SUBSONIC FLOWS FOR THE FULL EULER EQUATIONS IN HALF PLANE

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Abstract. We study the subsonic flows governed by full Euler equations in the half plane bounded below by a piecewise smooth curve asymptotically approaching x_1-axis. Nonconstant conditions in the far field are prescribed to ensure the real Euler flows. The Euler system is reduced to a single elliptic equation for the stream function. The existence, uniqueness and asymptotic behaviors of the solutions for the reduced equation are established by Schauder fixed point argument and some delicate estimates. The existence of subsonic flows for the original Euler system is proved based on the results for the reduced equation, and their asymptotic behaviors in the far field are also obtained.

1. Introduction

In this paper, we study subsonic polytropic flows governed by two-dimensional steady, full Euler equations:

$$\begin{cases} \nabla \cdot \mathbf{m} = 0, \\ \nabla \cdot \left(\dfrac{\mathbf{m} \otimes \mathbf{m}}{\rho} \right) + \nabla p = 0, \\ \nabla \cdot (\mathbf{m}(E + p/\rho)) = 0, \end{cases} \tag{1.1}$$

where ∇ is the gradient in $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, $\mathbf{m} = (m_1, m_2)$ the momentum, ρ the density, p the pressure, and

$$E = \frac{|\mathbf{m}|^2}{2\rho^2} + \frac{p}{(\gamma - 1)\rho}$$

the energy with adiabatic exponent $\gamma > 1$. The sonic speed of the flow is defined by

$$c = \sqrt{\gamma p/\rho}.$$

The flow is said to be subsonic if $|\mathbf{m}/\rho| < c$.

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To my best knowledge, no theoretical result was obtained for subsonic flows governed by full Euler system in unbounded domain. There are rich literatures of subsonic potential flows, which is a simplified model for Euler flows. Shiffman obtained the first existence result in [31], using variational method. In [1], Bers used complex analysis to show existence and uniqueness for the subsonic potential flows. Finn and Gilbarg [17, 18] solved the problem by PDE approach. Recently, Chen-Dafermos-Slemord-Wang [4] pushed the subsonic flows to the sonic limit, using the framework of compensated compactness. With in the same framework, Chen-Slemord-Wang [12] obtained transonic solutions by a vanishing viscosity method. Other results for subsonic or transonic flows of various models can be found in [2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 25, 20, 29, 30, 32, 33, 34], and [15, 16, 19] provide related background and introduction.

The domain we study is the upper half plane with piecewise smooth boundary asymptotically flattened as $|x_1| \to \infty$. This setting can be viewed as a symmetric airfoil problem. For a airfoil symmetric about x_1-axis, we cut the exterior domain in half along the symmetry axis, and the upper part becomes our domain. More generally, we allow the boundary to be curved away from the profile. In this way, our setting also includes the model for the wind glancing the landscape.

Let $U = (\mathbf{m}, p, \rho)$ be the solution for the subsonic flow. We prescribe an asymptotic limit U_∞, close to a constant subsonic state U_0, for the flow in the far field. Unlike the setting for potential flows, the asymptotic behavior U_∞ is not a constant state. Otherwise, the full Euler system can be reduced to a potential flow (cf. Proposition 3.1). To guarantee the convergence of the flows to U_∞ in the far field, we need to obtain some decay property for $\psi - l$, which is the difference between the stream function and its limit behavior. For the whole plane, one knows that the fundamental solution for a Laplace equation has the form $\log |\mathbf{x}|$. Therefore, in general, we can not expect the decay of a solution for an elliptic equation as $|\mathbf{x}| \to \infty$ in the exterior domain of the whole plane. This is the main technical obstacle for us to obtaining the subsonic flows in the whole plane. However, when flows are restricted in the half plane, we can exclude the logarithmic growth of solutions by prescribing proper decay condition at the infinity.

We reduce the Euler system to a single elliptic equation (3.19) for a stream function ψ by capturing some conservation properties of the system. More precisely, three properties are contained in (3.19): (1) existence of ψ represents conservation of mass; (2) we use the fact p/ρ^γ is constant along streamlines during the reduction, which implies entropy is conserved on each streamline. (3) to solve for ρ in terms of $\psi, \nabla\psi$, we use Bernoulli's law, which relates to the conservation of energy. Usually, stagnation points occur in various situations and cause major difficulties (for instance, regular reflection for Euler equations in self-similar coordinates). Our reduction process enables us to bypass the difficulty and to obtain the existence of solutions. However, we do not have uniqueness result due to the existence of stagnation points and complex behaviors of streamlines. More details are explained in Remark 6.1.

Once the Euler system is reduced to the elliptic equation (3.19), the remaining work is to solve this nonlinear equation. In detail, we first truncate the original domain Ω by ball $B_R(O)$ centered at the origin with radius R. We solve (3.19) in bounded domain $\Omega_R = \Omega \cap B_R(O)$ with properly prescribed boundary condition. It is a standard method that we linearize (3.19), construct a map T by solving the linearized equation, and prove the existence of a fixed point for T by Schauder fixed point theorem. The fixed point ψ_R is the solution for (3.19) in domain Ω_R.

To take the limit of $\{\psi_R\}$ as $R \to \infty$ and obtain the solution in Ω, the estimates should be independent of the radius R. It makes the estimates complicated that there is no sign for the coefficient b_0 in the linear equation (4.1). By choosing a proper barrier and using maximum principle with no restriction on the sign of b_0 (Lemma 4.2), we obtain uniform estimates, independent of R, for ψ_R. The barrier function we construct only works for the half plane, not the whole plane. Whether one can find a suitable barrier function for the whole plane is unclear at this moment.

The rest of the paper is organized as follows. In Section 2, we set up the subsonic flow problem, introduce the weighted norms, and state the main result. Section 3 explains the reduction of the full Euler system to a single elliptic equation for the stream function ψ. In Section 4, we prove a technical lemma for the linearized equation and obtain crucial esitmates. In Section 5, we construct a iteration scheme to solve the nonlinear equation (3.19) in truncated domain Ω_R. Schauder fixed point argument is used to prove the existence of the solution. In Section 6, we take the limit of subsequence of solutions ψ_R in Ω_R and obtain the solution in the half plane Ω. The relation between the original Euler system and the reduced system is explained.

2. Setup of the Subsonic Flow Problem

In order to describe the conditions and results of our subsonic problem, we need to use the following weighed Hölder norms: for any

$\mathbf{x}, \mathbf{x}'$ in a two-dimensional domain Ω and for a subset P of $\partial\Omega$, define $\delta_{\mathbf{x}} := \min(\operatorname{dist}(\mathbf{x}, P), 1)$, $\delta_{\mathbf{x},\mathbf{x}'} := \min(\delta_{\mathbf{x}}, \delta_{\mathbf{x}'}, 1)$, $\Delta_{\mathbf{x}} := \max(|\mathbf{x}|, 1)$ and $\Delta_{\mathbf{x},\mathbf{x}'} := \max(|\mathbf{x}|, |\mathbf{x}'|, 1)$. Let $\alpha \in (0,1)$, $\sigma, \beta \in \mathbb{R}$, and k be a nonnegative integer. Let $\mathbf{k} = (k_1, k_2)$ be a integer-valued vector, where $k_1, k_2 \geq 0$, $|\mathbf{k}| = k_1 + k_2$ and $D^{\mathbf{k}} = \partial_{x_1}^{k_1}\partial_{x_2}^{k_2}$. We define

$$
\begin{aligned}
[u]^{(\sigma;P)}_{k,0;(\beta);\Omega} &= \sup_{\substack{\mathbf{x}\in\Omega\\ |\mathbf{k}|=k}} \left(\delta_{\mathbf{x}}^{\max(k+\sigma,0)} \Delta_{\mathbf{x}}^{\beta+k} |D^{\mathbf{k}} u(\mathbf{x})|\right),\\
[u]^{(\sigma;P)}_{k,\alpha;(\beta);\Omega} &= \sup_{\substack{\mathbf{x},\mathbf{x}'\in\Omega\\ \mathbf{x}\neq\mathbf{x}'\\ |\mathbf{k}|=k}} \left(\delta_{\mathbf{x},\mathbf{x}'}^{\max(k+\alpha+\sigma,0)} \Delta_{\mathbf{x},\mathbf{x}'}^{\beta+k+\alpha} \frac{|D^{\mathbf{k}} u(\mathbf{x}) - D^{\mathbf{k}} u(\mathbf{x}')|}{|\mathbf{x}-\mathbf{x}'|^{\alpha}}\right),\\
\|u\|^{(\sigma;P)}_{k,\alpha;(\beta);\Omega} &= \sum_{i=0}^{k} [u]^{(\sigma;P)}_{i,0;(\beta);\Omega} + [u]^{(\sigma;P)}_{k,\alpha;(\beta);\Omega}.
\end{aligned}
\tag{2.1}
$$

For a vector-valued function $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, we define

$$
\|\mathbf{u}\|^{(\sigma;P)}_{k,\alpha;(\beta);\Omega} = \sum_{i=1}^{n} \|u_i\|^{(\sigma;P)}_{k,\alpha;(\beta);\Omega}.
$$

Remark 2.1. In the definition of the weighted norms, the lower index in the parenthesis represents the weigh at the infinity and the upper index represents the weight to the set P, which will be the set of some corner points on the boundary in the paper.

Define

$$
C^{(\sigma;P)}_{k,\alpha;(\beta)}(\Omega) = \{u : \|u\|^{(\sigma;P)}_{k,\alpha;(\beta);\Omega} < \infty\}. \tag{2.2}
$$

For the weighted norms of functions in one-dimensional space $\Gamma = (a,b)$, with either $a = -\infty$ or $b = \infty$, we define

$$
\begin{aligned}
[f]_{k,0;(\beta);\Gamma} &= \sup_{x\in\Gamma} (|x|+1)^{k+\beta} |f^{(k)}(x)|\\
[f]_{k,\alpha;(\beta);\Gamma} &= \sup_{x,x'\in\Gamma, x\neq x'} (\max(|x|,|x'|)+1)^{k+\alpha+\beta} \frac{|f^{(k)}(x) - f^{(k)}(x')|}{|x-x'|^{\alpha}}\\
\|f\|_{k,\alpha;(\beta);\Gamma} &= \sum_{i=0}^{k} [f]_{k,0;(\beta);\Gamma} + [f]_{k,\alpha;(\beta);\Gamma}.
\end{aligned}
\tag{2.3}
$$

Our domain Ω is the upper half plane bounded below by a piecewise smooth curve consisting of three parts:

$$
\partial\Omega = \Gamma_- \cup \mathcal{A} \cup \Gamma_+. \tag{2.4}
$$

The following is the description of the three parts for $\partial\Omega$ (see figure 1). Let $\Gamma_\pm = \{x_2 = f_\pm(x_1)\}$, where f_- is defined on $(-\infty, -1)$ and f_+ is

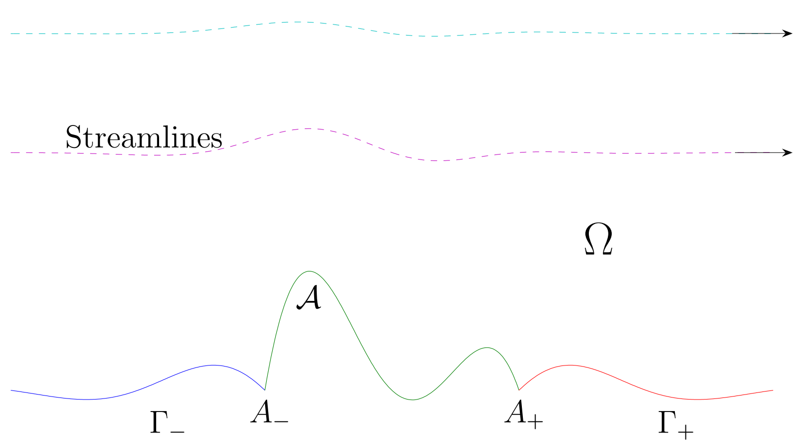

FIGURE 1. Domain Ω for Subsonic Flows

defined on $(1, \infty)$. Both Γ_- and Γ_+ approach x_1-axis as $|x_1|$ tends to ∞. More precisely, we let

$$\|f_-\|_{2,\alpha;(\alpha+\beta);(-\infty,-1)} \leq 1, \tag{2.5}$$

$$\|f_+\|_{2,\alpha;(\alpha+\beta);(1,\infty)} \leq 1. \tag{2.6}$$

Let $A_- = (-1, f_-(-1))$ and $A_+ = (1, f_+(1))$ be the end points of Γ_-, Γ_+, respectively. The arch $\mathcal{A}$ connecting $\Gamma_\pm$ at $A_\pm$ can be parameterized by

$$\mathbf{f}(s) = (f_1(s), f_2(s)), \quad s \in (-1, 1), \tag{2.7}$$

where $\mathbf{f}(-1) = A_-, \mathbf{f}(1) = A_+$ and f_1, f_2 are $C^{2,\alpha}$ smooth functions.

We assume the angles $\theta^0_\pm$ between $\Gamma_\pm$ and $\mathcal{A}$ at points $A_\pm$ satisfies:

$$\delta < \theta^0_\pm < \pi - \delta, \tag{2.8}$$

for a fixed constant δ.

Remark 2.2. The above condition guarantees that stream function ψ for the flow is $C^{1,\alpha}$ up to the corner points, which means the flow U is C^α up to the corners. If we allow corner angles $\theta^0_\pm \geq \pi$, ψ will be C^α up to the corners, and U will blow up at corners. We exclude the latter situation just to avoid unimportant details.

Without loss of generality, we may also assume that

$$\mathcal{A} \subset B_{D_0}(0), \tag{2.9}$$

i.e., $\mathcal{A}$ is contained in the ball of radius D_0 centered at 0, and

$$f_-(x_1) > -\frac{1}{2}, \quad f_+(x_1) > -\frac{1}{2}, \quad f_2(s) > -\frac{1}{2}. \tag{2.10}$$

This means that domain Ω is above the line $x_2 = -\frac{1}{2}$.

We prescribe slip condition on the boundary:

$$\mathbf{m} \cdot \nu|_{\partial\Omega} = 0, \tag{2.11}$$

where ν is the outer normal on boundary $\partial\Omega$.

Let $(m_*, 0, p_0, \rho_0)$ be a constant subsonic solution for (1.1), i.e., m_*/ρ_0 less than $\sqrt{p_0/\rho_0}$. Fix constants p_0, ρ_0 and let $m_0 \le m_*$ be a sufficiently small constant to be determined later. So $U_0 = (m_0, 0, p_0, \rho_0)$, as our background state, is also a subsonic solution for (1.1). We define a vector-valued function $U_\infty = (m_\infty, 0, p_0, \rho_\infty)$ of variable x_2 as the asymptotic state for our solution $U = (\mathbf{m}, p, \rho)$ at the far field. We assume that U_∞ is a small perturbation of the background solution U_0:

$$\|U_\infty - U_0\|_{2,\alpha;(0);(0,\infty)} \le \varepsilon m_0, \tag{2.12}$$

where $0 < \varepsilon < \frac{1}{2}$ is a small parameter to be determined later.

Set $P = \{A_-, A_+\}$ as the set for the weight.

Now we state our main theorem about the existence of the subsonic flow in the half plane Ω:

Theorem 2.1. *Suppose the boundary $\partial\Omega$ satisfies (2.5)–(2.10) and U_∞ satisfies (2.12). We fix $0 < \beta < \alpha < 1$, depending on δ in (2.8). For sufficiently small m_0, depending on $m_*, p_0, \rho_0, \delta, \alpha, \beta$ and the profile $\mathcal{A}$, and sufficiently small ε, depending on $m_*, p_0, \rho_0, \delta, \alpha, \beta, \mathcal{A}$ and m_0, there exists a subsonic solution $U \in C^{(-\alpha;P)}_{1,\alpha;(\beta)}(\Omega)$ for (1.1) with boundary condition (2.11), such that*

$$\|U - U_\infty\|^{(-\alpha;P)}_{1,\alpha;(\beta);\Omega} \le C m_0, \tag{2.13}$$

where C is a constant only depending on $m_*, p_0, \rho_0, \delta, \alpha, \beta$ and the profile $\mathcal{A}$, but independent of m_0 and ε.

Remark 2.3. Estimate (2.13) immediately gives the asymptotic behavior of the flow U. That is U approaches U_∞ in C^0 norm at the rate $|\mathbf{x}|^{-\beta}$ as $|\mathbf{x}| \to \infty$.

3. Reduction of the Euler System

In this section, we use the conservation properties of the Euler equations (1.1) to reduce the four-equation system to one elliptic equation.

By the conservation of mass (first equation of (1.1)), we can find a potential function ψ for the vector field $(-m_2, m_1)$, i. e.,

$$\psi_{x_1} = -m_2, \quad \psi_{x_2} = m_1. \tag{3.1}$$

From (1.1), we can derive

$$(m_1\partial_{x_1} + m_2\partial_{x_2})(\gamma \ln \rho - \ln p) = 0, \tag{3.2}$$

which implies that the quantity ρ^γ/p is constant along streamlines, provided that the solution is C^1 smooth. This constant only depends on the stream function ψ . Thus, we have

$$p = \frac{\gamma - 1}{\gamma} A(\psi)\rho^\gamma. \tag{3.3}$$

We will determine function A later by U_∞.

From (1.1), we can also derive the Bernoulli's law:

$$\frac{|\mathbf{m}|}{2\rho^2} + \frac{\gamma p}{(\gamma - 1)\rho} = B \tag{3.4}$$

along the streamlines, where B is the Bernoulli constant depending on ψ. With equation (3.3) and (3.1), the Bernoulli's law (3.4) can be written as

$$\frac{1}{2}|\nabla\psi|^2 + A(\psi)\rho^{\gamma+1} = B(\psi)\rho^2, \tag{3.5}$$

In the subsonic region, we have

$$|\nabla\psi|^2 < c^2\rho^2 = (\gamma - 1)A(\psi)\rho^{\gamma+1}. \tag{3.6}$$

Inequality (3.6) and the Bernoulli's law (3.5) implies

$$\rho^{\gamma-1} > \frac{2B}{(\gamma + 1)A}. \tag{3.7}$$

Let $\chi = \frac{1}{2}|\nabla\psi|^2$ and $h(\rho, \psi) = B(\psi)\rho^2 - A(\psi)\rho^{\gamma+1}$. Therefore, in subsonic region,

$$\frac{\partial h}{\partial \rho} = \rho A \left(\frac{2B}{(\gamma + 1)A} - \rho^{\gamma-1} \right) < 0.$$

Hence, we can uniquely solve

$$\chi = h(\rho, \psi) \equiv B(\psi)\rho^2 - A(\psi)\rho^{\gamma+1} \tag{3.8}$$

for $\rho = \rho(\chi, \psi)$ by implicit function theorem.

From (3.8), we can easily calculate

$$\rho_\chi = -\frac{1}{(\gamma+1)A\rho^\gamma - 2B\rho}, \tag{3.9}$$

$$\rho_\psi = \frac{B'\rho - A'\rho^\gamma}{(\gamma+1)A\rho^{\gamma-1} - 2B}. \tag{3.10}$$

Therefore, we compute

$$\begin{aligned}\rho_{x_1} &= \rho_\chi(\psi_{x_1}\psi_{x_1x_1} + \psi_{x_2}\psi_{x_1x_2}) + \rho_\psi\psi_{x_1}\\ &= \frac{-\psi_{x_1}\psi_{x_1x_1} - \psi_{x_2}\psi_{x_1x_2} + \psi_{x_1}(B'\rho^2 - A'\rho^{\gamma+1})}{(\gamma+1)A\rho^\gamma - 2B\rho},\end{aligned} \tag{3.11}$$

$$\begin{aligned}\rho_{x_2} &= \rho_\chi(\psi_{x_1}\psi_{x_1x_2} + \psi_{x_2}\psi_{x_2x_2}) + \rho_\psi\psi_{x_2}\\ &= \frac{-\psi_{x_1}\psi_{x_1x_2} - \psi_{x_2}\psi_{x_2x_2} + \psi_{x_2}(B'\rho^2 - A'\rho^{\gamma+1})}{(\gamma+1)A\rho^\gamma - 2B\rho}.\end{aligned} \tag{3.12}$$

Now we can reduce the Euler system into one equation. We replace $\mathbf{m}$ in the second equation of (1.1) with $(-\psi_{x_2}, \psi_{x_1})$ according to (3.1). Multiplying the second equation of (1.1) by $(\gamma+1)A\rho^\gamma - 2B\rho$, and using the expressions (3.11) and (3.12), we obtain the following equation:

$$\psi_{x_1}(a_{ij}(\psi, \nabla\psi)\psi_{x_ix_j} - F(\psi, \nabla\psi)) = 0, \tag{3.13}$$

where

$$a_{11}(\psi, \nabla\psi) = (\gamma-1)A(\psi)\rho^{\gamma+1} - \psi_{x_2}^2 \tag{3.14}$$

$$a_{12}(\psi, \nabla\psi) = a_{21}(\psi, \nabla\psi) = \psi_{x_1}\psi_{x_2} \tag{3.15}$$

$$a_{22}(\psi, \nabla\psi) = (\gamma-1)A(\psi)\rho^{\gamma+1} - \psi_{x_1}^2 \tag{3.16}$$

$$F(\psi, \nabla\psi) = \frac{\gamma-1}{\gamma}\rho^{\gamma+3}(\gamma AB' - 2A'B + AA'\rho^{\gamma-1}). \tag{3.17}$$

Similarly, the third equation of (1.1) gives rise to

$$\psi_{x_2}(a_{ij}(\psi, \nabla\psi)\psi_{x_ix_j} - F(\psi, \nabla\psi)) = 0. \tag{3.18}$$

For a system without stationary points, i.e., $\nabla\psi$ is nowhere $\mathbf{0}$, the original Euler system (1.1) can be reduced to the following equation for subsonic flows:

$$a_{ij}(\psi, \nabla\psi)\psi_{x_ix_j} = F(\psi, \nabla\psi). \tag{3.19}$$

Equation (3.19) can be written in divergence form:

$$\nabla \cdot \left(\frac{\nabla\psi}{\rho}\right) = B'\rho - \frac{1}{\gamma}A'\rho^\gamma. \tag{3.20}$$

Now we use the limit function U_∞ to determine A, B and the limit function $l(x_2)$ of the stream function ψ as $|x_1| \to \infty$.

Define

$$l(x_2) = \int_0^{x_2} m_\infty(s)ds. \tag{3.21}$$

By (2.12), we know that

$$\frac{1}{2}m_0 < m_\infty = l' < 2m_0,$$

which implies that l is invertible and

$$\frac{1}{2}m_0 x_2 < l(x_2) < 2m_0 x_2. \tag{3.22}$$

Let

$$\begin{aligned} \bar{A}(x_2) &= \frac{\gamma p_0}{(\gamma-1)\rho_\infty(x_2)} \\ \bar{B}(x_2) &= \frac{m_\infty^2(x_2)}{2\rho_\infty^2(x_2)} + \frac{\gamma p_0}{\rho_\infty(x_2)}. \end{aligned}$$

Then define

$$A(s) = \bar{A}(l^{-1}(s)), \quad B(s) = \bar{B}(l^{-1}(s)). \tag{3.23}$$

To describe the properties of A and B, we need to modify the weighted norm in (2.3) as follows:

$$\begin{aligned} [f]'_{k,0;(\beta);\Gamma} &= \sup_{x\in\Gamma}(|x|+m_0)^{k+\beta}|f^{(k)}(x)| \\ [f]'_{k,\alpha;(\beta);\Gamma} &= \sup_{x,x'\in\Gamma, x\neq x'}(\max(|x|,|x'|)+m_0)^{k+\alpha+\beta}\frac{|f^{(k)}(x)-f^{(k)}(x')|}{|x-x'|^\alpha} \\ \|f\|'_{k,\alpha;(\beta);\Gamma} &= \sum_{i=0}^{k}[f]'_{k,0;(\beta);\Gamma} + [f]'_{k,\alpha;(\beta);\Gamma}. \end{aligned} \tag{3.24}$$

Basically, we replace 1 in the weight in (2.3) with m_0 for the scaling reason.

Set

$$A_0 = \frac{\gamma p_0}{(\gamma-1)\rho_0^\gamma}, \quad B_0 = \frac{m_0^2}{2\rho_0^2} + \frac{\gamma p_0}{(\gamma-1)\rho_0}.$$

By (2.12), we conclude that

$$\|A - A_0\|'_{2,\alpha;(0);(0,\infty)} \le C_0\varepsilon m_0, \tag{3.25}$$

$$\|B - B_0\|'_{2,\alpha;(0);(0,\infty)} \le C_0\varepsilon m_0, \tag{3.26}$$

where C_0 is a constant depending only on m_*, p_0, ρ_0.

Let us discuss the asymptotic behavior of U as $|x_1| \to \infty$. We do not expect constants states at the infinity for general subsonic flows governed by full Euler equations. Actually, if the flow is uniform at the infinity, we only get a potential flow. This fact is described by the following proposition.

Proposition 3.1. *Suppose U is a C^1 solution of* (1.1) *with no stagnation points (* $\mathbf{m}$ *is nowhere* $\mathbf{0}$*). If U_∞ is a constant state, the flow U is potential, i.e., the velocity $\mathbf{u} = \mathbf{m}/\rho$ is irrotational.*

Proof. If U_∞ is constant, we immediately get A, B are constants by the procedure of obtaining A, B. Hence, we have $A' = B' = 0$. Equation (3.20), which is equivalent to (1.1) under the assumption in the proposition, becomes

$$\nabla \cdot \left(\frac{\nabla \psi}{\rho} \right) = 0.$$

Since

$$\psi_{x_1} = -m_2 = -\rho u_2, \quad \psi_{x_2} = m_1 = \rho u_1,$$

the above equation is just the irrotationality condition for the velocity $(u_1)_{x_2} - (u_2)_{x_1} = 0$. Therefore, we have a potential flow. □

In general, the Euler system (1.1) and equation (3.19) are not equivalent, because the streamlines may not be nice enough for us to do the reduction of the system. However, the solution of (3.19) guarantees the existence of the solution for Euler equations (1.1). Therefore, we only need to solve (3.19) in order to prove Theorem 2.1.

By the definition of the stream function ψ, the slip condition (2.11) becomes the Dirichlet boundary condition for equation (3.19):

$$\psi|_{\partial\Omega} = 0. \tag{3.27}$$

We define Σ as a set for the solutions of (3.19):

$$\Sigma = \{u : \|u - l\|^{(-\alpha-1;P)}_{2,\alpha;(\beta);\Omega} \leq C^* m_0\}, \tag{3.28}$$

where constant C^*, depending on $m_*, p_0, \rho_0, \alpha, \beta, \delta, \mathcal{A}$, will be determined later in the estimates.

We state the following theorem, which implies Theorem 2.1:

Theorem 3.1. *For sufficiently small m_0, depending on $m_*, p_0, \rho_0, \alpha, \beta, \delta, \mathcal{A}$, and sufficiently small ε, depending on $m_*, p_0, \rho_0, \alpha, \beta, \delta, \mathcal{A}$ and m_0, there exists a unique solution for equation* (3.19) *with boundary condition* (3.27) *in the set Σ defined in* (3.28).

We split the proof of Theorem 3.1 into the following steps:

(1) Use bounded domain $\Omega_R := \Omega \cap B_R(O)$ to approach Ω, where $B_R(O)$ is the ball with radius R and centered at the origin. We linearize equation (3.19) and solve the linear equation in Ω_R.

(2) With proper estimates for the linear equation, we solve the nonlinear equation (3.19) in bounded domain Ω_R using Schauder fixed point theorem.
(3) Let $R \to \infty$, we prove the existence of solution for (3.19) in Ω.
(4) Estimate the difference of any two solutions in Ω_R and then let $R \to \infty$ to obtain the uniqueness of the solution for (3.19).

Henceafter, we will use C to denote generic constants, depending on the fixed data $m_*, p_0, \rho_0, \alpha, \beta, \delta$, and the profile $\mathcal{A}$, but independent of m_0, R.

4. Estimates of a linear equation

In this section, we study a linear elliptic equation

$$a_{ij}(\mathbf{x})u_{x_i x_j} + b_i(\mathbf{x})u_{x_i} + b_0(\mathbf{x})u = 0, \tag{4.1}$$

in domain Ω_R. The estimates of this equation will be used later for the linearized equation. For equation (4.1), we have the following assumptions for the coefficients:

$$a_{ij}(\mathbf{x})\xi_i\xi_j \geq \lambda(\xi_1^2 + \xi_2^2) \quad \text{for any } \xi_i \in \mathbb{R}, \tag{4.2}$$

$$\|a_{ij} - e\delta_{ij}\|_{C^\alpha(\Omega_R)} + \sum_{i=1}^{3} \|b_i\|_{C^\alpha(\Omega_R)} \leq Cm_0, \tag{4.3}$$

$$(x_2+1)(|b_1|+|b_2|) + (x_2+1)^2|b_0| \leq Cm_0, \tag{4.4}$$

where λ, e are constants depending on p_0, ρ_0, and $\delta_{ij} = 1$ for $i = j$, otherwise $\delta_{ij} = 0$.

We let $R > D_0 + 1$ so that the boundary of Ω_R includes the whole profile $\mathcal{A}$. Let $S_R = \{|\mathbf{x}| = R\} \cap \partial\Omega_R$. Now we not only have the corner points A_-, A_+, but also have additional corners as the intersection of S_R with $\partial\Omega$. These two points are denoted by

$$S_-^R = \Gamma_- \cap S_R, \quad S_+^R = \Gamma_+ \cap S_R.$$

Then we define the set of boundary points for the weight as:

$$\tilde{P} = \{A_-, A_+, S_-^R, S_+^R\}.$$

The boundary condition for (4.1) is:

$$u|_{\partial\Omega_R} = g, \tag{4.5}$$

where

$$\|g\|_{2,\alpha;(\alpha+\beta);\Omega_R}^{(-1-\alpha;\tilde{P})} \leq Cm_0. \tag{4.6}$$

We then have the following lemma

Lemma 4.1. *Suppose $u \in C^{2,\alpha}(\Omega_R) \cap C(\overline{\Omega_R})$ is a solution for (4.1) with boundary condition (4.5) and assumptions (4.2)–(4.6) hold. For sufficiently small m_0 independent of R, we have the following estimate for u:*

$$\|u\|^{(-1-\alpha;\tilde{P})}_{2,\alpha;(\beta);\Omega_R} \leq C^* m_0, \tag{4.7}$$

where C^* is a constant independent of m_0, R.

To prove Lemma 4.1, we need a maximum principle for the elliptic equation (4.1) without restriciton on the sign for b_0. We take the following lemma from [24] (Theorem 2.11):

Lemma 4.2. *Let elliptic operator $L = a_{ij}\partial_{x_i}\partial_{x_j} + b_i\partial_{x_i} + b_0$. For any bounded connected domain $\mathcal{D}$, assume $a_{ij}, b_i \in C^0(\overline{\mathcal{D}})$ and a_{ij} satisfies ellipticity condition (4.2). Suppose there exists a function $v \in C^2(\mathcal{D}) \cap C^1(\overline{\mathcal{D}})$ such that $v > 0$ in $\overline{\mathcal{D}}$ and $Lv \leq 0$ in $\mathcal{D}$. Suppose $u \in C^2(\mathcal{D}) \cap C(\overline{\mathcal{D}})$ satisfies $Lu \geq 0$ in $\mathcal{D}$. Then $\frac{u}{v}$ achieves its nonnegative maximum on the boundary $\partial\mathcal{D}$.*

With this maximum principle, we start to prove Lemma 4.1:

Proof. Notice that there are two different weights in the norm in (4.7) (See definition of weighted norm (2.1)) : the weight with upper index $-\alpha - 1$ is for the small scale near the corner points A_-, A_+, S^R_-, S^R_+ and the weight with lower index β is for the large scale away from the origin. We split the proof into three parts: **Part 1** is for the estimate of maximum norm of u in the whole domain Ω_R; **Part 2** is for the region near $\mathcal{A}$; and **Part 3** is for the region far away from the profile $\mathcal{A}$. Let $D = 2D_0 + 1$, where D_0 is the radius to bound the profile $\mathcal{A}$. Let $\Omega_D = \Omega_R \cap \{|\mathbf{x}| \leq D\}$ be the region for **Part 2**, and $\Omega^c_D = \Omega_R \cap \{|\mathbf{x}| > D - 1\}$ for **Part 3**.

Part 1. In this part, we first construct a comparison function v and use the maximum principle (Lemma 4.2) in the whole domain Ω_R to obtain the control of the maximum norm of u.

Define the comparison function v by

$$v(\mathbf{x}) = r^{-\alpha-\beta}(x_2 + 1)^\alpha, \tag{4.8}$$

where $r = \sqrt{x_1^2 + (x_2 + 1)^2}$.

We now verify the fact that $Lv < 0$.

First, it is easy to compute

$$\begin{aligned} \Delta v &= (\beta^2 - \alpha^2) r^{-\alpha-\beta-2}(x_2 + 1)^\alpha \\ &\quad -\alpha(1 - \alpha) r^{-\alpha-\beta}(x_2 + 1)^{\alpha-2} \\ &\leq -\alpha(1 - \alpha) r^{-\alpha-\beta}(x_2 + 1)^{\alpha-2}, \end{aligned} \tag{4.9}$$

noticing that $0 < \beta < \alpha < 1$. Also, one can verify that

$$|Dv| \le Cr^{-\alpha-\beta}(x_2+1)^{\alpha-1}, \tag{4.10}$$

$$|D^2 v| \le Cr^{-\alpha-\beta}(x_2+1)^{\alpha-2}. \tag{4.11}$$

We rewrite Lv as

$$Lv = (L - e\Delta)v + e\Delta v.$$

By assumptions (4.3) and (4.4), together with (4.10) and (4.11), we have

$$|(L - e\Delta)v| \le Cm_0 r^{-\alpha-\beta}(x_2+1)^{\alpha-2}.$$

The above estimate and (4.9) imply that $Lv < 0$ in Ω_R, provided m_0 is small enough. Obviously, v is positive. Hence, by the maximum principle (Lemma 4.2), we conclude that

$$\frac{u}{v} \le \max_{\partial\Omega_R} \frac{|g|}{v}.$$

By replacing v with $-v$ and using Lemma 4.2 again, we have

$$|u/v| \le \max_{\partial\Omega_R} |g/v|.$$

This, with assumption (4.6), implies

$$|u(\mathbf{x})| \le Cm_0 r^{-\beta}. \tag{4.12}$$

Part 2. For the region near the profile $\mathcal{A}$, we need to take care of the corner points A_-, A_+. We use the weight up to P and drop the lower index β for the weight away from $\mathcal{A}$. We treat the corner A_- first, and A_+ can be dealt in the same way. For convenience, we move A_- to the origin O. Assume the angle between Γ_- and x_1-axis at O (original A_-) is θ_-, and the angle between $\mathcal{A}$ and x_1-axis at O is θ_0. Therefore, the tangential directions of Γ_- and $\mathcal{A}$ at O are

$$\nu_- = (\cos\theta_-, \sin\theta_-), \quad \nu_0 = (\cos\theta_0, \sin\theta_0),$$

respectively. Let $\bar{u} = u - g(O) - c_1 x_1 - c_2 x_2$, where c_1, c_2 are linear combinations of $\frac{\partial g}{\partial \nu_-}(O)$ and $\frac{\partial g}{\partial \nu_0}(O)$ through solving the linear system

$$\begin{cases} (c_1, c_2)\cdot \nu_- = \frac{\partial g}{\partial \nu_-}(O) \\ (c_1, c_2)\cdot \nu_0 = \frac{\partial g}{\partial \nu_0}(O). \end{cases}$$

Hence

$$|c_1| + |c_2| \le C|Dg(O)|,$$

and

$$\bar{u}(O) = 0, \quad D\bar{u}(O) = (0, 0). \tag{4.13}$$

Choose $r_0 > 0$ small enough, such that $\Omega_D \cap B_{r_0}(O)$ is connected. Thus, for this fixed radius $r_0 < \min(D_0, 1)$,

$$\bar{u}(\mathbf{x})|_{\partial\Omega_D \cap B_{r_0}(O)} \leq C m_0 |\mathbf{x}|^{1+\alpha}.$$

We know $\bar{u}$ satisfies the following equation

$$\bar{L}\bar{u} \equiv a_{ij}\partial_i\partial_j\bar{u} + b_i\partial_i\bar{u} = F_0, \tag{4.14}$$

where $F_0 = -b_i c_i - b_0 u$. By estimate (4.12) and condition (4.4), we have

$$|F_0| \leq C m_0.$$

Notice the elliptic operator in (4.14) does not contain $b_0\bar{u}$ term. So we can use standard maximum principle to control $\bar{u}$. The comparison function for $\bar{u}$ is defined in polar coordinates (r, θ) by

$$v_1(r, \theta) = C m_0 r^{1+\alpha} \sin(\tau + (\theta - \theta_0)),$$

for small positive τ depending on $\theta_- - \theta_0$ and α. One can check that

$$\bar{L}v_1 < -C m_0 < F_0 = \bar{L}\bar{u}.$$

Also the boundary condition satisfies

$$v_1|_{\partial(\Omega_D \cap B_{r_0}(O))} > C m_0 r^{1+\alpha} > \bar{u}|_{\partial(\Omega_D \cap B_{r_0}(O))}.$$

By maximum principle, we conclude that

$$|\bar{u}| \leq C m_0 r^{1+\alpha},$$

for $|\mathbf{x}| < r_0$.

Once we have the above estimate near the corner A_-, we use Schauder estimates with proper scaling to obtain the estimate near the corner $A_-(O)$:

$$\|u\|^{(-1-\alpha;\{A_-\})}_{2,\alpha;\Omega_D \cap B_{\frac{r_0}{2}}(A_-)} \leq C m_0. \tag{4.15}$$

The procedure is standard and related details can be found in chapter 6 of [22]. One can also refer to [5] (Lemma 4.2) for similar scaling argument. We sketch the proof as follows.

For any $\mathbf{x}_0 \in \Omega_D \cap B_{r_0/2}$, let the angle between Γ_- and the ray $A_-\mathbf{x}_0$ be $\theta_{\mathbf{x}_0}$, and the angle between Γ_- and $\mathcal{A}$ be θ_-^0. Consider two cases: **Case 1**, $\theta_{\mathbf{x}_0} > \pi/6$ and $\theta_- - \theta_{\mathbf{x}_0} > \pi/6$; **Case 2**, otherwise. For **Case 1**, we know that $\bar{u}$ satisfies equation

$$L\bar{u} = -b_i c_i - b_0(g(O) + c_1 x_1 + c_2 x_2).$$

Take the ball $B_{|\mathbf{x}_0|/2}(\mathbf{x}_0) \subset \Omega_D \cap B_{\frac{r_0}{2}}$ as the domain and by Schauder interior estimate (Theorem 6.2, [22]), we have

$$\|\bar{u}\|^{(0)}_{2,\alpha;B_{\frac{|\mathbf{x}_0|}{2}}(\mathbf{x}_0)} \leq C m_0 |\mathbf{x}_0|^{\alpha+1}. \tag{4.16}$$

Here the upper index (0) is understood as the weight up to $\partial B_{\frac{|\mathbf{x}_0|}{2}}(\mathbf{x}_0)$.

For **Case 2**, let $\mathbf{x}^*$ be a boundary point with the shortest distance from $\mathbf{x}_0$, and $d^* = |\mathbf{x}_0|\sin(\frac{3}{4}\theta_-^0)$. Hence, $B_{\frac{7d^*}{8}}(\mathbf{x}^*)$ still contains $\mathbf{x}^*$. We use Schauder boundary estimate (Lemma 6.4, [22]) in the domain $\Omega_D \cap B_{d^*}(\mathbf{x}^*)$ to obtain the estimate:

$$\|\bar{u}\|^{(0)}_{2,\alpha;B_{d^*}(\mathbf{x}^*)} \le Cm_0|\mathbf{x}_0|^{\alpha+1}. \tag{4.17}$$

Combining (4.16) and (4.17) gives the corner estimate (4.15). The other corner A_+ is treated in the same way. Together with standard Schauder estimates away from the corners, we conclude the estimate in Ω_D:

$$\|u\|^{(-\alpha-1;P)}_{2,\alpha;\Omega_D} \le Cm_0. \tag{4.18}$$

Part 3. For the domain Ω_D^c, we also consider two kinds of estimates: one is near the corner points S_-^R, S_+^R, the other is away from the corners.

The corner estimates are similar to those in **Part 2**. In brief, consider the corner S_-^R for instance. If $\mathbf{x}_0 \in \Omega_D^c \cap B_{r_0}(S_-^R)$, we have the following the estimate

$$\|u\|^{(-1-\alpha;\{S_-^R\})}_{2,\alpha;\Omega_D^c\cap B_{\frac{r_0}{2}}(S_-^R)} \le Cm_0R^{-\beta}. \tag{4.19}$$

If $\mathbf{x}_0 \in \Omega_D^c$ and $|\mathbf{x}_0| < R/2$, the ball $B_{\frac{|\mathbf{x}_0|}{2}}(\mathbf{x}_0)$ has no intersection with the outer boundary $S_R = \{|\mathbf{x}| = R\}$ or the profile $\mathcal{A}$. Using conditions (4.2),(4.3), (4.6) and estimate (4.12), by Schauder interior estimates (see Theorem 6.2 in [22]), we have

$$\|u\|_{2,\alpha;(\beta);B_{\frac{|\mathbf{x}_0|}{4}}(\mathbf{x}_0)} \le Cm_0, \tag{4.20}$$

where no upper index in the norm means no weight up to $\tilde{P}$. For $|\mathbf{x}_0| \ge R/2$, we use Schauder boundary estimates (Lemma 6.4 in [22]) with the boundary condition (4.6) and estimate (4.19) to obtain

$$\|u\|^{(-1-\alpha;\{S_-^R,S_+^R\})}_{2,\alpha;(\beta);B_{\frac{|\mathbf{x}_0|}{4}}(\mathbf{x}_0)\cap\Omega_D^c} \le Cm_0, \tag{4.21}$$

Estimates (4.18), (4.20) and (4.21) imply estimate (4.7) in the lemma. $\square$

By the continuity method, one can prove the existence of solutions for (4.1) with estimate (4.7). The uniqueness is simply the result of the maximum principle, Lemma 4.2, with the aid of the comparison function v constructed in Lemma 4.1. Since the procedure is standard, we omit the proof and only state the result as follows.

Lemma 4.3. *Assume* (4.2)–(4.6) *holds. For sufficiently small* m_0, *equation* (4.1) *with boundary condition* (4.5) *admits a unique solution* $u \in C^2(\Omega_R) \cap C(\overline{\Omega_R})$.

5. Nonlinear Equation in Bounded Domain

In this section, we will solve equation (3.19) in the bounded domain Ω_R with boundary condition given below.

We want to prescribe the boundary data for ψ such that (3.27) hold on $\partial\Omega \cap \Omega_R$ and $\psi - l$ vanishes away from $\partial\Omega$. We will define function g such that

$$\psi - l = g \quad \text{on} \quad \partial\Omega_R. \tag{5.1}$$

First it is easy to construct a smooth cutoff function $\eta(s)$ such that $\eta(s) = 1$ for $|s| \le D_0$ and $\eta(s) = 0$ for $|s| \ge D_0 + 1$. We also assume that $\|\eta\|_{C^{2,\alpha}(\mathbb{R})} \le 10$. Let

$$g(\mathbf{x}) = -\eta(x_2)\left((1-\eta(x_1))\, l(f_{\mathrm{sign}(x_1)}(x_1)) + \eta(x_1)\, l(x_2)\right), \tag{5.2}$$

where $\mathrm{sign}(x_1) = -$ for $x_1 < 0$ and $\mathrm{sign}(x_1) = +$ for $x_1 > 0$.

It is easy to check that $g|_{\partial\Omega} = -l|_{\partial\Omega}$, $g = 0$ for $x_2 > D_0 + 1$, and also g satisfies condition (4.6). Now, let constant C^* in (3.28) be the same as in estimate (4.7) in Lemma 4.1. The set for solutions of (3.19) in Ω_R is

$$\Sigma_R = \{u : \|u - l\|^{(-\alpha-1;\tilde{P})}_{2,\alpha;(\beta);\Omega_R} \le C^* m_0\}. \tag{5.3}$$

We state our lemma for the solution of (3.19) in Ω_R:

Lemma 5.1. *Equation* (3.19) *in* Ω_R *with boundary condition* (5.1) *admits a unique solution in the set* Σ_R, *provided* m_0, ε *are sufficiently small.*

Proof. To prove the lemma, we first linearize the nonlinear equation (3.19). By solving the linearized equation, we construct a map T in the set Σ_R . The solution of the nonlinear equation (3.19) is a fixed point of T.

We start the proof with the linearization of (3.19). We know that the limit function l satisfies (3.19) by its definition. That means the following equation holds:

$$a_{ij}(l, 0, l')\, l_{x_i x_j} = F(l, 0, l'), \tag{5.4}$$

where a_{ij}, F are defined in (3.14)–(3.16). Taking the difference of equations (3.19) and (5.4) leads to

$$\begin{aligned} & a_{ij}(\psi, \nabla\psi)(\psi - l)_{x_i x_j} + l''(a_{22}(\psi, \nabla\psi) - a_{22}(l, 0, l')) \\ = \ & F(\psi, \nabla\psi) - F(l, 0, l'). \end{aligned} \tag{5.5}$$

Denote $(l+s(\psi-l),\nabla(l+s(\psi-l)))$ by $\mathbf{t}_s^\psi$, and let

$$\begin{aligned} a_{ij}^\psi &= a_{ij}(\psi,\nabla\psi), && (5.6)\\ b_0^\psi &= l''\int_0^1 (a_{22})_\psi(\mathbf{t}_s^\psi)d\,s-\int_0^1 F_\psi(\mathbf{t}_s^\psi)d\,s, && (5.7)\\ (b_1^\psi,b_2^\psi) &= l''\int_0^1 (a_{22})_{\nabla\psi}(\mathbf{t}_s^\psi)d\,s-\int_0^1 F_{\nabla\psi}(\mathbf{t}_s^\psi)d\,s. && (5.8)\end{aligned}$$

Then we linearize equation (5.5) as follows:

$$a_{ij}^\psi(\tilde\psi-l)_{x_ix_j}+b_i^\psi(\tilde\psi-l)+b_0^\psi(\tilde\psi-l)=0. \tag{5.9}$$

We solve the above equation by applying Lemma 4.1. In the following, we will check the conditions (4.2)–(4.4) for $\psi\in\Sigma_R$. In fact, (4.2) will be satisfied if (4.3) holds with sufficiently small m_0. We let e in (4.3) be $a_{11}(m_0x_2,0,m_0)=a_{22}(m_0x_2,0,m_0)=\gamma p_0\rho_0$. By the expressions for a_{ij}, (3.14)–(3.16), and (3.10), (3.9), it is not hard to verify that

$$\|(a_{ij})_\psi(\mathbf{t}_s^\psi)\|_{C^\alpha(\Omega_R)}\le\frac{\varepsilon}{m_0}CC^*,$$
$$\|(a_{ij})_{\nabla\psi}(\mathbf{t}_s^\psi)\|_{C^\alpha(\Omega_R)}\le\frac{\varepsilon}{m_0}CC^*,$$

for $\psi\in\Sigma$ and m_0 small.

In the proof of this lemma, the generic constants C are independent of C^*.

Let $\varepsilon<\frac{m_0}{(C^*)^2}$, we have

$$\begin{aligned} &\|a_{ij}^\psi-e\delta_{ij}\|_{C^\alpha\Omega_R}\\ \le\ &\int_0^1\|(a_{ij})_\psi(\mathbf{t}_s^\psi)\|_{C^\alpha(\Omega_R)}ds\|\psi-m_0x_2\|_{C^\alpha(\Omega_R)}\\ &+\int_0^1\|(a_{ij})_{\nabla\psi}(\mathbf{t}_s^\psi)\|_{C^\alpha(\Omega_R)}ds\|\nabla(\psi-m_0x_2)\|_{C^\alpha(\Omega_R)}\\ \le\ &\varepsilon C(C^*)^2\\ \le\ &Cm_0.\end{aligned}$$

In the same manner, we can obtain

$$\|b_i\|_{C^\alpha(\Omega_R)}\le Cm_0,\quad i=0,1,2.$$

The above estimates lead to condition (4.3). Now we verify condition (4.4). For b_0^ψ, by its expression, we need to estimate $(a_{22})_\psi$ and F_ψ. By the definition of F, (3.17), and estimates (3.25), (3.26), we have

$$|F_\psi(\psi,\nabla\psi)|\le\frac{\varepsilon m_0C}{(m_0+|\psi|)^2}.$$

Let $u_s = l + s(\psi - l)$. For any $\psi \in \Sigma$, we consider two cases: **Case 1**, $|\psi - l| \leq \frac{1}{4}m_0x_2$; **Case 2**, otherwise. For **Case 1**,

$$u_s \geq l - \frac{1}{4}m_0x_2 \geq \frac{1}{4}m_0x_2,$$

noticing (3.22). Therefore,

$$|F_\psi(\mathbf{t}_s^\psi)| \;\; = \;\; |F_\psi(u_s, \nabla u_s)| \leq \frac{\varepsilon m_0 C}{(m_0 + |u_s|)^2} \leq \frac{\varepsilon C m_0}{(1+x_2)^2},$$

for $\varepsilon < {m_0}^2$. For **Case 2**, i.e., $|\psi - l| > \frac{1}{4}m_0x_2$, since $\psi \in \Sigma_R$, we have

$$\frac{1}{4}m_0x_2 < |\psi - l| \leq \frac{C^* m_0}{|\mathbf{x}|^\beta}.$$

This implies that $x_2 < (4C^*)^{\frac{1}{1+\beta}} \equiv R_0$. Hence, for $\varepsilon < ((m_0/(1+R_0))^2$, we have

$$|F_\psi(\mathbf{t}_s^\psi)| \leq \frac{\varepsilon C}{m_0} \leq \frac{\varepsilon C m_0 (1+R_0)^2}{(m_0)^2(1+x_2)^2} < \frac{C m_0}{(1+x_2)^2}.$$

The above analysis about both **Case1** and **Case 2** gives rise to

$$\left| \int_0^1 F_\psi(\mathbf{t}_s^\psi) d\, s \right| \leq \frac{C m_0}{(1+x_2)^2}. \tag{5.10}$$

Similarly, we have

$$|(a_{22})_\psi(\mathbf{t}_s^\psi)| \leq \frac{C m_0}{1+x_2}.$$

Together with the fact that $|l''(x_2)| < \frac{\varepsilon C m_0}{1+x_2}$, we conclude that

$$|b_0^\psi| < \frac{C m_0}{(1+x_2)^2}.$$

Same arguments apply to the estimates for b_1^ψ, b_2^ψ:

$$|b_1^\psi| + |b_2^\psi| < \frac{C m_0}{1+x_2}.$$

Therefore, we have verified condition (4.4). Thus, we apply Lemma 4.1 to solve (5.9) for $\tilde{\psi}$, where $u = \tilde{\psi} - l$ in Lemma 4.1. By estimate (4.7), we know that $\tilde{\psi} \in \Sigma_R$. Therefore, we can define a map T from Σ_R to itself by $T\psi \equiv \tilde{\psi}$. It is obvious that a solution for the nonlinear equation (3.19) is a fixed point of T. In order to prove the existence of a fixed point of T, we apply Schauder fixed point theorem, which says: if Σ_R is a compact convex set of a Banach space $\mathcal{B}$, and $T : \Sigma_R \to \Sigma_R$ is continuous in $\mathcal{B}$, T has a fixed point.

Now we let $\mathcal{B} = C^{(-1-\alpha';\tilde{P})}_{2,\alpha';(\beta)}(\Omega_R)$ (cf. definition (2.2)), where $0 < \alpha' < \alpha$. Obviously, Σ_R is compact and convex in $\mathcal{B}$. We only need to verify T is continuous in Σ_R. We prove this by contradiction argument.

Suppose T is not continuous. Then there exists a sequence $\{\psi_n\} \subset \Sigma_R$ such that $\psi_n \to \psi$ in $\mathcal{B}$, but $T\psi_n$ does not converge to $\tilde{\psi} = T\psi$. This implies that we can find a subsequence $\{T\psi_{n_k}\}$ such that

$$\|T\psi_{n_k} - \tilde{\psi}\|_{\mathcal{B}} \geq c_0 > 0, \tag{5.11}$$

where $\|\cdot\|_{\mathcal{B}}$ denotes the weighed norm $\|\cdot\|^{(-1-\alpha';\tilde{P})}_{2,\alpha';(\beta);\Omega_R}$ and c_0 is a fixed constant. Since $\{T\psi_{n_k}\}$ is compact in $\mathcal{B}$, there exists a subsequence, still denoted by $\{T\psi_{n_k}\}$, convergent to $\bar{\psi} \in \Sigma_R$. On the other hand, $\psi_n \to \psi$ in $\mathcal{B}$ implies that $a^{\psi_{n_k}}_{ij} \to a^{\psi}_{ij}(i,j=1,2)$, $b^{\psi_{n_k}}_i \to b^{\psi}_i(i=0,1,2)$ in $C^{\alpha'}$ norm. Let $k \to \infty$ and we see that $\bar{\psi}$ is also a solution of (5.9) with the same boundary condition (4.5). Inequality (5.11) implies that $\|\tilde{\psi} - \bar{\psi}\|_{\mathcal{B}} \geq c_0 > 0$, which means $\tilde{\psi}, \bar{\psi}$ are two distinct solutions for (5.9). This contradicts with the uniqueness of the solution for (5.9). Hence, we verified the continuity of T in Σ_R.

By Schauder fixed point theorem, there exists a fixed point ψ_R for T. So ψ_R is a solution for the nonlinear equation (3.19) in Ω_R with boundary condition (5.1).

The uniqueness of the solution for (3.19) is proved by the maximum principle Lemma 4.2 as follows.

For any two solutions $\psi_1, \psi_2 \in \Sigma_R$ of (3.19), (5.1), we take the difference of the two equations and obtain

$$a^{\psi_1}_{ij}(\psi_1 - \psi_2)_{x_ix_j} + (\psi_2)_{x_ix_j}(a^{\psi_1}_{ij} - a^{\psi_2}_{ij}) = F(\psi_1, \nabla\psi_1) - F(\psi_2, \nabla\psi_2).$$

Similar to the notations in (5.5)–(5.8), we set

$$\mathbf{t}_s = (\psi_2 + s(\psi_1 - \psi_2), \nabla(\psi_2 + s(\psi_1 - \psi_2))),$$

and let $u = \psi_1 - \psi_2$. Then we derive the following equation:

$$a^{\psi_1}_{ij}u_{x_ix_j} + b_iu_{x_i} + b_0u = 0, \tag{5.12}$$

where

$$b_0 = (\psi_2)_{x_ix_j}\int_0^1 (a_{ij})_{\psi}(\mathbf{t}_s)ds - \int_0^1 F_{\psi}(\mathbf{t}_s)ds, \tag{5.13}$$

$$(b_1, b_2) = (\psi_2)_{x_ix_j}\int_0^1 (a_{ij})_{\nabla\psi}(\mathbf{t}_s)ds - \int_0^1 F_{\nabla\psi}(\mathbf{t}_s)ds. \tag{5.14}$$

Notice that the factor $(\psi_2)_{x_ix_j}$ in (5.13) and (5.14) blows up at the corner points in $\tilde{P} = \{A_-, A_+, S^R_-, S^R_+\}$. Lemma 4.2 requires continuity of coefficients a_{ij}, b_i up to the boundary. Therefore, in order to apply the lemma, we truncate small neighborhoods around corners from Ω_R. Define

$$B_{\mathcal{P},r} = \bigcup_{I\in\mathcal{P}} B_r(I), \tag{5.15}$$

where $\mathcal{P}$ is a set of boundary points on Ω_R. Let

$$\Omega_R^{\mathcal{P},r} = \Omega_R \backslash \overline{B_{\mathcal{P},r}}, \quad S_{\mathcal{P},r} = \partial B_{\mathcal{P},r} \cap \Omega_R. \tag{5.16}$$

Now, we know $b_i \in C\left(\overline{\Omega_R^{\tilde{P},r}}\right)$, $i = 0, 1, 2$. Define $r_I = |\mathbf{x} - I|$, for any corner point $I \in \tilde{P}$. Then we have the following estimates for b_i:

$$(x_2+1)(|b_1|+|b_2|) + (x_2+1)^2|b_0| \le Cm_0(1 + \sum_{I\in\tilde{P}} r_I^{\alpha-1}). \tag{5.17}$$

We construct a comparison function $\tilde{v}$ as follows: Let

$$v_I = r_I^{-\frac{3}{4}\beta}(x_2+1)^{\frac{\beta}{2}}. \tag{5.18}$$

Define $\tilde{v} = \sum_{I\in\tilde{P}} v_I$. By (4.9)–(4.11), we can verify that

$$a_{ij}^{\psi_1}(v_I)_{x_ix_j} \le -c_0(r_I^{-\frac{3}{4}\beta-2}(x_2+1)^{\frac{\beta}{2}} + r_I^{-\frac{3}{4}\beta}(x_2+1)^{\frac{\beta}{2}-2}), \tag{5.19}$$

$$\left|\sum_{i=1,2} b_i(v_I)_{x_i} + b_0 v_I\right| \le Cm_0(1 + \sum_{I\in\tilde{P}} r_I^{\alpha-1}) r_I^{-\frac{3}{4}\beta}(x_2+1)^{\frac{\beta}{2}-2}, \tag{5.20}$$

where $c_0 > 0$ is a constant only depending on β. Inequalities (5.19) and (5.20) directly imply that

$$a_{ij}^{\psi_1}\tilde{v}_{x_ix_j} + b_i\tilde{v}_{x_i} + b_0\tilde{v} < 0.$$

By Lemma 4.2, we have

$$\sup_{\Omega_R^{\tilde{P},r}} \frac{|u|}{\tilde{v}} = \max_{\partial\Omega_R^{\tilde{P},r}} \frac{|u|}{\tilde{v}}. \tag{5.21}$$

Since $u|_{\partial\Omega_R} = 0$ and $u \in C^{1,\alpha}(\Omega_R)$, we know that

$$\max_{\partial\Omega_R^{\tilde{P},r}} \frac{|u|}{\tilde{v}} = \max_{S_{\tilde{P},r}} \frac{|u|}{\tilde{v}} \le Cr^\alpha R^{\frac{3}{4}\beta}.$$

Hence, we have

$$\sup_{\Omega_R^{\tilde{P},r}} |u| \le CR^{\frac{3}{4}\beta} r^\alpha \sup_{\Omega_R^{\tilde{P},r}} \tilde{v} \le CR^{\frac{3}{4}\beta} r^\alpha (1 + r^{-\frac{3}{4}\beta}).$$

Therefore, by letting $r \to 0$, the above inequality implies $\sup_{\Omega_R} |u| \le 0$. This shows the uniqueness of the solution for (3.19) in Ω_R. Hence, the proof of this lemma is complete.

$\square$

6. Subsonic Flow in Half Plane

After we solve (3.19) in Ω_R, we let R tend to infinity to prove Theorem 3.1 as follows.

Proof of Theorem 3.1. By Lemma 5.1, for a given radius R, we can find a unique solution $\psi_R \in \Sigma_R$. By a diagonal process, one can choose proper sequence $R_n \to \infty$ as $n \to \infty$, such that ψ_{R_n} converges to some function ψ in $\|\cdot\|^{(-\alpha'-1;P)}_{2,\alpha';(\beta);\Omega_Q}$ norm, for any fixed $Q > D_0 + 1$. Since $\psi_{R_n} \in \Sigma_{R_n}$, we have

$$\|\psi_{R_n} - l\|^{(-\alpha-1;P)}_{2,\alpha;(\beta);\Omega_{\frac{R_n}{2}}} \le C^* m_0, \tag{6.1}$$

for any $R_n > 2(D_0 + 1)$. Let $n \to \infty$ in (6.1), we obtain estimate

$$\|\psi - l\|^{(-\alpha-1;P)}_{2,\alpha;(\beta);\Omega} \le C^* m_0, \tag{6.2}$$

which implies $\psi \in \Sigma$. This completes the existence of solutions for (3.19).

To prove the uniqueness of the solution, we will use the asymptotic behavior of solutions described by set Σ. We still use the truncated domain Ω_R and follow the same strategy as in the uniqueness part of Lemma 5.1. Now, the situation here is slightly different from that in Lemma 5.1: (1) we have no singularity for $(\psi_2)_{x_i x_j}$ at corners S^R_-, S^R_+; (2) $u = \psi_1 - \psi_2$ does not vanish on the boundary portion $S_R = \{|\mathbf{x}| = R\} \cap \Omega$.

Similarly as in Lemma 5.1, we have the same equation (5.12) for u. We define the comparison function $\bar{v}$ by $\bar{v} = v_{A_-} + v_{A_+}$, where v_{A_-}, v_{A_+} are defined in (5.18). The domain we consider here is $\Omega^{P,r}_R$ defined by (5.15). By the same computation as in Lemma 5.1, we have

$$a^{\psi_1}_{ij} \bar{v}_{x_i x_j} + b_i \bar{v}_{x_i} + b_0 \bar{v} < 0.$$

We know that $|u| \le C R^{-\beta}$ on S_R by the definition of Σ. Also, we see

$$|u|_{S_{P,r}}| \le C r^{\alpha}, \quad \bar{v}|_{S_R} \ge R^{-\frac{3}{4}\beta}, \quad \bar{v}|_{S_{P,r}} \ge r^{-\frac{3}{4}\beta}.$$

Therefore, we conclude that

$$\sup_{\Omega^{P,r}_R} \frac{|u|}{\tilde{v}} \le C \max(r^{\alpha + \frac{3}{4}\beta}, R^{-\frac{1}{4}\beta}). \tag{6.3}$$

Let $r = R^{-\frac{1}{6}}$ and we obtain

$$\sup_{\Omega^{P,r}_R} |u| \le C R^{-\frac{1}{4}\beta} \sup_{\Omega^{P,r}_R} \bar{v} \le C R^{-\frac{1}{4}\beta}(1 + R^{\frac{\beta}{8}}) \le C R^{-\frac{\beta}{8}}.$$

Letting $R \to \infty$ gives rise to $\sup_\Omega |u| \leq 0$, which implies the uniqueness of the solution for (3.19) in Σ. This finishes the proof of Theorem 3.1. $\square$

Once we proved Theorem 3.1, define $U = (\mathbf{m}, p, \rho)$ by

$$\mathbf{m} = (\psi_{x_2}, -\psi_{x_1}), \quad \rho = \rho(\psi, \nabla\psi), \quad p = \frac{\gamma - 1}{\gamma} A(\psi)\rho^\gamma,$$

where ρ is uniquely solved from Bernoulli's law (3.5). Hence, by (3.13), (3.18) and (3.5), we can recover the original Euler equations (1.1). It is easy to check that U satisfies (2.13) with the aid of estimate (6.2).

Remark 6.1. The uniqueness of the solution for Euler system (1.1) can not be obtained from Theorem 3.1 due to two obstacles. One is the existence of stagnation points, which disqualifies equivalence between equation (3.19) and the two momentum equations (3.13), (3.18). The corners A_-, A_+ on $\partial\Omega$ are necessarily stagnation points, because $\nabla\psi$ is continuous up to the corners. Whether or what kind of stagnation points may appear inside domain Ω is not clear. The other problem is about the complexity of streamline topology. During the reduction in section 3, we assume streamlines have simple topology, which means that streamlines in Ω extend from $-\infty$ to ∞ in x_1, so that information about A, B can be carried along streamlines and reach the whole domain Ω. However, the geometry of profile $\mathcal{A}$ may be complicated and cause nontrivial topology of streamlines, such as closed orbits or intersection of streamlines at stagnation points. The above reasons prevent us from obtaining the uniqueness for the Euler flows out of Theorem 3.1.

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