\alpha_2}{\alpha_3}\int_0^t e^{-\frac{m}{2(\lambda+2\mu)}(t-s)}\left(\operatorname{osc}\rho(s,\cdot)+\langle\rho(s,\cdot)\rangle_{\alpha(s)}\right)\,ds \\ \leq c(E(0)+Y(0)) + \frac{C}{m}\frac{\mu}{\lambda+3\mu}\sup_{s\in[0,t]}\left(\operatorname{osc}\rho(s,\cdot)+\langle\rho(s,\cdot)\rangle_{\alpha(s)}\right). \end{aligned} \quad (64)$$

Above, $C$ depends only on $\alpha_0=\alpha(0)$. An inequality then can be derived from (63):

$$\begin{aligned} \langle\rho(t,\cdot)\rangle_{\alpha(t)} \leq c\langle\rho_0\rangle_{\alpha_0} + c\operatorname{osc}\rho_0 + c(E(0)+Y(0))^{\frac{1}{2}} + c(E(0)+Y(0)) \\ + \frac{CM}{m}\frac{\mu}{\lambda+3\mu}\sup_{s\in[0,t]}\left(\operatorname{osc}\rho(s,\cdot)+\langle\rho(s,\cdot)\rangle_{\alpha(s)}\right). \end{aligned} \quad (65)$$

We consider the system of inequalities (62) and (65). It is straightforward to verify that if

$$\frac{CM}{m}\frac{\mu}{\lambda+3\mu} < 1,$$

then we can choose $E(0)+Y(0)+\operatorname{osc}\rho_0+\langle\rho_0\rangle_{\alpha_0}$ so small, depending on $\lambda,\mu,\bar\rho,\alpha_0$, that Hypotheses $\mathbb{H}_0$ – $\mathbb{H}_2$ hold and (60) is verified. □

## 0.10 Proof of the existence

Consider now the sequence of data of the problem

$$\rho_0^n,\, \mathbf{u}_0^n \in C^\infty\left(\mathbb{R}^3_+\right)\times C_0^\infty\left(\mathbb{R}^3_+\right)^3,$$

which approximates the given initial data in the space $L^6_{loc}(\mathbb{R}^3_+)\times L^6\left(\mathbb{R}^3_+\right)^3$. Moreover, we require that $M > \rho_0^n > m > 0$, and smallness condition (7) as required by analysis of the previous sections. Such a sequence, clearly exists. We can take $\rho_0^n(x) = (\rho_0(x)+n^{-1})*\omega_{n^{-1}}(x)$, $\mathbf{u}_0^n = (\mathbf{u}_0(x))*\omega_{n^{-1}}(x)$, where $\omega_\epsilon$ is the standard mollifier. Accordingly, let $\rho^n,\mathbf{u}^n$ be the sequence of smooth solutions of the problem with $\rho_0^n,\mathbf{u}_0^n$ as the initial data. The existence of such solutions follows from the local existence result [11] and *a priori* estimates we obtained in the previous sections. In particular, we established that the following norms are bounded with bounds independent of $n$.

$$\{\rho^n\} \quad \text{bounded in} \quad L^\infty\left(\mathbb{R}_+ \times \mathbb{R}^3_+\right), \tag{66}$$

$$\{\sqrt{\rho^n}\mathbf{u}^n\} \quad \text{bounded in} \quad L^\infty\left(\mathbb{R}_+ : L^2(\mathbb{R}^3_+)\right), \tag{67}$$

$$\{\nabla\mathbf{u}^n\} \quad \text{bounded in} \quad L^2\left(\mathbb{R}_+ \times \mathbb{R}^3_+\right), \tag{68}$$

$$\{D^2\mathbf{u}^n\} \quad \text{bounded in} \quad L^2\left(\mathbb{R}_+ \times \mathbb{R}^3_+\right). \tag{69}$$

By the weak stability result of P.-L. Lions, see Theorem 5.1 of [8], bounds (66)–(68) imply the existence of an accumulation point $(\rho, \mathbf{u})$ of the sequence $(\rho^n, \mathbf{u}^n)$ in the weak topology of $L^6_{\text{loc}}(\mathbb{R}^3_+) \times L^6(\mathbb{R}^3_+)$ which is a weak solution of (1)–(4). Moreover, the bounds in the spaces from (66)–(69) hold for this $(\rho, \mathbf{u})$.

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