On the Convergence of Viscous Approximations After Shock Interactions

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Abstract: We consider a piecewise smooth solution to a scalar conservation law, with possibly interacting shocks. We show that, after the interactions have taken place, vanishing viscosity approximations can still be represented by a regular expansion on smooth regions and by a singular perturbation expansion near the shocks, in terms of powers of the viscosity coefficient.

1 - Introduction

Consider a strictly hyperbolic system of conservation laws

$$u_t + f(u)_x = 0, \tag{1.1}$$

together with its viscous approximations

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon\,. \tag{1.2}$$

For a fixed initial data with small total variation

$$u(0,\cdot) = \bar u(\cdot), \tag{1.3}$$

the convergence $u^\varepsilon \to u$, as $\varepsilon \to 0+$, was proved in [BB2]. An estimate on the convergence rate

$$\big\|u^\varepsilon(t) - u(t)\big\|_{\mathbf{L}^1(I\!\!R)} = \mathcal{O}(1)\cdot(t+1)\sqrt{\varepsilon}\,\ln\varepsilon,$$

was later provided in [BY]. In the scalar case, more detailed results can be found in [NT], [TT], and [TZ]. For related results on the stability of viscous shocks we refer to [FSe], [Ho], [MZ] and [Z].

Also for computational purposes, it is interesting to examine whether viscous approximations admit a power series expansion in the viscosity coefficient ε. In the case of a Hamilton-Jacobi equation on a bounded open domain $\Omega \subset I\!\!R^m$, Fleming and Souganidis [FS] showed that the solutions of the elliptic problem

$$\begin{cases} -\varepsilon\Delta u^\varepsilon + H(x, Du^\varepsilon) + u^\varepsilon = 0, & \text{for } x\in\Omega\,, \\ u^\varepsilon(x) = 0, & \text{for } x\in\partial\Omega\,, \end{cases} \tag{1.4}$$

admit an asymptotic expansion of the form

$$u^\varepsilon = u + \varepsilon v_1 + \varepsilon^2 v_2 + \cdots + \varepsilon^k v_k + o(\varepsilon^k). \tag{1.5}$$

Here the leading term u is the viscosity solution of the first order equation, formally obtained by setting $\varepsilon = 0$ in (1.4). The expansion (1.5) is valid restricted to suitable subsets of the domain Ω, where the limit solution u is smooth and can be constructed by the method of characteristics. This result was later used in [SD] to derive a higher order numerical method for Hamilton-Jacobi equations.

The recent paper [SX] has established a similar result in the context of a scalar conservation law. Namely, assume that the limit solution u of (1.1), (1.3) is smooth on a region Ω in the t-x plane bounded by two characteristics, say,

$$\Omega \doteq \Big\{(t,x)\,;\quad t\in[0,T]\,,\quad a+f'\big(\bar u(a)\big)\,t < x < b+f'\big(\bar u(b)\big)\,t\Big\}\,,$$

with $a<b$. Then one can determine functions v_j such that the expansion (1.5) is uniformly valid on every compact subset of Ω. Indeed, the analysis on [SX] shows that the presence of an arbitrary number of (possibly interacting) shocks outside the domain Ω does not affect the validity of the expansion in the region where u is smooth.

For discontinuous solutions, the viscous approximations clearly cannot converge uniformly on a neighborhood of a shock. As shown by the analysis of Goodman and Xin [GX], to represent the u^ε one needs to introduce a shock layer, described in terms of a stretched variable $\eta = \varepsilon^{-1}\big(x-\xi(t)\big)$. The viscous solution u^ε is obtained by matching the outer expansion (1.5) with an inner expansion of the form

$$u(t,\eta) = U_0(t,\eta) + \varepsilon U_1(t,\eta) + U_2(t,\eta) + \cdots. \tag{1.6}$$

Here $U_0(t,\cdot)$ is the unique viscous shock profile connecting the states $u\big(t,\xi(t)-\big)$, $u\big(t,\xi(t)+\big)$ to the right and to the left of the shock. The analysis in [GX] applies to isolated, non-interacting shocks. It is of interest to understand whether a similar inner and outer expansion can still be performed after several shock interactions have occurred. The present paper provides a positive answer in the case of a scalar conservation law.

More precisely, we consider a solution u to the conservation law (1.1) which contains arbitrarily many shock interactions, until at a certain time τ an isolated shock emerges. In addition, we consider a second solution $\tilde u$ containing one single shock, choosing the initial data $\tilde u(0,\cdot)$ in such a way that $\tilde u = u$ for $t>\tau$. Then we show that for $t>\tau$ the viscous approximations u^ε become exponentially close to $\tilde u^\varepsilon$ as $\varepsilon\to 0$. Indeed,

$$\big\|u^\varepsilon(t,\cdot) - \tilde u^\varepsilon(t,\cdot)\big\|_{\mathcal{C}^\nu} = o(\varepsilon^k),$$

for every $k,\nu\geq 1$. As a corollary, since $\tilde u^\varepsilon(t)$ admits a singular perturbation expansion, so does $u^\varepsilon(t)$ for all $t>\tau$.

2 - The main result

Let the scalar conservation law (1.1) have a smooth, convex flux, so that $f''(u)\geq k>0$ for all u. For a given time $\tau>0$, consider a bounded solution $u=u(t,x)$ which contains an arbitrary number of interacting shocks for $t<\tau$, but is piecewise smooth with one single shock for $t>\tau$, say located along the curve $x=\xi(t)$. We write

$$u^\pm(t) \doteq \lim_{x\to\xi(t)\pm} u(t,x),$$

for the left and right limits of u across this shock, and let

$$x^-(t) = \xi(\tau) - f'\big(u^-(\tau)\big)(\tau - t)\,, \qquad\qquad x^+(t) = \xi(\tau) - f'\big(u^+(\tau)\big)(\tau - t)\,,$$

be the minimal and maximal backward characteristics through the point $\big(\tau,\, \xi(\tau)\big)$. More precisely (see fig. 1), we assume that u is piecewise smooth outside the triangular domain bounded by the two backward characteristics impinging on the shock at time $t = \tau$:

$$\Lambda \doteq \Big\{(t,x)\,;\quad 0 \le t \le \tau\,,\ x^-(t) < x < x^+(t)\Big\}\,.$$

By suitably changing the initial data, we can then construct a second solution $\tilde u$ which is piecewise smooth with one single shock for all times $t \ge 0$, and moreover it coincides with u for $t > \tau$. Indeed, this can be achieved by choosing a suitable piecewise smooth initial condition $\tilde u(0,x)$ such that

$$\tilde u(0,x) = u(0,x), \qquad\qquad x \notin \big[x^-(0),\ x^+(0)\big], \tag{2.1}$$

$$\int_{x^-(0)}^{x^+(0)} \tilde u(0,x)\,dx = \int_{x^-(0)}^{x^+(0)} u(0,x)\,dx\,. \tag{2.2}$$

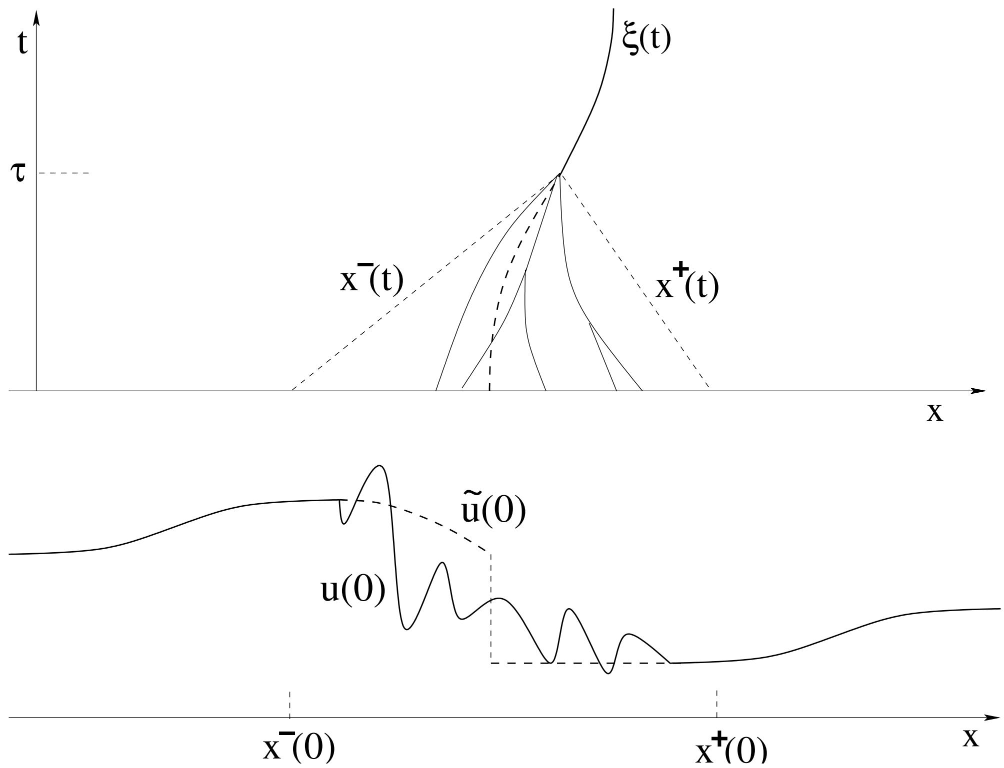

Figure 1: The solutions with initial data $u(0,\cdot)$ and $\tilde u(0,\cdot)$ coincide after time $t = \tau$.

The following theorem shows that, for any time $t > \tau$, the viscous approximations to the two solutions u and $\tilde u$ are extremely close. In particular, any singular perturbation expansion valid for $\tilde u^\varepsilon$ remains valid for u^ε as well.

Theorem 1. *In the above setting, let u^ε and $\tilde u^\varepsilon$ be the solutions to the viscous conservation law*

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \tag{2.3}$$

with initial data

$$u^\varepsilon(x,0) = u(x,0)\,, \qquad\qquad \tilde u^\varepsilon(x,0) = \tilde u(x,0)\,, \tag{2.4}$$

related as in (2.1)-(2.2). Let τ be the time when the single shock forms in the limit solution u. Then, for any integers $k, \nu \geq 0$, one has the high order convergence

$$\lim_{\varepsilon\to 0+} \varepsilon^{-k}\cdot \big\| u^\varepsilon - \tilde u^\varepsilon\big\|_{\mathcal{C}^\nu(\Omega)} = 0, \tag{2.5}$$

uniformly on every compact domain $\Omega \subset\subset \big\{(t,x)\,;\quad t>\tau\,,\ x\in I\!\!R^n\big\}$.

We sketch here the main ideas in the proof. Details will be worked out in Section 3. Call

$$U^- \doteq u\big(\tau,\,\xi(\tau)-\big)\,, \qquad\qquad U^+ \doteq u\big(\tau,\,\xi(\tau)+\big), \tag{2.6}$$

the left and right limits of the non-viscous solution u across the shock, at time $t=\tau$. By possibly performing the linear rescaling of coordinates

$$x' = x - \frac{f(U^+)-f(U^-)}{U^+-U^-}\,t\,,$$

and adding a constant to the flux f, it is not restrictive to assume that

$$f(U^+) = f(U^-) = 0\,, \tag{2.7}$$

so that the velocity of the shock at time $t=\tau$ is $\dot\xi(\tau)=0$. In the t-x plane we consider a rectangle of the form

$$Q = [\tau,\ \tau_4]\times\big[\xi(\tau)-\delta_0\,,\ \xi(\tau)+\delta_0\big]\,,$$

with $\tau_\ell = \tau + \ell\cdot C_0\,\delta_0$, for $\ell = 1,2,3,4$. Notice that, since $\dot\xi(\tau)=0$, given any constant $C_0>0$ we can choose $\delta_0>0$ small enough so that

$$a \doteq \xi(\tau)-\delta_0 < \xi(t) < \xi(\tau)+\delta_0 \doteq b\,. \tag{2.8}$$

for all $t\in[\tau,\ \tau+4C_0\delta_0]$. We recall that the asymptotic convergence result proved in [SX] shows that the solutions u^ε and $\tilde u^\varepsilon$ are extremely close, away from the shock. In particular, for every $\nu, k\geq 1$ one has

$$\sup_{t\in[\tau,\tau_4]} \big\|u^\varepsilon(t,\cdot)-\tilde u^\varepsilon(t,\cdot)\big\|_{\mathcal{C}^\nu(I\!\!R\setminus[a,b])} \ =\ \mathcal{O}(1)\cdot\varepsilon^k. \tag{2.9}$$

Here and in the sequel, we use the Landau symbol $\mathcal{O}(1)$ to denote a uniformly bounded quantity.

To estimate the distance $u^\varepsilon - \tilde u^\varepsilon$ inside the interval $[a,b]$, we shall use a homotopy method. Define $u^{\varepsilon,\theta}$ as the solution of (2.3) with interpolated initial data

$$u^{\varepsilon,\theta}(0,x) = \theta u^\varepsilon\big(0,x\big) + (1-\theta)\,\tilde u^\varepsilon\big(0,x\big)\,. \tag{2.10}$$

Moreover, call

$$z^{\varepsilon,\theta} \doteq \frac{\partial}{\partial\theta}\,u^{\varepsilon,\theta}\,.$$

A key step in the proof is to establish the asymptotic estimates

$$\int_a^b \left|z^{\varepsilon,\theta}(\tau_3,x)\right| dx \leq C_k\varepsilon^k, \tag{2.11}$$

for $\tau_3 = \tau + 3C_0\delta_0$ as above and every integer $k \geq 1$. Integrating w.r.t. $\theta \in [0,1]$, from (2.11) it follows

$$\begin{aligned} \int_a^b \left|u^\varepsilon(\tau_3,x) - \tilde u^\varepsilon(\tau_3,x)\right| dx &\leq \int_a^b \left|\int_0^1 \frac{\partial}{\partial\theta} u^{\varepsilon,\theta}(\tau_3,x)\, d\theta\right| dx \\ &\leq \sup_{\theta\in[0,1]} \int_a^b \left|z^{\varepsilon,\theta}(\tau_3,x)\right| dx \leq C_k\varepsilon^k. \end{aligned} \tag{2.12}$$

Using the regularity of the solutions $u^{\varepsilon,\theta}$, from the family of integral estimates (2.11), at the later time $\tau_4 > \tau_3$ on can derive pointwise estimates of the form

$$\left\|z^{\varepsilon,\theta}(\tau_4,\cdot)\right\|_{\mathcal{C}^\nu([a,b])} = \mathcal{O}(1)\cdot\varepsilon^k, \tag{2.13}$$

for every $k, \nu \geq 1$. Again integrating w.r.t. $\theta \in [0,1]$, these bounds in turn imply

$$\left\|u^\varepsilon(\tau_4,\cdot) - \tilde u^\varepsilon(\tau_4,\cdot)\right\|_{\mathcal{C}^\nu([a,b])} = \mathcal{O}(1)\cdot\varepsilon^k\,. \tag{2.14}$$

Given the compact domain Ω in the t-x plane, we can now choose $\delta_0 > 0$ so that

$$\Omega \subset [\tau_4, \infty[\,\times I\!\!R\,. \tag{2.15}$$

The bounds

$$\left\|u^\varepsilon(\tau_4,\cdot) - \tilde u^\varepsilon(\tau_4,\cdot)\right\|_{\mathcal{C}^\nu(I\!\!R)} = \mathcal{O}(1)\cdot\varepsilon^k\,, \tag{2.16}$$

which follow from (2.9) and (2.14), will finally imply (2.5).

Observing that each $z^{\varepsilon,\theta}$ provides a solution to the linearized conservation law

$$z_t + \left[f'(u^{\varepsilon,\theta})z\right]_x = \varepsilon\, z_{xx}\,, \tag{2.17}$$

to prove the key estimate (2.11) we consider the time intervals with extremal points $\tau < \tau_1 < \tau_2 < \tau_3$, as illustrated in fig. 2.

During the first interval $[\tau,\, \tau_1]$, following the analysis in [BD], we show that a viscous shock is formed. Hence, for all $t \in [\tau_1, \tau_3]$ and $\varepsilon > 0$ small enough, each solution $u^{\varepsilon,\theta}(t,\cdot)$ already contains one large viscous shock, say located around the point $\xi^{\varepsilon,\theta}(t)$. We can identify a thin region around the shock, of the form

$$\Lambda_{\varepsilon,\theta} \doteq \Big\{(t,x)\,;\quad t \in [\tau_1, \tau']\,,\quad x \in \left[\xi^{\varepsilon,\theta}(t) - C\varepsilon\,,\quad \xi^{\varepsilon,\theta}(t) + C\varepsilon\right]\Big\}\,,$$

such that, for $(t,x) \in [\tau_1,\, \tau_3] \times [a,b]$, outside this region we have

$$\left|u^{\varepsilon,\theta}(t,x) - U^-\right| \leq \frac{|U^- - U^+|}{7} \qquad \text{if } x < \xi^{\varepsilon,\theta}(t) - C\varepsilon\,, \tag{2.18}$$

$$\left|u^{\varepsilon,\theta}(t,x) - U^+\right| \leq \frac{|U^- - U^+|}{7} \qquad \text{if } x > \xi^{\varepsilon,\theta}(t) + C\varepsilon\,. \tag{2.19}$$

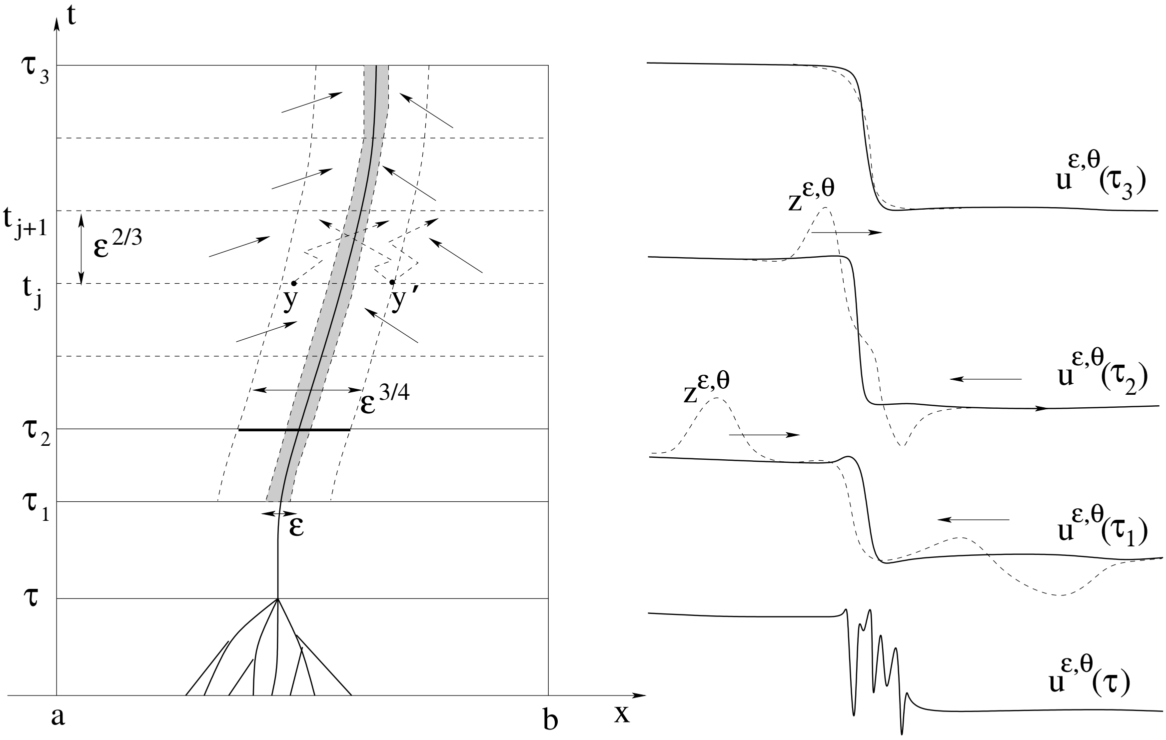

Figure 2: A viscous shock solution $u^{\varepsilon,\theta}$ and an infinitesimal perturbation $z^{\varepsilon,\theta}$. At time $t = \tau_1$ a viscous shock has formed. At $t = \tau_2$ most of the perturbation lies inside a small interval $I(\tau_2)$ of length $2\varepsilon^\gamma$. When $t = \tau_3$ nearly all the positive part of the perturbation $z^{\varepsilon,\theta}$ has cancelled with the negative part.

Next, we examine the behavior of the perturbation $z = z^{\varepsilon,\theta}$ during the remaining time interval $[\tau_1, \tau_3]$. By (2.18)-(2.19), the characteristics point strictly toward the strip $\Lambda_{\varepsilon,\theta}$. Indeed,

$$f'\big(u^{\varepsilon,\theta}(t,x)\big) \approx f'(U^-) > 0 \qquad\qquad \text{for} \quad x < \xi^{\varepsilon,\theta}(t) - C\varepsilon\,,$$

$$f'\big(u^{\varepsilon,\theta}(t,x)\big) \approx f'(U^+) < 0 \qquad\qquad \text{for} \quad x > \xi^{\varepsilon,\theta}(t) + C\varepsilon\,.$$

After some time, for $t \geq \tau_2$, we can show that almost all the perturbation is contained inside a strip of width $2\varepsilon^\gamma$ around the viscous shock, with $\gamma = 3/4$. Namely, introducing the interval

$$I(t) \doteq \big[\xi^{\varepsilon,\theta}(t) - \varepsilon^\gamma\,,\ \xi^{\varepsilon,\theta}(t) + \varepsilon^\gamma\big]\,,$$

around the point $\xi^{\varepsilon,\theta}$, for any $k \geq 1$ we have

$$\int_{I\!R\setminus I(t)} \big|z(t,x)\big|\,dx = \mathcal{O}(1)\cdot\varepsilon^k. \tag{2.20}$$

It now remains to understand what happens inside the interval $I(t)$ containing the shock. According to (2.1)-(2.2), the difference between the two solutions u^ε and $\tilde{u}^\varepsilon$ has zero total mass. This implies

$$\int_{-\infty}^{\infty} z(t,x)\,dx = 0\,. \tag{2.21}$$

We claim that, during the interval $[\tau_2, \tau_3]$, almost all the positive mass in $z = z^{\varepsilon,\theta}$ gets cancelled with the negative mass. To prove this, we divide $[\tau_2, \tau_3]$, into equal subintervals of length $\varepsilon^{2/3}$, inserting the points

$$t_j = \tau_2 + j \cdot \varepsilon^{2/3}, \qquad j = 0, 1, \ldots, N_\varepsilon . \tag{2.22}$$

A key step in the proof is to show that

$$\int_a^b \big|z(t_{j+1}, x)\big|\, dx \leq \alpha \cdot \int_a^b \big|z(t_j, x)\big|\, dx , \tag{2.23}$$

for some constant $\alpha < 1$ and all $j = 0, 1, \ldots, N_\varepsilon - 1$. From (2.23) it follows

$$\int_a^b \big|z(\tau_3, x)\big|\, dx \leq \alpha^{N_\varepsilon} \int_a^b \big|z(\tau_2, x)\big|\, dx \leq \alpha^{N_\varepsilon} \big\|u(0,\cdot) - \tilde u(0,\cdot)\big\|_{\mathbf{L}^1(I\!\!R)} = \mathcal{O}(1) \cdot \varepsilon^k, \tag{2.24}$$

for every $k \geq 1$. Indeed, $N_\varepsilon = (\tau_3 - \tau_2)/\varepsilon^{2/3}$, hence α^{N_ε} is an infinitesimal of higher order w.r.t. ε^k for any $k \geq 1$.

We conclude this section with some intuitive explanation about the inequalities (2.23). Calling $\Gamma(t, x, s, y)$ the fundamental solution of the linear parabolic equation (2.17), we can write

$$z(t_{j+1}, x) = \int \Gamma(t_{j+1}, x, t_j, y)\, z(t_j, y)\, dy .$$

Notice that $\Gamma(t, \cdot, s, y)$ can be interpreted as the probability density at time t of a random particle which is located at the point y at the initial time s. The motion of the particle is governed by the stochastic diffusion process

$$dY = f'\big(u^{\varepsilon,\theta}(t, Y(t))\big)\, dt + \sqrt{2\varepsilon}\, dB , \tag{2.25}$$

where B denotes a Brownian motion.

Consider the two sets

$$A_j^+ \doteq \Big\{x \in I(t_j)\,, \quad z(t_j, x) > 0\Big\}, \qquad A_j^- \doteq \Big\{x \in I(t_j)\,, \quad z(t_j, x) < 0\Big\}.$$

Since $z(t_j, \cdot)$ has zero total mass, and almost all of this mass is concentrated inside $I(t_j)$, we can write

$$z(t_{j+1}, x) \approx \int_{A_j^+} \Gamma(t_{j+1}, x, t_j, y)\, \big|z(t_j, y)\big|\, dy - \int_{A_j^-} \Gamma(t_{j+1}, x, t_j, y')\, \big|z(t_j, y')\big|\, dy' . \tag{2.26}$$

For any two points $y, y' \in I(t_j)$ we now have the key inequality

$$\begin{aligned} &\int \Big|\Gamma(t_{j+1}, x, t_j, y) - \Gamma(t_{j+1}, x, t_j, y')\Big|\, dx \\ &\qquad \leq 2\Big(1 - \text{Prob.}\Big\{Y(t) = Y'(t) \ \text{ for some } t \in [t_j,\, t_{j+1}]\Big\}\Big), \\ &\qquad \leq 2\alpha \end{aligned} \tag{2.27}$$

for some constant $\alpha < 1$. Here Y, Y' are two independent random paths of the diffusion process (2.25), starting from the points $y, y' \in I(t_j)$ respectively. Applying (2.27) to the case where $y \in A_j^+$

and $y' \in A_j^-$, from (2.26) we see that a nontrivial amount of cancellation occurs within each time interval $[t_j,\ t_{j+1}]$. Indeed, neglecting terms which are exponentially small as $\varepsilon \to 0$, we have

$$\int \big|z(t_{j+1},x)\big|\,dx \le \alpha \left(\int_{A_j^+} \big|z(t_j,y)\big|\,dy + \int_{A_j^-} \big|z(t_j,y')\big|\,dy' \right) .$$

Together with (2.20), this yields the estimate (2.23).

3 - Proof of the theorem

The proof of Theorem 1 will be given in several steps. As remarked in the previous section, we can assume that (2.7) holds, so that the shock has zero speed at the initial time $t = \tau$ when it is formed.

1. Fix times $\tau_\ell = \tau + \ell T$, with $\ell = 1, 2, 3, 4$, choosing $T = C_0\delta_0 > 0$ so that

$$\tau < \tau_4 < \min\ \big\{t;\quad (t,x) \in \Omega \quad \text{for some } x \in I\!R\big\}\,. \tag{3.1}$$

The precise values of the constants C_0, δ_0 will be determined later.

It is convenient to rescale coordinates, and consider $t' = (t-\tau)/\varepsilon$, $x' = (x - \xi(\tau))/\varepsilon$. Observe that the function $v^\varepsilon(t,x) \doteq u^\varepsilon\big(\tau + \varepsilon t\,,\ \xi(\tau) + \varepsilon x\big)$ provides a solution to the uniformly parabolic Cauchy problem

$$v_t + f(v)_x = v_{xx}\,, \tag{3.2}$$

$$v^\varepsilon(0,x) = u^\varepsilon\big(\tau\,,\ \xi(\tau) + \varepsilon x\big)\,. \tag{3.3}$$

It is useful to keep in mind that, as $\varepsilon \to 0$, the derivatives of the functions u^ε become arbitrarily large: $\|u^\varepsilon_x\|_{\mathbf{L}^\infty}\,,\ \|u^\varepsilon_{xx}\|_{\mathbf{L}^\infty} \to \infty\,.$ However, the derivatives of the rescaled functions v^ε remain uniformly bounded.

2. As in [BB1, BB2], in connection with any solution of (3.2) one can consider the planar curve

$$\gamma(t,x) = \begin{pmatrix} v(t,x) \\ w(t,v) \end{pmatrix} \doteq \begin{pmatrix} v(t,x) \\ f\big(v(t,x)\big) - v_x(t,x) \end{pmatrix}. \tag{3.4}$$

This curve evolves in time, moving in the direction of its curvature. Indeed, along each branch where $v_x = f(v) - w$ has constant sign, the function $w = w(t,v)$ satisfies the parabolic equation

$$w_t = \big(w - f(v)\big)^2 w_{vv}\,. \tag{3.5}$$

Observe that, if v is a viscous travelling wave solution for the equation (3.2), then the corresponding curve γ is a straight line, and does not vary in time. The speed of the travelling wave is given by the constant slope $\partial w/\partial v$. More generally, given any solution $v = v(t,x)$ of (3.2), for a fixed value v_0, the speed of the level set $t \mapsto x_0(t)$ implicitly defined by

$$v\big(t, x_0(t)\big) = v_0,$$

is given by

$$\frac{d}{dt}x_0(t) = \frac{\partial}{\partial v} w(t,v_0). \tag{3.6}$$

3. As in (2.6), let U^-, U^+ be the left and right limits of the inviscid solution u across the shock, at time $t = \tau$. Since we are assuming that the flux function is strictly convex, we can find intermediate states

$$U^+ < V^+ < V_0 < V^- < U^-,$$

and a constant $\eta_0 > 0$ such that

$$f'(V_0) = 0\,, \qquad\qquad \begin{cases} f'(u) \leq -2\eta_0 & \text{if} \quad u \leq V^+\,, \\ f'(u) \geq 2\eta_0 & \text{if} \quad u \geq V^-\,. \end{cases} \tag{3.7}$$

Since the equation (3.5) is uniformly parabolic when w is bounded away from $f(v)$, we can find $\eta_1 > 0$ such that the following holds. If $w = w(t, v)$ is any solution of (3.5) such that

$$\big|w(t', v)\big| \leq \eta_1 \qquad\qquad \text{for all } t' \in [t-1,\, t]\,, \quad v \in [V^+,\, V^-]\,, \tag{3.8}$$

then

$$\left|\frac{\partial}{\partial v} w(t, V_0)\right| \leq \eta_0\,. \tag{3.9}$$

4. Given two families of viscous solutions u^ε, $\tilde u^\varepsilon$, to estimate the distance between the corresponding rescaled solutions $v^\varepsilon, \tilde v^\varepsilon$ we shall use a homotopy method. Define $v^{\varepsilon,\theta}$ as the solution of (3.2) with interpolated initial data

$$v^{\varepsilon,\theta}(0, x) = \theta u^\varepsilon\big(\tau,\ \xi(\tau) + \varepsilon x\big) + (1-\theta)\,\tilde u^\varepsilon\big(\tau,\ \xi(\tau) + \varepsilon x\big)\,. \tag{3.10}$$

Moreover, call

$$z^{\varepsilon,\theta} \doteq \frac{\partial}{\partial\theta}\, v^{\varepsilon,\theta}\,. \tag{3.11}$$

Then $z = z^{\varepsilon,\theta}$ satisfies the linear equation

$$z_t + \big[f'(v^{\varepsilon,\theta})\, z\big]_x = z_{xx}, \tag{3.12}$$

together with the initial condition (independent of θ)

$$z(0, x) = u^\varepsilon\big(\tau,\ \xi(\tau) + \varepsilon x\big) - \tilde u^\varepsilon\big(\tau,\ \xi(\tau) + \varepsilon x\big)\,. \tag{3.13}$$

Observe that, for all $t \geq 0$ and all ε, θ, the assumptions (2.1)-(2.2) imply

$$\int_{-\infty}^{\infty} z^{\varepsilon,\theta}(t, x)\, dx \;=\; \int_{-\infty}^{\infty} z^{\varepsilon,\theta}(0, x)\, dx \;=\; 0\,. \tag{3.14}$$

5. We recall that the stretching of time and space variables defined in Step 1 transforms the domain

$$\Big\{(t, x)\,;\quad t \in [\tau, \tau + 4T]\,, \quad x \in \big[\xi(\tau) - \delta_0\,,\ \xi(\tau) + \delta_0\big]\Big\},$$

into the domain

$$\Big\{(t, x)\,;\quad t \in [0, 4T_\varepsilon]\,, \quad |x| \leq \frac{\delta_0}{\varepsilon}\Big\},$$

with $T_\varepsilon \doteq T/\varepsilon$.

Away from the shock, i.e. for $|x| \geq \delta_0/\varepsilon$ in the stretched coordinates, the result in [SX] guarantees the high order convergence $v^\varepsilon - \tilde{v}^\varepsilon = \mathcal{O}(\varepsilon^k)$ for every $k \geq 1$. The heart of the matter is to show that

$$\int_{-\delta_0/\varepsilon}^{\delta_0/\varepsilon} \big|z^{\varepsilon,\theta}(3T_\varepsilon, x)\big|\, dx = \mathcal{O}(\varepsilon^k), \tag{3.15}$$

for every integer $k \geq 1$ and uniformly for $\theta \in [0,1]$. As soon as the integral estimate (3.15) is proved, one can easily achieve similar pointwise estimates, because the coefficients $f'(v^{\varepsilon,\theta})$ of the equation are uniformly smooth. The strategy for proving the bounds (3.15) can be outlined as follows.

(i) At time $t = T_\varepsilon \doteq T/\varepsilon$ each solution $v^{\varepsilon,\theta}$ develops a large viscous shock. In the (t,v) variables, the graph of the corresponding curve γ at (3.4) becomes very close to a straight segment. More precisely, for $t > T_\varepsilon$ the functions $w^{\varepsilon,\theta}(t,v)$ satisfy

$$\big|w^{\varepsilon,\theta}(t,v)\big| \leq \eta_1 \qquad\qquad \text{for all } t \geq T_\varepsilon\,, \quad v \in [V^+,\, V^-]\,, \tag{3.16}$$

Hence, by (3.8)-(3.9)

$$\Big|\frac{\partial}{\partial v} w^{\varepsilon,\theta}(t, V_0)\Big| \leq \eta_0\,. \tag{3.17}$$

We can thus define the approximate location $\xi^{\varepsilon,\theta}(t)$ of the viscous shock, in terms of the identity

$$v^{\varepsilon,\theta}\big(t,\ \xi^{\varepsilon,\theta}(t)\big) = V_0. \tag{3.18}$$

According to (3.17), this viscous shock moves with speed

$$\big|\dot{\xi}^{\varepsilon,\theta}(t)\big| \leq \eta_0\,. \tag{3.19}$$

(ii) At time $t \geq 2T_\varepsilon$, nearly all the mass in $z^{\varepsilon,\theta}$ is located within the strip

$$I^{\varepsilon,\theta}(t) \doteq \big[\xi^{\varepsilon,\theta}(t) - \varepsilon^{-1/4}\ ,\ \xi^{\varepsilon,\theta}(t) + \varepsilon^{-1/4}\big]\,. \tag{3.20}$$

(iii) During the time interval $[2T_\varepsilon\ ,\ 3T_\varepsilon]$ nearly all the positive mass of $z^{\varepsilon,\theta}$ is cancelled with the negative mass. As a consequence, at time $t = 3T_\varepsilon$ the asymptotic estimates (3.15) hold.

6. We show here that each solution $v^{\varepsilon,\theta}$ develops one large viscous shock within time T_ε. This process of shock formation has been analyzed in detail in [BD]. The main differences between the situation analyzed in [BD] and the present one are the following: (i) Here we are assuming a strictly convex flux. This simplifies the proof, because we can use the classical one-sided Oleinik estimates on the gradient of the solution. (ii) We are not assuming anything about the behavior of the solution $v^{\varepsilon,\theta}(t,x)$ for $x \to \pm\infty$. Instead, we know that, at the endpoints of the interval $[-\delta_0/\varepsilon\,,\ \delta_0/\varepsilon]$, the function $v^{\varepsilon,\theta}$ takes values very close to U^-, U^+. Moreover its derivative is

$$v_x^{\varepsilon,\theta} \;=\; \varepsilon\, u_x^{\varepsilon,\theta} \;=\; \mathcal{O}(1)\cdot\varepsilon\,. \tag{3.21}$$

In the following, η_1 is the constant introduced at (3.8). We choose $\delta_1 > 0$ small enough so that

$$|w| \leq \frac{\eta_1}{4}, \tag{3.22}$$

whenever $|w - f(u)| \le \delta_1$ for some u such that either $|u - U^-| \le \delta_1$ or $|u - U^+| \le \delta_1$. Of course this is possible because $f(U^-) = f(U^+) = 0$.

As in (2.8), consider an interval $[a, b]$ containing the point $\xi(\tau)$ in its interior, and define the stretched interval $I_\varepsilon = [a_\varepsilon, b_\varepsilon]$ according to

$$a_\varepsilon \doteq \frac{a - \xi(\tau)}{\varepsilon} = -\frac{\delta_0}{\varepsilon}, \qquad b_\varepsilon \doteq \frac{b - \xi(\tau)}{\varepsilon} = \frac{\delta_0}{\varepsilon}.$$

Choosing $\delta_2 > 0$ and the interval $[a, b]$ small enough, in the rescaled variables we shall have

$$\left|v^{\varepsilon,\theta}(t, a_\varepsilon) - U^-\right| + \left|v^{\varepsilon,\theta}(t, b_\varepsilon) - U^+\right| \le \frac{\delta_1}{2}, \qquad \text{for all } t \in [0, \delta_2/\varepsilon]. \tag{3.23}$$

Since we are assuming $f'' \ge \kappa > 0$, after time τ in the original variables the function $u^{\varepsilon,\theta}$ satisfies $u_x^{\varepsilon,\theta} \le (\kappa\tau)^{-1}$. Hence, in the stretched variables, $v_x^{\varepsilon,\theta} \le \varepsilon(\kappa\tau)^{-1}$. Together with (3.23), this yields

$$U^+ - \frac{\delta_1}{2} - \frac{2\delta_0}{\kappa\tau} \le v^{\varepsilon,\theta}(t, x) \le U^- + \frac{\delta_1}{2} + \frac{2\delta_0}{\kappa\tau}, \tag{3.24}$$

valid for $0 \le t \le \delta_2/\varepsilon$ and $|x| \le \delta_0/\varepsilon$. Choosing δ_0 sufficiently small, we can thus achieve

$$v^{\varepsilon,\theta}(t, x) \in [U^+ - \delta_1\,,\ U^- + \delta_1]. \tag{3.25}$$

Next, we claim that, if (2.8) holds, then the curve $\gamma = \gamma^{\varepsilon,\theta}$ in (3.4) corresponding to $v^{\varepsilon,\theta}$ satisfies

$$\gamma(t, x) \in \Lambda_{\delta_1} \doteq \overline{co}\Big\{\big(u\,,\ f(u) + \xi\big)\,;\ \ u \in [U^+ - \delta_1\,,\ U^- + \delta_1],\ \ |\xi| \le \frac{\eta_1}{3}\Big\}, \tag{3.26}$$

whenever

$$t \in \Big[\frac{C_0\delta_0}{2\varepsilon}\,,\ \frac{C_0\delta_0}{\varepsilon}\Big], \qquad x \in I_\varepsilon \doteq \Big[-\frac{\delta_0}{\varepsilon}\,,\ \frac{\delta_0}{\varepsilon}\Big],$$

and $\varepsilon > 0$ is sufficiently small.

Indeed, for $x = \pm\delta_0/\varepsilon$ we have $v_x^{\varepsilon,\theta} = \mathcal{O}(\varepsilon)$, and the estimate (3.26) follows from (3.25). To prove that (3.26) holds for all intermediate values of x, we need to construct suitable upper and lower solutions for the parabolic equation (3.5).

We first observe that, since $f'' \ge \kappa > 0$, the function

$$w^-(t, v) \doteq f(v) - \frac{1}{\kappa t},$$

is a lower solution of (3.5). Hence every branch of the curve γ satisfies

$$w(t, v) \ge w^-(t, v) = f(v) - \frac{1}{\kappa t} \ge f(v) - \frac{\eta_1}{4}, \tag{3.27}$$

for $t \ge 4(\kappa\eta_1)^{-1}$. For any choice of C_0, δ_0, this is certainly true when $t \ge C_0\delta_0/2\varepsilon$, with ε sufficiently small.

Next, let f^+ be the affine function which coincides with f at the two points $v = U^+ - \delta_1$ and $v = U^- + \delta_1$. Moreover, consider the polynomial $p(v) = A + Bv - (v^2/2)$, choosing the constants A, B so that

$$p(U^+ - \delta_1) = p(U^- + \delta_1) = 1.$$

For $v \in [U^+ - \delta_1\,,\ U^- + \delta_1]$, consider a function of the form

$$w^+(t,v) = f^+(v) + \frac{\eta_1}{4} + \beta(t)\,p(v).$$

Computing

$$w_t^+ = \dot\beta(t)\,p(v)\,, \qquad \big(w^+ - f(v)\big)^2 w_{vv}^+ \leq -\big(\beta(t)\,p(v)\big)^2\beta(t)\,,$$

we deduce that the function w^+ is an upper solution of (3.5) provided that

$$\dot\beta(t)\,p(v) \geq -\beta^3(t)\,p^2(v) \qquad t \geq 0\,, \quad v \in [U^+ - \delta_1\,,\ U^- + \delta_1]\,.$$

Since $p(v) \geq 1$, this is certainly the case if $\dot\beta \geq -\beta^3$, hence if $\beta(t) = t^{-1/2}$.

Concerning the endpoints, when $x = \pm\delta_0/\varepsilon$ we already know that $f(v) - w = v_x^{\varepsilon,\theta} = \mathcal{O}(1)\cdot\varepsilon$. By a comparison argument, the portion of the curve $w = w(t,v)$ corresponding to the solution $v^{\varepsilon,\theta}$ as $x \in I_\varepsilon$ lies entirely below the upper solution w^+. For t sufficiently large we thus have

$$w(t,v) \leq w^+(t,v) \leq f^+(v) + t^{-1/2}\cdot\max\,\big\{p(u)\,;\ u \in I\!R\big\} + \frac{\eta_1}{4}\,. \tag{3.28}$$

In particular, this is true when $t \geq C_0\delta_0/2\varepsilon$, for ε sufficiently small. This achieves the proof of (3.26).

We now analyze the second phase of shock formation. We claim that, if (2.8) continues to hold, then the curve $\gamma = \gamma^{\varepsilon,\theta}$ corresponding to $v^{\varepsilon,\theta}$ satisfies

$$\gamma(t,x) \in \Lambda'_{\delta_1} \doteq \Big\{(u,w)\,;\ \ u \in [U^+ - \delta_1\,,\ U^- + \delta_1]\,,\ \ |w| \leq \eta_1\Big\}\,. \tag{3.29}$$

for all $t \geq \big[C_0\delta_0/\varepsilon\,,\ 4C_0\delta_0/\varepsilon\big]$ and $x \in I_\varepsilon \doteq [-\delta_0/\varepsilon\,,\ \delta_0/\varepsilon]$. Indeed, this result follows from the analysis in [BD], which we briefly recall here. Let $\eta_1 > 0$ be given and assume that the curve γ already lies in the convex set Λ_{δ_1} at (3.26) and that the values of γ at the endpoints $x = \pm\delta_0/\varepsilon$ are sufficiently close to $\big(U^\pm, f(U^\pm)\big)$. Then, according to Lemma 5 in [BD], the additional length of time Δt needed to achieve the inclusion (3.29) grows linearly with the length of the interval $I_\varepsilon = [-\delta_0/\varepsilon\,,\ \delta_0/\varepsilon]$, say $\Delta t \leq C\cdot 2\delta_0/\varepsilon$.

To achieve the desired estimate (3.29) for all $\varepsilon > 0$ sufficiently small, we thus choose the constants in the following order: η_1, δ_1, C_0, and finally δ_0, in such a way that (2.8) is satisfied for all $t \in [\tau\,,\ \tau + 4C_0\delta_0]$.

For future purpose, it is convenient to choose here the constant C_0 large enough so that it satisfies the additional inequality

$$C_0 \geq 8/\eta_0\,. \tag{3.30}$$

7. Having proved that, after time $T_\varepsilon = C_0\delta_0/\varepsilon$, each solution $v^{\varepsilon,\theta}$ contains a large viscous shock, we now study the behavior of the first order perturbations $z^{\varepsilon,\theta}$. The solution of the linear equation (3.12) can be expressed in term of the fundamental solutions. Indeed, for $0 < s < t$ one has

$$z(t,x) = \int \Gamma^{\varepsilon,\theta}(t,x,s,y)\,z(s,y)\,dy\,. \tag{3.31}$$

Here $\Gamma^{\varepsilon,\theta}(t,x,s,y)$ is the fundamental solution of (3.12) corresponding to a Dirac mass initially located the point y at time s. It is useful here to observe that, for $t \in [\tau,\ \tau + 4C_0\delta_0]$, the location of the shock in the original solutions $u = \tilde{u}$ remains strictly inside the fixed interval $[-\delta_0, \delta_0]$. By the analysis in [SX] we know that, for $t \in [0, 4T_\varepsilon]$, nearly all the perturbation lies within a bounded interval:

$$\int_{|x|>(\delta_0-c)/\varepsilon} \left|z^{\varepsilon,\theta}(t,x)\right| dx = o(\varepsilon^k), \tag{3.32}$$

for every $k \geq 1$. We can thus assume that, in all rescaled solutions $v^{\varepsilon,\theta}$, the viscous shocks are centered at points $\xi^{\varepsilon,\theta}$ such that

$$a_\varepsilon + c\varepsilon^{-1} \;=\; -\frac{\delta_0 - c}{\varepsilon} \;<\; \xi^{\varepsilon,\theta}(t) < \frac{\delta_0 - c}{\varepsilon} \;=\; b_\varepsilon - c\varepsilon^{-1}, \tag{3.33}$$

for some constant $0 < c < \delta_0$. In this step we show that, as $\varepsilon \to 0$, for $t \in [2T_\varepsilon\,,\ 4T_\varepsilon]$ we have the stronger asymptotic estimate

$$\int_{|x-\xi^{\varepsilon,\theta}(t)|>\varepsilon^{-1/4}} \left|z^{\varepsilon,\theta}(t,x)\right| dx = o(\varepsilon^k)\,. \tag{3.34}$$

This shows that nearly all of the perturbation $z^{\varepsilon,\theta}$ is concentrated in a narrow strip around the viscous shock in $v^{\varepsilon,\theta}$. To establish (3.34), consider any point $y \in [a_\varepsilon, b_\varepsilon]$. Because of (3.31)-(3.32), it suffices to show that, as $\varepsilon \to 0$,

$$\int_{|x-\xi^{\varepsilon,\theta}(t)|>\varepsilon^{-1/4}} \Gamma^{\varepsilon,\theta}(2T_\varepsilon, x, T_\varepsilon, y)\, dx = o(\varepsilon^k)\,. \tag{3.35}$$

uniformly as the initial point y varies in the interval $\big[a_\varepsilon + c\varepsilon^{-1}\,,\ b_\varepsilon - c\varepsilon^{-1}\big]$.

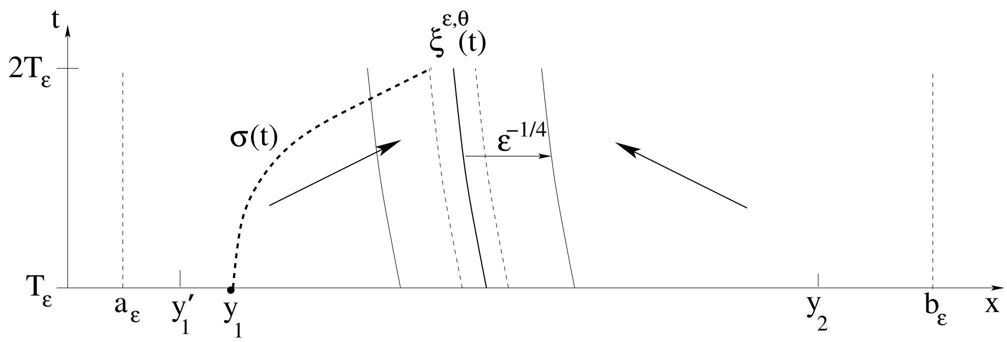

Figure 3: Outside a small strip of width $\mathcal{O}(1)$ the characteristic speed points toward the shock.

We now recall that, by the choice of V^+, V^- and η_0, η_1 at (3.7)–(3.9), we have

$$\left|\dot{\xi}^{\varepsilon,\theta}(t)\right| \leq \eta_0\,, \tag{3.36}$$

$$\lambda^{\varepsilon,\theta}(t,x) \doteq f'\big(v^{\varepsilon,\theta}(t,x)\big) \begin{cases} \geq\ 2\eta_0 & \text{if } x \in \big[a_\varepsilon\,,\ \xi^{\varepsilon,\theta} - \rho\big]\,, \\ \leq -2\eta_0 & \text{if } x \in \big[\xi^{\varepsilon,\theta} + \rho\,,\ b_\varepsilon\big]\,. \end{cases} \tag{3.37}$$

for some constant ρ and all $t \in [T_\varepsilon\,,\ 4T_\varepsilon]$.

To prove (3.35), we use the representation $\Gamma^{\varepsilon,\theta} = Z_x$, where Z provides the solution to the linear parabolic Cauchy problem

$$Z_t + \lambda^{\varepsilon,\theta}(t,x)\, Z_x = Z_{xx} \qquad\qquad Z(0,x) = \begin{cases} 0 & \text{if} \quad x < y\,, \\ 1 & \text{if} \quad x > y\,. \end{cases} \tag{3.38}$$

We begin by examining the special case where $y = y_1 \doteq a_\varepsilon + c\varepsilon^{-1}$. Thanks to (3.30), we can construct a smooth path $t \mapsto \sigma(t)$ such that (see fig. 3)

$$\sigma(T_\varepsilon) = y_1\,, \qquad \sigma(2T_\varepsilon) = \xi^{\varepsilon,\theta}(2T_\varepsilon) - \rho\,, \qquad \dot\sigma(t) \in [0\,,\ \eta_0/2], \qquad \text{for all } t \in [T_\varepsilon\,,\ 2T_\varepsilon]\,. \tag{3.39}$$

For $t \in [T_\varepsilon\,,\ 2T_\varepsilon]$, consider the two functions (see fig. 4)

$$Z_1(t,x) = \begin{cases} e^{(\eta_0/2)(x-\sigma(t))} + \beta T_\varepsilon & \text{if} \quad x < \sigma(t)\,, \\ 1 + \beta T_\varepsilon & \text{if} \quad x \geq \sigma(t)\,, \end{cases}$$

$$Z_2(t,x) = \begin{cases} \beta(t - T_\varepsilon) & \text{if} \quad x \leq a_\varepsilon\,, \\ \beta(t - T_\varepsilon) + \beta(x - a_\varepsilon)^2/2 & \text{if} \quad x > a_\varepsilon\,, \end{cases}$$

with

$$\beta \doteq \exp\left\{ -\frac{\eta_0}{2} \cdot \frac{c}{2\varepsilon} \right\}.$$

Set $y_1' \doteq (a_\varepsilon + y_1)/2$. In connection with the parabolic equation in (3.38), a straightforward computation now shows that

- Z_1 is an upper solution for $x \in [a_\varepsilon\,,\ \infty[\,$,
- Z_2 is an upper solution for $x \in\,]-\infty\,,\ y_1]$,
- $Z_2(t, a_\varepsilon) < Z_1(t, a_\varepsilon)$, while $Z_1(t, y_1') < Z_2(t, y_1')$.

We conclude that the function

$$Z^+(t,x) \doteq \min\big\{ Z_1(t,x)\,,\ Z_2(t,x) \big\},$$

is an upper solution of the Cauchy problem (3.38). In particular, as $\varepsilon \to 0$ it satisfies the asymptotic estimate

$$\begin{aligned} Z\big(2T_\varepsilon\,, \xi^{\varepsilon,\theta}(2T_\varepsilon) - \rho - \varepsilon^{-1/4}\big) &\leq Z^+\big(2T_\varepsilon\,,\ \sigma(2T_\varepsilon) - \varepsilon^{-1/4}\big) \\ &= \exp\left\{ -\frac{\eta_0}{2} \cdot \varepsilon^{-1/4} \right\} + \exp\left\{ -\frac{\eta_0}{2} \cdot \frac{c}{2\varepsilon} \right\} \cdot \frac{C_0 \delta_0}{\varepsilon} = o(\varepsilon^k)\,. \end{aligned}$$

for any positive integer k. Next, consider the other extreme case where $y = y_2 \doteq b_\varepsilon - c\varepsilon^{-1}$. An entirely similar estimate yields

$$Z\big(2T_\varepsilon\,,\ \xi^{\varepsilon,\theta}(2T_\varepsilon) + \rho + \varepsilon^{-1/4}\big) \geq 1 - o(\varepsilon^k)\,,$$

for any $k \geq 1$.

Finally, consider any initial point $y \in [y_1,\, y_2]$. By comparison with the two above cases, we conclude that the corresponding solution of the Cauchy problem (3.39) satisfies

$$Z\big(2T_\varepsilon\,,\ \xi^{\varepsilon,\theta}(2T_\varepsilon) - \rho - \varepsilon^{-1/4}\big) = o(\varepsilon^k)\,, \qquad Z\big(2T_\varepsilon\,,\ \xi^{\varepsilon,\theta}(2T_\varepsilon) + \rho + \varepsilon^{-1/4}\big) \geq 1 - o(\varepsilon^k)\,. \tag{3.40}$$

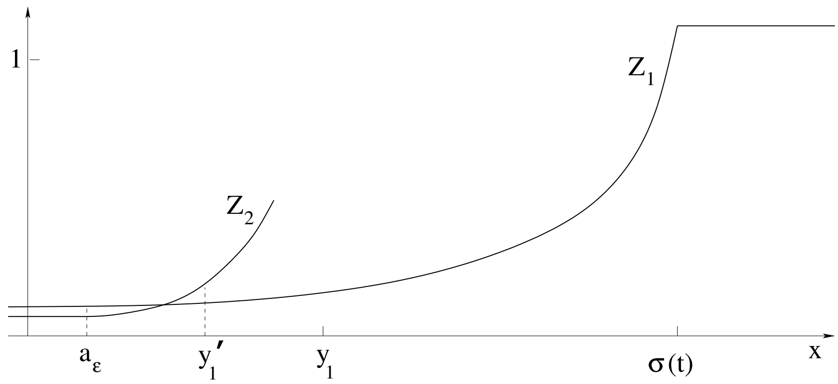

Figure 4: Two upper solutions for the parabolic equation (3.38).

Recalling that $\Gamma^{\varepsilon,\theta} = Z_x$, from (3.40) we deduce (3.35), as claimed.

8. In preparation for the next comparison estimate, we study a specific Cauchy problem. Let $M > \eta_0 > 0$ and $\rho > 0$ be given. Define the points

$$P_\varepsilon \doteq \varepsilon^{-1/4}, \qquad\qquad Q_\varepsilon = 2\varepsilon^{-1/4} + \rho,$$

and consider the equation

$$W_t + \lambda(x)\, W_x = W_{xx}, \qquad\qquad \lambda(x) = \begin{cases} \eta_0 & \text{if} \quad x \in [0,\, 2P_\varepsilon], \\ -M & \text{if} \quad x \notin [0,\, 2P_\varepsilon], \end{cases} \tag{3.41}$$

with initial condition

$$W(0,x) = \begin{cases} 0 & \text{if} \quad x < P_\varepsilon, \\ 1 & \text{if} \quad x > P_\varepsilon. \end{cases} \tag{3.42}$$

We claim that, as $\varepsilon \to 0$, at time $t = \varepsilon^{-1/3}$ the solution satisfies

$$W(\varepsilon^{-1/3},\, 0) = o(\varepsilon^k), \qquad\qquad W(\varepsilon^{-1/3},\, Q_\varepsilon) \le \alpha < 1, \tag{3.43}$$

for some constant α independent of ε and any $k \ge 1$.

The first estimate in (3.43) is proved by constructing a suitable supersolution, as in the previous step. Set

$$\sigma(t) \doteq \begin{cases} P_\varepsilon + (\eta_0 t/2) & \text{if} \quad t \in [0,\, 2P_\varepsilon/\eta_0], \\ 2P_\varepsilon & \text{if} \quad t \in [2P_\varepsilon/\eta_0\,,\ \varepsilon^{-1/3}]. \end{cases}$$

For $t \in [0\,,\ \varepsilon^{-1/3}]$, consider the two functions (see fig. 4)

$$W_1(t,x) = \begin{cases} e^{(\eta_0/2)(x-\sigma(t))} + \beta\varepsilon^{-1/3} & \text{if} \quad x < \sigma(t), \\ 1 + \beta\varepsilon^{-1/3} & \text{if} \quad x \ge \sigma(t), \end{cases}$$

$$W_2(t,x) = \begin{cases} \beta t & \text{if} \quad x \le 0, \\ \beta t + \beta x^2/2 & \text{if} \quad x > 0, \end{cases}$$

with

$$\beta \doteq \exp\Big\{ -\frac{\eta_0}{2} \cdot \varepsilon^{-1/4} \Big\}.$$

In connection with the parabolic equation (3.41), a straightforward computation now shows that

- W_1 is an upper solution for $x \in [0\,,\ \infty[\,$,
- W_2 is an upper solution for $x \in]-\infty\,,\ 2P_\varepsilon]$,
- $W_2(t,0) < W_1(t,0)$, while $W_1(t,P_\varepsilon) < W_2(t,P_\varepsilon)$.

We conclude that the function

$$W^+(t,x) \doteq \min\,\big\{W_1(t,x)\,,\ W_2(t,x)\big\},$$

is an upper solution of the Cauchy problem (3.41). Therefore, as $\varepsilon \to 0$ we have the asymptotic estimate

$$W(\varepsilon^{-1/3}\,,\ 0) \leq W^+(\varepsilon^{-1/3}\,,\ 0) \ = \ \varepsilon^{-1/3} \cdot \exp\Big\{-\frac{\eta_0}{2}\cdot\varepsilon^{-1/4}\Big\} \ = \ o(\varepsilon^k)\,.$$

for any positive integer k.

To prove the second inequality in (3.43) we observe that, when $t \geq 2P_\varepsilon/\eta_0$ and $\sigma(t) = 2P_\varepsilon$, one has

$$W^+(t,\,2P_\varepsilon - 1) \ \leq\ W_1(t,\,2P_\varepsilon - 1) \ = \ e^{-\eta_0/2} + \beta\varepsilon^{-1/3} \ < \ \frac{1+e^{-\eta_0/2}}{2}\,,$$

for all $\varepsilon > 0$ sufficiently small. On the domain

$$\mathcal{D} \doteq \Big\{(t,x)\,;\ \ t \in [2P_\varepsilon/\eta_0\,,\ \varepsilon^{-1/3}]\,,\ \ x \geq 2P_\varepsilon - 1 + \frac{M}{2}(t-\varepsilon^{-1/3})\Big\}\,,$$

consider the function

$$W^\sharp(t,x) \doteq 1 + \varepsilon - \beta^* \exp\left\{-2M\Big(x - (2P_\varepsilon - 1) - \frac{M}{2}(t-\varepsilon^{-1/3})\Big)\right\}\,.$$

with $\beta^* = (1-e^{-\eta_0/2})/4$. For $\varepsilon > 0$ sufficiently small, one checks that the function $W^\sharp$ provides an upper solution to (3.41) on the domain $\mathcal{D}$. Moreover, $W^\sharp > W^+$ on the parabolic boundary of $\mathcal{D}$. We thus conclude that

$$W(t,x) \leq \min\,\big\{W^+(t,x)\,,\ W^\sharp(t,x)\big\},$$

for all $(t,x) \in \mathcal{D}$. In particular, this implies

$$W(\varepsilon^{-1/3}, Q_\varepsilon) \ \leq\ W^\sharp(\varepsilon^{-1/3}, Q_\varepsilon) \ = \ 1 + \varepsilon - \beta^* \cdot e^{-2M(1+\rho)} \ \leq\ \alpha\,, \tag{3.44}$$

with

$$\alpha = 1 - \frac{1-e^{-\eta_0/2}}{5}\, e^{-2M(1+\rho)} \ < \ 1,$$

and for all $\varepsilon > 0$ sufficiently small.

9. We now divide the time interval $[2T_\varepsilon,\,3T_\varepsilon]$ into equal subintervals, inserting the times

$$t_j \doteq 2T_\varepsilon + j\cdot\varepsilon^{-1/3}, \qquad\qquad j = 0,1,\ldots,N_\varepsilon\,.$$

We also define the intervals

$$I_j \doteq \left[\xi^{\varepsilon,\theta}(t_j) - \rho - \varepsilon^{-1/4}\,,\ \xi^{\varepsilon,\theta}(t_j) + \rho + \varepsilon^{-1/4}\right]. \tag{3.45}$$

We claim that, for each j and every couple of points $y, y' \in I_{j-1}$ one has

$$\int_{I\!R\setminus I_j} \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y)\, dx = o(\varepsilon^k), \tag{3.46}$$

for all $k \geq 1$, and moreover

$$\int_{I\!R} \left|\Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y) - \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y')\right| dx \ \leq\ 2\alpha\,. \tag{3.47}$$

To prove (3.46), define

$$Y(t,x) \ \doteq\ \int_{-\infty}^{x+\xi^{\varepsilon,\theta}(t_{j-1}+t)-\rho-2\varepsilon^{-1/4}} \Gamma^{\varepsilon,\theta}(t_{j-1}+t\,,\ x'\,,\ t_{j-1}\,,\ y)\, dx'.$$

Observe that Y is a lower solution of the Cauchy problem (3.41)-(3.42). Hence $Y \leq W$. In particular, taking $x = 0$ we conclude

$$\int_{-\infty}^{\xi^{\varepsilon,\theta}(t_j)-\rho-2\varepsilon^{-1/4}} \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y)\, dx \ =\ Y(\varepsilon^{-1/3}, 0) \leq W(\varepsilon^{-1/3}, 0) = o(\varepsilon^k),$$

for any $k \geq 1$. By reversing the direction of the x-axis we obtain the symmetric estimate

$$\int_{\xi^{\varepsilon,\theta}(t_j)+\rho+2\varepsilon^{-1/4}}^{\infty} \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y)\, dx = o(\varepsilon^k)\,.$$

Together, the two above estimates yield (3.46).

To prove (3.47), assume $y < y'$ and consider the function

$$\Gamma^*(t,x) \doteq \Gamma^{\varepsilon,\theta}(t_{j-1}+t, x, t_{j-1}, y') - \Gamma^{\varepsilon,\theta}(t_{j-1}+t, x, t_{j-1}, y).$$

Observe that

$$\int_{-\infty}^{\infty} \Gamma^*(t,x)\, dx = 0.$$

Moreover, for each $t > 0$ this function has exactly one intersection with the x-axis, say located at $x = \zeta(t)$, so that

$$\begin{cases} \Gamma^*(t,x) > 0 & \text{if } \ x > \zeta(t_{j-1}+t)\,, \\ \Gamma^*(t,x) < 0 & \text{if } \ x < \zeta(t_{j-1}+t)\,. \end{cases}$$

At time $t = \varepsilon^{-1/3}$ we consider two cases. If $\zeta(t_j) \leq \xi^{\varepsilon,\theta}(t_j)$, then

$$\begin{aligned}\int \left|\Gamma^*(\varepsilon^{-1/3}, x)\right| dx \ &=\ 2\int_{-\infty}^{\zeta(t_j)} \left|\Gamma^*(\varepsilon^{-1/3}, x)\right| dx \ \leq\ 2\int_{-\infty}^{\zeta(t_j)} \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y')\, dx \\ &\leq\ 2\int_{-\infty}^{\xi^{\varepsilon,\theta}(t_j)} \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y')\, dx \ \leq\ 2Y(\varepsilon^{-1/3},\ \rho+2\varepsilon^{-1/4}) \\ &\leq\ 2W(\varepsilon^{-1/3},\ \rho+2\varepsilon^{-1/4}) \ \leq\ 2\alpha\,,\end{aligned}$$

because of (3.44). The alternative case, where $\zeta(t_j) \geq \xi^{\varepsilon,\theta}(t_j)$, can be handled in an entirely similar way, reversing the direction of the x-axis.

Because of the representation

$$z^{\varepsilon,\theta}(t_j, x) = \int \Gamma^{\varepsilon,\theta}(t_j, x, t_{j-1}, y)\, dy\,,$$

the two estimates (3.46)-(3.47) show that, during each time interval $[t_{j-1}\,,\ t_j]$, the amount of mass $z^{\varepsilon,\theta}$ that creeps out of the interval I_j at (3.45) is asymptotically $o(\varepsilon^k)$, for every $k \geq 1$. Moreover,

$$\int_{I_j} z^{\varepsilon,\theta}(t_j,\, x)\, dx \leq \alpha \int_{I_{j-1}} z^{\varepsilon,\theta}(t_{j-1},\, x)\, dx\,.$$

Since the total number of subintervals is $N_\varepsilon \sim \varepsilon^{-2/3}$, we conclude that at time $t = 3T_\varepsilon$ one has the asymptotic estimate

$$\int_{-\infty}^{\infty} \left| z^{\varepsilon,\theta}(3T_\varepsilon,\, x) \right| dx = o(\varepsilon^k), \tag{3.48}$$

for any $k \geq 1$.

10. Working still in the stretched variables, from the representation formula

$$z(t+1, x) = \int \Gamma^{\varepsilon,\theta}(t+1, x, t, y)\, z(t, y)\, dy,$$

it follows the estimate

$$\left\| \frac{\partial^{m+n}}{\partial x^m \partial t^n} z^{\varepsilon,\theta}(t+1, \cdot) \right\|_{\mathbf{L}^\infty(\mathbb{R})} \leq \left\| \frac{\partial^{m+n}}{\partial x^m \partial t^n} \Gamma^{\varepsilon,\theta}(t+1, \cdot, t, y) \right\|_{\mathbf{L}^\infty(\mathbb{R})} \cdot \left\| z^{\varepsilon,\theta}(t, \cdot) \right\|_{\mathbf{L}^1(\mathbb{R})}.$$

Observing that the map $t \mapsto \left\| z^{\theta,\varepsilon}(t, \cdot) \right\|_{\mathbf{L}^1}$ is non-increasing, and using the uniform bounds

$$\left| \frac{\partial^{m+n}}{\partial x^m \partial t^n} \Gamma(t+1, x, t, y) \right| \leq C_{m,n},$$

for suitable constants $C_{m,n}$, we deduce

$$\left| \frac{\partial^{m+n}}{\partial x^m \partial t^n} v^\varepsilon(t, x) - \frac{\partial^{m+n}}{\partial x^m \partial t^n} \tilde{v}^\varepsilon(t, x) \right| \ \leq\ C_{m,n} \left\| z^{\varepsilon,\theta}(3T_\varepsilon, \cdot) \right\|_{\mathbf{L}^1(\mathbb{R})} \ =\ \mathcal{O}(1) \cdot \varepsilon^k\,,$$

for $t \geq 4T_\varepsilon > 3T_\varepsilon + 1$ and for any positive integer k. Returning to the original variables, for $t \geq \tau_4$ we have

$$\left| \frac{\partial^{m+n}}{\partial x^m \partial t^n} u^\varepsilon(t, x) - \frac{\partial^{m+n}}{\partial x^m \partial t^n} \tilde{u}^\varepsilon(t, x) \right| = \mathcal{O}(1) \cdot \varepsilon^k\, \varepsilon^{-(m+n)}\,.$$

Since the integers $k, m, n \geq 0$ are arbitrary, this achieves the proof. □

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