STABILITY OF VISCOUS SHOCKS ON FINITE INTERVALS

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Abstract

Consider the Cauchy problem for a system of viscous conservation laws with a solution consisting of a thin, viscous shock layer connecting smooth regions. We expect the time dependent behavior of such a solution to involve two processes. One process consists of the large scale evolution of the solution. This process is well modeled by the corresponding inviscid equations. The other process is the adjustment in shape and position of the shock layer to the large scale solution. The time scale of the second process is much faster than the first, $1/\nu$ compared to 1. The second process can be divided into two parts, adjustment of the shape and of the position. During this adjustment the end states are essentially constant.

In order to answer the question of stability we have developed a technique where the two processes can be separated. To isolate the fast process, we consider the region in the vicinity of the shock layer. The equations are augmented with special boundary conditions which reflect the slow change of the end states. We show that, for the isolated fast process, the perturbations decay exponentially in time.

1 Introduction

Consider a system of viscous conservation laws,

$$u_\tau + f(u)_\xi = \nu u_{\xi\xi}, \quad \xi \in \mathbb{R}, \quad \tau \geq 0 \ . \tag{1}$$

Here $0 < \nu \ll 1$ denotes the viscosity, $u = u(\xi, \tau)$ is a vector function with n components and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth function of the state vector

u. We are interested in solutions consisting of a viscous shock layer which connects smooth regions. We expect the time dependent behavior of such a solution to involve two processes. One process consists of the large scale evolution of the solution. Diffusion acts weakly in this process, and it is well modeled by the corresponding inviscid equations. To first approximation this process determines the position of the shock via the Rankine-Hugoniot relations. The other process is the adjustment in shape and position of the shock layer due to perturbations in the shock layer itself. Here the diffusive effects are large. The time scale of the second process is much faster than the time scale of the first, $1/\nu$ to 1. On the fast time scale, the large scale of the solution is essentially constant.

In order to study the stability of the shock profile to *local* perturbations, we focus on the fast process and will not include effects of the slower large scale behavior or the effect of global perturbations. The focus on local perturbations is, among others, of interest from a numerical point of view because errors in numerical computations are mainly generated in the shock layer, and one wants to know if these numerical errors can cause the solution to change character.

This paper contains a discussion of three different models that can be used to understand stability of shock profiles. We present a new eigenvalue result for systems on bounded domains and a new non-linear stability result.

2 Stability of Traveling Waves

As a model, one has studied the stability of traveling wave solutions of the Cauchy problem

$$\begin{aligned} v_t + f(v)_x &= v_{xx}, \quad x \in \mathbb{R}, \quad t \geq 0 \ , \\ v(-\infty, t) &= U_L, \quad v(\infty, t) = U_R, \\ v(x, 0) &= v_0(x). \end{aligned} \tag{2}$$

See, for instance, [5] and [7, 8, 9, 2]. In (2) the standard scaling of space and time eliminates the parameter ν from the equation. Without loss of generality one can assume the wave speed to be zero.

We make the following standard assumptions:

Assumption 1: *There is a steady solution $U(x)$, i.e.,*

$$f(U(x))_x = U_{xx}(x) \ ,$$

where

$$\lim_{x\to\infty} U(x) = U_R, \quad \lim_{x\to-\infty} U(x) = U_L, \quad U_L \neq U_R \ .$$

To study the stability of $U(x)$ we consider (2) with initial data

$$v(x,0) = U(x) + u_0(x),$$

where u_0 is a small perturbation of U with compact support. Linearizing (2) at $U(x)$ gives

$$u_t + (A(x)u)_x = u_{xx}, \quad x \in \mathbb{R}, \quad u(\cdot,t) \in L_2. \tag{3}$$

Here $A(x) = J(U(x))$ where $J = \partial f/\partial u$ is the Jacobian of f. For the Jacobian evaluated at the end states we use the notations $A_R = J(U_R)$ and $A_L = J(U_L)$. The eigenvalue problem corresponding to (3) is

$$\lambda\phi + (A(x)\phi)_x = \phi_{xx}, \quad x \in \mathbb{R}, \quad \phi \in L_2. \tag{4}$$

Clearly, $s = 0, \phi = U_x$ is an eigensolution.

Assumption 2: *There is no non–trivial solution of (4) with Re$\,\lambda \geq 0$, $\lambda \neq 0$, and, for $\lambda = 0$, all eigenfunctions in L_2 are multiples of $\phi = U_x$.*

Assumption 3: *The matrices A_L and A_R have real, distinct, non–zero eigenvalues, and there are $n+1$ ingoing characteristics, i.e., the total number of positive eigenvalues of A_L and negative eigenvalues of A_R is $n+1$.*

Assumption 4: *The $n \times n$ matrix*

$$M = (U_L - U_R, S_R^{II}, S_L^{I}) \tag{5}$$

is nonsingular. Here the columns of S_L^I and S_R^{II} are the eigenvectors corresponding to outgoing characteristics to the left and to the right, respectively.

Remark 1: It follows that the convergence of $U(x)$ to the end states as $x \to \pm\infty$ is exponential. Therefore, we can use the representation

$$A(x) = A_R + e^{-\beta x} B_R(x), \qquad x \geq l_0, \tag{6}$$
$$A(x) = A_L + e^{-\beta |x|} B_L(x), \qquad x \leq -l_0, \tag{7}$$

with $\beta > 0$ for some sufficiently large l_0; here $B_{R,L}(x)$ are bounded matrix functions. In our analysis below we will assume that $l_0 > 0$ is a fixed constant for which $e^{-\beta l_0}$ is sufficiently small.

Remark 2: The eigenvalues of $A(x)$ are only required to be real at the limit states. This is useful when considering two phase flows, for example.

Remark 3: Assumptions 3 and 4 are standard also in an inviscid setting.

The first nonlinear stability result for shocks of arbitrary strength was given in [5] where stability to small zero–mass perturbations was established. The result is: [1]

Theorem 1: *If the initial data is of the form*

$$v(x,0) = U(x) + \varepsilon h_x, \quad \|h\|_1 + \|h\|_2 = 1,$$

then, for sufficiently small ε,

$$\lim_{t\to\infty} |v(\cdot,t) - U(\cdot)|_\infty = 0,$$

i.e., U *is non-linearly stable to small zero–mass perturbations.*

In [5] the question of stability was reduced to the study of estimates for the resolvent equation,

$$s\hat{u} + (A(x)\hat{u})_x = \hat{u}_{xx} + \hat{F}(x).$$

There are two difficulties in obtaining the necessary estimates. First, since the coefficients of (2) do not explicitly depend on x and t, the steady state solution $U(x)$ is not unique. Any shifted profile, $U(x+\alpha)$ with α fixed, is also a stationary solution of (2). Correspondingly, the function

$$U_x(x+\alpha)$$

satisfies

$$(A(x+\alpha)U_x(x+\alpha))_x = (U_x(x+\alpha))_{xx},$$

which means that U_x is an eigenfunction of the eigenvalue problem

$$s\phi + (A(x)\phi)_x = \phi_{xx}, \quad x \in \mathbb{R}, \quad \phi \in L_2(\mathbb{R}),$$

with eigenvalue $s = 0$. Thus, generally one cannot expect convergence of $v(x,t)$ to $U(x)$ as $t \to \infty$. Non–uniqueness of the steady solution does not have to be a problem. In [4] we treated this difficulty in a slightly different

[1] We use the notations $\|u\|_1 = \int_{-\infty}^{\infty} |u(x)|dx$, $\|u\|_2^2 = \int_{-\infty}^{\infty} |u(x)|^2 dx$, $|u|_\infty = \sup_x |u(x)|$.

setting. Instead of proving convergence to $U(x)$, we showed that $v(x,t)$ converges to a shifted profile, $U(x+\alpha)$. Technically, we make the ansatz

$$v(x,t) = U(x+\alpha(t)) + w(x,t)$$

and choose $\alpha(t)$ so that $w(x,t)$ never has a component in the direction of the eigenfunction, $U_x(x+\alpha(t))$. We will use the same technique here.

Second, for general systems, local perturbations of strong shocks will not remain local. Along the outgoing characteristics diffusion waves are formed. A typical behavior is given by

$$\frac{1}{\sqrt{x+t}}\, e^{-\frac{(x-t)^2}{t}}.$$

Perturbations of this type decay in time only like $1/\sqrt{t}$, and the L_2–norm of the perturbation over space and time is not finite. Thus the resolvent technique cannot be used to prove convergence. Also, through nonlinear interaction, signals are sent back to the shock layer via the ingoing characteristics. This slows down the convergence also in the shock layer.

In [5] we considered only zero–mass perturbations. Then the convergence rate is improved by a factor $1/\sqrt{t}$ and, as we have shown, the resolvent technique can be applied. Another advantage of the zero–mass assumption is that $v(x,t)$ converges to $U(x)$ without shift.

In this paper we are interested in the stability of the shock layer to *local* perturbations, without including interaction with the large scale variations of the solution. With the original scaling of time and space, the amplitude of the diffusion waves is of order $\mathcal{O}(\sqrt{\nu})$. Thus, the slowly decaying diffusion waves vary on time and space scales comparable to the large scale variations that we have already excluded from our study, but with a much smaller amplitude. The mathematical difficulties caused by the slow convergence of the diffusion waves are therefore irrelevant and we need to formulate a different model problem.

We analyze instead the stability of $U(x)$ as a solution of an initial–boundary value problem formulated on a finite interval, $|x| \le l$.

3 Stability on a Bounded Interval

The seemingly most natural choice is to use $U(\pm l)$ as boundary data at $x = \pm l$:

$$
\begin{aligned}
u_t + f(u)_x &= u_{xx}, \quad |x| \le l, \quad t \ge 0 \ , \\
u(x,0) &= U(x) + y_0(x), \quad |x| \le l \ , \\
u(\pm l, t) &= U(\pm l), \quad t > 0 \ .
\end{aligned} \tag{8}
$$

Later we will modify the boundary conditions by allowing a time–dependent shift of U in the boundary data. Clearly, $U(x)$ is a steady solution of (8).

In [3] we investigated the initial–boundary value problem for Burgers' equation,

$$
\begin{aligned}
u_\tau + (\frac{u^2}{2})_\xi &= = \nu u_{\xi\xi}, \quad -1 \le \xi \le 1, \quad \tau \ge 0, \qquad (9) \\
u(-1,\tau) &= 1, \quad u(1,\tau) = -1, \\
u(\xi, 0) &= u_0(\xi).
\end{aligned}
$$

Here $0 < \nu \ll 1$. There is the unique, anti-symmetric, steady solution, $U(\xi) = -U(-\xi)$, which is the restriction of a shock profile on the whole line with end states $U(-\infty) = 1 + \mathcal{O}(e^{-\frac{1}{\nu}})$ and $U(\infty) = -1 + \mathcal{O}(e^{-\frac{1}{\nu}})$.

In computations we observe the following: The solution rapidly approaches a shifted profile $U(\xi + \alpha)$, where α depends on the initial function. However, this is not the correct steady solution. In a second phase the solution converges exponentially slowly to the correct steady solution, $U(\xi)$.

There is no corresponding behavior in the infinite–line case or in the inviscid case; see Abarbanell et. al. [6]. In the inviscid case one considers

$$u_\tau + (\frac{u^2}{2})_\xi = 0, \quad -1 \le \xi \le 1.$$

Then the discontinuous function

$$U(\xi) = \begin{cases} 1, & \xi < 0, \\ -1, & \xi > 0 \end{cases}$$

is a steady solution with a shock at $\xi = 0$. With perturbed initial data,

$$u(\xi, 0) = U(\xi) + h(\xi),$$

the solution will in general converge to a shifted profile,

$$\lim_{\tau\to\infty} u(\xi,\tau) = U(\xi+\alpha).$$

By conservation,

$$\frac{d}{d\tau}\int_{-1}^{1} u d\xi = 0.$$

Therefore, the shift α is determined by

$$\alpha = \int_{-1}^{1} h(\xi)d\xi.$$

In the viscous case we have

$$\frac{d}{d\tau}\int_{-1}^{1} u d\xi = \nu(u_\xi(1,\tau) - u_\xi(-1,\tau)).$$

Thus, there is no conservation. The exponentially slow convergence in the viscous case can be explained by considering the corresponding eigenvalue problem

$$\lambda\phi + (U(\xi)\phi)_\xi = \nu\phi_{\xi\xi}, \quad -1 \le \xi \le 1, \quad \phi(\pm 1) = 0. \tag{10}$$

All eigenvalues λ_j are real and negative and there is one algebraically simple eigenvalue λ_0 and corresponding eigenfunction ϕ_0 with

$$\lambda_0 = \mathcal{O}(e^{-1/\nu}), \quad \phi_0(\xi) = U_\xi(\xi) + \mathcal{O}(e^{-\frac{1}{\nu}}).$$

All other eigenvalues satisfy

$$\lambda_j \le -\frac{c}{\nu}, \quad j = 1, 2, \ldots$$

where $c > 0$ is independent of ν. The perturbation equation, linearized at U, predicts what happens. After a short initial phase, the only part of the perturbation that remains is in the direction of ϕ_0. Such a perturbation corresponds to shifting the profile. This remaining perturbation will only decay exponentially slowly to zero, i.e., the shift will disappear only very slowly. Clearly, the small eigenvalue λ_0 controls the speed at which the shock layer moves to the correct position in the second phase.

For systems of equations, the linearized problem on the bounded domain reads

$$\begin{aligned} u_t + (Au)_x &= u_{xx}, \quad |x| \leq l, \quad t \geq 0\ , \\ u(x,0) &= y_0(x), \quad |x| \leq l\ , \\ u(\pm l, t) &= 0, \quad t > 0 \end{aligned} \tag{11}$$

and the corresponding eigenvalue problem is

$$s\varphi + (A\varphi)_x \;=\; \varphi_{xx}, \quad |x| \leq l, \quad \varphi(\pm l) = 0. \tag{12}$$

The result for Burgers' equation can be generalized to systems. In the next section we will prove Theorem 2 formulated below. It is assumed that l_0 is a fixed, sufficiently large, constant. More precisely, the term $e^{-\beta l_0}$ appearing in (6) and (7) is assumed to be small.

Theorem 2: *For sufficiently large l, the eigenvalue problem (12) has a simple, isolated eigenvalue s_0 with corresponding eigenfunction φ_0, satisfying*

$$s_0 = \mathcal{O}(e^{-\gamma(l-l_0)}), \quad |\varphi_0(x) - U_x(x)| \leq Ce^{-\gamma l}, \quad |x| \leq l. \tag{13}$$

Here $\gamma > 0$ is independent of l. Further, there is a constant $\delta > 0$ so that all other eigenvalues satisfy

$$Re\, s_j \leq -\delta, \quad j = 1, 2, \ldots$$

The constant δ is independent of l.

The theorem implies at best an exponentially slow approach to the steady profile. In fact, there is no reason for the exponentially small eigenvalue s_0 to have negative real part. Thus (8) can be mildly unstable even though (1) is stable.

For the linear problem (11) we can use an eigenfunction expansion to represent the solution. All components, except the one in the direction of φ_0, will converge to zero exponentially fast. Thus, after a short time we have

$$u(x,t) \approx U(x) + \alpha(t)U_x(x) \approx U(x + \alpha(t)),$$

which corresponds to the observed behavior in the case of Burgers' equation.

4 Analysis of the Eigenvalue Problem

In this section we will prove Theorem 2. The proof relies on comparing the finite–interval eigenvalue problem

$$s\bar{\varphi} + (A\bar{\varphi})_x = \bar{\varphi}_{xx} \quad |x| \le l, \quad \bar{\varphi}(\pm l) = 0 \,, \tag{14}$$

with the infinite–line eigenvalue problem,

$$s\varphi + (A(x)\varphi)_x = \varphi_{xx} \quad x \in \mathbb{R}, \quad \varphi \in L_2. \tag{15}$$

The main idea is to reformulate both eigenvalue problems, (14) and (15), as problems on a fixed interval, $-l_0 \le x \le l_0$, where $l \gg l_0 \gg 1$. First note that one can show, see Section 2 of [5], that there is a constant C_0 so that there are no eigenvalues s of (15) or of (14) with $|s| \ge C_0$, $\mathrm{Re}\, s \ge -1$. Therefore, we only need to consider $|s| \le C_0$.

A first step is to show that the infinite–line eigenvalue problem (15) can be reformulated as an eigenvalue problem on a bounded domain, $|x| \le l_0$, with non–standard boundary conditions:

$$\begin{aligned} s\varphi + (A(x)\varphi)_x &= \varphi_{xx}, \quad |x| \le l_0 \\ \varphi_x &= Q_R(s)\varphi \quad \text{at} \quad x = l_0 \\ \varphi_x &= Q_L(s)\varphi \quad \text{at} \quad x = -l_0 \end{aligned} \tag{16}$$

Here $Q_R(s)$ and $Q_L(s)$ are analytic matrix functions of s which, at first, are only defined for $\mathrm{Re}\, s \ge 0$, $s \ne 0$. We will show that the matrix functions $Q_{R,L}(s)$ can be extended analytically to all $|s| \le C_0$, $\mathrm{Re}\, s > -\delta$, for some positive δ. However, the problems (15) and (16) are equivalent only for $\mathrm{Re}\, s \ge 0$, $s \ne 0$, in the following sense: If (s, φ) solves (15), then the restriction of φ to $|x| \le l_0$ solves (16). Further, if (s, φ) solves (16) then there is a unique smooth extension of $\varphi(x)$ to all of $\mathbb{R}$ by solving $s\varphi + (A\varphi)_x = \varphi_{xx}$ for $|x| > l_0$, with $\varphi \in L_2(\mathbb{R})$. No equivalence, in this sense, generally holds for the problems (15) and (16) if $\mathrm{Re}\, s < 0$.

Also, the finite–interval eigenvalue problem (14) on $-l \le x \le l$ can be reformulated as an eigenvalue problem on $-l_0 \le x \le l_0$. In this case, the original problem (14) and the reduced problem are equivalent for all values of s. The only differences between the two reduced problems occurs in the boundary conditions, and we will show that this difference is exponentially small, i.e., of order $\mathcal{O}(e^{-\gamma(l-l_0)})$.

To carry out the reductions, we use the representations (6) and (7). By assumption, the matrix A_R has n distinct, real eigenvalues $\lambda_j \neq 0$, and there exists a nonsingular matrix S_R with

$$S_R^{-1} A_R S_R = \Lambda_R = \begin{pmatrix} -\Lambda_R^I & 0 \\ 0 & \Lambda_R^{II} \end{pmatrix}. \tag{17}$$

Here Λ_R^I and Λ_R^{II} are diagonal matrices with r and $n-r$ positive diagonal entries, respectively. Similarly, for $x \leq -l_0$, we have the representation (7), and there is a non–singular matrix S_L with

$$S_L^{-1} A_L S_L = \begin{pmatrix} -\Lambda_L^I & 0 \\ 0 & \Lambda_L^{II} \end{pmatrix}. \tag{18}$$

Here Λ_L^I and Λ_L^{II} are diagonal matrices with $r-1$ and $n+1-r$ positive diagonal entries, respectively.

Before giving a proof of Theorem 2, we note that if

$$A(x) = A_R, \quad x \geq l_0,$$

then for $x \geq l_0$ the components of $\tilde{\varphi} = (S_R)^{-1}\varphi$ solve uncoupled equations:

$$s\tilde{\varphi}_j + \lambda_j \tilde{\varphi}_{jx} = \tilde{\varphi}_{jxx}, \quad j = 1, \ldots, n. \tag{19}$$

For the solutions, κ_{1j} and κ_{2j}, of the characteristic equation

$$\kappa^2 - \lambda_j \kappa - s = 0 \tag{20}$$

the following holds. (See Lemma 4.1 in [5].)

Lemma 1. *The solutions of (20),*

$$\kappa_{1j} = \frac{\lambda_j}{2}\left(1 + \sqrt{1 + \frac{4s}{\lambda_j^2}}\right), \quad \kappa_{2j} = \frac{\lambda_j}{2}\left(1 - \sqrt{1 + \frac{4s}{\lambda_j^2}}\right), \tag{21}$$

satisfy:

1. *For* $Re\, s > -\frac{1}{4}\lambda_j^2$, *the functions depend analytically on* s.

2. $Re\, \kappa_{ij} \neq 0,\ i = 1, 2,$ *for* $Re\, s \geq 0,\ s \neq 0$.

3. $\kappa_{1j} = \lambda_j + \mathcal{O}(s), \quad \kappa_{2j} = -\frac{s}{\lambda_j} + \frac{s^2}{\lambda_j^3} + \mathcal{O}(s^3)$ *for* $|s| \ll 1$.

4. *sign(Re* κ_{1j}*)* $=$*sign* λ_j, *sign(Re* κ_{2j}*)* $= -$*sign* λ_j, *for* $Re\, s \geq 0,\ s \neq 0$.

5. *For every* $\delta > 0$ *there is a constant* $\rho > 0$ *so that for all* s *with* $|s| \geq \delta,\ Re\, s \geq 0$,

   (a) $|Re\ \kappa_{ij}| \geq \rho,\ i = 1, 2.$

   (b) $|(\kappa_{ij} - \kappa_{kl})| \geq \rho \quad if \quad (i,j) \neq (k,l).$

Proof: Property 1 is clear if we define the square root by the usual analytic continuation. Next, assume that $\kappa_{kj} = i\tau, \tau$ real, $\tau \neq 0$. By (20),

$$-\tau^2 - i\tau\lambda_j = s, \quad \text{i.e.,} \quad \mathrm{Re}\, s < 0.$$

This proves Property 2. Property 3 follows directly from (21). Property 4 follows from Property 3 for small $|s|$ and then from Properties 1 and 2 for all s with $\mathrm{Re}\, s \geq 0,\ s \neq 0$. For $|s| >> 1$, Property 5 follows directly from (21). Since Re $\kappa_{ij} \neq 0$, we can use a compactness argument to prove 5a. The formula $\kappa_{1j} - \kappa_{2j} = \lambda_j\sqrt{1 + \frac{4s}{\lambda_j^2}}$ shows that 5b holds if $j = l$. If κ solves

$$\kappa^2 - \lambda_j\kappa - s = 0, \quad \kappa^2 - \lambda_l\kappa - s = 0, \quad j \neq l,$$

then

$$(\lambda_j - \lambda_l)\ \mathrm{Re}\ \kappa = 0, \quad \text{i.e.,} \quad \mathrm{Re}\ \kappa = 0,$$

which violates Property 2. This proves the lemma.

The general solution of (19) is

$$\tilde{\varphi}_j(x) = \sigma_1 e^{\kappa_{1j}(x-l_0)} + \sigma_2 e^{\kappa_{2j}(x-l_0)}.$$

Here κ_{1j} and κ_{2j} are given by (21). Note that $\kappa_{ij} = \kappa_{ij}(s)$. For ease of notation we will in general not write out the s-dependence. For $\mathrm{Re}\, s \geq 0,\ s \neq 0$, the requirement $\tilde{\varphi}_j \in L_2(l_0, \infty)$ yields the condition

$$\sigma_1 = 0 \quad \text{if} \quad \lambda_j > 0, \quad \sigma_2 = 0 \quad \text{if} \quad \lambda_j < 0\ .$$

Equivalently,

$$\tilde{\varphi}_{jx}(l_0) = \begin{cases} \kappa_{2j}\tilde{\varphi}_j(l_0) & \text{if} \quad \lambda_j > 0, \\ \kappa_{1j}\tilde{\varphi}_j(l_0) & \text{if} \quad \lambda_j < 0. \end{cases} \tag{22}$$

In terms of the original variable $\varphi = S_R\tilde{\varphi}$, we obtain from (22):

$$\varphi_x(l_0) = S_R\tilde{Q}_R(s)S_R^{-1}\varphi(l_0). \tag{23}$$

Here $\tilde{Q}_R(s)$ is the diagonal matrix with entries $\kappa_{1j}(s), 1 \leq j \leq r$, and $\kappa_{2j}(s), r+1 \leq j \leq n$, according to (22). We note that

$$\tilde{Q}_R(s) = \begin{pmatrix} -\Lambda_R^I & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(s).$$

If $A(x) = A_R$ for $x \geq l_0$, then the matrix $Q_R(s) = S_R\tilde{Q}_R(s)S_R^{-1}$ is the coefficient matrix in the boundary condition at $x = l_0$ for the reduced problem, (16). Note that, since the coefficient matrix depends on s, this is a non–standard boundary condition.

4.1 Reduction of the All–Line Problem

The roots, κ_{1j} and κ_{2j}, of the characteristic equation (20) are important also in the general case. Consider the half–line $x \geq l_0$ and use the representation

$$A(x) = A_R + \varepsilon e^{-\beta(x-l_0)}B_R(x) \quad \text{for } x \geq l_0, \quad \varepsilon = e^{-\beta l_0}\ . \tag{24}$$

To simplify notation, we assume that A_R is diagonal with elements λ_j. To begin with we consider only $\operatorname{Re} s \geq 0$, $C_0 \geq |s| > 0$.

Introducing the notation

$$\theta = A\varphi - \varphi_x, \tag{25}$$

we can write the equation $s\varphi + (A\varphi)_x = \varphi_{xx}$ as a first–order system:

$$\begin{pmatrix} \varphi \\ \theta \end{pmatrix}_x = \begin{pmatrix} A & -I \\ -sI & 0 \end{pmatrix}\begin{pmatrix} \varphi \\ \theta \end{pmatrix}. \tag{26}$$

In the simplified case, with $A(x) = \Lambda_R$, (26) decouples into 2×2 systems

$$\begin{pmatrix} \varphi_j \\ \theta_j \end{pmatrix}_x = \begin{pmatrix} \lambda_j & -1 \\ -s & 0 \end{pmatrix}\begin{pmatrix} \varphi_j \\ \theta_j \end{pmatrix}. \tag{27}$$

The eigenvalues of the matrix on the right side are κ_{1j} and κ_{2j}, and the system (27) is diagonalized by the transformation

$$\begin{pmatrix} \varphi_j \\ \theta_j \end{pmatrix} = S_j \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix}, \quad S_j = \begin{pmatrix} 1 & -\frac{1}{s}\kappa_{2j} \\ -\frac{s}{\kappa_{1j}} & 1 \end{pmatrix}. \tag{28}$$

Here

$$S_j^{-1} = \frac{1}{\kappa_{1j} - \kappa_{2j}} \begin{pmatrix} \kappa_{1j} & \frac{1}{s}\kappa_{1j}\kappa_{2j} \\ s & \kappa_{1j} \end{pmatrix}. \tag{29}$$

Note that S_j and S_j^{-1} are analytic functions of s, and can be extended analytically to all s with $\operatorname{Re} s > -\lambda_j^2/4$. The diagonalized system is

$$\begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix}_x = \begin{pmatrix} \kappa_{1j} & 0 \\ 0 & \kappa_{2j} \end{pmatrix} \begin{pmatrix} \phi_j \\ \psi_j \end{pmatrix}. \tag{30}$$

If we require the solution to belong to $L_2(l_0, \infty)$, there can be no growing components, and the solution must satisfy

$$\phi_j(l_0) = 0 \text{ if } \lambda_j > 0 \ , \quad \psi_j(l_0) = 0 \text{ if } \lambda_j < 0. \tag{31}$$

This is the boundary condition at $x = l_0$ for the reduced problem. By inverting the transformations (28) and (25) we recover (23).

In the general case, i.e., for $\varepsilon \neq 0$, the transformation (28) will yield a coupled system. Let the components of φ be ordered so that $\varphi^I = (\varphi_1, \ldots, \varphi_r)^T$ corresponds to eigenvalues $\lambda_j < 0$ and $\varphi^{II} = (\varphi_{r+1}, \ldots \varphi_n)^T$ corresponds to $\lambda_j > 0$. We partition the vectors ϕ and ψ correspondingly, i.e., we set

$$\phi^I = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_r \end{pmatrix}, \ \phi^{II} = \begin{pmatrix} \phi_{r+1} \\ \vdots \\ \phi_n \end{pmatrix}, \ \psi^I = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_r \end{pmatrix}, \ \psi^{II} = \begin{pmatrix} \psi_{r+1} \\ \vdots \\ \psi_n \end{pmatrix}.$$

Finally, let

$$\Phi = \begin{pmatrix} \phi \\ \psi \end{pmatrix}$$

and apply (28) to the components of the first–order system (26), yielding

$$\Phi_x = \left(\mathcal{K} + \varepsilon e^{-\beta(x-l_0)} B(x)\right) \Phi, \quad x \geq l_0. \tag{32}$$

Here

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_1 & 0 \\ 0 & \mathcal{K}_2 \end{pmatrix}, \quad \mathcal{K}_i = \begin{pmatrix} \mathcal{K}_i^I & 0 \\ 0 & \mathcal{K}_i^{II} \end{pmatrix} = \begin{pmatrix} \kappa_{i1} & & \\ & \ddots & \\ & & \kappa_{in} \end{pmatrix},$$

is diagonal, and we have partitioned according to sign of real part. By Lemma 1 we have that for $\operatorname{Re} s \geq 0$, $s \neq 0$, the elements of $\mathcal{K}_1^I$ and $\mathcal{K}_2^{II}$ have negative real parts, while the elements of $\mathcal{K}_1^{II}$ and $\mathcal{K}_2^{I}$ have positive real parts. The matrices $\mathcal{K}_i^I$ and $\mathcal{K}_i^{II}$ are of size $r \times r$ and $(n-r) \times (n-r)$, respectively. Furthermore,

$$B(x) := S^{-1} \begin{pmatrix} B_R(x) & 0 \\ 0 & 0 \end{pmatrix} S$$

where the matrix S collects the transformations S_j.

For $C_0 \geq |s| \geq c_0 > 0$, $\mathrm{Re}\, s \geq 0$, the diagonal elements of $\mathcal{K}$ are distinct with real parts bounded away from zero. For sufficiently large l_0, the value $\varepsilon = e^{-\beta l_0}$ is small compared with c_0, and the system (32) can be diagonalized by a transformation, analytic in s, which is an $\mathcal{O}(\varepsilon e^{-\beta(x-l_0)})$ perturbation of the identity. The diagonalization separates growing components from decaying components. It is easy to show that the resulting boundary conditions at $x = l_0$ are $\mathcal{O}(\varepsilon)$ perturbations of the boundary conditions in the simplified case ($\varepsilon = 0$) and that they can be analytically extended to $\mathrm{Re}\, s \geq -\delta_0$, for some $\delta_0 > 0$.

For $|s| \leq c_0 \ll 1$ we have

$$\begin{array}{ll} \mathcal{K}_1^I = -\Lambda_R^I + \mathcal{O}(s), & \mathcal{K}_1^{II} = \Lambda_R^{II} + \mathcal{O}(s), \\ \mathcal{K}_2^I = s(\Lambda_R^I)^{-1} + \mathcal{O}(s^2), & \mathcal{K}_2^{II} = -s(\Lambda_R^{II})^{-1} + \mathcal{O}(s^2). \end{array} \tag{33}$$

The analysis in this case is more delicate since the elements of $\mathcal{K}_2$ are no longer uniformly separated. Consider the first–order system

$$\Phi_x = \left(\mathcal{K} + \varepsilon e^{-\beta(x-l_0)} B(x)\right) \Phi, \quad x \geq l_0, \quad \Phi \in L_2(l_0, \infty), \tag{34}$$

$$\phi^I(l_0) = y_1, \quad \psi^{II}(l_0) = y_2, \tag{35}$$

for $\mathrm{Re}\, s > 0$, $|s| \leq c_0$. By Lemma B3, the system (34) with side condition (35) has a unique bounded solution, and (34) determines a linear mapping

$$\begin{pmatrix} \phi^{II}(l_0) \\ \psi^I(l_0) \end{pmatrix} = \varepsilon Q(s) \begin{pmatrix} \phi^I(l_0) \\ \psi^{II}(l_0) \end{pmatrix}. \tag{36}$$

The equation (36) is the boundary condition at $x = l_0$ for the all–line problem reduced to $[-l_0, l_0]$. The matrix $Q(s)$ depends analytically on s and can be extended to a full neighborhood of $s = 0$ using analytic continuation; see Lemma B3. Also, note that for $\varepsilon = 0$ condition (36) reduces to (31).

Similar considerations yield a boundary condition at $x = -l_0$. Thus this procedure defines the reduction to $-l_0 \leq x \leq l_0$ of the all-line eigenvalue problem (16) for all s with $\mathrm{Re}\, s \geq -\delta$, $|s| \leq C_0$ for some $\delta > 0$. In the following theorem we relate the resolvent result from [5] to the reduced all-line eigenvalue problem. The proof is given in Appendix A.

Theorem 3: *The problem (16), which is the reduction of the all–line eigenvalue problem (15) to $|x| \leq l_0$, has no eigenvalue s with $Re\, s \geq 0$, $s \neq 0$. There is an algebraically simple, eigenvalue at $s = 0$.*

Remark 4: *The statement "$s = 0$ is an algebraically simple eigenvalue" is defined in the following way. To determine which s are eigenvalues we calculate the general solution of the differential equation and determine whether there is a nontrivial solution satisfying the boundary conditions or not. This leads to a linear system of equations, $B(s)\sigma = 0$. There is a nontrivial solution if and only if the determinant $D(s) := \det B(s)$ satisfies $D(s) = 0$. In our case, $B(s)$ and $D(s)$ are analytic functions for $Re\, s > -\delta$. We define $s = 0$ as an algebraically simple eigenvalue if $D(0) = 0$ and $D'(0) \neq 0$.*

An immediate consequence is:

Theorem 4: *There is a constant $\delta > 0$ so that the eigenvalue problem (16) has no eigenvalue other than $s = 0$ for $Re\, s \geq -\delta$.*

Proof: As already noted, there are no eigenvalues with $|s| \geq C_0$, $\mathrm{Re}\, s \geq -1$. From Theorem 3 we have $D'(0) \neq 0$, and it follows that there exists a δ_1 such that $s = 0$ is the only eigenvalue with $|s| \leq \delta_1$. For $\delta_1 \leq |s| \leq C_0$, $\mathrm{Re}\, s \geq 0$ it follows from Assumption 2 that $D(s) \neq 0$. Since D is a continuous function of s there is a $\delta_2 > 0$ such that $D(s) \neq 0$ when $\mathrm{Re}\, s \geq -\delta_2$. This completes the proof.

4.2 Reduction of the Bounded–Interval Problem

We will compare the reduced infinite–line eigenvalue problem (16) with the bounded–interval eigenvalue problem (14) reduced to the interval $[-l_0, l_0]$. Here $1 \ll l_0 \ll l$. The only difference between the two reduced problems occurs in the boundary conditions at $x = \pm l_0$.

First we consider the simplified case with $\varepsilon = 0$, i.e, with $A(x) = A_R$ for $x \geq l_0$. The boundary conditions at $x = l_0$ for the reduced infinite–line eigenvalue problem are given by (31) (or, equivalently, by (23)). We will now reduce problem (14) to $|x| \leq l_0$ and show that the resulting boundary conditions at $x = l_0$ are exponentially small (as a function of $l - l_0$) perturbations of (31). Consider (30) on $l_0 \leq x \leq l$, but call the variables $\bar{\phi}_j$ and $\bar{\psi}_j$ instead of ϕ_j and ψ_j. The solution for $l_0 \leq x \leq l$ will consist of growing and decaying components, and the relation between the components is determined by the boundary condition $\bar{\varphi}_j(l) = 0$. In terms of the new variables, $\bar{\phi}_j$ and $\bar{\psi}_j$, this condition reads

$$\bar{\phi}_j(l) - \frac{\kappa_{2j}}{s}\bar{\psi}_i(l) = 0. \tag{37}$$

Since $\bar{\phi}_{jx} = \kappa_{1j}\bar{\phi}_j$ and $\bar{\psi}_{jx} = \kappa_{2j}\bar{\psi}_j$ equation (37) is equivalent to the following condition at $x = l_0$:

$$e^{\kappa_{1j}(l-l_0)}\bar{\phi}_j(l_0) - \frac{\kappa_{2j}}{s}e^{\kappa_{2j}(l-l_0)}\bar{\psi}_j(l_0) = 0. \tag{38}$$

By (21),

$$\kappa_{1j} - \kappa_{2j} = \begin{cases} +\sqrt{\lambda_j^2 + 4s} & \text{if } \lambda_j > 0, \\ -\sqrt{\lambda_j^2 + 4s} & \text{if } \lambda_j < 0. \end{cases}$$

Therefore we write (38) as

$$\bar{\phi}_j(l_0) = \frac{\kappa_{2j}}{s}e^{-(\kappa_{1j}-\kappa_{2j})(l-l_0)}\bar{\psi}_j(l_0) \quad \text{if } \lambda_j > 0, \tag{39}$$

$$\bar{\psi}_j(l_0) = \frac{s}{\kappa_{2j}}e^{-(\kappa_{2j}-\kappa_{1j})(l-l_0)}\bar{\phi}_j(l_0) \quad \text{if } \lambda_j < 0. \tag{40}$$

It follows that, as long as $\operatorname{Re} s \geq -\frac{1}{8}\min\lambda_j^2$, (40) and (39) is a perturbation of order $\mathcal{O}(e^{-\gamma(l-l_0)})$ of (31), with $\gamma = \min_j|\lambda_j|/\sqrt{2}$ independent of s. This result is a consequence of the fact that (37) couples one growing (with x) and one decaying component. It is important that, even when $|s|$ is small, at least one of the exponential rates is essentially independent of s.

For the general case, i.e., for $\varepsilon \neq 0$ in (24), we again only need to consider the case $|s| \leq C_0$. When discussing (14) we introduce $\bar{\theta}$, $\bar{\phi}$, $\bar{\psi}$ and $\bar{\Phi}$, defined correspondingly to θ, ϕ, ψ and Φ. Corresponding to (37), the boundary condition $\bar{\varphi}(l) = 0$ in terms of $\bar{\phi}$ and $\bar{\psi}$ is

$$\bar{\phi}(l) - \frac{1}{s}\mathcal{K}_2\bar{\psi}(l) = 0,$$

or, equivalently,

$$\begin{pmatrix} \bar{\phi}^{II}(l) \\ \bar{\psi}^{I}(l) \end{pmatrix} = D(s)\begin{pmatrix} \bar{\psi}^{II}(l) \\ \bar{\phi}^{I}(l) \end{pmatrix}, \quad D(s) = \begin{pmatrix} \frac{1}{s}\mathcal{K}_2^{II} & \\ & s(\mathcal{K}_2^{I})^{-1} \end{pmatrix} \tag{41}$$

For $\operatorname{Re} s \geq -\min\lambda_j^2/8$, $|s| \leq C_0$, D is a diagonal, analytic matrix function of s. We will apply Lemma B4 with

$$u^I = \begin{pmatrix} \bar{\phi}^I \\ \bar{\psi}^{II} \end{pmatrix}, \; u^{II} = \begin{pmatrix} \bar{\psi}^I \\ \bar{\phi}^{II} \end{pmatrix}, \; \Lambda^I = \begin{pmatrix} -\mathcal{K}_1^I & \\ & -\mathcal{K}_2^{II} \end{pmatrix}, \; \Lambda^{II} = \begin{pmatrix} \mathcal{K}_2^I & \\ & \mathcal{K}_1^{II} \end{pmatrix}.$$

The result is that for all $\mathrm{Re}\, s \geq 0$, $C_0 \geq |s| > 0$, there is a unique bounded solution $\bar{\Phi}(x, s)$ of

$$\begin{aligned} \bar{\Phi}_x &= \left(\mathcal{K} + \varepsilon e^{-\beta(x-l_0)} B(x)\right) \bar{\Phi}, \quad l_0 \leq x \leq l, & (42) \\ & \bar{\phi}^I(l_0) = y_1,\ \bar{\psi}^{II}(l_0) = y_2. & (43) \end{aligned}$$

augmented with (41).

The boundary value problem (41) – (43) determines a mapping

$$\begin{pmatrix} \bar{\phi}^{II}(l_0) \\ \bar{\psi}^I(l_0) \end{pmatrix} = \varepsilon \bar{Q}(s) \begin{pmatrix} \bar{\phi}^I(l_0) \\ \bar{\psi}^{II}(l_0) \end{pmatrix}, \tag{44}$$

where $\bar{Q}(s)$ is analytic and can be extended to all s with $\mathrm{Re}\, s \geq -\delta$, $|s| \leq C_0$, by analytic continuation.

In the all–line case, the mapping (36) is the boundary condition at $x = l_0$ for the all–line problem reduced to $[-l_0, l_0]$. In order to compare (36) with (44) in the case $|s| \leq c_0$ we first use Lemma B3 for $\mathrm{Re}\, s > 0$ to replace (34), (35) by a problem on $[l_0, l]$:

$$\begin{aligned} \Phi_x &= \left(\mathcal{K} + \varepsilon e^{-\beta(x-l_0)} B(x)\right) \Phi, \quad l_0 \leq x \leq l, & (45) \\ \phi^I(l_0) &= y_1,\ \psi^{II}(l_0) = y_2, \quad \begin{pmatrix} \phi^{II}(l) \\ \psi^I(l) \end{pmatrix} = \varepsilon e^{-\beta(l-l_0)} P(s) \begin{pmatrix} \phi^I(l) \\ \psi^{II}(l) \end{pmatrix}, & (46) \end{aligned}$$

Here $P(s)$ is analytic and can be extended to a full neighborhood of $s = 0$ using analytic continuation.

The following lemma states that Q and $\bar{Q}$ differ from each other only by exponentially small terms. The result is valid in a full neighborhood of $s = 0$.

Lemma 2: *Let $\bar{\Phi}(x, s)$ and $\Phi(x, s)$ be the solutions of (41)–(43) and (45), (46), respectively. There is a $\delta > 0$ such that for $|s| \leq \delta$ we have*

$$|\Phi(l_0) - \bar{\Phi}(l_0)| \leq c\varepsilon e^{-\eta(l-l_0)}(|y_1| + |y_2|).$$

Here η and c are positive constants, independent of s and l.

Proof: Introduce $\delta\Phi = \Phi - \bar{\Phi}$ and the corresponding $\delta\phi$ and $\delta\psi$. Then we obtain

$$\begin{aligned} \delta\Phi_x &= \left(\mathcal{K} + \varepsilon e^{-\beta(x-l_0)} B(x)\right) \delta\Phi, \\ \begin{pmatrix} \delta\phi^I(l_0) \\ \delta\psi^{II}(l_0) \end{pmatrix} &= 0, \\ \begin{pmatrix} \delta\phi^{II}(l) \\ \delta\psi^I(l) \end{pmatrix} &= D \begin{pmatrix} \delta\psi^{II}(l) \\ \delta\phi^I(l) \end{pmatrix} + \varepsilon e^{-\beta(l-l_0)} P(s) \begin{pmatrix} \phi^I(l) \\ \psi^{II}(l) \end{pmatrix} - D \begin{pmatrix} \psi^{II}(l) \\ \phi^I(l) \end{pmatrix}. \end{aligned}$$

We need to show that $\delta\phi^{II}(l_0)$ and $\delta\psi^{I}(l_0)$ are exponentially small. To begin with we shall estimate the inhomogeneous terms in the boundary condition at $x = l$. For $\varepsilon = 0$ we have

$$\phi^I(l) = e^{\mathcal{K}_1^I(l-l_0)}y_1, \quad \psi^{II}(l) = e^{\mathcal{K}_2^{II}(l-l_0)}.$$

If $|s| \le c_0$ where c_0 is sufficiently small, then $|\mathrm{Re}\,\kappa_{2j}| \le 2c_0/|\lambda_j| < \beta/4$ and $|\mathrm{Re}\,\kappa_{1j}| \ge \gamma$. (Here, as before, $\gamma = \min|\lambda_j|/\sqrt{2}$.) Note that the elements of $\mathcal{K}_1^I$ have negative real parts. Therefore,

$$|e^{-\frac{\beta}{4}(l-l_0)}\psi^{II}(l)| \le |y_2|, \quad |\phi^I(l)| \le e^{-\gamma(l-l_0]}|y_1|.$$

If $\varepsilon \ne 0$ we can apply Lemma B3 with $\delta = \beta/4$ to (34), (35) and obtain

$$|e^{-\frac{\beta}{4}(l-l_0)}\psi^{II}(l)| \le |y_2| + 2\varepsilon\frac{|B|_\infty}{\beta}(|y_1| + |y_2|). \tag{47}$$

To estimate the components of $\phi^I(l)$ for $\varepsilon \ne 0$, we apply Lemma B2 to (34), (35). It follows that there is a constant c such that

$$|\phi^I(l)| \le e^{-\gamma(l-l_0)}|y_1| + c\,e^{-\frac{\tilde\beta}{2}(l-l_0)}\varepsilon|B|_\infty(|y_1| + |y_2|), \quad \tilde\beta = \min(\beta, |\lambda_j|). \tag{48}$$

Thus all the inhomogeneous terms in the boundary condition at $x = l$ are exponentially small, except for $D^I\psi^{II}(l)$. Here D^I is the upper diagonal block of D. In the case $\varepsilon = 0$ the effect of this term is

$$\delta\phi^{II}(x) = e^{\mathcal{K}_1^{II}(x-l)}D^I\psi^{II}(l).$$

Clearly, the corresponding part of the solution decays rapidly away from the boundary at $x = l$ and is exponentially small at $x = l_0$. If $\varepsilon \ne 0$ Lemma B5 yields a similar estimate. By Lemma B4 the effect of the remaining terms are estimated, completing the proof of the lemma.

If $C_0 \ge |s| \ge c_0 > 0$, $\mathrm{Re}\,s \ge 0$, the elements of $\mathcal{K}$ are distinct with real parts bounded away from zero. The full system can be diagonalized by a transformation, analytic in s, which is an $\mathcal{O}(\varepsilon e^{-\beta(x-l_0)})$ perturbation of the identity. It is easy to show that the resulting boundary conditions for the bounded–interval and for the infinite–line problems differ by $\mathcal{O}(\varepsilon e^{-\beta(l-l_0)})$, and that the result can be extended to $\mathrm{Re}\,s \ge -\delta_0$ for some $\delta_0 > 0$.

Proof of Theorem 2. We only have to consider $|s| \le C_0$. We have shown above that the bounded–interval and the all–line eigenvalue problems, ((15)

and (14)), reduced to $[-l_0, l_0]$ differ only by exponentially small terms in the boundary conditions at $x = \pm l_0$. The derivation of the boundary conditions was done for $|s| \le C$, $\mathrm{Re}\, s \ge 0$, $s \ne 0$, but the coefficient matrices are analytic in s, and can be extended to $\mathrm{Re}\, s > -\delta$ for some $\delta > 0$ by analytic continuation. Correspondingly, for $\mathrm{Re}\, s > -\delta$, $|s| \le C_0$, the two corresponding determinants, $D(s)$ and $\bar{D}(s)$, are analytic functions, and $\bar{D}$ is an exponentially small perturbation of D. The eigenvalues in the two cases are the zeros $D(s)$ and $\bar{D}(s)$, respectively. By Theorem 4, $s = 0$ is a simple root of $D(s) = 0$, and the only root with $\mathrm{Re}\, s \ge -\tilde{\delta}$ for some $\tilde{\delta} > 0$. Since $\bar{D}$ is an exponentially small perturbation of D, the eigenvalue result follows by a continuity argument.

For $|x| \le l_0$ the estimate for the eigenfunction follows by the same argument. We must also discuss the eigenfunction for $l_0 \le |x| \le l$. For $l_0 \le x \le l$, introduce $\bar{\varphi}_0 - \varphi_0 = \tilde{\varphi}$, and obtain

$$\bar{s}_0 \tilde{\varphi} + (A\tilde{\varphi})_x = \tilde{\varphi}_{xx} + \bar{s}_0 \varphi_0, \quad l_0 \le x \le l.$$

Note that the difference in eigenvalues and in eigenfunctions in $[-l_0, l_0]$ is $\mathcal{O}(\varepsilon e^{-\eta(l-l_0)})$. Also, $\bar{\varphi}_0(l) = 0$, and $\varphi_0(x)$ decays exponentially. This yields that

$$|\tilde{\varphi}(l_0)| + |\tilde{\varphi}_x(l_0)| + |\tilde{\varphi}(l)| + \|\bar{s}_0 \varphi_0\|_{1,[l_0,l]} = \mathcal{O}(\varepsilon e^{-\eta(l-l_0)}).$$

We can then use Lemma B4 to prove that $|\tilde{\varphi}|_\infty$ is exponentially small. This completes the proof of Theorem 2.

4.3 Computing Eigenvalues

One can solve the eigenvalue problem (14) to check if Assumption 2 for (15) is satisfied. (See, for example, [1] for results relating the spectra of finite–interval and all–line eigenvalue problems.) For large l, problem (14) becomes increasingly ill-conditioned. However, one can avoid this by a change of variables: Let $C(x)$ denote the matrix function satisfying

$$C_x = \frac{1}{2} AC, \quad C(0) = I \ ,$$

and introduce into (14) a new variable ψ by

$$\phi = C(x)\psi.$$

We obtain

$$s\psi + C^{-1}C_{xx}\psi = \psi_{xx}, \quad \psi(\pm l) = 0. \tag{49}$$

Since A converges exponentially fast to constant matrices as $x \to \pm\infty$ it follows that (49) is a well behaved eigenvalue problem.

5 Nonlinear Stability on a Bounded Interval with Modified Boundary Conditions

In this section we consider nonlinear systems

$$\begin{aligned} u_t + f(u)_x &= u_{xx}, \quad -l \le x \le l, \quad t \ge 0, \\ u(x,0) &= u_0(x). \end{aligned} \tag{50}$$

To begin with, the boundary conditions are

$$u(\pm l, t) = U(\pm l) \tag{51}$$

where $U(x)$ is a steady solution of the all–line problem, i.e., $f(U)_x = U_{xx}, x \in \mathbb{R}$. Assume the initial data to be a small perturbation of $U(x)$, that is

$$u(x,0) = U(x) + \varepsilon y_0(x), \quad |\varepsilon| \ll 1, \quad |y|_\infty = 1.$$

Define $y(x,t)$ by

$$u(x,t) = U(x) + \varepsilon y(x,t). \tag{52}$$

Then y satisfies

$$y_t = Ly + \varepsilon(h(y))_x, \quad -l \le x \le l,\ t \ge 0, \tag{53}$$

$$\begin{aligned} y(x,0) &= y_0(x), \\ y(\pm l, t) &= 0. \end{aligned} \tag{54}$$

Here $Ly = -(Ay)_x + y_{xx}$ where $A(x)$ is the Jacobian of f evaluated at $U(x)$. Furthermore,

$$\varepsilon h(y) := -\frac{1}{\varepsilon}\Big(f(U(x) + \varepsilon y) - f(U(x)) - \varepsilon A(x) y\Big)\ ,$$

which vanishes quadratically at $y = 0$. We may assume that

$$|h(y)| \le c|y|^2$$

for some constant c. (The function h also depends on x and ε, but this is unimportant and suppressed in our notation.)

5.1 Analysis of the Linearized Problem

The linearized problem corresponding to (53), (54) is

$$\begin{aligned} y_t = Ly, \quad -l \le x \le l, \quad t \ge 0, \\ y(\pm l, t) = 0, \quad y(x,0) = y_0(x), \end{aligned} \tag{55}$$

and the corresponding eigenvalue problem is

$$s\varphi = L\varphi, \quad -l \le x \le l, \quad \varphi(\pm l) = 0. \tag{56}$$

By the previous section, we know that there is a simple eigenvalue s_0, $|s_0| \ll 1$, and that all other eigenvalues satisfy $\operatorname{Re} s_i \le -\delta$. The eigenfunction corresponding to s_0, denoted $\varphi_0(x)$, is close to the eigenfunction of the infinite–line problem, that is

$$|\varphi_0(x) - U_x(x)| \le ce^{-\beta l}.$$

Therefore, the projection

$$P_0 = -\frac{1}{2\pi i}\int_{|s|=\frac{\delta}{4}} (sI - L)^{-1} ds,$$

which projects $L_2(-l,l)$ onto the subspace spanned by φ_0, is a bounded linear operator from $L_2(-l,l)$ to $H_2(-l,l)$.

Without restriction we may assume that

$$P_0 y_0 = 0. \tag{57}$$

Let us show that (57) is, in fact, not a restriction: If $P_0 y_0 = \varrho\varphi_0$, $\varrho \ne 0$, we change the ansatz (52) and the boundary conditions (51) to

$$u(x,t) = U(x+\varepsilon\alpha) + \varepsilon y(x,t), \quad u(\pm l, t) = U(\pm l + \varepsilon\alpha), \tag{58}$$

respectively. Here α will be determined as we will now explain. With $y(x,t)$ defined in (58), the initial condition for y becomes $y(x,0) = y_1(x)$ with

$$u(x,0) = U(x+\varepsilon\alpha) + \varepsilon y_1(x), \quad y_1(x) = y_0(x) - (U(x+\varepsilon\alpha) - U(x))/\varepsilon. \tag{59}$$

We will use the notation

$$\begin{aligned} L_\alpha y &= y_{xx} - (A_\alpha y)_x, \quad A_\alpha(x) = A(x+\varepsilon\alpha), \\ P_\alpha &= -\frac{1}{2\pi i}\int_{|s|=\frac{\delta}{4}} (sI - L_\alpha)^{-1} ds. \end{aligned}$$

Further, denote by s_0^α and φ_0^α the eigenvalue close to zero and the corresponding eigenfunction, respectively, of L_α with boundary condition $\varphi^\alpha(\pm l) = 0$. We have

$$\begin{aligned} P_\alpha y_1 &= P_\alpha y_0 - \frac{1}{\varepsilon} P_\alpha (U(x+\varepsilon\alpha) - U(x)) \\ &= P_\alpha y_0 - \alpha P_\alpha (U_x(x+\varepsilon\alpha) + \mathcal{O}(\varepsilon\alpha)) \\ &= P_\alpha y_0 - \alpha(1 + \mathcal{O}(\varepsilon\alpha) + \mathcal{O}(e^{-\beta l}))\varphi_0^\alpha . \end{aligned}$$

Since

$$\|P_\alpha - P_0\| = \mathcal{O}(\varepsilon\alpha)$$

we obtain

$$P_\alpha y_0 = (\varrho + \mathcal{O}(\varepsilon\alpha))\phi_0^\alpha .$$

Therefore,

$$P_\alpha y_1 = \Big(\varrho - \alpha\Big(1 + \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\alpha) + \mathcal{O}(e^{-\beta l})\Big)\Big)\varphi_0^\alpha .$$

By the implicit function theorem, we can choose $\alpha \sim \rho$ so that $P_\alpha y_1 = 0$ provided l is sufficiently large and ε is sufficiently small. Therefore, we may assume (57) to hold.

For the linear problem (55) the equality (57) implies

$$P_0 y(\cdot, t) = 0, \quad t \geq 0.$$

Thus the solution $y(x,t)$ of the linear problem (55) converges exponentially fast to zero as $t \to \infty$.

5.2 Analysis of the Nonlinear Problem with Modified Boundary Conditions

Consider the nonlinear equation $u_t + f(u)_x = u_{xx}$ with initial condition $u(x,0) = U(x) + \varepsilon y_0(x)$. As explained above, we may assume $P_0 y_0 = 0$ without restriction. In the nonlinear case, there is no guarantee, however, that the perturbation $y(x,t)$ (defined by (52)) remains without component in the direction of φ_0. To leading order, a non–zero component in this direction corresponds to a shifted profile, $U(x + \varepsilon\alpha)$, and by modifying the boundary conditions for $u(x,t)$ we allow shifting of the profile to occur.

With $\alpha(t)$ a smooth function that needs to be determined and satisfies $\alpha(0) = 0$, let

$$u(x,t) = U(x + \varepsilon\alpha(t)) + \varepsilon y(x,t).$$

Here $y(x,t)$ is a new perturbation variable. The nonlinear equation $u_t + f(u)_x = u_{xx}$ becomes

$$\begin{aligned} \alpha_t U_x^\alpha + y_t &= L_\alpha y + \varepsilon h_\alpha(y)_x, \quad -l \le x \le l, \quad t \ge 0. \\ y(x,0) &= y_0(x). \end{aligned} \tag{60}$$

Here h_α and U_x^α are defined by

$$\varepsilon h_\alpha(y) := -\frac{1}{\varepsilon}\Big(f(U(x+\varepsilon\alpha)+\varepsilon y) - f(U(x+\varepsilon\alpha)) - \varepsilon A_\alpha(x) y\Big) \tag{61}$$

and

$$U_x^\alpha(x) = U_x(x + \varepsilon\alpha). \tag{62}$$

As discussed in Section 3, the boundary conditions must allow for a shift in order to avoid an eigenvalue which is exponentially close to zero. Thus we consider the boundary conditions

$$u(\pm l, t) = U(\pm l + \varepsilon\alpha(t)), \tag{63}$$

where $\alpha(t)$ will be determined so that

$$P_{\alpha(t)} y(\cdot, t) = 0. \tag{64}$$

We will now prove that it is possible to determine $\alpha(t)$ by the side condition (64) and that, for the new problem, $y(x,t)$ converges exponentially fast to zero (in maximum norm) as $t \to \infty$.

Introduce

$$y^I = (I - P_\alpha)y, \quad y^{II} = P_\alpha y.$$

Note that $y^{II}(\cdot, 0) = 0$ and

$$(P_\alpha)_t = -\frac{\varepsilon\alpha_t}{2\pi i}\int_{|s|=\frac{\delta}{4}} (sI - L_\alpha)^{-1}\frac{\partial L_\alpha}{\partial\alpha}(sI - L_\alpha)^{-1} ds =: \varepsilon\alpha_t R_\alpha. \tag{65}$$

As in [4], R_α is a bounded operator from $L_2(-l,l)$ to $H_1(-l,l)$. Differentiating $y^I = (I - P_\alpha)y$, we obtain

$$(I - P_\alpha)y_t = (y^I)_t + \varepsilon\alpha_t R_\alpha y \tag{66}$$

and, similarly,

$$P_\alpha y_t = (y^{II})_t + P_\alpha (y^I)_t - \varepsilon \alpha_t R_\alpha y^{II}. \tag{67}$$

Also,

$$(I - P_\alpha) U_x^\alpha = e^{-\beta l} r^I, \quad \|r^I\| \sim 1, \tag{68}$$

and

$$P_\alpha U_x^\alpha = (1 + \mathcal{O}(e^{-\beta l})) \varphi_0^\alpha. \tag{69}$$

Applying the projectors $I - P_\alpha$ and P_α to (60) yields

$$\begin{aligned} \alpha_t (I - P_\alpha) U_x^\alpha + (I - P_\alpha) y_t &= L_\alpha y^I + \varepsilon (h_{\alpha,x})^I, \\ \alpha_t P_\alpha U_x^\alpha + P_\alpha y_t &= L_\alpha y^{II} + \varepsilon (h_{\alpha,x})^{II}. \end{aligned}$$

Using (66)–(69) one obtains

$$\begin{aligned} (y^I)_t &= L_\alpha y^I + \varepsilon (h_{\alpha,x})^I - \alpha_t e^{-\beta l} r^I - \varepsilon \alpha_t R_\alpha y, & (70) \\ (y^{II})_t &= L_\alpha y^{II} + \varepsilon (h_{\alpha,x})^{II} - \alpha_t (1 + \mathcal{O}(e^{-\beta l})) \phi_0^\alpha & (71) \\ &\quad + \varepsilon \alpha_t P_\alpha R_\alpha y + \varepsilon \alpha_t R_\alpha y^{II}. & (72) \end{aligned}$$

If ε is small enough and l is large enough we can determine $\alpha(t)$ as a smooth function so that

$$\alpha_t \Big((1 + \mathcal{O}(e^{-\beta l})) \varphi_0^\alpha - \varepsilon P_\alpha R_\alpha y^I \Big) = \varepsilon (h_{\alpha,x})^{II}. \tag{73}$$

Then we obtain:

$$(y^{II})_t = L_\alpha y^{II} + \varepsilon \alpha_t (P_\alpha + I) R_\alpha y^{II}. \tag{74}$$

By construction, $y^{II}(x,0) = 0$, and a unique solubility argument for (74) implies

$$y^{II}(x,t) = 0, \quad t \geq 0. \tag{75}$$

Therefore (70) becomes

$$\begin{aligned} (y^I)_t &= L_\alpha y^I - \varepsilon \alpha_t R_\alpha y^I - \alpha_t e^{-\beta l} r^I + \varepsilon (h_\alpha ((y^I)_x))^I, & (76) \\ y^I(x,0) &= y_0(x). \end{aligned}$$

In [4] we have considered systems of the form (76) and proved exponential convergence to zero when the coefficients of L_α do not depend on t and the eigenvalues have negative real parts. Therefore, we rewrite (76) in terms of $v^I = (I - P_0)y^I$ and $v^{II} = P_0 y^I$ and replace $L_\alpha y^I$ by Lv^I. We have

$$L_\alpha = L + \varepsilon\alpha\tilde{L}, \quad P_\alpha = P_0 + \varepsilon\alpha\tilde{P}. \tag{77}$$

Here $\tilde{P}$ is a bounded operator from L_2 into H_1 which depends smoothly on α and t, and $\tilde{L}$ is a first–order differential operator with smooth coefficients. We have

$$v^{II} = P_0 y^I = (P_\alpha - \varepsilon\alpha\tilde{P})y^I = -\varepsilon\alpha\tilde{P}(v^I + v^{II}).$$

Therefore,

$$v^{II} = \varepsilon\alpha\tilde{\tilde{P}}v^I, \quad \tilde{\tilde{P}} = -(I + \varepsilon\alpha\tilde{P})^{-1}\tilde{P}, \tag{78}$$

and

$$y^I = v^I + v^{II} = (I + \varepsilon\alpha\tilde{\tilde{P}})v^I. \tag{79}$$

Now we can write (76) in the form

$$v_t^I = Lv^I + (\varepsilon + e^{-\beta l})(F(x,t,v^I) + G_x(x,t,v^I)). \tag{80}$$

The technique in [4] is now directly applicable, and convergence at an exponential rate, $y^I(\cdot,t) \to 0$ as $t \to \infty$, follows. Reconsidering (73), one obtains that $\alpha(t)$ approaches a constant. This completes the proof of the following result:

Theorem 5: *Let Assumptions 1 to 4 be satisfied. For sufficiently small ε and sufficiently large l, there is a smooth function $\alpha(t)$ and a constant α_1 with*

$$\lim_{t\to\infty} \alpha(t) = \alpha_1,$$

$$\lim_{t\to\infty} u(x,t) = U(x + \varepsilon\alpha_1).$$

The rate of convergence is exponential.

6 Appendix A

In this appendix we prove Theorem 3.

For the analytic matrix function $B(s)$, introduced in Remark 4 following Theorem 3, we use the representation

$$B(s) = B_0 + sB_1 + \mathcal{O}(s^2) \tag{81}$$

and recall the notation $D(s) = det\, B(s)$.

Assumption 2 implies that B_0 is singular, and the eigenvalue $s = 0$ of B_0 is geometrically simple. In particular, $D(0) = 0$. Also, in [5] we have proved that the resolvent for the infinite–line problem is $\sim 1/|s|$ for all $\mathrm{Re}\, s > 0$, which yields

$$|B(s)^{-1}| \le \frac{K}{|s|}, \quad \mathrm{Re}\, s > 0. \tag{82}$$

We now show that this implies $D'(0) \neq 0$: After applying a similarity transformation to B_0, we may assume

$$B(s) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B_{22}^{(0)} & \\ 0 & & & \end{pmatrix} + s \begin{pmatrix} b_{11}^{(1)} & b_{12}^{(1)} & \dots & \dots \\ b_{21}^{(1)} & & & \\ \vdots & & B_{22}^{(1)} & \\ \vdots & & & \end{pmatrix} + \mathcal{O}(s^2), \quad det\, B_{22}^{(0)} \neq 0\,.$$

There is a transformation

$$S(s) = I + \mathcal{O}(s),$$

so that

$$S^{-1}(s) B(s) S(s) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B_{22}^{(0)} & \\ 0 & & & \end{pmatrix} + s \begin{pmatrix} b_{11}^{(1)} & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & B_{22}^{(1)} & \\ 0 & & & \end{pmatrix} + \mathcal{O}(s^2)\ .$$

The estimate (82) yields that $b_{11}^{(1)} \neq 0$, and then

$$D(s) = s b_{11}^{(1)} det\, B_{22}^{(0)} + \mathcal{O}(s^2)$$

implies $D'(0) \neq 0$.

7 Appendix B

In this appendix we collect some results for first–order differential systems

$$u_x = (\Lambda + \varepsilon e^{-\beta x} B(x))u, \quad x \geq 0. \tag{83}$$

Here $\beta > 0$ is a constant, $B(x)$ is a continuous, bounded matrix function, and Λ is a diagonal matrix of the form

$$\Lambda = \begin{pmatrix} -\Lambda^I & 0 \\ 0 & \Lambda^{II} \end{pmatrix}, \quad \operatorname{Re} \Lambda^I > 0, \quad \operatorname{Re} \Lambda^{II} > 0. \tag{84}$$

Partition $u(x)$ correspondingly. In our application, the matrices Λ^I and Λ^{II} both have size $n \times n$, and in the following we will make this assumption.

For $\varepsilon = 0$, the solution formula for (83) makes estimates straightforwardly available. Our results below are for the $\varepsilon \neq 0$ case, and they are perturbations of corresponding $\varepsilon = 0$ results.

Lemma B1: *Consider (83) with boundary condition*

$$u^I(0) = y_0^I. \tag{85}$$

If

$$\frac{\varepsilon |B|_\infty}{\beta} \leq \frac{1}{4}, \tag{86}$$

then there is a unique bounded solution of (83), (85). It satisfies

$$|u^I|_\infty \leq (1 + \frac{2\varepsilon |B|_\infty}{\beta})|y_0^I|, \quad |u^{II}|_\infty \leq \frac{2\varepsilon |B|_\infty}{\beta} |y_0^I|. \tag{87}$$

Proof: To begin with, let $\varepsilon = 0$. The unique bounded solution is given by

$$y^I(x) = e^{-\Lambda^I x} y_0^I, \quad |y^I|_\infty \leq |y_0^I|, \tag{88}$$

$$y^{II}(x) \equiv 0. \tag{89}$$

To treat the case $\varepsilon \neq 0$ we let $u = y + v$. Then v satisfies

$$v_x = \Lambda v + \varepsilon e^{-\beta x} B(x)(v + y), \quad v^I(0) = 0. \tag{90}$$

Assume first that a bounded solution v exists and consider the exponentially decaying terms as forcing terms. We have

$$v^I(x) = \varepsilon \int_0^x e^{-\Lambda^I (x-\xi) - \beta\xi} B_1(\xi)(v(\xi) + y(\xi)) d\xi, \tag{91}$$

$$v^{II}(x) = -\varepsilon \int_x^\infty e^{-\Lambda^{II} (\xi - x) - \beta\xi} B_2(\xi)(v(\xi) + y(\xi)) d\xi. \tag{92}$$

Here B_1 and B_2 consist of the first and last n rows of B, respectively. From (91) and (92) we derive

$$|v|_\infty \le \frac{\varepsilon}{\beta}|B|_\infty(|v|_\infty + |y|_\infty).$$

The estimate and uniqueness follows for $\varepsilon|B|_\infty/\beta \le 1/4$.

To prove existence we let $v^0 \equiv 0$ and consider the iteration

$$v_x^{n+1} = \Lambda v^{n+1} + \varepsilon e^{-\beta x} B(x)(v^n + y), \quad (v^{n+1})^I(0) = 0.$$

If

$$\varepsilon|B|_\infty/\beta \le \frac{1}{4},$$

then it is easy to show that $w^n = v^n - v^{n-1}$ satisfies

$$|w^n|_\infty \le \frac{2\varepsilon}{\beta}|B|_\infty |w^{n-1}|_\infty.$$

Thus the iteration converges uniformly to the unique bounded solution. This concludes the proof.

In the following, we always make the smallness assumption (86) for ε.

In the next lemma we sharpen the estimates. Denote the components of u^I and u^{II} by u_j^I and u_j^{II}, respectively, and the diagonal elements of Λ^I and Λ^{II} by λ_j^I and λ_j^{II}, respectively. By (88) and (89):

$$|y_j^I(x)| \le e^{-Re\lambda_j^I x}|y_0^I|, \quad y_j^{II}(x) \equiv 0.$$

By (91):

$$|v_j^I(x)| \le \varepsilon|B|_\infty(|v|_\infty + |y|_\infty)\int_0^x |e^{-\lambda_j^I(x-\xi)-\beta\xi}|d\xi \ .$$

By partitioning the integral at $x/2$, we see that it is exponentially small with exponent $\gamma_j^I = \min(Re\lambda_j^I, \beta)/2$. Thus

$$|v_j^I(x)| \le c\ \varepsilon|B|_\infty|y_0^I|e^{-\gamma_j^I x}$$

for some constant c. Correspondingly, by (92):

$$|v_j^{II}(x)| \le \varepsilon|B|_\infty(|v|_\infty + |y|_\infty)\int_x^\infty |e^{\lambda_j^{II}(x-\xi)-\beta\xi}|d\xi \le c\ \varepsilon|B|_\infty|y_0^I|e^{-\beta x}.$$

This proves:

Lemma B2: *The bounded solution of (83), (85) satisfies the more precise estimates*

$$|u_j^I(x)| \le (1 + const\,\varepsilon|B|_\infty)e^{-\gamma_j^I x}|y_0^I|, \quad |u_j^{II}(x)| \le const\,\varepsilon|B|_\infty e^{-\beta x}|y_0^I|,$$

with $\gamma_j^I = \min(Re\lambda_j^I, \beta)/2$.

As shown above, for every $u^I(0) = y_0^I$ the system (83) on $0 \le x < \infty$) has a unique bounded solution $u(x)$. In particular, $u^{II}(0)$ is determined uniquely by $u^I(0)$. In this way, the system (83) determines a linear mapping:

$$u^{II}(0) = \varepsilon Q u^I(0), \quad |Q| \le 2\frac{|B|_\infty}{\beta}.$$

Next we let $B = B(x,s)$ and $\Lambda = \Lambda(s)$ be analytic functions of s defined in some open, connected subset S_0 of the $s-$plane. Assume $\varepsilon|B|_\infty/\beta \le 1/4$, $\mathrm{Re}\Lambda^I > 0$, $\mathrm{Re}\Lambda^{II} > 0$ for all $s \in S_0$. Then $u = u(x,s)$ is also an analytic function of s, bounded by (87), and there is a unique analytic mapping:

$$u^{II}(0) = \varepsilon Q(s) u^I(0), \quad |Q(s)| \le 2\frac{|B|_\infty}{\beta}. \tag{93}$$

Even if the real part of some of the eigenvalues of $\Lambda(s)$ change sign in S_0, under suitable assumptions we can still define an analytic mapping $Q(s)$ by using analytic continuation. To be specific, we will consider a neighborhood S_0 of $s = 0$, $S_0 = \{s : |s| < \sigma\}$, where $\sigma > 0$ is fixed, and assume that Λ and B are analytic for $s \in S_0$, and that for a constant $\delta > 0$

$$\mathrm{Re}\,\Lambda^I(s) + \delta I > 0, \quad \mathrm{Re}\,\Lambda^{II}(s) + \delta I > 0. \tag{94}$$

Lemma B3: *Consider (83) and (85). Assume (94) with* $0 < \delta \le \beta/4$ *for all* $s \in S_0$*. Then there is a unique solution* $u(x,s) = y(x,s) + v(x,s)$*, analytic in* s *for each* x*, satisfying*

$$\begin{aligned} y^I(x,s) &= e^{-\Lambda^I x} y_0^I, \quad |e^{-\delta x} y^I|_\infty \le |y_0^I|, \\ y^{II}(x,s) &\equiv 0, \\ |e^{-\delta x} v^I|_\infty + |v^{II}|_\infty &\le \frac{8\varepsilon|B|_\infty}{\beta}|y_0^I|. \end{aligned}$$

for $s \in S_0$. In particular the mapping $Q = Q(s)$ in (93) is an analytic function of s.

Proof: The transformed variable

$$\tilde{u}(x) = \begin{pmatrix} e^{-\delta x} I & \\ & e^{\delta x} I \end{pmatrix} u(x) \tag{95}$$

satisfies

$$\tilde{u}_x = \tilde{\Lambda}\tilde{u} + \varepsilon e^{-\frac{\beta}{2}x}\tilde{B}(x)\tilde{u}, \quad \tilde{u}^I(0) = y_0^I, \quad \tilde{\Lambda} = \begin{pmatrix} -(\Lambda^I + \delta I) & \\ & \Lambda^{II} + \delta I \end{pmatrix}.$$

Here $|\tilde{B}|_\infty \le |B|_\infty$, and $\tilde{\Lambda}$ satisfies (84) for all $s \in S_0$. Now we can apply Lemma B1, and by (95) the claims follow.

Next we consider (83) for $0 \le x \le l$ with $2n$ boundary conditions

$$u^I(0) = y_0^I, \quad u^{II}(l) = Du^I(l) + y_l^{II}, \tag{96}$$

where D is a diagonal matrix.

Lemma B4: *Consider (83),(96) where Λ satisfies (84). If*

$$\frac{\varepsilon |B|_\infty}{\beta}(1 + |D|) \le \frac{1}{4},$$

then the unique solution satisfies

$$\begin{aligned} |u^I|_\infty &\le |y_0^I| + \frac{2\varepsilon|B|_\infty}{\beta}\Big((1+|D|)|y_0^I| + |y_l^{II}|\Big), \\ |u^{II}|_\infty &\le |D||y_0^I| + |y_l^{II}| + \frac{2\varepsilon|B|_\infty}{\beta}(1+|D|)\Big((1+|D|)|y_0^I| + |y_l^{II}|\Big) \end{aligned}$$

Also, if $y_l^{II} = 0$ and $\Lambda = \Lambda(s)$, $B = B(x,s)$, $D = D(s)$ are analytic functions of $s \in S_0$ and if (84) is valid, then the boundary value problem

$$u_x = (\Lambda + \varepsilon e^{-\beta x}B)u, \; 0 \le x \le l, \quad u^I(0) = y_0^I, \quad u^{II}(l) = Du^I(l),$$

determines a unique analytic mapping:

$$u^{II}(0) = \bar{P}(s)u^I(0), \quad |\bar{P}(s)| \le |e^{-(\Lambda^I + \Lambda^{II})l}||D| + \varepsilon\frac{2(1+|D|)|B|_\infty}{\beta}. \tag{97}$$

Remark: In our application, (84) is satisfied for $Res > 0$. By analytic continuation there is a unique analytic mapping in a full neighborhood of the origin, $|s| < \sigma$, for some $\sigma > 0$.

Proof: To begin with, let $s \in S_0$ be fixed. Existence of a solution of (83),(96) follows once the estimates are shown. As above, let y be the solution of the corresponding problem with $\varepsilon = 0$:

$$
\begin{aligned}
y^I(x) &= e^{-\Lambda^I x} y_0^I, && (98)\\
y^{II}(x) &= e^{-\Lambda^{II}(l-x)} \left(D y^I(l) + y_l^{II} \right). && (99)
\end{aligned}
$$

It follows that

$$
|y^I|_\infty \le |y_0^I|,\ |y^{II}|_\infty \le |D||y_0^I| + |y_l^{II}|, \quad |y|_\infty \le (1+|D|)|y_0^I| + |y_l^{II}|. \tag{100}
$$

We also note that

$$
y^{II}(0) = e^{-(\Lambda^I + \Lambda^{II})l} D y_0^I \quad \text{if} \quad y_l^{II} = 0 \ . \tag{101}
$$

For $v = u - y$ we proceed as in the proof of Lemma B1. The solution formula

$$
\begin{aligned}
v^I(x) &= \varepsilon \int_0^x e^{-\Lambda^I(x-\xi)} e^{-\beta\xi} B_1(\xi)(v+y)d\xi, && (102)\\
v^{II}(x) &= -\varepsilon \int_x^l e^{-\Lambda^{II}(\xi-x)} e^{-\beta\xi} B_2(\xi)(v+y)d\xi + e^{-\Lambda^{II}(l-x)} D v^I(l) && (103)
\end{aligned}
$$

yields

$$
\begin{aligned}
|v^I|_\infty &\le \frac{\varepsilon |B|_\infty}{\beta}(|y|_\infty + |v|_\infty),\\
|v^{II}|_\infty &\le \frac{\varepsilon |B|_\infty}{\beta}(|y|_\infty + |v|_\infty) + |D||v^I(l)|\\
&\le \frac{\varepsilon |B|_\infty}{\beta}(1+|D|)(|y|_\infty + |v|_\infty).
\end{aligned}
$$

The estimates follow if $(\varepsilon|B|_\infty/\beta)(1+|D|) \le 1/2$. Existence of the analytic mapping for each $s \in S_0$ follows as before.

We shall also give an estimate in the special case where the simplified problem ((83) with $\varepsilon = 0$) only contains terms that decay rapidly in $0 \le x \le l$ away from the boundary at $x = l$.

Lemma B5: *Consider (83), (96) with* $y_0^I = 0$*. If there is a constant* $\gamma > 0$ *such that for all* $s \in S_0$ *the solution of the simplified problem (with* $\varepsilon = 0$*) satisfies*

$$|y^{II}(x)| = |e^{\Lambda^{II}(x-l)}y_l^{II}| \leq e^{\gamma(x-l)}|y_l^{II}|,$$

and if $\varepsilon|B|_\infty$ *is sufficiently small (as compared with* β *and* γ*), then there is a constant* c *such that*

$$|u^{II}(0)| \leq c\ e^{-\tilde{\gamma}l}|y_l^{II}|, \quad \tilde{\gamma} = \min(\gamma, \beta)/2.$$

Proof: The simplified problem has the solution given by (98) and (99) with $y^I \equiv 0$. By (102) and (103):

$$\begin{aligned} |v^I(x)| &\leq \varepsilon|B|_\infty(\frac{|v|_\infty}{\beta} + |y_l^{II}|\int_0^l e^{-\beta\xi+\gamma(\xi-l)}d\xi) \\ &\leq \frac{\varepsilon|B|_\infty}{\beta}(|v|_\infty + c\ e^{-\tilde{\gamma}l}|y_l^{II}|), \qquad (104) \\ |v^{II}(x)| &\leq \frac{\varepsilon|B|_\infty}{\beta}(|v|_\infty + c\ e^{-\tilde{\gamma}l}|y_l^{II}|) + |D||v^I(l)| \\ &\leq \frac{\varepsilon|B|_\infty}{\beta}(1+|D|)(c\ e^{-\tilde{\gamma}l}|y_l^{II}| + |v|_\infty). \end{aligned}$$

The lemma follows.

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