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Euler Incognito

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Abstract

The nonlinear flow equations discussed recently by Bender and Feinberg are all reduced to the well-known Euler equation after change of variables.

Consider for $u(x,t)$ the class of nonlinear PDEs introduced and discussed by Bender and Feinberg [1].

$$u_t = (u_x)^k \, u \tag{1}$$

where k is a parameter. (Including phases for the variables requires only minor changes in the discussion to follow.) We are actually interested only in $0 \neq k \neq 1$, since $k = 0$ and $k = 1$ are easily understood and well-known. So, upon changing dependent variable to

$$v = \frac{k-1}{k} \, u^{k/(k-1)} \tag{2}$$

equation (1) becomes

$$v_t = (v_x)^k \ . \tag{3}$$

Differentiating this with respect to x and making a further change of variable to

$$w = k\,(v_x)^{k-1} = k\,(u_x)^{k-1} u \tag{4}$$

we find the familiar Euler-Monge equation in canonical form

$$w_t = w \, w_x \ . \tag{5}$$

Thus for any $k \neq 0$ and $k \neq 1$ the original equation (1) is reduced to Euler's through a change of dependent variable, although for technical reasons that are more or less obvious from the explicit constructions, it is often useful to assume $k > 1$. (For $k = 1$ of course, (1) is already the Euler-Monge equation *without* any change of variable.)

As is well-known (cf. [1] or [2] for references) the general solution for w is given implicitly by

$$w = F\,(x + wt) \tag{6}$$

where F is an arbitrary differentiable function. By using the previous changes of variables and integrating once with respect to x, solutions for v, and hence u, follow from those for w. For example, if F is linear, $w = \frac{x-x_0}{t_0-t}$, $v = \frac{k-1}{k}\frac{(x-x_0)^{k/(k-1)}}{[k(t_0-t)]^{1/(k-1)}}$, and $u = \frac{x-x_0}{[k(t_0-t)]^{1/k}}$.

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Moreover, (1) leads to two infinite families of local conserved currents whose time and space components are powers of u and u_x, but not higher derivatives. The first family is quickly seen to be

$$\left(J_0^{(n)}\ ,\ J_1^{(n)}\right)=\left(u^n u_x\ ,\ u^{n+1}\left(u_x\right)^k\right) \tag{7}$$

for any n, not necessarily integer. Obviously, all these currents have simple topological charges. On the solution set of (1), $(n+1)\,J_\mu^{(n)}=\varepsilon_{\mu\nu}\partial^\nu u^{n+1}$, and $\partial^\mu J_\mu^{(n)}=0$ immediately follows. The second family of currents may be obtained from the known (non-topological) conserved currents for w, namely $\left((n+1)\,w^n\ ,\ nw^{n+1}\right)$, just by changing variables. Thus[1]

$$\begin{aligned}\left(K_0^{(n)}\ ,\ K_1^{(n)}\right)&=\left((n+1)\left(v_x\right)^{n(k-1)}\ ,\ nk\left(v_x\right)^{(n+1)(k-1)}\right)\\&=\left((n+1)\left(u_x\right)^{n(k-1)}u^n\ ,\ nk\left(u_x\right)^{(n+1)(k-1)}u^{n+1}\right)\ .\end{aligned} \tag{8}$$

On the solution set of (1), or equivalently (3), it is straightforward to show $\partial^\mu K_\mu^{(n)}=0$.

Finally, the linearization of (5) as given in [2] can be used to relate the spatial derivative of (3), or equivalently of (1), to a linear equation. Define

$$\psi\left(a,x,t\right)\equiv\frac{1}{a}\left(\exp\left(ak\left(v_x\right)^{k-1}\right)-1\right)\ . \tag{9}$$

Technically, it is useful to assume $k>1$ here, especially for slowly varying v. It follows that

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial a\partial x}\right)\psi=\left(v_t-\left(v_x\right)^k\right)_x\times k\left(k-1\right)\left(v_x\right)^{k-2}\exp\left(ak\left(v_x\right)^{k-1}\right)\ . \tag{10}$$

Thus $\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial a\partial x}\right)\psi=0$ iff $\left(v_t-\left(v_x\right)^k\right)_x=0$. (If the second factor on the RHS of (10) were to vanish, for both positive and negative a, this would require $k>2$, $v_x=0$, and hence also $\left(v_t-\left(v_x\right)^k\right)_x=0$.) Encoding initial data for the nonlinear system in the form (9) therefore allows the data to be evolved linearly. Given a well-behaved solution to the linear equation for ψ, we may then extract the nonlinear data at other times just by taking the limit $v_x\left(x,t\right)=\left(\frac{1}{k}\lim_{a\to 0}\psi\left(a,x,t\right)\right)^{1/(k-1)}$. Integrating with respect to x modulo a function of only the time variable yields v, hence u.

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References

[1] C Bender and J Feinberg, "Does the complex deformation of the Riemann equation exhibit shocks?" arXiv:0709.2727v1 [hep-th].

[2] T Curtright and D Fairlie, "Extra Dimensions and Nonlinear Equations" J.Math.Phys. 44 (2003) 2692-2703, arXiv:math-ph/0207008v1.

[1]In terms of u, as $k\to 1$ we note that $(n+1)\,(n+2)\,J_\mu^{(n)}\to\partial_x K_\mu^{(n+1)}\to\left((n+2)\left(u^{n+1}\right)_x\ ,\ (n+1)\left(u^{n+2}\right)_x\right)$.