Existence and strong pre-compactness properties for entropy solutions of a first-order quasilinear equation with discontinuous flux [1]

E.Yu. Panov

Abstract

Sequences of entropy solutions of a non-degenerate first-order quasilinear equation are shown to be strongly pre-compact in the general case of a Caratheodory flux vector. Existence of the weak and entropy solution to Cauchy problem for such equation is also established. The proofs are based on general localization principle for H-measures corresponding to sequences of measure-valued functions.

§ 1. Introduction

We consider a first-order quasilinear equation

$$\operatorname{div}_x \varphi(x,u) + \psi(x,u) = 0. \tag{1}$$

Here $\varphi(x,u) = (\varphi_1(x,u),\dots,\varphi_n(x,u))$, $u = u(x)$, $x = (x_1,\dots,x_n) \in \Omega$, where Ω is an open subset of $\mathbb{R}^n$; the flux vector $\varphi(x,u)$ is assumed to be a Caratheodory vector (i.e. it is continuous with respect to u and measurable with respect to x) such that for some $q > 2$ the functions

$$\alpha_M(x) = \max_{|u|\le M} |\varphi(x,u)| \in L^q_{loc}(\Omega) \quad \forall M > 0 \tag{2}$$

(here and below $|\cdot|$ stands for the Euclidean norm of a finite-dimensional vector). We also assume that for any fixed $p \in \mathbb{R}$ the distribution

$$\operatorname{div}_x \varphi(x,p) = \gamma_p \in \mathrm{M}_{loc}(\Omega), \tag{3}$$

where $\mathrm{M}_{loc}(\Omega)$ is the space of locally finite Borel measures on Ω with the standard locally convex topology generated by semi-norms $p_\Phi(\mu) = \operatorname{Var}(\Phi\mu)$, $\Phi = \Phi(x) \in C_0(\Omega)$.

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The function $\psi(x,u)$ is assumed to be a Caratheodory function on $\Omega\times\mathbb{R}$ such that

$$\beta_M(x) = \max_{|u|\le M} |\psi(x,u)| \in L^1_{loc}(\Omega) \quad \forall M>0. \tag{4}$$

Let $\gamma_p = \gamma_p^r + \gamma_p^s$ be the decomposition of the measure γ_p into the sum of the regular and the singular measures, so that $\gamma_p^r = \omega_p(x)dx$, $\omega_p(x) \in L^1_{loc}(\Omega)$, and γ_p^s is a singular measure (supported on a set of zero Lebesgue measure). We denote by $|\gamma_p^s|$ the variation of the measure γ_p^s, which is a non-negative locally finite Borel measure on Ω. Denote, as usual,

$$\operatorname{sign} u = \begin{cases} 1 & , \quad u>0, \\ -1 & , \quad u<0, \\ 0 & , \quad u=0. \end{cases}$$

Now, we introduce the notion of entropy solution of (1).

Definition 1. A measurable function $u(x)$ on Ω is called an entropy solution of equation (1) if $\varphi(x,u(x)) \in L^1_{loc}(\Omega,\mathbb{R}^n)$, $\psi(x,u(x)) \in L^1_{loc}(\Omega)$, and for all $p\in\mathbb{R}$ the Kruzhkov-type entropy inequality (see [9]) holds

$$\begin{aligned} &\operatorname{div}_x [\operatorname{sign}(u(x)-p)(\varphi(x,u(x))-\varphi(x,p))] + \\ &\operatorname{sign}(u(x)-p)[\omega_p(x)+\psi(x,u(x))] - |\gamma_p^s| \le 0 \end{aligned} \tag{5}$$

in the sense of distributions on Ω (in the space $\mathcal{D}'(\Omega)$); that is, for all non-negative functions $f(x)\in C_0^\infty(\Omega)$

$$\begin{aligned} &\int_\Omega [\operatorname{sign}(u(x)-p)\,(\varphi(x,u(x))-\varphi(x,p),\nabla f(x)) - \\ &\operatorname{sign}(u(x)-p)(\omega_p(x)+\psi(x,u(x)))f(x)]\,dx + \int_\Omega f(x)d|\gamma_p^s|(x) \ge 0 \end{aligned}$$

(here $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^n$).

Our definition extends the notion of weak entropy solution introduced for the case of one space variable in [6, 7]. Notice also that we do not require that $u(x)$ is a weak solution of (1).

We assume that the flux vector $\varphi(x,u)$ is non-degenerate in the sense of the following definition.

Definition 2. A vector $\varphi(x,u)$ is said to be *non-degenerate* if for almost all $x\in\Omega$ for all $\xi\in\mathbb{R}^n$, $\xi\ne 0$ the functions $\lambda\to(\xi,\varphi(x,\lambda))$ are not constant on non-degenerate intervals.

In this paper we shall establish the strong pre-compactness property for sequences of entropy solutions. This result generalizes the previous results of [10, 11, 12, 13] to the case when flux vector may be discontinuous with respect to spatial variables while entropy solutions may be generally unbounded.

Theorem 1. *Suppose that u_k, $k \in \mathbb{N}$ is a sequence of entropy solutions of (1) with non-degenerate flux vector $\varphi(x,u)$, such that $|\varphi(x,u_k(x))| + |\psi(x,u_k(x))| + \rho(u_k(x))$ is bounded in $L^1_{loc}(\Omega)$, where $\rho(u)$ is a nonnegative super-linear function (i.e. $\rho(u)/u \to \infty$ as $u \to \infty$). Then there exists a subsequence of u_k, which converges in $L^1_{loc}(\Omega)$ to some entropy solution $u(x)$.*

Now, we consider the evolutionary equation

$$u_t + \mathrm{div}_x \varphi(t,x,u) = 0, \tag{6}$$

$u = u(t,x)$, $(t,x) \in \Pi = \mathbb{R}_+ \times \mathbb{R}^n$, where $\mathbb{R}_+ = (0,+\infty)$. We assume that $\varphi(t,x,u)$ is a Caratheodory vector on $\Pi \times \mathbb{R}$ such that $\varphi(t,x,\cdot) \in C^1(\mathbb{R},\mathbb{R}^n)$ for each fixed $(t,x) \in \Pi$. We also assume that the vector $(u,\varphi(t,x,u)) \in \mathbb{R}^{n+1}$ is non-degenerate. The latter means that for a.e. $(t,x) \in \Pi$ for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ the functions $u \to (\xi, \varphi(t,x,u))$ are not affine on non-degenerate intervals. We also suppose that for some $a, b \in \mathbb{R}$, $a < b$ $\varphi(\cdot,a) = \varphi(\cdot,b) \equiv 0$, $\max\limits_{u\in[a,b]} |\varphi(\cdot,u)| \in L^q_{loc}(\bar{\Pi})$, $q > 2$, $\bar{\Pi} = [0,+\infty) \times \mathbb{R}^n$, and

$$\mathrm{div}_x \varphi(\cdot,p) = \gamma_p = \omega_p(t,x)dtdx + \gamma_p^s \in \mathrm{M}_{loc}(\bar{\Pi}),$$

here γ_p^s is a singular part of the measure γ_p.

We underline that equations like (1), (6) occur in various applications, for instance in porous media, sedimentation processes, traffic flow, radar shape-from-shading problems, blood flow, and have been widely studied in recent years.

We shall study the Cauchy problem for equation (6) with initial condition

$$u(0,x) = u_0(x), \tag{7}$$

where $u_0(x) \in L^\infty(\mathbb{R}^n)$, $a \le u_0(x) \le b$.

Definition 3. A function $u = u(t,x) \in L^\infty(\Pi)$ is called an entropy solution of problem (6), (7), if $\forall p \in \mathbb{R}$, $\forall f = f(t,x) \in C_0^\infty(\bar{\Pi})$, $f \geq 0$

$$\int_\Pi [|u-p|f_t + \operatorname{sign}(u-p)\left(\varphi(t,x,u) - \varphi(t,x,p), \nabla_x f\right) - \operatorname{sign}(u-p)\omega_p(t,x)f(t,x)]\,dtdx + \int_\Pi f(t,x)d|\gamma_p^s|(t,x) + \int_{\mathbb{R}^n} |u_0(x)-k|f(0,x)dx \geq 0. \tag{8}$$

A function $u(t,x) \in L^\infty(\Pi)$ is called a weak solution if $u(t,x)$ satisfies (6) in the sense of distribution.

Theorem 2. *Under the above assumptions there exist a weak and entropy solution $u(t,x)$ of (6), (7) such that $a \leq u(t,x) \leq b$.*

Observe that the statement of Theorem 2 covers results of [8], where existence of weak solution is proved for the two-dimensional equation

$$u_t + f(k,u)_x + g(l,y)_y = 0$$

with fixed BV-functions $k = k(x,y)$, $l = l(x,y)$ and sufficiently smooth flux functions f, g.

Theorems 1,2 will be proved in the last section. The proof is based on general localization properties for H-measures corresponding to bounded sequences of measure-valued functions.

In next section 2 we describe the main concepts, in particular the concept of measure-valued functions. In sections 3,4 we introduce the notion of H-measure and prove the localization property. Finally, in the last section 5 these results are applied to prove Theorems 1 and 2.

§ 2. Main concepts

Recall that a *measure-valued* function on Ω is a weakly measurable map $x \to \nu_x$ of the set Ω into the space of probability Borel measures with compact support in $\mathbb{R}$. The weak measurability of ν_x means that for each continuous function $f(\lambda)$ the function $x \to \int f(\lambda)d\nu_x(\lambda)$ is Lebesgue-measurable on Ω.

Remark 1. If ν_x is a measure-valued function then, as was shown in [11], the functions $\int g(\lambda)d\nu_x(\lambda)$ are measurable in Ω for all bounded Borel functions $g(\lambda)$. More generally, if $f(x,\lambda)$ is a Caratheodory function and $g(\lambda)$ is a bounded Borel function then the function $\int f(x,\lambda)g(\lambda)d\nu_x(\lambda)$ is measurable. This follows from the fact that any Caratheodory function is strongly measurable as a map $x \to f(x,\cdot) \in C(\mathbb{R})$ (see [5], Chapter 2) and, therefore, is a pointwise limit of step functions $f_m(x,\lambda) = \sum\limits_i g_{mi}(x)h_{mi}(\lambda)$ so that for $x \in \Omega$ $f_m(x,\cdot) \underset{m\to\infty}{\to} f(x,\cdot)$ in $C(\mathbb{R})$.

A measure-valued function ν_x is said to be bounded if there exists $M > 0$ such that $\operatorname{supp}\nu_x \subset [-M, M]$ for almost all $x \in \Omega$. We denote the smallest value of M with this property by $\|\nu_x\|_\infty$.

Finally, measure-valued functions of the form $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $\delta(\lambda - u)$ is the Dirac measure concentrated at u are said to be *regular*; we identify them with the corresponding functions $u(x)$. Thus, the set $MV(\Omega)$ of bounded measure-valued functions on Ω contains the space $L^\infty(\Omega)$. Note that for a regular measure-valued function $\nu_x(\lambda) = \delta(\lambda - u(x))$ the value $\|\nu_x\|_\infty = \|u\|_\infty$. Extending the concept of boundedness in $L^\infty(\Omega)$ to measure-valued functions we shall say that a subset A of $MV(\Omega)$ is *bounded* if $\sup_{\nu_x\in A}\|\nu_x\|_\infty < \infty$.

We define below the weak and the strong convergence of sequences of measure-valued functions

Definition 4. Let $\nu_x^k \in MV(\Omega)$, $k \in \mathbb{N}$, and let $\nu_x \in MV(\Omega)$. Then
1) the sequence ν_x^k converges weakly to ν_x if for each $f(\lambda) \in C(\mathbb{R})$,

$$\int f(\lambda)d\nu_x^k(\lambda) \underset{k\to\infty}{\to} \int f(\lambda)d\nu_x(\lambda) \ \text{ in the weak-}* \text{ topology of } L^\infty(\Omega);$$

2) the sequence ν_x^k converges to ν_x *strongly* if for each $f(\lambda) \in C(\mathbb{R})$,

$$\int f(\lambda)d\nu_x^k(\lambda) \underset{k\to\infty}{\to} \int f(\lambda)d\nu_x(\lambda) \ \text{ in } L^1_{loc}(\Omega).$$

The next result was proved in [16] for regular functions ν_x^k. The proof can easily be extended to the general case, as was done in [11].

Theorem 3. *Let ν_x^k, $k \in \mathbb{N}$ be a bounded sequence of measure-valued functions. Then there exist a subsequence $\nu_x^r = \nu_x^k$, $k = k_r$, and a measure-valued function $\nu_x \in MV(\Omega)$ such that $\nu_x^r \to \nu_x$ weakly as $r \to \infty$.*

Theorem 3 shows that bounded sets of measure-valued functions are weak1y precompact.

We shall study the strong pre-compactness property using Tartar's techniques of H-measures.

Let $F(u)(\xi)$, $\xi \in \mathbb{R}^n$, be the Fourier transform of a function $u(x) \in L^2(\mathbb{R}^n)$, $S = S^{n-1} = \{ \xi \in \mathbb{R} \mid |\xi| = 1 \}$ be the unit sphere in $\mathbb{R}^n$. Denote by $u \to \overline{u}$, $u \in \mathbb{C}$ the complex conjugation.

The concept of an H-measure corresponding to some sequence of vector-valued functions bounded in $L^2(\Omega)$ was introduced by Tartar [17] and Gerard [4] on the basis of the following result. For $l \in \mathbb{N}$ let $U_k(x) = \big(U_k^1(x), \dots, U_k^l(x)\big) \in L^2(\Omega, \mathbb{R}^l)$ be a sequence weakly convergent to the zero vector.

Proposition 1 (see [17], Theorem 1.1). *There exists a family of complex Borel measures $\mu = \{\mu^{ij}\}_{i,j=1}^l$ in $\Omega \times S$ and a subsequence $U_r(x) = U_k(x)$, $k = k_r$, such that*

$$\langle \mu^{ij}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(U_r^i\Phi_1)(\xi)\overline{F(U_r^j\Phi_2)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi$$

for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$.

The family $\mu = \{\mu^{ij}\}_{i,j=1}^l$ is called the H-measure corresponding to $U_r(x)$.

The concept of H-measure has been extended in [11] (see also [12, 13]) to sequences of measure-valued functions. We study the properties of such H-measures in the next section.

§ 3. *H*-measures corresponding to bounded sequences of measure-valued functions

Let $\nu_x^k \in MV(\Omega)$ be a bounded sequence of measure-valued functions weakly convergent to a measure-valued function $\nu_x^0 \in MV(\Omega)$. For $x \in \Omega$ and $p \in \mathbb{R}$ we set

$$u_k(x,p) = \nu_x^k((p,+\infty)), \quad u_0(x,p) = \nu_x^0((p,+\infty)).$$

Then, as mentioned in Remark 1, for $k \in \mathbb{N} \cup \{0\}$ and $p \in \mathbb{R}$ the functions $u_k(x,p)$ are measurable in $x \in \Omega$; thus, $u_k(x,p) \in L^\infty(\Omega)$ and $0 \le u_k(x,p) \le 1$. Let

$$E = E(\nu_x^0) = \left\{ p_0 \in \mathbb{R} \;\middle|\; u_0(x,p) \underset{p\to p_0}{\longrightarrow} u_0(x,p_0) \text{ in } L^1_{loc}(\Omega) \right\}.$$

We have the following result, the proof of which can be found in [11].

Lemma 1. *The complement $\bar{E} = \mathbb{R} \setminus E$ is at most countable and if $p \in E$ then $u_k(x,p) \underset{k\to\infty}{\to} u_0(x,p)$ in the weak-$*$ topology in $L^\infty(\Omega)$.*

Let $U_k^p(x) = u_k(x,p) - u_0(x,p)$. Then, by Lemma 1, $U_k^p(x) \to 0$ as $k \to \infty$ weakly-$*$ in $L^\infty(\Omega)$ for $p \in E$.

The next result, similar to Proposition l, has also been established in [11].

Proposition 2. 1) *There exists a family of locally finite complex Borel measures $\{\mu^{pq}\}_{p,q\in E}$ in $\Omega \times S$ and a subsequence $U_r(x) = \{U_r^p(x)\}_{p\in E}$, $U_r^p(x) = U_k^p(x)$, $k = k_r$ such that for all $\Phi_1(x), \Phi_2(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$*

$$\langle \mu^{pq}, \Phi_1(x)\overline{\Phi_2(x)}\psi(\xi)\rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi. \quad (9)$$

2) *The correspondence $(p,q) \to \mu^{pq}$ is a continuous map from $E \times E$ into the space $\mathrm{M}_{loc}(\Omega \times S)$.*

Definition 5. *We call the family of measures $\{\mu^{pq}\}_{p,q\in E}$ the H-measure corresponding to the subsequence $\nu_x^r = \nu_x^k$, $k = k_r$.*

We point out the following important properties of an H-measure.

Lemma 2. 1) *$\mu^{pp} \geq 0$ for each $p \in E$;* 2) *$\mu^{pq} = \overline{\mu^{qp}}$ for all $p,q \in E$;* 3) *for $p_1, \dots, p_l \in E$ and $g_1, \dots, g_l \in C_0(\Omega \times S)$ the matrix $A = a_{ij} = \langle \mu^{p_i p_j}, g_i \overline{g_j}\rangle$, $i,j = 1, \dots, l$ is positive-definite.*

Proof. We prove 3). First let the functions $g_i = g_i(x,\xi)$ be finite sums of functions of the form $\Phi(x)\psi(\xi)$, where $\Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$. Then it follows from (9) that

$$a_{ij} = \lim_{r\to\infty} \int_{\mathbb{R}^n} H_r^i(\xi)\overline{H_r^j(\xi)} d\xi, \quad (10)$$

where $H_r^i(\xi) = F(g_i(\cdot, \xi/|\xi|)U_r^{p_i})(\xi)$. Hence setting $g_i(x,\xi) = g(x,\xi) = \sum_{k=1}^{m} \Phi_k(x)\psi_k(\xi)$ we obtain

$$H_r^i(\xi) = \sum_{k=1}^{m} F(\Phi_k U_r^{p_i})(\xi)\psi_k\left(\frac{\xi}{|\xi|}\right).$$

It immediately follows from (10) that $a_{ji} = \overline{a_{ij}}$, $i,j = 1, \dots, l$, which shows that A is a Hermitian matrix. Further, for $\alpha_1, \dots, \alpha_l \in \mathbb{C}$ we have

$$\sum_{i,j=1}^{l} a_{ij}\alpha_i\overline{\alpha_j} = \lim_{r\to\infty} \int_{\mathbb{R}^n} |H_r(\xi)|^2 d\xi \geq 0, \quad H_r(\xi) = \sum_{i=1}^{l} H_r^i(\xi)\alpha_i$$

which means that A is positive-definite.

In the general case of $g_i \in C_0(\Omega \times S)$ one carries out the proof of 3) by approximating the functions g_i, $i = 1, \dots, l$ in the uniform norm by finite sums of functions of the form $\Phi(x)\psi(\xi)$.

Assertions 1) and 2) are easy consequences of 3). For setting $l = 1$, $p_1 = p$ and $g_1 = g$ we obtain the relation $\langle \mu^{pp}, |g|^2 \rangle \geq 0$, which holds for all $g \in C_0(\Omega \times S)$, thus showing that μ^{pp} is real and non-negative. To prove 2) we represent an arbitrary function $g = g(x, \xi)$ with compact support in the form $g = g_1\overline{g_2}$. Let $l = 2$, $p_1 = p$ and $p_2 = q$. In view of 3),

$$\langle \mu^{pq}, g \rangle = \langle \mu^{pq}, g_1\overline{g_2} \rangle = \overline{\langle \mu^{qp}, g_2\overline{g_1} \rangle} = \overline{\langle \mu^{qp}, \overline{g} \rangle} = \langle \overline{\mu^{qp}}, g \rangle$$

and $\mu^{pq} = \overline{\mu^{qp}}$. The proof is complete.

We consider now a countable dense index subset $D \subset E$.

Proposition 3. *There exists a family of complex finite Borel measures μ_x^{pq} in the sphere S with $p, q \in D$, $x \in \Omega'$, where Ω' is a subset of Ω of full measure, such that $\mu^{pq} = \mu_x^{pq} dx$ that is, for all $\Phi(x, \xi) \in C_0(\Omega \times S)$ the function*

$$x \to \langle \mu_x^{pq}(\xi), \Phi(x, \xi) \rangle = \int_S \Phi(x, \xi) d\mu_x^{pq}(\xi)$$

is Lebesgue-measurable on Ω, bounded, and

$$\langle \mu^{pq}, \Phi(x, \xi) \rangle = \int_\Omega \langle \mu_x^{pq}(\xi), \Phi(x, \xi) \rangle dx.$$

Moreover, for $p, p', q \in D$, $p' > p$

$$\operatorname{Var} \mu_x^{pq} \leq 1 \quad \textit{and} \quad \operatorname{Var} (\mu_x^{p'q} - \mu_x^{pq}) \leq 2 \left(\nu_x^0((p, p')) \right)^{1/2}. \tag{11}$$

Proof. We claim that $\mathrm{pr}_\Omega \operatorname{Var} \mu^{pq} \leq \mathrm{meas}$ for $p, q \in E$, where meas is the Lebesgue measure on Ω. Assume first that $p = q$. By Lemma 2, the measure μ^{pp} is non-negative. Next, in view of relation (9) with $\Phi_1(x) = \Phi_2(x) = \Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \equiv 1$,

$$\langle \mu^{pp}, |\Phi(x)|^2 \rangle = \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi U_r^p)(\xi)\overline{F(\Phi U_r^p)(\xi)}d\xi =$$
$$\lim_{r\to\infty} \int_\Omega |U_r^p(x)|^2 |\Phi(x)|^2 dx \le \int_\Omega |\Phi(x)|^2 dx$$

(we use here Plancherel's equality and the estimate $|U_r^p(x)| \le 1$). Thus, we see that that $\mathrm{pr}_\Omega \mu^{pp} \le \mathrm{meas}$.

Let $p, q \in E$, A be a bounded open subset of Ω, and $g = g(x,\xi) \in C_0(A \times S)$, $|g| \le 1$. Let also $g_1 = g/\sqrt{|g|}$ (we set $g_1 = 0$ for $g = 0$) and $g_2 = \sqrt{|g|}$. Then $g_1, g_2 \in C_0(A \times S)$, $g = g_1\overline{g_2}$, $|g_1|^2 = |g_2|^2 = |g|$ and the matrix

$$\begin{pmatrix} \langle \mu^{pp}, |g| \rangle & \langle \mu^{pq}, g \rangle \\ \overline{\langle \mu^{pq}, g \rangle} & \langle \mu^{qq}, |g| \rangle \end{pmatrix}$$

is positive-definite by Lemma 2; in particular,

$$|\langle \mu^{pq}, g \rangle| \le (\langle \mu^{pp}, |g| \rangle \langle \mu^{qq}, |g| \rangle)^{1/2} \le (\mu^{pp}(A\times S)\mu^{qq}(A\times S))^{1/2} \le \mathrm{meas}(A).$$

We take account of the inequalities $\mathrm{pr}_\Omega \mu^{pp} \le \mathrm{meas}$ and $\mathrm{pr}_\Omega \mu^{qq} \le \mathrm{meas}$ to obtain the last estimate. Since g can be an arbitrary function in $C_0(A\times S)$, $|g| \le 1$, we obtain the inequality $\mathrm{Var}\, \mu^{pq}(A \times S) \le \mathrm{meas}(A)$, The measure μ^{pq} is regular, therefore this estimate holds for all Borel subsets A of Ω and

$$\mathrm{pr}_\Omega \mathrm{Var}\, \mu^{pq} \le \mathrm{meas}\,. \tag{12}$$

It follows from (12) that for all $\psi(\xi) \in C(S)$ we have

$$\mathrm{Var}\, \mathrm{pr}_\Omega \left(\psi(\xi)\mu^{pq}(x,\xi)\right) \le \|\psi\|_\infty \cdot \mathrm{pr}_\Omega \mathrm{Var}\, \mu^{pq} \le \|\psi\|_\infty \cdot \mathrm{meas}\,. \tag{13}$$

In view of (13) the measures $\mathrm{pr}_\Omega(\psi(\xi)\mu^{pq}(x,\xi))$ are absolutely continuous with respect to Lebesgue measure, and the Radon-Nikodym theorem shows that

$$\mathrm{pr}_\Omega \left(\psi(\xi)\mu^{pq}(x,\xi)\right) = h_\psi^{pq}(x) \cdot \mathrm{meas},$$

where the densities $h_\psi^{pq}(x)$ are measurable on Ω and, as seen from (13),

$$\|h_\psi^{pq}(x)\|_\infty \le \|\psi\|_\infty. \tag{14}$$

We now choose a non-negative function $K(x) \in C_0^\infty(\mathbb{R}^n)$ with support in the unit ball such that $\int K(x)dx = 1$ and set $K_m(x) = m^n K(mx)$ for $m \in \mathbb{N}$. Clearly, the sequence of K_m converges in $\mathcal{D}'(\mathbb{R}^n)$ to the Dirac δ-function (that is, this sequence is an approximate unity).

Let $B-\lim\limits_{m\to\infty} c_m$ be a generalized Banach limit on the space l_∞ of bounded sequences $c = \{c_m\}_{m\in\mathbb{N}}$, i.e. $L(c) = B - \lim\limits_{m\to\infty} c_m$ is a linear functional on l_∞ with the property:

$$\varliminf_{m\to\infty} c_m \le L(c) \le \varlimsup_{m\to\infty} c_m$$

(in particular for convergent sequences $c = \{c_m\}$ $L(c) = \lim\limits_{m\to\infty} c_m$). For complex sequences $c_m = a_m + ib_m$ the Banach limits is defined by complexification: $B - \lim\limits_{m\to\infty} c_m = L(a) + iL(b)$, where $a = \{a_m\}$, $b = \{b_m\}$ are real and imaginary parts of the sequence $c = \{c_m\}$, respectively. Modifying the densities $h_\psi^{pq}(x)$ on subsets of measure zero, for instance, replacing them by the functions

$$B - \lim_{m\to\infty} \int_\Omega h_\psi^{pq}(y) K_m(x-y)dy$$

(obviously, the value $h_\psi^{pq}(x)$ does not change for any Lebesgue point x of the function h_ψ^{pq}), we shall assume that for all $x \in \Omega$ we have

$$h_\psi^{pq}(x) = B - \lim_{m\to\infty} \int_\Omega h_\psi^{pq}(y) K_m(x-y)dy. \tag{15}$$

Let Ω' be the set of common Lebesgue points of the functions $h_\psi^{pq}(x)$, $u_0(x,p) = \nu_x^0((p,+\infty))$, and $u_0^-(x,p) = \nu_x^0([p,+\infty)) = \lim\limits_{q\to p-} u_0(x,q)$, where $p,q \in D$ and ψ belongs to F, some countable dense subset of $C(S)$. The family of (p,q,ψ) is countable, therefore Ω' is of full measure.

The dependence of the h_ψ^{pq} on ψ, regarded as a map from $C(S)$ into $L^\infty(\Omega)$, is clearly linear and continuous (in view of (14)), therefore it follows from the density of F in $C(S)$ that $x \in \Omega'$ is a Lebesgue point of the functions $h_\psi^{pq}(x)$ for all $\psi(\xi) \in C(S)$ and $p,q \in D$ (here we also take account of (15)).

For $p,q \in D$ and $x \in \Omega'$ the equality $l(\psi) = h_\psi^{pq}(x)$ defines a continuous linear functional in $C(S)$; moreover, $\|l\| \le 1$ in view of (14). By the Riesz-Markov theorem this functional can be defined by integration with respect to some complex Borel measure $\mu_x^{pq}(\xi)$ in S and $\operatorname{Var}\mu_x^{pq} = \|l\| \le 1$. Hence

$$h_\psi^{pq}(x) = \langle \mu_x^{pq}(\xi), \psi \rangle = \int_S \psi(\xi) d\mu_x^{pq}(\xi) \tag{16}$$

for all $\psi(\xi) \in C(S)$.

Equality (16) shows that the functions $x \to \int_S \psi(\xi) d\mu_x^{pq}(\xi)$ are bounded and measurable for all $\psi(\xi) \in C(S)$. Next, for $\Phi(x) \in C_0(\Omega)$ and $\psi(\xi) \in C(S)$ we have

$$\int_\Omega \left(\int_S \Phi(x)\psi(\xi) d\mu_x^{pq}(\xi) \right) dx = \int_\Omega \Phi(x) h_\psi^{pq}(x) dx =$$
$$\int_\Omega \Phi(x) d\mathrm{pr}_\Omega \left(\psi(\xi)\mu^{pq}\right) = \int_{\Omega\times S} \Phi(x)\psi(\xi) d\mu^{pq}(x,\xi). \tag{17}$$

Approximating an arbitrary function $\Phi(x,\xi) \in C_0(\Omega \times S)$ in the uniform norm by linear combinations of functions of the form $\Phi(x)\psi(\xi)$ we derive from (17) that the integral $\int_S \Phi(x,\xi) d\mu_x^{pq}(\xi)$ is Lebesgue-measurable with respect to $x \in \Omega$, bounded, and

$$\int_\Omega \left(\int_S \Phi(x,\xi) d\mu_x^{pq}(\xi) \right) dx = \int_{\Omega\times S} \Phi(x,\xi) d\mu^{pq}(x,\xi)$$

that is, $\mu^{pq} = \mu_x^{pq} dx$. Recall that $\operatorname{Var}\mu_x^{pq} \le 1$.

It remains to prove the last estimate in (11). Let $p, p', q \in D$, $p' > p$ and $x \in \Omega'$. We set $\Phi_m = \sqrt{K_m} \in C_0(\mathbb{R}^n)$, $m \in \mathbb{N}$, where the sequence of kernels K_m is as defined above. Starting from some index m the function $\Phi_m(x-y)$ (of the y-variable) belongs to $C_0(\Omega)$ and, in view of Proposition 2, for all $\psi(\xi) \in C(S)$ we have

$$\left| \int_\Omega K_m(x-y) \left(h_\psi^{p'q}(y) - h_\psi^{pq}(y) \right) dy \right| =$$
$$\left| \langle (\mu^{p'q} - \mu^{pq})(y,\xi), K_m(x-y)\psi(\xi) \rangle \right| =$$
$$\lim_{r\to\infty} \left| \int_{\mathbb{R}^n} F(\Phi_m(U_r^{p'} - U_r^p))(\xi) \overline{F(\Phi_m U_r^q)(\xi)} \psi\left(\frac{\xi}{|\xi|}\right) d\xi \right| \le$$
$$\|\psi\|_\infty \varlimsup_{r\to\infty} \left[\left(\int_{\mathbb{R}^n} |F(\Phi_m(U_r^{p'} - U_r^p))(\xi)|^2 d\xi \right)^{1/2} \right.$$

$$\times \left(\int_{\mathbb{R}^n} |F(\Phi_m U_r^q)(\xi)|^2 d\xi \right)^{1/2} \Bigg] =$$

$$= \|\psi\|_\infty \varlimsup_{r\to\infty} \left[\left(\int_\Omega K_m(x-y)(U_r^{p'}(y) - U_r^p(y))^2 dy \right)^{1/2} \right.$$

$$\left. \times \left(\int_\Omega K_m(x-y)(U_r^q(y))^2 dy \right)^{1/2} \right]. \tag{18}$$

Note that $|U_r^q| \le 1$, $\int_\Omega K_m(x-y)dy = 1$ and, therefore,

$$\int_\Omega K_m(x-y)(U_r^q(y))^2 dy \le 1. \tag{19}$$

Further,

$$\int_\Omega K_m(x-y)(U_r^{p'}(y) - U_r^p(y))^2 dy \le$$
$$2\int_\Omega K_m(x-y)|U_r^{p'}(y) - U_r^p(y)| dy \le$$
$$2\int_\Omega K_m(x-y)(u_r(y,p) - u_r(y,p'))dy +$$
$$2\int_\Omega K_m(x-y)(u_0(y,p) - u_0(y,p'))dy \tag{20}$$

(note that $u_r(y,p) - u_r(y,p') \ge 0$ for $r \in \mathbb{N}\cup\{0\}$). Since $p, p' \subset E$, it follows from Lemma 1 that $u_r(y,p) - u_r(y,p') \underset{r\to\infty}{\to} u_0(y,p) - u_0(y,p')$ in the weak-$*$ topology in $L^\infty(\Omega)$, therefore

$$\lim_{r\to\infty} \int_\Omega K_m(x-y)(u_r(y,p) - u_r(y,p'))dy =$$
$$\int_\Omega K_m(x-y)(u_0(y,p) - u_0(y,p'))dy,$$

and by (20),

$$\varlimsup_{r\to\infty} \left(\int_\Omega K_m(x-y)(U_r^{p'}(y) - U_r^p(y))^2 dy \right)^{1/2} \le$$
$$2\left(\int_\Omega K_m(x-y)(u_0(y,p) - u_0(y,p'))dy \right)^{1/2}. \tag{21}$$

From (18), in view of (19), (21), we obtain the estimate

$$\left| \int_\Omega K_m(x-y)(h_\psi^{p'q}(y) - h_\psi^{pq}(y))dy \right| \le$$

$$2\|\psi\|_\infty \left(\int_\Omega K_m(x-y)(u_0(y,p) - u_0(y,p'))dy \right)^{1/2}$$

and passing to the limit as $m \to \infty$, since $x \in \Omega'$ is a Lebesgue point of the functions $h_\psi^{p'q}$, h_ψ^{pq}, and $u_0(\cdot\,,p')$, we obtain the inequality

$$\left| h_\psi^{p'q}(x) - h_\psi^{pq}(x) \right| \le \|\psi\|_\infty \left(u_0(x,p) - u_0(x,p')\right)^{1/2},$$

that is, for all $\psi(\xi) \in C(S)$ we have

$$\left| \langle \mu_x^{p'q} - \mu_x^{pq}, \psi \rangle \right| \le 2\|\psi\|_\infty \left(u_0(x,p) - u_0(x,p')\right)^{1/2},$$

and therefore

$$\mathrm{Var}\,(\mu_x^{p'q} - \mu_x^{pq}) \le 2\left(u_0(x,p) - u_0(x,p')\right)^{1/2} = 2\left(\nu_x^0((p,p'])\right)^{1/2}. \qquad (22)$$

Now we demonstrate that for $x \in \Omega'$ $\nu_x(\{p\}) = 0$ for each $p \in D$. Indeed, $\nu_x^0(\{p\}) = u_0^-(x,p) - u_0(x,p)$ and since $p \in D \subset E$ is a continuity point of the map $p \to u_0(x,p)$ in $L^1_{loc}(\Omega)$ we conclude that $u_0^-(x,p) - u_0(x,p) = 0$ a.e. in Ω. By the construction $x \in \Omega'$ is a common Lebesgue point of this function, therefore $\nu_x^0(\{p\}) = u_0^-(x,p) - u_0(x,p) = 0$, as required. In particular $\nu_x^0(\{p'\}) = 0$ and we can replace the segment $(p,p']$ in estimate (22) by the interval (p,p'). The proof is complete.

Corollary 1. *The correspondences $p \to \mu_x^{pq}$ and $q \to \mu_x^{pq}$ are continuous maps of the set D into the space* $\mathrm{M}(S)$ *of finite complex Borel measures in S (with norm* Var *).*

Proof. The continuity of the map $p \to \mu_x^{pq}$ is an immediate consequence of estimate (11). In the case of the map $q \to \mu_x^{pq}$ we must take into account the equality $\mu_x^{pq} = \overline{\mu_x^{qp}}$, which is an easy consequence of Lemma 2(2).

Remark 2. a) Since the H-measure is absolutely continuous with respect to x-variables identity (9) is satisfied for $\Phi_1(x), \Phi_2(x) \in L^2(\Omega)$. Indeed, by Proposition 3 we can rewrite this identity in the form: $\forall \Phi_1(x), \Phi_2(x) \in C_0(\Omega)$, $\psi(\xi) \in C(S)$

$$\int_\Omega \Phi_1(x)\overline{\Phi_2(x)}\langle \psi(\xi), \mu_x^{pq}(\xi)\rangle dx =$$

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r^p)(\xi)\overline{F(\Phi_2 U_r^q)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi. \qquad (23)$$

Both sides of this identity are continuous with respect to $(\Phi_1(x), \Phi_2(x))$ in $L^2(\Omega) \times L^2(\Omega)$ and since $C_0(\Omega)$ is dense in $L^2(\Omega)$ we conclude that (23) is satisfied for each $\Phi_1(x), \Phi_2(x) \in L^2(\Omega)$;

b) if $x \in \Omega'$ is a Lebesgue point of a function $\Phi(x) \in L^2(\Omega)$ then

$$\Phi(x)\langle \mu_x^{pq}, \psi(\xi)\rangle = \lim_{m\to\infty} \lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi\Phi_m U_r^p)(\xi)\overline{F(\Phi_m U_r^q)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi \tag{24}$$

for all $\psi(\xi) \in C(S)$, where $(\Phi\Phi_m U_r^p)(y) = \Phi(y)\Phi_m(x-y)U_r^p(y)$ and $(\Phi_m U_r^q)(y) = \Phi_m(x-y)U_r^q(y)$.

Indeed, it follows from (23) that

$$\lim_{r\to\infty} \int_{\mathbb{R}^n} F(\Phi\Phi_m U_r^p)(\xi)\overline{F(\Phi_m U_r^q)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) d\xi = \int_\Omega h_\psi^{pq}(y)\Phi(y)K_m(x-y)dy. \tag{25}$$

Now, since $x \in \Omega'$ is a Lebesgue point of the functions $h_\psi^{pq}(y)$ and $\Phi(y)$, and the function $h_\psi^{pq}(y)$ is bounded, x is also a Lebesgue point for the product of these functions. Therefore,

$$\lim_{m\to\infty} \int_\Omega h_\psi^{pq}(y)\Phi(y)K_m(x-y)dy = \Phi(x)h_\psi^{pq}(x) = \Phi(x)\langle \mu_x^{pq}, \psi(\xi)\rangle,$$

and (24) follows from (25) in the limit as $m \to \infty$;

c) for $x \in \Omega'$ and each families $p_i \in D$, $\psi_i(\xi) \in C(S)$, $i = 1, \ldots, l$ the matrix $\langle \mu_x^{p_ip_j}, \psi_i\overline{\psi_j}\rangle$, $i,j = 1,\ldots,l$ is positive definite. Indeed, as follows from Lemma 2(3), for $\alpha_1, \ldots, \alpha_l \in \mathbb{C}$

$$\sum_{i,j=1}^{l} \langle \mu_x^{p_ip_j}, \psi_i\overline{\psi_j}\rangle \alpha_i\overline{\alpha_j} =$$

$$\lim_{m\to\infty} \sum_{i,j=1}^{l} \langle \mu^{p_ip_j}(y,\xi), \Phi_m(x-y)\psi_i(\xi)\overline{\Phi_m(x-y)\psi_j(\xi}\rangle \alpha_i\overline{\alpha_j} \ge 0.$$

Taking in the above property $l = 2$, $p_1 = p$, $p_2 = q$, $\psi_1(\xi) = \psi(\xi)/\sqrt{|\psi(\xi)|}$ ($\psi_1 = 0$ for $\psi = 0$) and $\psi_2(\xi) = \sqrt{|\psi(\xi)|}$, $\psi(\xi) \in C(S)$, we obtain, as in the proof of Proposition 3, that the matrix $\begin{pmatrix} \langle \mu_x^{pp}, |\psi|\rangle & \langle \mu_x^{pq}, \psi\rangle \\ \overline{\langle \mu_x^{pq}, \psi\rangle} & \langle \mu_x^{qq}, |\psi|\rangle \end{pmatrix}$ is positive definite. In particular,

$$|\langle \mu_x^{pq}, \psi\rangle| \le (\langle \mu_x^{pp}, |\psi|\rangle \cdot \langle \mu_x^{qq}, |\psi|\rangle)^{1/2}$$

and this easily implies that for any Borel set $A \subset S$

$$\operatorname{Var} \mu_x^{pq}(A) \leq \left(\mu_x^{pp}(A)\mu_x^{qq}(A)\right)^{1/2}. \tag{26}$$

We now fix $x \in \Omega'$, $p_0 \in D$. Let $L(p) \subset \mathbb{R}^n$ be the smallest linear subspace containing $\operatorname{supp} \mu_x^{pp_0}$, $p \in D$, and let $L = L(p_0)$, $l = \dim L$.

Lemma 3. *There exists positive δ such that $L(p) = L$ for each $p \in [p_0 - \delta, p_0 + \delta] \cap D$.*

Proof. Remark firstly that, as it directly follows from (26), $\operatorname{supp} \mu_x^{pp_0} \subset \operatorname{supp} \mu_x^{p_0p_0} \subset L$ and, therefore $L(p) \subset L$. For positive r we denote $V_r = [p_0 - r, p_0 + r] \cap D$, $L_r = \bigcap_{p \in V_r} L(p)$. Clearly, $L_r \subset L$ is a decreasing (with respect to inclusion) family of linear subspaces of the finite-dimensional space L, therefore starting from some quantity $r = \delta > 0$ for all $r \in (0, \delta]$ we have $L_r = \tilde{L} \subset L$. To prove the lemma it suffices to show that $\tilde{L} = L$. For in that case $L \subset L(p) \subset L$ and the equality $L(p) = L$, $p \in V_\delta$ follows. We carry out the proof of the equality $\tilde{L} = L$ by contradiction. Thus, we assume that $\tilde{L} \neq L$. Then $m = \dim \tilde{L} < l = \dim L$. We fix $\varepsilon > 0$. By Corollary 1 there exists $r \in (0, \delta]$ such that for $p \in V_r$ we have

$$\operatorname{Var}\left(\mu_x^{pp_0} - \mu_x^{p_0p_0}\right) < \varepsilon. \tag{27}$$

By the definition of the space L_r we can choose a strictly decreasing finite sequence of subspaces L'_i, $i = 0, \ldots, k$, such that $L'_0 = L$, $L'_k = L_\delta = \tilde{L}$, and $L'_i = L'_{i-1} \cap L(p_i)$, where $p_i \in V_r$, $i = 1, \ldots, k$. Clearly, $k \leq \dim L - \dim \tilde{L} = l - m$. By the definition of the $L(p)$ we have $\operatorname{supp} \mu_x^{p_ip_0} \subset L(p_i)$. Hence $\operatorname{Var}\left(\mu_x^{p_ip_0}(CL(p_i)\right) = 0$, where CA for $A \subset \mathbb{R}^n$ is the difference $S \setminus A$. It now follows from (27) that

$$\mu_x^{p_0p_0}(CL(p_i)) < \varepsilon, \quad i = 1, \ldots, k.$$

Since $\tilde{L} = \bigcap_{i=1}^k L(p_i)$, it follows that $C\tilde{L} = \bigcup_{i=1}^k CL(p_i)$ and

$$\mu_x^{p_0p_0}(C\tilde{L}) \leq \sum_{i=1}^k \mu_x^{p_0p_0}(CL(p_i)) \leq k\varepsilon.$$

Since ε is an arbitrary positive number, it follows that $\mu_x^{p_0p_0}(C\tilde{L}) = 0$ and $\operatorname{supp} \mu_x^{p_0p_0} \subset \tilde{L}$. Further, L is the smallest subspace such that $\operatorname{supp} \mu_x^{p_0p_0} \subset L$, therefore $L \subset \tilde{L}$, which is a contradiction. This completes the proof.

We consider now the complex linear subspace

$$R(p) = \left\{ \int \psi(\xi)\xi d\mu_x^{ppo}(\xi) : \ \psi(\xi) \in C(S) \right\} \subset \mathbb{C}^n.$$

Lemma 4. *We have the equality $R(p) = \bar{L}(p)$, where $\bar{L}(p) = L(p) + iL(p) \subset \mathbb{C}^n$ is the complex linear subspace spanned by $L(p)$.*

Proof. The relation

$$\left(\int \psi(\xi)\xi d\mu_x^{ppo}(\xi), \nu \right) = \int \psi(\xi)(\xi, \nu) d\mu_x^{ppo}(\xi), \quad \nu \in \mathbb{C}^n, \quad \psi(\xi) \in C(S)$$

(here and below we consider the scalar products $(\cdot\,,\,\cdot)$ of vectors in $\mathbb{C}^n$) shows us that the orthogonal complements $(R(p))^\perp = (L(p))^\perp$ are the same (in $\mathbb{C}^n$), which means that $R(p) = \bar{L}(p)$. The proof is complete.

Suppose that $f(y, \lambda)$ is a Caratheodory vector-function on $\Omega \times \mathbb{R}$, i.e. $f(y, \cdot) \in C(\mathbb{R}, \mathbb{R}^n)$ for each $y \in \Omega$ and the functions $x \to f(x, \lambda)$ are Lebesgue measurable on Ω for every fixed $\lambda \in \mathbb{R}$. Assume also that the following estimate holds

$$\forall M > 0 \quad \|f(x, \cdot)\|_{M,\infty} = \max_{|\lambda| \le M} |f(x, \lambda)| \le \alpha_M(x) \in L^2_{loc}(\Omega). \tag{28}$$

Since the space $C(\mathbb{R}, \mathbb{R}^n)$ is separable with respect to the standard locally convex topology generated by seminorms $\| \cdot \|_{M,\infty}$, then, by the Pettis theorem (see [5], Chapter 3), the map $x \to F(x) = f(x, \cdot) \in C(\mathbb{R}, \mathbb{R}^n)$ is strongly measurable and in view of estimate (28) we see that $F(x), (F(x))^2 \in L^1_{loc}(\Omega, C(\mathbb{R}, \mathbb{R}^n))$. In particular (see [5], Chapter 3), a.e. $x \in \Omega$ are Lebesgue points both maps $F(x), (F(x))^2$, i.e.

$$\forall M > 0 \ \lim_{m \to \infty} \int K_m(x - y) \|F(x) - F(y)\|_{M,\infty} dy =$$
$$\lim_{m \to \infty} \int K_m(x - y) \|(F(x))^2 - (F(y))^2\|_{M,\infty} dy = 0.$$

Since, evidently,

$$\|F(x) - F(y)\|^2_{M,\infty} \le 2\|F(x) - F(y)\|_{M,\infty} \|F(x)\|_{M,\infty} + \|(F(x))^2 - (F(y))^2\|_{M,\infty},$$

from the above relation it follows that for a set $\Omega_f \subset \Omega$ of full measure of values x

$$\lim_{m \to \infty} \int K_m(x - y) \|F(x) - F(y)\|^2_{M,\infty} dy = 0 \quad \forall M > 0. \tag{29}$$

Clearly, each $x \in \Omega_f$ is a Lebesgue point of all functions $x \to f(x,\lambda)$, $\lambda \in \mathbb{R}$. Let $\Omega'' = \Omega' \cap \Omega_f$, $\gamma_x^r = \nu_x^r - \nu_x^0$. By $\theta(\lambda)$ we shall denote the Heaviside function:

$$\theta(\lambda) = \begin{cases} 1, & \lambda > 0, \\ 0, & \lambda \le 0. \end{cases}$$

Suppose that $x \in \Omega''$, $p_0 \in D$, and the subspace L and the segment $V = V_\delta = [p_0 - \delta, p_0 + \delta] \cap D$ are determined as in Lemma 3, $\chi(\lambda) = \theta(\lambda - p_1) - \theta(\lambda - p_2)$, where $p_1, p_2 \in V$. Assume also that $f(y,\lambda)$ takes its values in $L^\perp$. For a vector-function $h(y,\lambda)$ on $\Omega \times \mathbb{R}$, which is Borel and locally bounded with respect to the second variable, we denote $I_r(h)(y) = \int h(y,\lambda)d\gamma_y^r(\lambda)$. In view of the strong measurability of $F(x)$ and (28) we see that $I_r(f\cdot\chi)(y) \in L^2_{loc}(\Omega)$ (cf. Remark 1).

Proposition 4. *Under the above assumptions,*

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|}, F(\Phi_m I_r(f\cdot\chi))(\xi)\right)\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi = 0$$

for all $\psi(\xi) \in C(S)$. *Here* $\Phi_m = \Phi_m(x-y) = \sqrt{K_m(x-y)}$ *and* $I_r(f\cdot\chi), U_r^{p_0}$ *are functions of the variable* $y \in \Omega$.

Proof. Note that starting from some index m the supports of the $\Phi_m(x-y)$ lie in some compact subset B of Ω. Without loss of generality we can assume that $\operatorname{supp}\Phi_m \subset B$ for all $m \in \mathbb{N}$. Let $\tilde f(y,\lambda) = f(x,\lambda)$, $M = \sup\|\nu_y^r\|_\infty$. Then

$$|I_r((f-\tilde f)\cdot\chi)(y)| \le \int |f(y,\lambda) - f(x,\lambda)|d\mathrm{Var}\,\gamma_y^r(\lambda) \le 2\|F(y) - F(x)\|_{M,\infty}$$

and by Plancherel's identity

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|}, F(\Phi_m I_r(f\cdot\chi))(\xi)\right)\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi - \right.\\ &\left.\int_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|}, F(\Phi_m I_r(\tilde f\cdot\chi))(\xi)\right)\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right| = \\ &\left|\int_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|}, F(\Phi_m I_r((f-\tilde f)\cdot\chi))(\xi)\right)\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right| \le \\ &\|\psi\|_\infty\|\Phi_m I_r((f-\tilde f)\cdot\chi)\|_2\|\Phi_m U_r^{p_0}\|_2 \le C\|\Phi_m I_r((f-\tilde f)\cdot\chi)\|_2 \le \\ &2C\left(\int K_m(x-y)\|F(y)-F(x)\|^2_{M,\infty}dy\right)^{1/2}, \ C = \text{const.}\end{aligned}$$

Here we take account of the equality

$$\|\Phi_m\|_2 = \left(\int_\Omega K_m(x-y)dy\right)^{1/2} = 1.$$

From the above estimate and (29) it follows that

$$\lim_{m\to\infty}\lim_{r\to\infty}\left|\int_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|}, F(\Phi_m I_r(f\cdot\chi))(\xi)\right)\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi - \int_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|}, F(\Phi_m I_r(\tilde{f}\cdot\chi))(\xi)\right)\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right| = 0 \quad (30)$$

and it is sufficient to prove the proposition with f replaced by $\tilde{f}$. This function is continuous and does not depend on y. Therefore for any $\varepsilon > 0$ there exists a vector-valued function $g(\lambda)$ of the form $g(\lambda) = \sum\limits_{i=1}^{k} v_i\theta(\lambda - p_i)$, where $v_i \in L^\perp$ and $p_i \in V$ such that $\|\tilde{f}\cdot\chi - g\|_\infty \le \varepsilon$ on $\mathbb{R}$.

Using again Plancherel's identity and the fact that

$$\left|\int(\tilde{f}\cdot\chi - g)(\lambda)d\gamma_y^r(\lambda)\right| \le \int |(\tilde{f}\cdot\chi - g)(\lambda)|d\mathrm{Var}\,(\gamma_y^r)(\lambda) \le 2\varepsilon,$$

we obtain

$$\left|\int_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|}, F(\Phi_m I_r(\tilde{f}\cdot\chi - g))(\xi)\right)\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right| \le \|\Phi_m I_r(\tilde{f}\cdot\chi - g)\|_2\cdot\|\Phi_m U_r^{p_0}\|_2\cdot\|\psi\|_\infty \le c\varepsilon \quad (31)$$

for $\psi(\xi) \in C(S)$, where c is a constant independent of m.

Since

$$I_r(g)(y) = \int\left(\sum_{i=1}^{k} v_i\theta(\lambda - p_i)\right)d\gamma_y^r(\lambda) = \sum_{i=1}^{k} v_i U_r^{p_i}(y),$$

we obtain the limit relation

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|}, F(\Phi_m I_r(g))(\xi)\right)\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi = \sum_{i=1}^{k}\langle\mu_x^{p_i p_0}, (v_i,\xi)\psi(\xi)\rangle = 0. \quad (32)$$

The last equality is a consequence of the inclusion $\operatorname{supp}\mu_x^{p_ip_0}\subset L$, which holds by Lemma 3 for all $i=1,\dots,k$ (because $p_i\in V$), combined with the relation $v_i\perp L$. By (31) and (32),

$$\varlimsup_{m\to\infty}\varlimsup_{r\to\infty}\left|\int_{\mathbb{R}^n}\left(\frac{\xi}{|\xi|},F(\Phi_m I_r(\tilde f\cdot\chi))(\xi)\right)\overline{F(\Phi_m U_r^{p_0})(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right|\le \mathrm{const}\cdot\varepsilon,$$

and it suffices to observe that $\varepsilon>0$ can be arbitrary to complete the proof.

§ 4. Localization principle and strong pre-compactness of bounded sequences of measure-valued functions.

Let $\varphi(x,\lambda)=(\varphi_1(x,\lambda),\dots,\varphi_n(x,\lambda))$ be a Caratheodory vector on $\Omega\times\mathbb{R}$, such that for each $M>0$

$$\alpha_M(x)=\max_{|u|\le M}|\varphi(x,u)|\in L^2_{loc}(\Omega). \tag{33}$$

Consider a bounded sequence ν_x^k, $k\in\mathbb{N}$ of measure-valued functions, and suppose that for each $p\in\mathbb{R}$ the sequence of distributions

$$\operatorname{div}_x\left(\int\theta(\lambda-p)(\varphi(x,\lambda)-\varphi(x,p))d\nu_x^k(\lambda)\right)\quad\text{is pre-compact in } H^{-1}_{loc}(\Omega). \tag{34}$$

Here $\theta(u)$ is the Heaviside function and $H^{-1}_{loc}(\Omega)$ is the locally convex space of distributions $u(x)$ such that $uf(x)$ belongs to the Sobolev space H_2^{-1} for all $f(x)\in C_0^\infty(\Omega)$. The topology in $H^{-1}_{loc}(\Omega)$ is generated by the family of semi-norms $u\to\|uf\|_{H_2^{-1}}$, $f(x)\in C_0^\infty(\Omega)$.

We choose a subsequence $\nu_x^r=\nu_x^k$, $k=k_r$ weakly convergent to a bounded measure-valued function ν_x^0 such that the H-measure $\mu^{pq}=\mu_x^{pq}dx$, $p,q\in D$ is well defined.

Define the measures $\gamma_x^r=\nu_x^r-\nu_x^0$ and set of full measure $\Omega''=\Omega'\cap\Omega_\varphi$ as in the previous section.

The following Theorem shows that $\operatorname{supp}\mu_x^{pp}$ consists of $\xi\in S$ such that the function $(\varphi(x,\lambda),\xi)=\sum_{i=1}^n\varphi_i(x,\lambda)\xi_i$ is constant in a vicinity of p.

Theorem 4 (localization principle). *Suppose that $x\in\Omega''$ and $\mu_x^{p_0p_0}\ne 0$ for some $p_0\in D$. Then there exists $\delta>0$ such that $(\varphi(x,\lambda),\xi)=$ const on the interval $\lambda\in(p_0-\delta,p_0+\delta)$ for all $\xi\in\operatorname{supp}\mu_x^{p_0p_0}$.*

Proof. Throughout the proof we use the notation of § 3. Let $V = V_\delta = [p_0 - \delta, p_0 + \delta] \cap D$ be an interval chosen in accordance with Lemma 3, L be a linear span of $\operatorname{supp} \mu_x^{p_0 p_0}$. As follows from (34) and the weak convergence $\nu_y^r \to \nu_y^0$,

$$\mathcal{L}_p^r(y) = \operatorname{div}_y \left(\int \theta(\lambda - p)(\varphi(y,\lambda) - \varphi(y,p)) d\gamma_y^r(\lambda) \right) \underset{r\to\infty}{\to} 0 \text{ in } H_{loc}^{-1}(\Omega). \tag{35}$$

For $p \in V$ we consider the sequence of distributions

$$\mathcal{L}_p^r - \mathcal{L}_{p_0}^r = \operatorname{div}_y(Q_r^p(y)), \quad r \in N,$$

where the vector-valued functions $Q_r^p(y)$ are as follows:

$$\begin{aligned} Q_r^p(y) = \int (\varphi(y,\lambda) - \varphi(y,p))\theta(\lambda - p) d\gamma_y^r(\lambda) - \\ \int (\varphi(y,\lambda) - \varphi(y,p_0))\theta(\lambda - p_0) d\gamma_y^r(\lambda) = \\ \int (\varphi(y,p) - \varphi(y,\lambda))\chi(\lambda) d\gamma_y^r(\lambda) - \\ \int (\varphi(y,p) - \varphi(y,p_0))\theta(\lambda - p_0) d\gamma_y^r(\lambda) = \\ \int (\varphi(y,p) - \varphi(y,\lambda))\chi(\lambda) d\gamma_y^r(\lambda) - (\varphi(y,p) - \varphi(y,p_0)) U_r^{p_0}(y); \end{aligned} \tag{36}$$

here $\chi(\lambda) = \theta(\lambda - p_0) - \theta(\lambda - p)$.

As already noted, $\operatorname{div}_y(Q_r^p(y)) \underset{r\to\infty}{\to} 0$ in $H_{loc}^{-1}(\Omega)$ and if $\Phi(y) \in C_0^\infty(\Omega)$ then

$$\operatorname{div}_y(Q_r^p \Phi(y)) \underset{r\to\infty}{\to} 0 \ \text{ in } H_2^{-1}. \tag{37}$$

Using the Fourier transformation, from (37) we obtain

$$|\xi|^{-1}(\xi, F(Q_r^p \Phi)(\xi)) \to 0 \ \text{ in } L^2(\mathbb{R}^n) \tag{38}$$

as $r \to \infty$. Indeed, as follows from the definition of H_2^{-1} (see, for instance, [1]), (37) is equivalent to the following condition:

$$(1 + |\xi|)^{-1}(\xi, F(Q_r^p \Phi)(\xi)) \underset{r\to\infty}{\to} 0 \ \text{ in } L^2(\mathbb{R}^n),$$

which shows that

$$|\xi|^{-1}(\xi, F(Q_r^p \Phi)(\xi)) \to 0 \ \text{ in } L^2(\mathbb{R}^n \setminus B) \tag{39}$$

as $r \to \infty$ (here B is the ball $\{\xi \in \mathbb{R}^n : |\xi| \le 1\}$). By (33) we have the uniform estimate $\|Q_r^p\Phi\|_1 \le 8\|\alpha_M\Phi\|_1 = \text{const}$, where $M = \max(\sup\|\nu_x\|_\infty, |p|, |p_0|)$. This estimate implies that the functions $|\xi|^{-1}(\xi, F(Q_r^p\Phi)(\xi))$ are bounded uniformly in $r \in \mathbb{N}$. By assumption, $\nu_y^r \underset{r\to\infty}{\to} \nu_y^0$ weakly in $MV(\Omega)$, which easily implies that $Q_r^p(y) \underset{r\to\infty}{\to} 0$ in $L^\infty(\Omega, \mathbb{R}^n)$ in the weak-$*$ topology, and the sequence $F(Q_r^p\Phi)(\xi)$ converges pointwise to zero as $r \to \infty$. Hence, it follows from Lebesgue's dominated convergence theorem that

$$|\xi|^{-1}(\xi, F(Q_r^p\Phi)(\xi)) \to 0 \quad \text{in } L^2(B)$$

as $r \to \infty$. Combined with (39) this yields relation (38) in an obvious way. Let $\psi(\xi) \in C(S)$. By (38), using the boundedness of the sequence $U_r^{p_0}\Phi(x)$ in $L^2(\mathbb{R}^n)$ we obtain

$$\int_{\mathbb{R}^n} |\xi|^{-1}(\xi, F(Q_r^p\Phi)(\xi))\overline{F(U_r^{p_0}\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right) \to 0$$

as $r \to \infty$, or in view of (36),

$$\lim_{r\to\infty}\left\{\int_{\mathbb{R}^n} |\xi|^{-1}(\xi, F(U_r^{p_0}f\Phi)(\xi))\overline{F(U_r^{p_0}\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi - \int_{\mathbb{R}^n} |\xi|^{-1}(\xi, F(V_r^p\Phi)(\xi))\overline{F(U_r^{p_0}\Phi)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi\right\} = 0, \tag{40}$$

where

$$f(y) = \varphi(y,p) - \varphi(y,p_0) \quad \text{and} \quad V_r^p(y) = \int (\varphi(y,p) - \varphi(y,\lambda))\chi(\lambda)d\gamma_y^r(\lambda).$$

We set in (40) $\Phi(y) = \Phi_m(x-y)$, where the functions Φ_m were defined in § 3 in the proof of Proposition 3, and pass to the limit as $m \to \infty$. By Remark 2 (see equality (24)) we obtain

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-1}(\xi, F(U_r^{p_0}f\Phi_m)(\xi))\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi = (\varphi(x,p) - \varphi(x,p_0), \langle\mu_x^{p_0p_0}, \xi\psi(\xi)\rangle),$$

therefore

$$(\varphi(x,p) - \varphi(x,p_0), \langle\mu_x^{p_0p_0}, \xi\psi(\xi)\rangle) = \lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-1}(\xi, F(V_r^p\Phi_m)(\xi))\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi. \tag{41}$$

Let π_1 and π_2 be orthogonal projections of $\mathbb{R}^n$ onto the subspaces L and $L^\perp$ respectively; let $\tilde{\varphi}(x,\lambda) = \pi_1(\varphi(x,\lambda))$, $\bar{\varphi}(x,\lambda) = \pi_2(\varphi(x,\lambda))$. Recall that L is the smallest subspace containing $\operatorname{supp}\mu_x^{p_0p_0}$. Hence

$$(\varphi(x,p) - \varphi(x,p_0), \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle) = (\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0), \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle)\,. \tag{42}$$

Further, $V_r^p(y) = \pi_1(V_r^p(y)) + \pi_2(V_r^p(y))$ and

$$\pi_1(V_r^p(y)) = \int (\tilde{\varphi}(y,p) - \tilde{\varphi}(y,\lambda))\,\chi(\lambda)d\gamma_y^r(\lambda),$$
$$\pi_2(V_r^p(y)) = \int (\bar{\varphi}(y,p) - \bar{\varphi}(y,\lambda))\,\chi(\lambda)d\gamma_y^r(\lambda).$$

In the notation of Proposition 4,

$$\pi_2(V_r^p(y)) = I_r(h\cdot\chi),$$

where $h(y,\lambda) = \bar{\varphi}(y,p) - \bar{\varphi}(y,\lambda)$ is a Caratheodory vector taking its values in $L^\perp$. Now, by Proposition 4 we obtain

$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-1}(\xi, F(\pi_2(V_r^p)\Phi_m)(\xi))\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi = 0. \tag{43}$$

Let $\tilde{V}_p^r(y) = \pi_1(V_p^r(y))$. From (41), in view of (42) and (43), we see that

$$(\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0), \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle) =$$
$$\lim_{m\to\infty}\lim_{r\to\infty}\int_{\mathbb{R}^n} |\xi|^{-1}(\xi, F(\tilde{V}_r^p\Phi_m)(\xi))\overline{F(U_r^{p_0}\Phi_m)(\xi)}\psi\left(\frac{\xi}{|\xi|}\right)d\xi,$$

which in turn, by Bunyakovskii inequality and Plancherel's equality, gives us the estimate

$$|(\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0), \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle)| \le \varlimsup_{m\to\infty}\varlimsup_{r\to\infty} \|\tilde{V}_r^p\Phi_m\|_2\cdot\|U_r^{p_0}\Phi_m\|_2\cdot\|\psi\|_\infty. \tag{44}$$

Next, for $M_p(y) = \max\limits_{|\lambda-p_0|\le|p-p_0|} |\tilde{\varphi}(y,p) - \tilde{\varphi}(y,\lambda)|$

$$|\tilde{V}_p^r(y)| \le M_p(y)\left|\int \chi(\lambda)d\left(\nu_y^r(\lambda) + \nu_y^0(\lambda)\right)\right| =$$
$$M_p(y)|u_r(y,p_0) - u_r(y,p) + u_0(y,p_0) - u_0(y,p)|$$

so that, in view of the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$ and the relation $|u_r(y,p_0) - u_r(y,p)| = \operatorname{sign}(p-p_0)(u_r(y,p_0) - u_r(y,p)) \le 1$, $r \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} \|\tilde{V}_r^p \Phi_m\|_2^2 \le 2 \int_\Omega (M_p(y))^2 ((u_r(y,p_0) - u_r(y,p))^2 + \\ (u_0(y,p_0) - u_0(y,p))^2) K_m(x-y)dy \le \\ 2 \operatorname{sign}(p-p_0) \int_\Omega (M_p(y))^2 (u_r(y,p_0) - u_r(y,p) + \\ u_0(y,p_0) - u_0(y,p)) K_m(x-y)dy. \end{aligned} \tag{45}$$

Since $p_0, p \in D \subset E$, it follows from Lemma 1 that

$$u_r(y,p_0) - u_r(y,p) \to u_0(y,p_0) - u_0(y,p)$$

as $r \to \infty$ in the weak-$*$ topology of $L^\infty(\Omega)$ and from (45) we now obtain the estimate

$$\varlimsup_{r\to\infty} \|\tilde{V}_r^p \Phi_m\|_2^2 \le 4 \int_\Omega (M_p(y))^2 |u_0(y,p_0) - u_0(y,p)| K_m(x-y)dy,$$

from which, passing to the limit as $m \to \infty$, we obtain

$$\varlimsup_{m\to\infty} \varlimsup_{r\to\infty} \|\tilde{V}_r^p \Phi_m\|_2^2 \le 4(M_p(x))^2 |u_0(x,p_0) - u_0(x,p)| \tag{46}$$

(here we bear in mind that x is a Lebesgue point of the functions $u_0(y,p_0)$, $u_0(y,p)$, and $(M_p(y))^2$ (the latter easily follows from the fact that $x \in \Omega_\varphi$ is a Lebesgue point of the maps $y \to \varphi(y,\cdot)$, $y \to (\varphi(y,\cdot))^2$ into the space $C(\mathbb{R})$). Further, we have $|U_r^{p_0}| \le 1$, therefore $\|U_r^{p_0}\Phi_m\|_2 \le \|\Phi_m\|_2 = 1$ and, in view of (44) and (46),

$$\begin{aligned} |\,(\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0), \langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle)\,| \le \\ \le 2\|\psi\|_\infty M_p(x)\omega(p), \\ \omega(p) = |u_0(x,p_0) - u_0(x,p)|^{1/2} \underset{p\to p_0}{\longrightarrow} 0 \end{aligned} \tag{47}$$

(remind that $p_0 \in D$ is a continuity point of the function $p \to u_0(x,p)$ for $x \in \Omega'$). Next, by Lemma 4, the set of vectors of the form $\langle \mu_x^{p_0p_0}, \xi\psi(\xi)\rangle$, $\psi(\xi) \in C(S)$ spans the subspace $\overline{L} = L + iL$. Hence we can choose functions $\psi_i(\xi) \in C(S)$, $i = 1, \dots, l$ such that the vectors $v_i = \langle \mu_x^{p_0p_0}, \xi\psi_i(\xi)\rangle$ make up an algebraic basis in L.

By (47), for $\psi(\xi) = \psi_i(\xi)$, $i = 1, \ldots, l$, we obtain

$$| (\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0), v_i) | \leq c_i \omega(p) M_p(x), \quad c_i = \text{const},$$

and since v_i, $i = 1, \ldots, l$ is a basis in L, these estimates show that for all $p \in V$

$$|\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0)| \leq c\omega(p) M_p(x) =$$
$$c\omega(p) \max_{|\lambda - p_0| \leq |p - p_0]} |\tilde{\varphi}(x,p) - \tilde{\varphi}(x,\lambda)|, \quad c = \text{const}. \tag{48}$$

Taking a smaller δ if necessary we can assume that $2c\omega(p) \leq \varepsilon < 1$ for all $p \in V$. Now, in view of (48),

$$|\tilde{\varphi}(x,p) - \tilde{\varphi}(x,p_0)| \leq \frac{\varepsilon}{2} \max_{|\lambda - p_0| \leq |p - p_0]} |\tilde{\varphi}(x,p) - \tilde{\varphi}(x,\lambda)|, \tag{49}$$

and since $\varphi(x,p)$ is continuous with respect to p and the set D is dense, the estimate (49) holds for all $p \in [p_0 - \delta, p_0 + \delta]$.

We claim that now $\tilde{\varphi}(x,p) = \tilde{\varphi}(x,p_0)$ for $p \in [p_0 - \delta, p_0 + \delta]$. Indeed, assume that for $p' \in [p_0 - \delta, p_0 + \delta]$

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| = \max_{|\lambda - p_0| \leq \delta} |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)|.$$

Then for $|\lambda - p_0| \leq |p' - p_0|$ we have

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,\lambda)| \leq |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)| +$$
$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| \leq 2|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|$$

and

$$\max_{|\lambda - p_0| \leq |p' - p_0]} |\tilde{\varphi}(x,p') - \tilde{\varphi}(x,\lambda)| \leq 2|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)|.$$

We now derive from (49) with $p = p'$ that

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| \leq \varepsilon|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0|,$$

and since $\varepsilon < 1$, this implies that

$$|\tilde{\varphi}(x,p') - \tilde{\varphi}(x,p_0)| = \max_{\lambda \in [p_0 - \delta, p_0 + \delta]} |\tilde{\varphi}(x,\lambda) - \tilde{\varphi}(x,p_0)| = 0.$$

We conclude that $\varphi(x,p) - \varphi(x,p_0) \in L^\perp$ for all $p \in (p_0-\delta, p_0+\delta)$, i.e. $(\varphi(x,\lambda),\xi) = (\varphi(x,p_0),\xi) = \text{const}$ on the interval $\lambda \in (p_0-\delta, p_0+\delta)$ for all $\xi \in L$. The proof is complete.

Theorem 5. *If the sequence ν_x^k converges as $k \to \infty$ weakly to ν_x^0 and satisfies (34) with non-degenerate vector $\varphi(x,u)$ then this sequence converges strongly to ν_x^0.*

Proof. Let $\nu_x^r = \nu_x^k$, $k = k_r$, be a subsequence such that the H-measure $\{\mu^{pq}\}_{p,q\in E}$ is well defined. As directly follows from the assertion of Theorem 4 and non-degeneracy condition in Definition 2, $\mu_x^{pp} = 0$ for a.e. $x \in \Omega$ and $p \in D$. Therefore, $\mu^{pp} = \mu_x^{pp} dx \equiv 0$ for $p \in D$. By Lemma 2,3) we see that $\mu^{pq} = 0$ for $p,q \in D$ and since D is dense and μ^{pq} is continuous in p,q (see Proposition 2) it follows that $\mu^{pq} \equiv 0$ for all $p,q \in E$. This implies that

$$u_r(x,p) \to u_0(x,p) \ \text{ in } \ L^2_{loc}(\Omega)$$

as $r \to \infty$. Indeed, it follows from the definition of an H-measure and Plancherel's equality that

$$\lim_{r\to\infty} \|U_r^p \Phi\|_2^2 = \langle \mu^{pp}, |\Phi(x)|^2 \rangle = 0$$

for all $\Phi(x) \in C_0(\Omega)$ and $p \in E$. Thus, for $p \in E$ we have

$$\int \theta(\lambda - p) d\nu_x^r(\lambda) \underset{r\to\infty}{\to} \int \theta(\lambda - p) d\nu_x^0(\lambda) \ \text{ in } L^2_{loc}(\Omega). \tag{50}$$

Any continuous function can be uniformly approximated on any compact subset by finite linear combinations of functions $\lambda \to \theta(\lambda - p)$, $p \in E$. Hence, it follows from (50) that for all $f(\lambda) \in C(\mathbb{R})$ we have

$$\int f(\lambda) d\nu_x^r(\lambda) \underset{r\to\infty}{\to} \int f(\lambda) d\nu_x^0(\lambda) \ \text{ in } L^2_{loc}(\Omega),$$

and therefore also in $L^1_{loc}(\Omega)$, that is, the subsequence ν_x^r converges to ν_x^0 strongly. Finally, for each admissible choice of the subsequence ν_x^r the limit measure-valued function is uniquely defined, therefore the original sequence ν_x^k is also strongly convergent to ν_x^0. The proof is now complete.

Taking account of Theorem 3 one can also give another formulation of Theorem 5: each bounded sequence of measure-valued functions satisfying

(34) is pre-compact in the sense of strong convergence. Observe that in the regular case $\nu_x^k(\lambda) = \delta(\lambda - u_k(x))$ condition (34) has the form: $\forall p \in \mathbb{R}$

$$\operatorname{div}_x[\theta(u_k(x) - p)(\varphi(x, u_k(x)) - \varphi(x, p))] \text{ is pre-compact in } H_{loc}^{-1}(\Omega). \quad (51)$$

In this case Theorem 5 yields the following

Corollary 2. *Each bounded sequence $u_k(x) \in L^\infty(\Omega)$ satisfying (51) with non-degenerate vector $\varphi(x,u)$ contains a subsequence convergent in $L^1_{loc}(\Omega)$.*

Proof. It only need to note that if the sequence $u_k(x)$ converges to a measure-valued function ν_x^0 strongly in $MV(\Omega)$, then by the definition of strong convergence

$$u_k(x) \underset{k\to\infty}{\to} u_0(x) = \int \lambda d\nu_x^0(\lambda) \text{ in } L^1_{loc}(\Omega)$$

(which also shows that $\nu_x^0(\lambda) = \delta((\lambda - u_0(x))$ is regular in Ω).

Remark 3. The statements of Theorems 4 and 5 remains true also for sequences of unbounded measure-valued (or usual) functions. For the proof we should apply cut-off functions $s_{a,b}(u) = \max(a, \min(u, b))$, $a, b \in \mathbb{R}$ and derive that bounded sequences of measure-valued functions $s_{a,b}^* \nu_x^k$ satisfy (34). Then, under non-degeneracy condition, we obtain strong pre-compactness property for these sequences.

For instance, consider the sequence $u_k(x) \in L^1_{loc}(\Omega)$, $k \in \mathbb{N}$. Let $\varphi(x,u)$ be a non-degenerate Caratheodory vector, satisfying (33). Suppose that $\varphi(x, u_k(x)) \in L^1_{loc}(\Omega)$ and condition (51) holds. Let $a, b \in \mathbb{R}$, $a < b$, $v_k = s_{a,b}(u_k) = \max(a, \min(u_k, b))$. Then $v_k = v_k(x)$ is a bounded sequence in $L^\infty(\Omega)$ and for each $p \in \mathbb{R}$

$$\operatorname{div}_x[\theta(v_k - p)(\varphi(x, v_k) - \varphi(x,p))] = \operatorname{div}_x[\theta(u_k - p')((\varphi(x, u_k) - \varphi(x, p'))] - \\ \operatorname{div}_x[\theta(u_k - b)((\varphi(x, u_k) - \varphi(x, b))] + \theta(p' - p)\operatorname{div}_x(\varphi(x, p') - \varphi(x,p)),$$

where $p' = s_{a,b}(p)$. From this identity and (51) it follows that the sequence $\operatorname{div}_x \theta(v_k - p)(\varphi(x, v_k) - \varphi(x,p))$ is pre-compact in $H^{-1}_{loc}(\Omega)$. By Corollary 2 the sequences $v_k(x) = s_{a,b}(u_k)$ are pre-compact in $L^1_{loc}(\Omega)$ for every $a, b \in \mathbb{R}$, $a < b$. Using the standard diagonal extraction we can choose a subsequence $u_r(x) = u_{k_r}(x)$ such that for each $m \in \mathbb{N}$ the sequence $s_{-m,m}(u_r)$ converges as $r \to \infty$ to some function $w_m(x)$ in $L^1_{loc}(\Omega)$. Obviously, a.e. in Ω

$$|w_m(x)| \le m, \text{ and } w_m(x) = s_{-m,m}(w_l(x)) \ \forall l > m.$$

This allows to define a unique (up to equality a.e.) measurable function $u(x) \in \mathbb{R} \cup \{\pm\infty\}$ such that $w_m(x) = s_{-m,m}(u(x))$ a.e. on Ω. If $a, b \in \mathbb{R}$, $a < b$ then for $m > \max(|a|, |b|)$

$$s_{a,b}(u_r) = s_{a,b}(s_{-m,m}(u_r)) \underset{r\to\infty}{\to} s_{a,b}(w_m) = s_{a,b}(s_{-m,m}(u)) = s_{a,b}(u) \text{ in } L^1_{loc}(\Omega).$$

In fact, we have proved the following general statement.

Theorem 6. *Suppose that the sequence of measurable functions $u_k(x)$ is such that for some non-degenerate Caratheodory vector $\varphi(x, u)$, which satisfies (33), for each $a, b \in \mathbb{R}$, $a < b$*

$$\operatorname{div}_x \varphi(x, s_{a,b}(u_k)) \text{ is pre-compact in } H^{-1}_{loc}(\Omega). \tag{52}$$

Then

a) there exists a measurable function $u(x) \in \mathbb{R} \cup \{\pm\infty\}$ such that, after extraction of a subsequence u_r, $r \in \mathbb{N}$, $s_{a,b}(u_r) \to s_{a,b}(u)$ $\forall a, b \in \mathbb{R}$, $a < b$.

b) If in addition the following estimates are satisfied

$$\int_K \rho(u_k(x))dx \le C_K, \tag{53}$$

for each compact set $K \subset \Omega$, where $\rho(u)$ is a positive Borel function, such that $\rho(u)/u \underset{u\to\infty}{\to} \infty$, then $u(x) \in L^1_{loc}(\Omega)$ and $u_r \to u$ in $L^1_{loc}(\Omega)$ as $r \to \infty$.

Proof. If $v_k = s_{a,b}(u_k)$ then for each $p \in \mathbb{R}$

$$\operatorname{div}_x[\theta(v_k - p)(\varphi(x, v_k) - \varphi(x, p))] = \operatorname{div}_x \varphi(x, s_{a',b}(u_k)) - \operatorname{div}_x \varphi(x, p),$$

where $a' = \max(a, p)$ (remark that in the case $b \le a'$ the above distribution is trivial). By (52) this distribution is compact in $H^{-1}_{loc}(\Omega)$. As we have already established this implies the assertion a). To prove b), observe that, extracting a subsequence, if necessary, we can assume that $s_{-m,m}(u_r) \to s_{-m,m}(u)$ as $m \to \infty$ a.e. in Ω for every $m \in \mathbb{N}$. This implies that $u_r \to u$ a.e. in Ω and by Fatou lemma from (53) it follows that

$$\int_K \rho(u(x))dx \le C_K.$$

In particular, $u(x) \in L^1_{loc}(\Omega)$. Now, fix a compact $K \subset \Omega$ and $\varepsilon > 0$. By the assumption $\rho(u)/u \underset{u\to\infty}{\to} \infty$ we can choose $m \in \mathbb{N}$ such that

$|u|/\rho(u) \le \varepsilon/(2C_K)$ for $|u| > m$. Then

$$\int_K |u_r(x) - u(x)|dx \le \int_K |s_{-m,m}(u_r(x)) - s_{-m,m}(u(x))|dx +$$
$$\int_K |u_r(x)|\theta(|u_r(x)| - m)dx + \int_K |u(x)|\theta(|u(x)| - m)dx$$
$$\le \int_K |s_{-m,m}(u_r(x)) - s_{-m,m}(u(x))|dx +$$
$$\frac{\varepsilon}{2C_K}\left(\int_K \rho(u_r(x))dx + \int_K \rho(u(x))dx\right) \le$$
$$\int_K |s_{-m,m}(u_r(x)) - s_{-m,m}(u(x))|dx + \varepsilon.$$

This implies that $\varlimsup_{r\to\infty} \int_K |u_r(x) - u(x)|dx \le \varepsilon$ and since $\varepsilon > 0$ is arbitrary we conclude that $\lim_{r\to\infty} \int_K |u_r(x) - u(x)|dx = 0$ for any compact $K \subset \Omega$, i.e. $u_r \to u$ in $L^1_{loc}(\Omega)$. The proof is complete.

§ 5. Proofs of Theorems 1,2.

We need the following simple

Lemma 5. *Suppose $u = u(x)$ is an entropy solution of (1). Then for all $a, b \in \mathbb{R}$, $a < b$*

$$\operatorname{div}\varphi(x, s_{a,b}(u)) = \zeta_{a,b} \quad in\ \mathcal{D}'(\Omega), \tag{54}$$

where $\zeta_{a,b} \in \mathrm{M}_{loc}(\Omega)$. Moreover, for each compact set $K \subset \Omega$ we have $\operatorname{Var}\zeta_{a,b}(K) \le C(K, a, b, I)$, *where $I = I(x) = |\varphi(x, u(x))| + |\psi(x, u(x))| \in L^1_{loc}(\Omega)$ and the map $I \to C(K, a, b, I)$ is bounded on $L^1_{loc}(\Omega)$.*

Proof. By known representation property for non-negative distributions we derive from (5) that

$$\operatorname{div}_x [\operatorname{sign}(u(x) - p)(\varphi(x, u(x)) - \varphi(x, p))] +$$
$$\operatorname{sign}(u(x) - p)[\omega_p(x) + \psi(x, u(x))] - |\gamma_p^s| = -\kappa_p \ \text{ in } \mathcal{D}'(\Omega),$$

where $\kappa_p \in \mathrm{M}_{loc}(\Omega)$, $\kappa_p \ge 0$. Besides, for a compact set $K \subset \Omega$ we have the estimate

$$\kappa_p(K) \le \int f_K(x)d\kappa_p(x) =$$

$$\int_\Omega [\mathrm{sign}(u(x)-p)\,(\varphi(x,u(x))-\varphi(x,p))\,,\nabla f_K(x))-$$

$$\mathrm{sign}(u(x)-p)(\omega_p(x)+\psi(x,u(x)))f_K(x)]\,dx+\int_\Omega f_K(x)d|\gamma_p^s|(x)\le$$

$$A(K,p,I)=\int_\Omega [I(x)\max(|\nabla f_K(x)|,|f_K(x)|)+|\varphi(x,p)|\cdot|\nabla f_K(x)|+$$

$$|\omega_p(x)|f_K(x)]dx+\int_\Omega f_K(x)d|\gamma_p^s|(x),$$

where $f_K(x)\in C_0^1(\Omega)$ is a non-negative function, which equals 1 on K. Hence,

$$\mathrm{div}_x\,[\mathrm{sign}(u(x)-p)(\varphi(x,u(x))-\varphi(x,p))]=\zeta_p, \tag{55}$$

where

$$\zeta_p=|\gamma_p^s|-\kappa_p-\mathrm{sign}(u(x)-p)[\omega_p(x)+\psi(x,u(x))]\in \mathrm{M}_{loc}(\Pi).$$

In particular, taking into account the equality $|\gamma_p^s|+|\omega_p(x)|dx=|\gamma_p|$ we obtain the estimates for measures ζ_p: $|\zeta_p|\le\kappa_p+|\gamma_p|+|\psi(x,u(x))|dx$.

Further, notice that

$$\varphi(x,s_{a,b}(u))=(\varphi(x,a)+\varphi(x,b))/2+$$
$$(\mathrm{sign}(u-a)(\varphi(x,u)-\varphi(x,a))-\mathrm{sign}(u-b)(\varphi(x,u)-\varphi(x,b))/2$$

and it follows from (55) that relation (54) holds with $\zeta_{a,b}=(\zeta_a-\zeta_b+\gamma_a+\gamma_b)/2$. Moreover, we have

$$\mathrm{Var}\,\zeta_{a,b}(K)\le C(K,a,b,I)=(A(K,a,I)+A(K,b,I))/2+$$
$$|\gamma_a|(K)+|\gamma_b|(K)+\int_K|\psi(x,u(x))|dx.$$

To complete the proof it remains to note that the dependence of $C(K,a,b,I)$ on the function $I(x)\in L^1_{loc}(\Omega)$ is evidently bounded.

Proof of Theorem 1. Taking into account that the sequence $I_k(x)=|\varphi(x,u_k(x))|+|\psi(x,u_k(x))|$ is bounded in $L^1_{loc}(\Omega)$, we derive from Lemma 5 that for all $a,b\in\mathbb{R}$

$$\mathrm{div}\varphi(x,s_{a,b}(u_k))=\zeta^k_{a,b}\ \text{ in } \mathcal{D}'(\Omega),$$

where $\zeta^k_{a,b}$ is a bounded sequence in $\mathrm{M}_{loc}(\Omega)$. Further, in view of condition (2) $|\varphi(x,s_{a,b}(u_k))|\in L^q_{loc}(\Omega)$, which implies that the sequence $\zeta^k_{a,b}$ is bounded in $H^{-1}_{q,loc}(\Omega)$. Using for instance Murat interpolation lemma (see [16], Lemma 28) we derive that the sequence $\zeta^k_{a,b}$ is pre-compact in H^{-1}_{loc}. Hence condition (52) is satisfied. By our assumption condition (53) is also satisfied. By Theorem 6 we conclude that some subsequence u_r converges as $r \to \infty$ to a limit function u in $L^1_{loc}(\Omega)$. Finally, passing to the limit as $r \to \infty$ in relation (5) with $u = u_r$ we conclude that the limit function $u = u(x)$ is an entropy solution of (1).

Remark 4. Based on relation (54), we can introduce the class of quasi-solutions, including, by Lemma 5, entropy solutions of (1), as well as entropy sub- and super-solutions of this equation, see [14, 15]. As is seen from the proof of Theorem 1, the statement of this Theorem remains true for more general case when $u_k(x)$ are quasi-solutions of equation (1).

Proof of Theorem 2. To prove Theorem 2 we use the approximation of the flux vector. We choose a non-negative function $\xi(s) \in C_0^\infty(\mathbb{R})$ with support in the segment $[-1,0]$ such that $\int \xi(s)ds = 1$ and set $\xi_m(s) = m\xi(ms)$ for $m \in \mathbb{N}$, $\alpha_m(\tau,y) = \xi_m(\tau)\prod_{i=1}^{n}\xi_m(y_i)$, $(\tau,y) \in \mathbb{R}\times\mathbb{R}^n$, so that the sequence α_m is an approximate unity on $\mathbb{R}^{n+1}$. Consider the averaged vector

$$\varphi_m(t,x,u) = (\varphi * \alpha_m)(t,x,u) = \int_{\mathbb{R}^{n+1}} \varphi(t-\tau, x-y, u)\alpha_m(\tau,y)d\tau dy.$$

Then, by known properties of averaged functions, $\varphi_m(t,x,u) \in C^\infty(\Pi, C^1(\mathbb{R}))$ and $\varphi_m(t,x,\cdot) \to \varphi(t,x,\cdot)$ in $L^q_{loc}(\bar{\Pi}, C^1(\mathbb{R}))$ as $m \to \infty$. In particular,

$$\max_{u\in[a,b]} |\varphi_m(t,x,u) - \varphi(t,x,u)| \underset{m\to\infty}{\to} 0 \ \text{ in } L^q_{loc}(\bar{\Pi}). \tag{56}$$

Notice also that $\varphi_m(t,x,a) = \varphi_m(t,x,b) = 0$.

Then, recall that $\operatorname{div}_x\varphi(t,x,p) = \gamma_p = \gamma_p^r + \gamma_p^s$, where $\gamma_p^r = \omega_p(t,x)dtdx$ and therefore

$$\operatorname{div}_x\varphi_m(t,x,p) = \gamma^r_{mp} + \gamma^s_{mp},$$

where $\gamma^r_{mp}, \gamma^s_{mp} \in C^\infty(\Pi)$,

$$\gamma^r_{mp} = \omega_p * \alpha_m \underset{m\to\infty}{\to} \omega_p \ \text{ in } L^1_{loc}(\bar{\Pi}), \tag{57}$$

$$|\gamma^s_{mp}| \le |\gamma^s_p| * \alpha_m \underset{m\to\infty}{\to} |\gamma^s_p| \ \text{ weakly in } M_{loc}(\bar{\Pi}).$$

From the latter relation it follows that for each $f(t,x) \in C_0(\bar{\Pi})$, $f(t,x) \geq 0$

$$\overline{\lim_{m\to\infty}} \int_{\bar{\Pi}} f(t,x)|\gamma^s_{mp}(t,x)|dtdx \leq \int_{\bar{\Pi}} f(t,x)d|\gamma^s_p|. \tag{58}$$

Observe also that $\gamma_p|_{t=0} = 0$ and therefore also $\gamma^s_p|_{t=0} = 0$ (hence we can replace in (58) the integration domain $\bar{\Pi}$ by Π). Indeed, if $f(x) \in C^1_0(\mathbb{R}^n)$ and $h > 0$ then

$$\int_{[0,h)\times\mathbb{R}^n} f(x)d\gamma_p(t,x) = -\int_{[0,h)\times\mathbb{R}^n} (\varphi(t,x,p), \nabla_x f)dtdx \to 0 \ \text{ as } h \to 0,$$

which implies that $\int_{\{0\}\times\mathbb{R}^n} f(x)d\gamma_p(t,x) = 0$ for all $f(x) \in C^1_0(\mathbb{R}^n)$ and, therefore, $\gamma_p|_{t=0} = 0$.

Since the flux $\varphi_m(t,x,u)$ is sufficiently smooth then by the classical Kruzhkov result [9] there exists an entropy solution $u_m(t,x)$ to the Cauchy problem

$$u_t + \mathrm{div}_x\varphi_m(t,x,u) = 0, \quad u(0,x) = u_0(x). \tag{59}$$

Recall that $a \leq u_0(x) \leq b$, and $\varphi_m(t,x,a) = \varphi_m(t,x,b) = 0$ (i.e. the constants a, b are entropy solutions of the approximate equations). By the maximum principle we see that $a \leq u_m(t,x) \leq b$. Taking $p = a, b$ in the relation

$$|u_m - p|_t + \mathrm{div}_x[\mathrm{sign}(u_m - p)(\varphi_m(t,x,u_m) - \varphi_m(t,x,p))] + \\ \mathrm{sign}(u_m - p)\mathrm{div}_x\varphi_m(t,x,p) \leq 0 \ \text{ in } \mathcal{D}'(\Pi), \tag{60}$$

we derive that $u_m = u_m(t,x)$ is a weak solution of the approximate equation that is

$$(u_m)_t + \mathrm{div}_x\varphi_m(t,x,u_m) = 0 \ \text{ in } \mathcal{D}'(\Pi). \tag{61}$$

This implies in particular that for each $p \in \mathbb{R}$

$$(u_m - p)_t + \mathrm{div}_x(\varphi_m(t,x,u_m) - \varphi_m(t,x,p)) + \mathrm{div}_x\varphi_m(t,x,p) = 0 \ \text{ in } \mathcal{D}'(\Pi). \tag{62}$$

Combining (60) and (62), we obtain

$$(\theta(u_m - p)(u_m - p))_t + \mathrm{div}_x[\theta(u_m - p)(\varphi_m(t,x,u_m) - \varphi_m(t,x,p))] + \\ \theta(u_m - p)\mathrm{div}_x\varphi_m(t,x,p) \leq 0 \ \text{ in } \mathcal{D}'(\Pi).$$

From this relation it follows, in the same way as in the proof of Theorem 1, that the sequence of distributions

$$\mathcal{L}_{1m} = (\theta(u_m - p)(u_m - p))_t + \mathrm{div}_x[\theta(u_m - p)(\varphi_m(t, x, u_m) - \varphi_m(t, x, p))]$$

is bounded in $M_{loc}(\Pi) \cap H^{-1}_{q,loc}(\Pi)$ and therefore pre-compact in $H^{-1}_{loc}(\Pi)$. Since in view of (56)

$$|\varphi_m(t, x, u_m) - \varphi(t, x, u_m)| \le \max_{u\in[a,b]} |\varphi_m(t, x, u) - \varphi(t, x, u)| \underset{m\to\infty}{\to} 0$$

in $L^q_{loc}(\bar{\Pi})$ and also $\varphi_m(t, x, p) - \varphi(t, x, p) \underset{m\to\infty}{\to} 0$ in $L^q_{loc}(\bar{\Pi})$ we see that the sequence

$$\mathcal{L}_{2m} = \mathrm{div}_x[\theta(u_m - p)(\varphi(t, x, u_m) - \varphi_m(t, x, u_m) - \varphi(t, x, p) + \varphi_m(t, x, p))]$$

converges to zero in $H^{-1}_{loc}(\Pi)$. Thus, the sequence

$$(\theta(u_m - p)(u_m - p))_t + \mathrm{div}_x[\theta(u_m - p)(\varphi(t, x, u_m) - \varphi(t, x, p))] = \mathcal{L}_{1m} + \mathcal{L}_{2m}$$

is pre-compact in $H^{-1}_{loc}(\Pi)$. By Corollary 2 we conclude that after extraction of a subsequence, if necessary, the sequence u_m converges in $L^1_{loc}(\Pi)$ to some function $u = u(t, x)$. Clearly, $a \le u(t, x) \le b$. Taking into account (56) we see that $\varphi_m(t, x, u_m) \to \varphi(t, x, u)$ as $m \to \infty$ in $L^1_{loc}(\bar{\Pi})$. Passing to the limit as $m \to \infty$ in relation (61), we obtain that

$$u_t + \mathrm{div}_x\varphi(t, x, u) = 0 \quad \text{in } \mathcal{D}'(\Pi),$$

i.e. $u(t, x)$ is a weak solution of (6). To show that $u(t, x)$ is also an entropy solution of this equation, remark that, as follows from (8) applied for the approximate equation, for each $p \in \mathbb{R}$, $f(t, x) \in C^1_0(\bar{\Pi})$

$$\int_\Pi [|u_m - p|f_t + \mathrm{sign}(u_m - p)\,(\varphi_m(t, x, u_m) - \varphi_m(t, x, p), \nabla_x f) - \\ \mathrm{sign}(u_m - p)\gamma^r_{mp}(t, x)f(t, x)\big]\,dtdx + \\ \int_\Pi f(t, x)|\gamma^s_{mp}(t, x)|dtdx + \int_{\mathbb{R}^n} |u_0(x) - p|f(0, x)dx \ge 0.$$

Passing in this relation to the limit as $m \to \infty$ and taking into account (57), (58), we derive

$$\int_\Pi [|u - p|f_t + \mathrm{sign}(u - p)\,(\varphi(t, x, u) - \varphi(t, x, p), \nabla_x f) - \\ \mathrm{sign}(u - p)\omega_p(t, x)f(t, x)]\,dtdx + \\ \int_\Pi f(t, x)d|\gamma^s_p|(t, x) + \int_{\mathbb{R}^n} |u_0(x) - p|f(0, x)dx \ge 0 \tag{63}$$

for such $p \in \mathbb{R}$ that the level set $u^{-1}(p)$ has zero Lebesgue measure (as is easy to see, then $\operatorname{sign}(u_m(t,x)-p) \to \operatorname{sign}(u(t,x)-p)$ as $m \to \infty$ a.e. on Π). Since the set P of such p has full measure and, therefore, is dense, for an arbitrary $p \in \mathbb{R}$ we can choose sequences $p_r^- < p < p_r^+$, $p_r^\pm \in P$, $r \in \mathbb{N}$ convergent to p. Taking a sum of relations (63) with $p = p_r^-$ and $p = p_r^+$ and passing to the limit as $r \to \infty$, with account of the point-wise relation $\operatorname{sign}(u-p_r^-) + \operatorname{sign}(u-p_r^+) \underset{r\to\infty}{\to} 2\operatorname{sign}(u-p)$, we obtain that (63) holds for all $p \in \mathbb{R}$, i.e. $u(t,x)$ is an entropy solution of the problem (6), (7). The proof is complete.

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