# TIME-ASYMPTOTIC BEHAVIOUR OF WEAK SOLUTIONS FOR A VISCOELASTIC TWO-PHASE MODEL WITH NONLOCAL CAPILLARITY

ALEXANDER DRESSEL & CHRISTIAN ROHDE

ABSTRACT. The aim of this paper is to study the time-asymptotic behaviour of weak solutions of an initial-boundary value problem for a viscoelastic two-phase material with capillarity in one space dimension. Therein, the capillarity is modelled via a nonlocal interaction potential. Based on the existence and regularity results of [5], we analyze the time-asymptotic convergence of the strain-velocity field. In particular, we will show that, in the time-asymptotic limit, the strain converges pointwise almost everywhere to a stationary solution. The results of this paper also apply for interaction potentials with non-vanishing negative part.

## 1. INTRODUCTION

This paper is concerned with the long time behaviour of weak solutions to the initial value problem

$$\begin{array}{rcl} w_t - v_x &=& 0, \\ v_t - [\sigma(w) + L_\epsilon w]_x &=& \mu v_{xx}, \end{array} \tag{1.1}$$

$$v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \tag{1.2}$$

where the real-valued unknown functions $v = v(x,t)$ and $w = w(x,t)$ of $(x,t) \in I \times [0,\infty)$ (for $I = [0,1]$) represent the velocity and the strain field in Lagrangian coordinates; we impose the Dirichlet boundary condition

$$\int_I w(y,t)dy = \int_I w_0(x)dx \quad \forall t \in [0,\infty). \tag{1.3}$$

In system (1.1), the diffusion coefficient $\mu > 0$ is fixed, and the deformation stress $w \mapsto \sigma(w)$ is given by

$$\sigma(w) = w^3 - w. \tag{1.4}$$

For $\lambda > 0$, the capillarity stress $L_\epsilon$ is defined by

$$L_\epsilon u = \lambda(\phi_\epsilon * u - u). \tag{1.5}$$

Therein, $u \mapsto \phi_\epsilon * u$ is the convolution operator given by

$$(1.6)\qquad (\phi_\epsilon * u)(x) = \int_I \phi_\epsilon(x-y)u(y)\,dy, \qquad \phi_\epsilon(x) = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right).$$

On the interaction kernel $\phi$ we assume that

$$(1.7)\qquad \phi \in L^\infty(\mathbb{R}),\;\; \phi(x) = \phi(-x)\;\; \forall x \in \mathbb{R}.$$

For the justification this class of models and further references, we refer to [5].

The function $\sigma : \mathbb{R} \to \mathbb{R}$ in (1.4) represents a toy example of a material law for a homogeneous viscoelastic two-phase medium. The stored energy function associated to $\sigma$ is

$$(1.8)\qquad S(w) = \int_0^w \sigma(u)\,du = \frac{1}{4}w^4 - \frac{1}{2}w^2.$$

It has the form of a double-well potential with minima at $\pm 1$. In case of $w \in (-\infty, -\sqrt{1/3})$ ($w \in (\sqrt{1/3}, \infty)$) a state $w$ is referred to be in the *low (high) strain phase*. On the interval $J = (-\sqrt{1/3}, \sqrt{1/3})$, the *transitional region*, the function $S' = \sigma$ is decreasing.

The evolution process governed by (1.1) dissipates the nonlocal energy

$$(1.9)\qquad \begin{aligned} \mathfrak{H}(v,w) &= \int_I S(w(x)) + \frac{v(x)^2}{2}\,dx + \mathfrak{E}(w),\\ \mathfrak{E}(w) &= \frac{\lambda}{4}\int_I\int_I \phi_\epsilon(x-y)(w(x)-w(y))^2\,dydx. \end{aligned}$$

The main result of this this paper is that, for any weak solution $(v,w) \in C^0([0,\infty), L^2)$ of the aforementioned initial-boundary value problem with $\mathfrak{H}(v_0,w_0) < \infty$, where $t \mapsto \mathfrak{H}(v(\cdot,t), w(\cdot,t))$ is non-increasing, in the limit $t \to \infty$, the strain field $w(\cdot,t)$ strongly converges $\bar{w}$ in $L^2(I)$, where $\bar{w}$ is a weak solution of the elastostatic equation

$$(1.10)\qquad [\sigma(\bar{w}) + L_\epsilon \bar{w}]_x \;=\; 0,$$

whose deformation field in case of the displacement boundary condition (1.3) fulfills the following relation

$$(1.11)\qquad \int_I \bar{w}(y)dy = \int_I w_0(x)dx.$$

For nonnegative $\phi_\epsilon$, the existence of stationary solutions for $\epsilon > 0$ fixed and the limit $\epsilon \to 0$ have already been studied (see [1] and [2]). In case of $\phi_\epsilon$

having a nonvanishing negative part there exists a solution with periodic microstructure (cf. [8]).

Note that, by means of the methods in [7], based on a transformation of the equations into a reaction-diffusion system, the case $\lambda = 0$ can be considered.

In order to address the case $\lambda > 0$, we have used a different proof strategy, which consists of the following main parts:

(i). Proof of *weak* time-asymptotic convergence of the strain field $w(\cdot, t)$ in $L^2(I)$.

(ii). A priori estimates on the stress $(x, t) \mapsto \sigma(w(x, t)) + (L_\epsilon w)(x, t)$ on $(0, 1) \times (0, \infty)$.

(iii). Proof of *strong* time-asymptotic convergence of the strain field $w(\cdot, t)$ in $L^2(I)$ by means of the results in parts (i) and (ii).

Note that, in addressing the case $\lambda > 0$, we essentially take into account the compactness of the convolutional part of $L_\epsilon$ on $L^2(I)$. With this compactness at hand, the convolutional part maps weakly convergent sequences into strongly convergent ones.

Our paper is organized as follows. In Section 2, we introduce some notation, recall the results in [5] our analysis is based on and formulate the main results. In Section 3, we perform the proofs of the main results.

## 2. Notation and main theorem

In this section, we introduce some notation and formulate the main result about the time asymptotic behaviour of weak solutions to system (1.1) with boundary conditions (1.3).

Recall that a locally integrable function $(v, w) : I \times [0, \infty) \to \mathbb{R}^2$ is referred to as a weak solution of system (1.1) iff $v$, $\sigma(w)$ and $L_\epsilon w$ are locally integrable on $I \times (0, \infty)$ and, for any $\psi \in C_0^\infty(I \times (0, \infty))$,

$$
\begin{aligned}
&\int_0^\infty \int_I -\psi_t w + \psi_x v \; dxdt = 0, \\
&\int_0^\infty \int_I -\psi_t v + \psi_x \left[\sigma(w) + L_\epsilon w\right] \; dxdt = \int_0^\infty \int_I \mu \psi_{xx} v \; dxdt.
\end{aligned}
\tag{2.1}
$$

A weak solution $(v, w)$ of system (1.1) is referred to as a stationary weak solution of system (1.1) iff it is a weak solution of system (1.1) and does not depend on $t$.

*Remark* 2.1. In particular, any stationary weak solution of system (1.1) fulfilling the boundary condition (1.3) has the form

$$(v,w) = (0,\bar{w}), \quad \text{where } \bar{w} = \bar{w}(x). \tag{2.2}$$

From definition (1.5) we know that

$$L_\epsilon w = -\lambda w + \hat{L}_\epsilon w, \quad \text{where } \hat{L}_\epsilon w = \lambda \phi_\epsilon * w. \tag{2.3}$$

From $\phi_\epsilon \in L^\infty(I)$ we conclude that

$$\hat{L}_\epsilon : L^2(I) \to L^2(I) \text{ is compact.} \tag{2.4}$$

Before we formulate the main result of this paper, we recall from Theorem 2.1 and Remark 3.1 in [5]:

**Theorem 2.1.** *Assume that, in system (1.1), the real numbers $\epsilon$ and $\mu$ are strictly positive, that the initial values $(v_0, w_0)$ are Lebesgue-measurable on $(0,1)$ and, for $t = 0$, the functional $\mathfrak{H}$ defined in (1.9) is bounded from above:*

$$\mathfrak{H}(v_0, w_0) < \infty.$$

*Then, the following claims are true:*

*(i). There exists a unique global weak solution*

$$(v,w) \in C^0([0,\infty), L^2(I) \times L^2(I)) \tag{2.5}$$

*of system (1.1) fulfilling the initial value condition (1.2) and the Dirichlet boundary condition (1.3) such that $t \mapsto \mathfrak{H}(v(\cdot,t), w(\cdot,t))$ is non-increasing.*
*(ii). For this weak solution there holds*

$$(v,w) \in L^\infty(\nu,\infty; H^{1,\infty}(I)) \times L^\infty(I \times (\nu,\infty)) \quad \forall \nu > 0, \tag{2.6}$$

$$v \in L^2(0,\infty; H_0^{1,2}(I)) \tag{2.7}$$

*and*

$$v_x = w_t \in L^2(I \times (0,\infty)). \tag{2.8}$$

*Furthermore, there exists a strictly positive real number $\gamma$ with the following property:*

$$\|v_x\|_{C^\gamma([\nu,\infty);L^2(I))} < \infty \quad \forall \nu \in (0,\infty). \tag{2.9}$$

*Remark* 2.2. *In fact, on the initial data* $(v_0, w_0)$, *we only assume that the Hamiltonian* $\mathfrak{H}(v_0, w_0)$ *is bounded. By means of appropriate time layer estimates , we have shown that, for any* $\nu > 0$, *the strain field* $w$ *s contained in* $L^\infty((0,1) \times (\nu, \infty))$ *(cf.* [5], *p. 15ff). In this sense, also for* $\lambda = 0$, *we do not need to assume uniform boundedness of the initial strain* $w_0$ *(as in* [3] *and* [7]*) in order to get uniform boundedness of the strain* $w(\cdot, t)$ *for strictly positive* $t$.

In the first part of this article, we prove weak time-asymptotic convergence of the strain field. More precisely, we will prove the following theorem:

**Theorem 2.2.** *Under the assumptions of Theorem 2.1, for the weak solution* $(v, w) \in C^0([0, \infty), L^2(I) \times L^2(I))$ *of the initial-boundary problem (1.1), there holds:*

$$v(\cdot, t) \to 0 \;\; in \;\; H^{1,2}(I) \;\; for \;\; t \to \infty, \tag{2.10}$$

*and, for some* $\bar{w} \in L^2(I)$, *we have*

$$w(\cdot, t) \rightharpoonup \bar{w} \;\; in \;\; L^2(I) \;\; for \;\; t \to \infty. \tag{2.11}$$

In the second part, we will show that, in the limit $t \to \infty$, the stress is constant:

**Theorem 2.3.** *Under the assumptions of Theorem 2.1 there exists a real number* $P$ *such that*

$$\lim_{t \to \infty} \sigma(w(x,t)) + (L_\epsilon w)(x,t) \equiv P \;\; for \; a.e. \;\; x \in I. \tag{2.12}$$

Then, based on Theorems 2.2 and 2.3, we will prove the main result of this paper:

**Theorem 2.4.** *Under the assumptions of Theorem 2.1, for the weak limit in (2.11), we have*

$$w(\cdot, t) \to \bar{w} \;\; in \;\; L^2(I) \;\; for \;\; t \to \infty. \tag{2.13}$$

*Furthermore,* $\bar{w}$ *is a weak solution of the elastostatic equation (1.10). More precisely: for the real number* $P$ *in Theorem 2.3, we have*

$$\sigma(\bar{w}(x)) + L_\epsilon(\bar{w})(x) = P \;\; for \; a.e. \;\; x \in (0,1). \tag{2.14}$$

The following corollary is a straightforward consequence of relations (1.3) and (2.13):

**Corollary 2.5.** *There holds for the time-asymptotic limit* $\bar{w}$ *in Theorem 2.4:*

$$\int_I w_0(x) = \int_I \bar{w}(x) dx.$$

## 3. Proof of main results

3.1. **Weak convergence of the strain.** In this section, we prove Theorem 2.2.

In order to show relation (2.10), we use the following lemma:

**Lemma 3.1.** *Assume that, for a strictly positive real number $\gamma$, there holds $g \in L^2(0,\infty)$ and*

$$\|g\|_{C^\gamma([0,\infty))} < \infty. \tag{3.1}$$

*Then there holds* $\lim_{t\to\infty} g(t) = 0$.

For convenience of the reader, the straightforward proof of this lemma will be given at the end of this section.

In order to prove relation (2.10), we note that, due to (2.8) and (2.9), for

$$g(t) := \|v_x(\cdot,t)\|_{L^2(0,1)}$$

the assumptions of Lemma 3.1 are fulfilled.

From Lemma 3.1 we obtain

$$\lim_{t\to\infty} v_x(\cdot,t) = 0 \ \text{ in } \ L^2(I). \tag{3.2}$$

From (2.8) and (3.2) relation (2.10) follows.

It remains to prove prove that, for some $\bar{w} \in L^2(I)$,

$$w(\cdot,t) \rightharpoonup \bar{w} \ \text{ in } \ L^2(I) \ \text{ for } \ t \to \infty. \tag{3.3}$$

Due to Theorem 2.1, we have

$$w \in C^0([0,\infty), L^2(I)) \cap L^\infty(0,\infty; L^2(I)).$$

In particular, there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ in $[0,\infty)$ with $\lim_{k\to\infty} t_k = \infty$ such that

$$w(\cdot,t_k) \rightharpoonup \bar{w} \ \text{ in } \ L^2(I) \ \text{ for } \ k \to \infty. \tag{3.4}$$

It remains to show that, for any pair $\{t_k^{(1)}\}_{k\in\mathbb{N}}, \{t_k^{(2)}\}_{k\in\mathbb{N}}$ of sequences in $[0,\infty)$ with $\lim_{k\to\infty} t_k^{(i)} = \infty$ for $i = 1,2$ such that

$$w(\cdot,t_k^{(i)}) \rightharpoonup \bar{w}^{(i)} \ \text{ in } \ L^2(I) \ \text{ for } \ k \to \infty,$$

we have $\bar{w}^{(1)} = \bar{w}^{(2)}$.

Assume this is not the case.

Then there exists a sequence $\{s_k\}_{k\in\mathbb{N}}$ in $[0,\infty)$ with $\lim_{k\to\infty} s_k = \infty$ and a function $u \in L^2(I)$ such that

$$\liminf_{k\to\infty} \left| \int_I u(x) \left[ w(x, s_{k+1}) - w(x, s_k) \right] dx \right| > 0,$$

or, in other words,

$$0 < \liminf_{k\to\infty} \left| \int_{s_k}^{s_{k+1}} \int_I u(x) w_t(x,t) dx dt \right| \overset{(1.1)}{=} \liminf_{k\to\infty} \left| \int_{s_k}^{s_{k+1}} \int_I u(x) v_x(x,t) dx dt \right| \overset{(3.2)}{=} 0,$$

a contradiction.

Hence, relation (3.4) follows. □

It remains to prove Lemma 3.1:

Assume, there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} t_k = \infty$ such that

$$\liminf_{k\to\infty} |g(t_k)| \geq \kappa > 0.$$

Then, due to (3.1), there exists a real number $\kappa \in (0,\infty)$ such that the following implication is true

$$|g(t)| \geq \frac{\kappa}{2} \forall t \in (t_k - \delta, t_k + \delta) \tag{3.5}$$

After eventual choice of a further subsequence of $\{t_k\}_{k\in\mathbb{N}}$ we can assume that

$$(t_k - \delta, t_k + \delta) \cap (t_l - \delta, t_l + \delta) \quad \text{for} \quad l \neq k. \tag{3.6}$$

From (3.5) and (3.6) it follows

$$\|g\|^2_{L^2(0,\infty)} = \int_0^\infty g(t)^2 dt \geq \sum_{k=1}^\infty \int_{t_k-\delta}^{t_k+\delta} g(t)^2 dt \geq \int_{t_k-\delta}^{t_k+\delta} \frac{\kappa^2}{4} dt = \infty.$$

This is a contradiction to $g \in L^2(0,\infty)$, and therefore, for any sequence $\{t_k\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} t_k = \infty$, we have

$$\lim_{k\to\infty} g(t_k) = 0,$$

and the claim of Lemma 3.1 follows. □

3.2. **Asymptotic behaviour of the stress.** In this section, we analyze the stress in order to prove Theorem 2.3.

Therein, our main objective is to show that, for a.e. $(x,y) \in I^2$ and any $\nu \in (0,\infty)$,

(3.1)
$$\tilde{g}(x,y,\cdot) := \sigma(w(x,\cdot)) + (L_\epsilon w)(x,\cdot) - \sigma(w(y,\cdot)) - (L_\epsilon w)(y,\cdot) \in H^{1,2}(\nu,\infty).$$

Once this relation has been shown, we know that, for a.e. $(x,y) \in I^2$, we have $\|\tilde{g}(x,y,\cdot)\|_{C^{\frac{1}{2}}([0,\infty))} < \infty$ and $\tilde{g}(x,y,\cdot) \in L^2(0,\infty)$.

This fact together with Lemma 3.1 (with $\gamma = \frac{1}{2}$ and $g = \tilde{g}(x,y,\cdot)$) gives for a.e. $(x,y) \in I^2$

$$\lim_{t\to\infty} \tilde{g}(x,y,t) = 0 \text{ for a.e. } (x,y) \in I^2.$$

Taking the definition in (3.1) into account, Theorem 2.3 then easily follows.

In order to prove relation (3.1) or a.e. $(x,y) \in I^2$ and any $\nu \in (0,\infty)$, we define auxiliary functions in terms of our initial-boundary value problem. More precisely, let $(v,w)$ be a weak solution of system (1.1) fulfilling the assumptions of Theorem 2.1, and, for $(y,z,t) \in I \times I \times [0,\infty)$, set

$$V(y,z,t) := \int_y^z v(x,t)dx, \tag{3.2}$$

and

$$f(y,z,t) := \sigma(w(z,t)) + (L_\epsilon w)(z,t) - \sigma(w(y,t)) - (L_\epsilon w)(y,t). \tag{3.3}$$

As, for any $\nu > 0$, the strain $w$ is uniformly bounded on $I \times [\nu,\infty)$, there holds

$$|f(y,z,t)| \le C(\nu) \ \ \forall (y,z,t) \in I \times I \times [\nu,\infty). \tag{3.4}$$

Recall that, due to the first equation in (1.1) and (2.6), for any $\nu > 0$, we have

$$w_t = v_x \in L^\infty(I \times (\nu,\infty)), \tag{3.5}$$

and, due to (2.8),

$$w_t \in L^2(I \times (0,\infty)). \tag{3.6}$$

Furthermore, for any $\nu > 0$, there exists a finite real number $K$ such that, for a.e. $(y,z,t) \in I \times I \times [\nu,\infty)$,

$$|f_t(y,z,t)| \le (|\sigma'(w(y,t))| + |\sigma'(w(z,t))| + K)\,(|w_t(y,t)| + |w_t(z,t)|)\,.$$

In particular, due to (3.5) and (3.6), there holds

$$f_t \in L^2(I \times I \times (\nu, \infty)) \;\; \forall \nu > 0. \tag{3.7}$$

Due to the second equation in (1.1) (after spatial integration over the interval $(z, y)$) we obtain

$$V_t = f + \mu V_{yy}. \tag{3.8}$$

The proof of Theorem 2.3 relies on the following lemma:

**Lemma 3.2.** *Under the assumptions of Theorem 2.3, for $V$ defined in (3.2), there holds*

$$\int_\nu^\infty \int_I \int_I V_t^2(y, z, t) dydzdt < \infty. \tag{3.9}$$

We will prove Lemma 3.2 at the end of this section.

First, based on relation (3.9), we will prove Theorem 2.3:

Due to definition (3.2), for $(z, t) \in [0, 1] \times [0, \infty)$, the function $y \mapsto V(y, z, t)$ is an antiderivative of $v(\cdot, t)$.

This fact together with (2.8) implies

$$V_{yy} \in L^2(I \times I \times [0, \infty)). \tag{3.10}$$

Relations (3.8), (3.9) and (3.10) imply that, for any $\nu > 0$,

$$\int_\nu^\infty \int_I \int_I f^2(y, z, t) dydzdt < \infty. \tag{3.11}$$

In other words, for a.e. $(x, y) \in I^2$, the function $t \mapsto f(x, y, t)$ is contained in $L^2(\nu, \infty)$.

Due to (3.7), for a.e. $(x, y) \in I^2$ and any $\nu \in (0, \infty)$, relation (3.1) follows. □

It remains to prove Lemma 3.2:

Due to (2.6), there holds

$$V \in L^\infty(I \times I \times [\nu, \infty)) \;\; \forall \nu > 0. \tag{3.12}$$

Multiplication of (3.8) with $V_t(y, z, t)$ and integration over $I \times I \times [\nu, T]$ gives for $0 < \nu < T < \infty$

$$\int_\nu^T \int_I \int_I V_t^2(y, z, t) dydzdt = I_1 + I_2, \tag{3.13}$$

where

$$I_1 = \int_\nu^T \int_I \int_I V_t(y,z,t) f(y,z,t) dydzdt$$

and

$$I_2 = \mu \int_\nu^T \int_I \int_I V_t(y,z,t) V_{yy}(y,z,t) dydzdt.$$

The integral $I_1$ can be rewritten as

$$\int_\nu^T \int_I \int_I V_t(y,z,t) f(y,z,t) dydzdt = \mathcal{I}_1 - \mathcal{I}_2 - \mathcal{I}_3,$$

where

$$\mathcal{I}_1 = \int_I \int_I V(y,z,T) f(y,z,T) dydz,$$

$$\mathcal{I}_2 = \int_I \int_I V(y,z,\nu) f(y,z,\nu) dydzdt$$

and

$$\mathcal{I}_3 = \int_\nu^T \int_I \int_I V(y,z,t) f_t(y,z,t) dydzdt.$$

Due to (3.4), (3.7) and (3.12), for any $\nu > 0$, there exists a constant $C = C(\nu)$ independent of $T$ such that

$$|I_1| \leq |\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_3| \leq C. \tag{3.14}$$

Furthermore,

$$|I_2| \leq \frac{1}{2} \int_\nu^T \int_I \int_I V_t^2(y,z,t) dydzdt + \frac{\mu^2}{2} \int_0^T \int_I \int_I V_{yy}^2(y,z,t) dydzdt$$

and therefore, after subtraction of the first integral on the right-hand side of the last inequality from (3.13) with the constant $C$ (which is independent of $T$) in (3.14), we obtain in the limit $T \to \infty$

$$\frac{1}{2} \int_\nu^\infty \int_I \int_I V_t^2(y,z,t) dydzdt \leq C + \frac{\mu^2}{2} \int_\nu^\infty \int_I \int_I V_{yy}^2(y,z,t) dydzdt \overset{(3.10)}{<} \infty. \tag{3.15}$$

Hence, Lemma 3.2 has been proven. □

3.3. **Proof of strong time-asymptotic convergence.** In order to prove Theorem 2.4, first note that, due to (2.4), the operator $\hat{L}_\epsilon$ maps weakly convergent sequences in $L^2(I)$ onto strongly convergent ones.

In particular, (3.4) implies now

$$\lim_{t\to\infty} (\hat{L}_\epsilon w)(\cdot,t) = \hat{L}_\epsilon \bar{w} \ \text{ in } \ L^2(I). \tag{3.1}$$

For any $\delta \in (0,\infty)$ set

(3.2)
$$M_\delta = \left\{(x,t) \in I \times [0,\infty) \,|\, \left|\sigma(w(x,t)) - \lambda w(x,t) + (\hat{L}_\epsilon \bar{w})(x) - P\right| > \delta\right\},$$

where $P$ is the real number in Theorem 2.3.

In the first step, we prove the following lemma:

**Lemma 3.3.** *For any $\delta > 0$ there exists set $\tilde{N}(\delta) \subset I$ of measure zero such that, for any $x \in I \setminus \tilde{N}(\delta)$, there exists a real number $\tilde{t}(\delta,x) \in (0,\infty)$ with the following property:*

$$t \geq \tilde{t}(\delta,x) \Rightarrow (x,t) \notin M_\delta. \tag{3.3}$$

*Proof:* We know that

$$\lim_{t\to\infty} \sigma(w(x,t)) - \lambda w(x,t) + (\hat{L}_\epsilon w)(x,t) = P \ \text{ for a.e. } \ x \in I.$$

In particular, there exists a set $\tilde{N}(\delta) \subset I$ of measure zero such that, for any $x \in I \setminus \tilde{N}(\delta)$, there exists a real number $\tilde{t}(\delta,x) \in (0,\infty)$ with the following property:

$$\left|\sigma(w(x,t)) - \lambda w(x,t) + (\hat{L}_\epsilon \bar{w})(x) - P\right| \leq \delta \ \ \forall t \geq \tilde{t}(\delta,x).$$

This is equivalent to the claim of the lemma. □

In the second step, we will perform an elementary curve sketching of the graph of $\sigma$.

Note that $\sigma'(z) = 3z^2 - 1$ and therefore,

$$\sigma'(\hat{z}_i) - \lambda = 3\hat{z}_i^2 - 1 - \lambda = 0,$$

where

$$\hat{z}_{1/2} = \pm\sqrt{\frac{\lambda+1}{3}}. \tag{3.4}$$

Furthermore, set

$$\sigma_1 := \sigma(\hat{z}_1) - \lambda\hat{z}_1 - P, \;\; \sigma_2 := \sigma(\hat{z}_2) - \lambda\hat{z}_2 - P, \tag{3.5}$$

and, for $\bar{\epsilon} > 0$, define

$$\mathcal{N}_{\bar{\epsilon}} := \mathbb{R} \times \{(\sigma_1 - \bar{\epsilon}, \sigma_1 + \bar{\epsilon}) \cup (\sigma_2 - \bar{\epsilon}, \sigma_2 + \bar{\epsilon})\}. \tag{3.6}$$

For $y \in I$, set

$$\mathfrak{M}(y) := \Big\{ z \in \mathbb{R} \,|\, \sigma(z) - \lambda z - P = -(\hat{L}_\epsilon \bar{w})(y) \Big\}. \tag{3.7}$$

If, for any $i \in \{1, 2\}$, we have $-(\hat{L}_\epsilon \bar{w})(y)) \neq \sigma_i$, then we have either

$$\mathfrak{M}(y) = \{z(y)\} \tag{3.8}$$

or

$$\mathfrak{M}(y) = \{z_1(y), z_2(y), z_3(y)\}, \tag{3.9}$$

where $z_i(y) \neq z_j(y)$ for $i \neq j$.

Our argument relies on the following lemma:

**Lemma 3.4.** *For any $y \in I$ and $\bar{\epsilon} > 0$, there exists $f(\bar{\epsilon}) > 0$ such that the following implication is true:*

(3.10)
$$\Big((w(y,t), -(\hat{L}_\epsilon \bar{w})(y)) \notin \mathcal{N}_{\bar{\epsilon}} \textit{ and } (y,t) \notin M_\delta(y)\Big) \Rightarrow \textit{dist}(w(y,t), \mathfrak{M}(y)) < f(\bar{\epsilon})\delta.$$

*Furthermore, there exists a function $\mu = \mu(\delta, \bar{\epsilon})$ with $\lim \mu_{(\delta,\bar{\epsilon})\to 0}\mu(\delta, \bar{\epsilon}) = 0$ and the following property:*

(3.11)
$$\Big((w(y,t), -(\hat{L}_\epsilon \bar{w})(y)) \in \mathcal{N}_{\bar{\epsilon}} \textit{ and } (y,t) \notin M_\delta(y)\Big) \;\Rightarrow\; \textit{min}_{i\in\{1,2\}}\{|w(y,t) - \hat{z}_i|\} < \mu(\delta, \bar{\epsilon}).$$

The elementary proof of this lemma will be given at the end of this section.

First, based on relations (3.10) and (3.11), we continue our proof.

Recall from Lemma 3.3 that, for any $\delta > 0$ and $y \in I \setminus \tilde{N}(\delta)$, there exists a strictly positive real number $\tilde{t}(\delta, y)$ such that

$$(y,t) \notin M_\delta(y) \;\; \forall t \geq \tilde{t}(\delta, y).$$

Due to Lemma 3.4, this relation implies, that, for any $(\delta, \bar{\epsilon}) \in (0, \infty)^2$ and $y \notin \tilde{N}(\delta)$, we have

$$\operatorname{dist}(w(y,t),\mathfrak{M}(y)) < f(\bar\epsilon)\delta \ \ \forall t \geq \tilde t(\delta,x)$$

(with $\mathfrak{M}(y)$ being defined in (3.7)) or

$$\Big((w(y,t),-(\hat L_\epsilon \bar w)(y)) \in \mathcal{N}_{\bar\epsilon} \text{ and } (y,t)\notin M_\delta(y)\Big) \ \ \forall t \geq \tilde t(\delta,y).$$

In particular, for a.e. $y \in I\setminus \tilde N(\delta)$ and any pair $(\delta,\bar\epsilon)\in(0,\infty)^2$, we have

$$|w(y,t)-z(y)| < f(\bar\epsilon)\delta \ \ \forall t \geq \tilde t(\delta,y), \tag{3.12}$$

$$\min_{i\in\{1,2,3\}}(|w(y,t)-z_i(y)|) < f(\bar\epsilon)\delta \ \ \forall t \geq \tilde t(\delta,y) \tag{3.13}$$

or

$$\Big((w(y,t),-(\hat L_\epsilon \bar w)(y)) \in \mathcal{N}_{\bar\epsilon} \text{ and } (y,t)\notin M_\delta(y)\Big) \ \ \forall t \geq \tilde t(\delta,y).$$

Therein, case (3.12) (resp. (3.13)) corresponds to case (3.8) (resp. case (3.9)).

Due to (2.8), there holds $w_t(y,\cdot)\in L^2(0,\infty)$ for a.e. $y\in I$.

In particular, we have

$$w(y,\cdot)\in C^0([0,\infty)) \ \text{ for a.e. } \ y\in I. \tag{3.14}$$

This together with (3.14) implies that, for a.e. $y \in I\setminus\tilde N(\delta)$ and any pair $(\delta,\bar\epsilon)\in(0,\infty)^2$ with $\delta>0$ being chosen such that

$$(z_i(y)-f(\bar\epsilon)\delta, z_i(y)+f(\bar\epsilon)\delta)\cap(z_j(y)-f(\bar\epsilon)\delta, z_j(y)+f(\bar\epsilon)\delta)=\emptyset \ \text{ for } \ i\neq j, \tag{3.15}$$

we have

$$|w(y,t)-z(y)| < f(\bar\epsilon)\delta \ \ \forall t \geq \tilde t(\delta,y), \tag{3.16}$$

$$\exists i\in\{1,2,3\}: \ \ |w(y,t)-z_i(y)| < f(\bar\epsilon)\delta \ \ \forall t \geq \tilde t(\delta,y) \tag{3.17}$$

or

$$\Big((w(y,t),-(\hat L_\epsilon \bar w)(y)) \in \mathcal{N}_{\bar\epsilon} \text{ and } (y,t)\notin M_\delta(y)\Big) \ \ \forall t \geq \tilde t(\delta,y). \tag{3.18}$$

With relation (3.11) at hand, we conclude from (3.14) and (3.18) that, for any $(\delta,\bar\epsilon)\in(0,\infty)^2$ with

$$(\hat z_1-\mu(\delta,\bar\epsilon),\hat z_1+\mu(\delta,\bar\epsilon))\cap(\hat z_2-\mu(\delta,\bar\epsilon),\hat z_2+\mu(\delta,\bar\epsilon))=\emptyset, \tag{3.19}$$

we have for $\hat z_i$ defined in (3.4)

$$(3.20) \qquad \exists i \in \{1,2\}: \quad |w(y,t) - \hat{z}_i| < \mu(\delta,\bar{\epsilon}) \;\; \forall t \geq \tilde{t}(\delta,y).$$

In other words: for any $(\delta,\bar{\epsilon}) \in (0,\infty)^2$ fulfilling relations (3.15) and (3.19) we have case (3.16), (3.17) or (3.20).

Now, choose a sequence $\{(\delta_k,\bar{\epsilon}_k)\}_{k\in\mathbb{N}}$ in $(0,\infty)^2$ with

$$\lim_{k\to\infty}(\delta_k,\bar{\epsilon}_k) = \lim_{k\to\infty} f(\bar{\epsilon}_k)\delta_k = \lim_{k\to\infty}\mu(\delta_k,\bar{\epsilon}_k) = 0$$

such that relations (3.15) and (3.19) are fulfilled.

Then, for any $k \in \mathbb{N}$, we have one of the cases (3.16), (3.17) and (3.20) with $(\delta,\bar{\epsilon}) = (\delta_k,\bar{\epsilon}_k)$.

In particular,

$$\exists \hat{w}(y) := \lim_{t\to\infty} w(y,t) \;\text{ for a.e. }\; y \in I,$$

and, due to the continuity of $\sigma$,

$$\lim_{t\to\infty}\sigma(w(y,t)) + (L_\epsilon w)(y,t) = \lim_{t\to\infty}\sigma(w(y,t)) - \lambda w(y,t) + (\hat{L}_\epsilon w)(y,t)$$

$$= \sigma(\hat{w}(y)) - \lambda\hat{w}(y) + (\hat{L}_\epsilon\bar{w})(y) = \sigma(\hat{w}(y)) + (L_\epsilon\bar{w})(y)$$

for a.e. $y \in I$.

With these relations at hand, the claim of the theorem is a direct consequence of the fact that $\hat{w} = \bar{w}$: if the strong time-asymptotic limit of $w(\cdot,t)$ exists in $L^2(I)$, the strong and the weak limit coincide. □

It remains to prove Lemma 3.4:

In order to prove relation (3.10), let $\Gamma \subset \mathbb{R}^2$ denote the graph of $z \mapsto \sigma(z) - \lambda z - P$ and, for $p \in \Gamma$, let $s(p)$ be the slope of the tangent of $\Gamma$ through $p$.

Then, for any $\bar{\epsilon} > 0$, the restriction of the function $|s(\cdot)| : \Gamma \to [0,\infty)$ on $\Gamma \setminus \mathcal{N}_{\bar{\epsilon}}$ (with $\mathcal{N}_{\bar{\epsilon}}$ being defined in 3.6) is bounded from below by a strictly positive constant.

This is due to the fact that $s(p) = 0$ iff $p = (\hat{z}_1,\sigma_1)$ or $p = (\hat{z}_2,\sigma_2)$, where $\hat{z}_i$ and $\sigma_i$ are defined in (3.4) and (3.5), that, for any $\bar{\epsilon} > 0$, these points are contained in $\mathcal{N}_{\bar{\epsilon}}$, that $\lim_{p\in\Gamma,|p|\to\infty}|s(p)| = \infty$ and that, for any $\bar{\epsilon} > 0$, the set $\Gamma \setminus \mathcal{N}_{\bar{\epsilon}}$ is closed.

In particular, if, for $z \in \mathbb{R}$, we have

$$|\sigma(z) - (1+\lambda)z + (\hat{L}_\epsilon \bar{w})(y) - P| < \delta \tag{3.21}$$

and

$$(z, -(\hat{L}_\epsilon \bar{w})(y)) \notin \mathcal{N}_{\bar{\epsilon}}$$

then, for the strictly positive real number $\delta$ in relation (3.21), we have

$$\mathrm{dist}(z, \mathfrak{M}(y)) \leq \left(\sup_{p \in \Gamma \setminus \mathcal{N}_{\bar{\epsilon}}} \{|s(p)|^{-1}\}\right) \delta$$

and relation (3.10) follows.

In order to prove relation relation (3.11), we first note that

$$\begin{aligned}
&\sigma(z) - \lambda(z) - P \\
&\quad = \sigma(\hat{z}_i) + \sigma'(\hat{z}_i)(z - \hat{z}_i) + \frac{1}{2}\sigma''(\hat{z}_i)(z - \hat{z}_i)^2 + \frac{1}{6}\sigma'''(\hat{z}_i)(z - \hat{z}_i)^3 \\
&\quad = \sigma_i + 3\hat{z}_i(z - \hat{z}_i)^2 + (z - \hat{z}_i)^3 \\
&\quad = \sigma_i \pm \sqrt{3(\lambda+1)}(z - \hat{z}_i)^2 + (z - \hat{z}_i)^3.
\end{aligned}$$

Furthermore, the following implication ist true:

$$(z, -(\hat{L}_\epsilon \bar{w})(y)) \in \mathcal{N}_{\bar{\epsilon}} \;\Rightarrow\; \left|(\hat{L}_\epsilon \bar{w})(y) - \sigma_i\right| \leq \bar{\epsilon}.$$

In particular, as long as, for some $i \in \{1, 2\}$,

$$|z - \hat{z}_i| \leq \frac{|3\hat{z}_i|}{2}$$

and

$$|\sigma(z) - \lambda z - P - \sigma_i| \leq \delta$$

we obtain

$$|3\hat{z}_i| - |z - \hat{z}_i| \geq \frac{|3\hat{z}_i|}{2} = \sqrt{\frac{3(\lambda+1)}{4}}$$

and therefore

$$\delta \geq |\sigma(z) - \lambda z - P - \sigma_i| - \bar{\epsilon} \geq \sqrt{\frac{3(\lambda+1)}{4}}(z - \hat{z}_i)^2 - \bar{\epsilon}.$$

In particular,

$$|z - \hat{z}_i| \leq \mu(\delta, \bar{\epsilon}),$$

for $\mu(\delta, \bar{\epsilon}) = \frac{\sqrt{\bar{\epsilon}+\delta}}{\gamma}$, where $\gamma = \sqrt{\frac{3(\lambda+1)}{4}}$.□

## References


[1] G. Alberti & G. Bellettini, A non-local anisotropic model for phase transitions: Asymptotic behaviour of rescaled energies, *European J. Appl. Math. 9*, No. 3, 261–284 (1998)

[2] G. Alberti, G. Bellettini, M. Cassandro, E. Presutti, Surface tension in Ising systems with Kac potentials, *J. Stat. Phys. 82*, 743–793 (1996)

[3] G. Andrews & J.M. Ball, Asymptotic behaviour and changes of phase in one-dimensional nonlinear viscoelasticity, *J. Differential Equations 44*, (1982), 316–341.

[4] P.W. Bates, X. Chen, A.J.J. Chmaij, Heteroclinic solutions of a van der Waals model with indefinite interactions, *Calc. Var. 24(3)*, 261-281 (2005)

[5] A. Dressel & C. Rohde, Global existence and uniqueness of solutions for a viscoelastic two-phase model with nonlocal capillarity, *http://www.iumj.indiana.edu/Preprints/3271.pdf*, to appear in *Indiana Univ. Math. J.*

[6] L. Modica, The gradient theory of phase transitions and the minimal interface criterion, *Arch. Rat. Mech. Anal. 98*, 123–142 (1987)

[7] R.L. Pego, Phase transitions in One-Dimensional Nonlinear Viscoelasticity: Admissibility and Stability, *Arch. Rat. Mech. Anal. 97*, 353–394 (1987)

[8] X. Ren & L. Truskinovsky, Finite scale microstructures in nonlocal elasticity, *Journal of Elasticity 59*, 319–355 (2000)