GLOBAL EXISTENCE OF WEAK SOLUTIONS FOR A VISCOUS TWO-PHASE MODEL

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Abstract. The purpose of this paper is to explore a viscous two-phase liquid-gas model relevant for well and pipe flow. Our approach relies on applying suitable modifications of techniques previously used for studying the single-phase isothermal Navier-Stokes equations. A main issue is the introduction of a novel two-phase variant of the potential energy function needed for obtaining fundamental a priori estimates. We derive an existence result for weak solutions in a setting where transition to single-phase flow is guaranteed not to occur when the initial state is a true mixture of both phases. Some numerical examples are also included in order to demonstrate characteristic behavior of solutions. In particular, we illustrate how two-phase flow is genuinely different compared to single-phase flow concerning the behavior of an initial mass discontinuity.

1. Introduction

We are interested in a one-dimensional two-phase liquid (ℓ) and gas (g) model composed of two separate mass equations and a common momentum equation in the following form:

$$\begin{aligned} m_t + (v_\ell m)_x &= 0, \\ n_t + (v_{\mathrm{g}} n)_x &= 0, \\ (mv_\ell + nv_{\mathrm{g}})_t + (mv_\ell^2 + nv_{\mathrm{g}}^2)_x + p(m,n)_x &= q_F + q_G + \mu(v_{\mathrm{mix}})_{xx}, \quad \mu > 0, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} m =& \alpha_\ell \rho_\ell, \qquad n = \alpha_{\mathrm{g}} \rho_{\mathrm{g}}, \\ q_F =& -f v_{\mathrm{mix}}, \qquad q_G = g \rho_{\mathrm{mix}}, \\ v_{\mathrm{mix}} = \alpha_\ell v_\ell +& \alpha_{\mathrm{g}} v_{\mathrm{g}}, \qquad \rho_{\mathrm{mix}} = \alpha_\ell \rho_\ell + \alpha_{\mathrm{g}} \rho_{\mathrm{g}}, \end{aligned} \tag{1.2}$$

where f and g are nonnegative constants. Here ρ denotes density, α is volume fraction, v is fluid velocity, and p is pressure. This model is supplemented with equations of state (EOS) for the two phases, here assumed to be of the form

$$\rho_\ell = \rho_{l,0} + \frac{p - p_{l,0}}{a_l^2}, \qquad \rho_{\mathrm{g}} = \frac{p}{a_g^2}, \tag{1.3}$$

where a_l and a_g are sonic speed, respectively, in liquid and gas, and $p_{l,0}$ and $\rho_{l,0}$ are reference pressure and density given as constants. Moreover, we have given the following basic constraint

$$\alpha_\ell + \alpha_{\mathrm{g}} = 1, \tag{1.4}$$

and an algebraic expression describing the relation between gas and liquid velocity of the form

$$v_{\mathrm{g}} = K v_{\mathrm{mix}} + S, \qquad K, S \ \text{ are constants.} \tag{1.5}$$

This model is often referred to as the drift-flux mixture model [2, 3, 4, 7, 8, 9, 10, 27]. Note that by combining (1.3) and (1.4), we get a nonlinear pressure law $p(m,n)$ of the form

$$p(m,n) = C(-b(m,n) + \sqrt{b(m,n)^2 + c(m,n)}), \tag{1.6}$$

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with $C = \frac{1}{2}a_l^2$ and $k_0 = \rho_0 - p_0/a_l^2 > 0$ and $a_0 = (a_g/a_l)^2$ and

$$\begin{aligned} b(m,n) &= k_0 - m - \Big(\frac{a_g}{a_l}\Big)^2 n = k_0 - m - a_0 n \\ c(m,n) &= 4k_0\Big(\frac{a_g}{a_l}\Big)^2 n = 4k_0 a_0 n. \end{aligned} \tag{1.7}$$

Development of good discrete methods for solving the inviscid variant of (1.1) has been a topic for many papers during the last decade, see for instance [2]–[16] and [29]–[34]. However, few results providing insight into qualitative (mathematical) properties of this model (and various simplified variants) seem to exist. Besides results relevant for existence, uniqueness, stability, and regularity of weak solutions, it is also of great interest to understand more precisely under what conditions formation of vacuum (transition to single-phase flow, i.e. $m = 0$ or $n = 0$) may occur.

Relation to other mixture models. It is interesting to compare the model (1.1) with other mixture models for which existence results of weak solutions have been provided. Here we would like to mention one direction represented by the work [17, 19]. This model is designed such that

(i) at each point of the space occupied by the mixture there are particles belonging to each component (i.e. transition to single-phase flow does no occur in this model);
(ii) basic energy-estimates are in place.

More precisely, the one-dimensional model takes the form

$$\begin{aligned} m_t + (v_\ell m)_x &= 0, \\ n_t + (v_{\mathrm{g}} n)_x &= 0, \\ (mv_\ell)_t + (mv_\ell^2)_x + p(m,n)_x &= \mu(v_\ell)_{xx} + J, \quad \mu > 0, \\ (nv_{\mathrm{g}})_t + (nv_{\mathrm{g}}^2)_x + q(m,n)_x &= \mu(v_{\mathrm{g}})_{xx} - J, \end{aligned} \tag{1.8}$$

where J involves interaction terms of the form

$$\begin{aligned} J &= a\Big(m,n,|v_\ell - v_{\mathrm{g}}|\Big)\cdot(v_\ell - v_{\mathrm{g}}) + \Psi'(c_1 m + c_2 n)\Big(c_2 n(c_1 m)_x - c_1 m(c_2 n)_x\Big), \\ p(m,n) &= c_1 m\rho\Psi'(\rho), \qquad \rho = c_1 m + c_2 n, \qquad c_1, c_2 > 0, \\ q(m,n) &= c_2 n\rho\Psi'(\rho). \end{aligned} \tag{1.9}$$

where $a(\cdot,\cdot,\cdot)$ is non-negative function and $\Psi(\rho)$ is a function whose choice characterizes the pressure law under consideration. One natural choice is for instance

$$\Psi(\rho) = \rho^{\gamma-1}, \qquad \gamma > 1.$$

This model is thermo-mechanically consistent in the sense that the following basic energy estimate is obtained:

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}} \rho\Psi(\rho)\,dx + \frac{d}{dt}\Big(\frac{1}{2}\int_{\mathbb{R}} m|v_\ell|^2\,dx\Big) + \frac{d}{dt}\Big(\frac{1}{2}\int_{\mathbb{R}} n|v_{\mathrm{g}}|^2\,dx\Big) \\ &+\mu\int_{\mathbb{R}} (v_\ell)_x^2 + (v_{\mathrm{g}})_x^2\,dx + \int_{\mathbb{R}} a\Big(m,n,|v_\ell - v_{\mathrm{g}}|\Big)|v_\ell - v_{\mathrm{g}}|^2\,dx = 0. \end{aligned} \tag{1.10}$$

Here the pressure laws p and q together with the interaction term J are carefully designed such that terms nicely can be grouped together resulting in the equation (1.10). Note, however that J involves non-conservative terms mn_x and nm_x which we do not have much information about. Thus, the authors in [17] neglect the second term of J. The price to pay for this simplification is that the general energy estimate (1.10) is lost.

On the other hand, alternative mixture models that use only one fluid velocity and one density for the whole mixture, have been studied. These models cannot take into account mutual interactions of the individual components. From a mathematical point of view, more is known about these models, see for instance [6, 36, 37].

The model (1.1) is somewhat different from the above two mentioned mixture models.

- First, it involves an additional variable, the volume fraction variable represented by α_g (or α_ℓ) which allows modeling of co-existent two-phase and single-phase regions. In other words, the model allows transition from single-phase to two-phase flow.
- The model involves two different fluid velocities whose difference $v_\ell - v_g = \Phi(\cdot)$ is governed by an algebraic expression and/or experimental data. A special case is the no slip flow condition $v_\ell = v_g$ corresponding to $\Phi = 0$.
- The model (1.1) is supplemented with a pressure law $p = p(m,n)$ depending on the masses m and n. Having specified an EOS for each of the two phases $\rho_\ell(p)$ and $\rho_g(p)$, the fundamental relation $\alpha_\ell + \alpha_g = 1$ then directly defines the pressure law $p(m,n)$. In particular, the resulting pressure law is normally not of the form given in (1.9). A main challenge here seems to be able to obtain a priori energy-type of estimates for such pressure laws, in particular, when fluid velocities are different. For the mixture model (1.10), the inclusion of a pressure related term Ψ in the interaction term J in (1.9) plays a crucial role in order to obtain the energy estimate (1.10).

Note also that although we apply linear EOSs for the gas and liquid phase in (1.3), the resulting pressure law $p(m,n)$ given in (1.6) for the two-phase mixture, becomes a nonlinear function. This reflects some of the additional complexity represented by two-phase modeling.

In order to compare the mixture model (1.8) with the mixture model (1.1) we add the two momentum equations in (1.8), where we for simplicity have used $c_1 = c_2 = 1$ in (1.9). This gives us

$$
\begin{aligned}
m_t + (v_\ell m)_x &= 0,\\
n_t + (v_g n)_x &= 0,\\
(m v_\ell + n v_g)_t + (m v_\ell^2 + n v_g^2)_x + P(m+n)_x &= \mu (v_{\mathrm{mix}})_{xx}, \quad \mu > 0,
\end{aligned}
$$

where $P(m+n) = (m+n)^2 \Psi'(m+n)$. Thus, a main difference compared to the model (1.1) is that the pressure $P = P(\rho)$ depends on one variable, the mixture mass $\rho = m + n$ instead of the two-variable pressure law $p(m,n)$ given by (1.6) and (1.7).

A simplified model. The model (1.1)–(1.5) is highly relevant, for example, in modeling of well and pipe flow processes [9]. Particularly, if we replace the density relations (1.3) with those used to represent more realistic fluids as well as apply more advanced slip relations than (1.5). The purpose of this paper is to focus on some basic aspects of this model and for that purpose we will deal with a simplified version of (1.1). The simplification we make is as follows:

(i) Due to the fact that the liquid phase is much heavier than the gas phase, typically to the order $\rho_\ell/\rho_g \sim 10^3$, we neglect the gas phase in the mixture momentum equation.
(ii) We restrict ourselves to a flow regime where we can assume that fluid velocities are equal, i.e., $v_\ell = v_g = u$.

Consequently, from now on we focus on the simplified model

$$
\begin{aligned}
m_t + (um)_x &= 0\\
n_t + (un)_x &= 0\\
(mu)_t + (mu^2)_x + p(m,n)_x &= \mu u_{xx}, \qquad \mu > 0,
\end{aligned}
\tag{1.11}
$$

with initial data

$$
(m(0,\cdot), n(0,\cdot), u(0,\cdot)) = (m_0, n_0, u_0). \tag{1.12}
$$

We may set $\mu = 1$ in the following. In this paper we also focus on a viscous dominated setting where we neglect the friction term q_F and where the flow is horizontal, i.e. $q_G = 0$.

It is also interesting to have in mind the model (1.11) described in Lagrangian coordinates $c = n/m$ and $v = 1/m$. The model then takes the form

$$
\begin{aligned}
c_t &= 0\\
v_t - u_x &= 0\\
u_t + p(c,v)_x &= \mu u_{xx}, \qquad \mu > 0.
\end{aligned}
\tag{1.13}
$$

Models of the form (1.13) with $\mu = 0$ have been studied more recently by several researchers. Lu has considered global weak solutions relying on a suitable variant of the compensated compactness method [28]. Fan has studied travelling waves and Riemann problems for a similar model relevant for liquid/vapor phase transition [12, 13], see also [1]. In addition, an inviscid variant of (1.13) with a friction term in the momentum equation has been studied in several works [25, 26, 30, 35]. They explore the damping mechanism from the friction term to the smoothness and large time behavior of the solutions. In [18] the authors consider the drift-flux model in Lagrangian coordinates with a slip-law similar to (1.5).

The idea of this paper. The main purpose of this paper is to explore some aspects of the two-phase model (1.11) concerning existence of weak solutions. One important flow scenario relevant for the two-phase model is a situation where we avoid transition to single-phase flow, that is, we can find upper and lower limits for the masses m and n of the form

$$C(T)^{-1} \le m, n \le C(T), \qquad (t, x) \in [0, T] \times \mathbb{R}. \tag{1.14}$$

A clearer understanding of the conditions that ensure such a behavior is clearly desirable, also from an application point of view. Such pointwise control has been a topic for much research within development of mathematical theory for single-phase Navier-Stokes flow, see for example [24].

In this paper we will explore the viscous two-phase model by applying techniques similar to those used by Hoff for the single-phase isothermal Navier-Stokes as described in [22, 23]. A main point here is that we need to define an appropriate two-phase variant of the potential function, in the following denoted as $Q(m, n)$, that will give us a basic energy estimate of the form

$$\frac{d}{dt}\int_{\mathbb{R}}\Big[\frac{1}{2}mu^2 + Q(m,n)\Big]\,dx + \mu\int_0^T\int_{\mathbb{R}}(u_x)^2\,dx\,dt = 0. \tag{1.15}$$

More precisely, we introduce a two-phase potential energy density function $Q(m, n, \overline{m}, \overline{n})$ of the form

$$Q(m, n|\overline{m}, \overline{n}) := m\int_{\overline{m}}^{m}\Big[\frac{p\left(s, \frac{n}{m}s\right) - p\left(\overline{m}, \overline{n}\right)}{s^2}\Big]\,ds + mg\Big(\frac{n}{m}, \overline{m}\Big),$$

where an appropriate choice of g must be made in order to ensure that Q becomes non-negative for positive $m, n, \overline{m}$ and $\overline{n}$. Here we also, in view of the two mass equations of (1.11), shall make use of the fact that m and n are related by

$$\frac{n(t,x)}{m(t,x)} = \Big(\frac{n_0}{m_0}\Big)(X_t^{-1}(x)),$$

where $X_t(x)$ is the characteristic emanating from x at time $t = 0$. Equipped with this two-phase energy potential function and the corresponding energy estimate (Lemma 1 and Corollary 1), we observe that gentle modifications of the single-phase arguments allow us to obtain the bound (1.14) (Lemma 2). Thus, we demonstrate that such single-phase techniques can provide useful insight into characteristic features relevant for the two-phase model (1.11).

Equipped with pointwise control of m and n, we then can derive several a priori higher order regularity estimates in L^p and Sobolev spaces of a sequence of approximate solutions $\{m^\delta, n^\delta, u^\delta\}$ obtained by applying a regularization of initial data m_0, n_0, u_0, see Lemma 3. These estimates yield some basic convergence results, however, not strong convergence of m^δ, n^δ as needed in order to recover the nonlinear pressure law $p(m, n)$. A strong convergence result is presented in Lemma 4 and is mainly based on continuity estimates for particle trajectories together with the higher order estimates of Lemma 3 which imply Holder type estimates for the effective viscous flux $u_x^\delta - p(m^\delta, n^\delta)$. This in turn gives rise to strong compactness of m^δ and n^δ. The conclusion of the various Lemmas is summed up in Theorem 1 given below.

The result of this paper. Along the line of [23] we work with initial data having possibly different limits at $x = \pm\infty$ in order to include general Riemann data in the analysis. More precisely, we assume (u_-, u_+) and (m_-, m_+), (n_-, n_+) (positive) and let $\overline{u}$, $\overline{m}$, and $\overline{n}$ be smooth monotone functions such that

$$\begin{aligned} \overline{m}(x) &= m_\pm, \qquad \text{when } \pm x \geq 1, \quad \overline{m}(x) > 0\ \forall x \in \mathbb{R} \\ \overline{n}(x) &= n_\pm, \qquad \text{when } \pm x \geq 1, \quad \overline{n}(x) > 0\ \forall x \in \mathbb{R} \\ \overline{u}(x) &= u_\pm, \qquad \text{when } \pm x \geq 1. \end{aligned} \tag{1.16}$$

Our main theorem is the following:

Theorem 1. *Assume that the initial data m_0, n_0, and u_0 satisfy*

$$\begin{aligned} &0 < \underline{\kappa}_0 \leq m_0 \leq \overline{\kappa}_0 < \infty, \qquad 0 < \underline{\iota}_0 \leq n_0 \leq \overline{\iota}_0 < \infty, \\ &m_0 - \overline{m} \in L^2(\mathbb{R}), \qquad n_0 - \overline{n} \in L^2(\mathbb{R}), \qquad u_0 - \overline{u} \in L^2(\mathbb{R}). \end{aligned} \tag{1.17}$$

A). *First, there exists a global weak solution (m, n, u) of (1.11) and (1.12) on $\mathbb{R}^+ \times \mathbb{R}$ such that*

$$m - \overline{m},\ n - \overline{n},\ mu - \overline{mu} \in C([0,\infty); H^{-1}(\mathbb{R})), \tag{1.18}$$

$$u - \overline{u} \in C((0,\infty); L^2(\mathbb{R})), \tag{1.19}$$

$$u(t,\cdot),\ \mu u_x(t,\cdot) - p(m(t,\cdot), n(t,\cdot)) + p(\overline{m}, \overline{n}) \in H^1(\mathbb{R}), \quad t > 0, \tag{1.20}$$

$$u_t(t,\cdot),\ \dot{u}(t,\cdot) \in L^2(\mathbb{R}), \quad t > 0. \tag{1.21}$$

Here $\dot{u} = \frac{du}{dt} = u_t + uu_x$. In addition, for a finite time $T > 0$, there exist constants $\overline{\kappa}(T), \underline{\kappa}(T)$ and $\overline{\iota}(T), \underline{\iota}(T)$ such that for $t \in [0,T]$

$$\begin{aligned} &0 < \underline{\kappa}(T) \leq m(t,x) \leq \overline{\kappa}(T) < \infty, \qquad a.e.\ in\ \mathbb{R}, \\ &0 < \underline{\iota}(T) \leq n(t,x) \leq \overline{\iota}(T) < \infty, \qquad a.e.\ in\ \mathbb{R}. \end{aligned} \tag{1.22}$$

B). *Second, for a finite interval $[0,T]$ more precise information about the results (1.18)–(1.21) can be given. That is, there is a constant $C(T)$ depending on upper bounds in the time interval $[0,T]$ for $\|m_0 - \overline{m}\|_2$, $\|n_0 - \overline{n}\|_2$, $\|u_0 - \overline{u}\|_2$, $\|m_0\|_\infty$, $\|m_0^{-1}\|_\infty$, $\|n_0\|_\infty$, $\|n_0^{-1}\|_\infty$ such that,*

$$\begin{aligned} \sup_{0<t\leq T} \Big[&\|m(t,\cdot) - \overline{m}\|_{L^2(\mathbb{R})} + \|u(t,\cdot) - \overline{u}\|_{L^2(\mathbb{R})} + \sigma(t)^{1/2} \|u_x(t,\cdot)\|_{L^2(\mathbb{R})} \\ &+ \sigma(t) \Big(\|\dot{u}(t,\cdot)\|_{L^2(\mathbb{R})} + \|\mu u_x(t,\cdot) - p(m(t,\cdot), n(t,\cdot)) + p(\overline{m}, \overline{n})\|_{L^2(\mathbb{R})} \Big) \Big] \leq C(T); \end{aligned} \tag{1.23}$$

$$\begin{aligned} \int_0^T \Big[&\|u(s,\cdot)\|^2_{L^2(\mathbb{R})} + \sigma(s) \|\dot{u}(s,\cdot)\|^2_{L^2(\mathbb{R})} + \sigma(s) \|\mu u_x(s,\cdot) - p(m(s,\cdot), n(s,\cdot)) + p(\overline{m}, \overline{n})\|^2_{L^2(\mathbb{R})} \\ &+ \sigma(s)^2 \|\dot{u}_x(s,\cdot)\|^2_{L^2(\mathbb{R})} \Big] \, ds \leq C(T); \end{aligned} \tag{1.24}$$

where $\sigma(t) = \min\{t, 1\}$. Moreover, for $0 < \tau < T$

$$\sigma(\tau)^{1/4} \|u\|_{L^\infty([\tau,T]\times\mathbb{R})} + \sigma(\tau)^{1/2} \langle u \rangle^{1/2,1/4}_{[\tau,T]\times\mathbb{R}} \leq C(T), \tag{1.25}$$

where $\langle u \rangle^{1/2,1/4}_{[\tau,T]\times\mathbb{R}}$ is the usual Hölder norm with exponent 1/2 in x and 1/4 in t.

The rest of this paper is organized as follows: In Section 2 we obtain a fundamental energy estimate, pointwise estimates for m and n, as well as higher order regularity estimates. Compactness and convergence to weak solutions is then discussed in Section 3, whereas some numerical results are included in Section 4 in order to demonstrate characteristic behavior of the viscous two-phase model.

2. Various estimates

2.1. **Preliminary.** First, for the derivation of a priori estimates we need an existence result of smooth solutions for small time. We here shall assume (without proof) that we have an existence result of the following form which is a direct generalization of a classical result for single-phase Navier-Stokes [38, 33].

Proposition 1 (classical solution in a small time interval)**.** *Let (m_0, n_0, u_0) satisfy (1.17), then there exists an $T_0 > 0$ depending on $\overline{\kappa}_0$, $\underline{\kappa}_0$ and $\overline{\iota}_0, \underline{\iota}_0$, and $\|m_0 - \overline{m}\|_{H^1}$, $\|n_0 - \overline{n}\|_{H^1}$, and $\|u_0 - \overline{u}\|_{H^1}$ such that the model has a unique solution on $(0, T_0)$ satisfying*

$$
\begin{aligned}
m - \overline{m} &\in L^\infty(0, T_1; H^1(\mathbb{R})), \qquad \partial_t m \in L^2((0, T_1) \times \mathbb{R}),\\
n - \overline{n} &\in L^\infty(0, T_1; H^1(\mathbb{R})), \qquad \partial_t n \in L^2((0, T_1) \times \mathbb{R}),\\
u - \overline{u} &\in L^2(0, T_1; H^2(\mathbb{R})), \qquad \partial_t u \in L^2((0, T_1) \times \mathbb{R}),
\end{aligned} \tag{2.1}
$$

for $T_1 < T_0$. Moreover, there exist some $\overline{\kappa}(t), \overline{\iota}(t) < \infty$ and $\underline{\kappa}(t), \underline{\iota}(t) > 0$ such that

$$
\underline{\kappa}(t) \le m(t,x) \le \overline{\kappa}(t), \qquad \underline{\iota}(t) \le n(t,x) \le \overline{\iota}(t) \tag{2.2}
$$

for all $t \in (0, T_0)$.

By applying the method of characteristics we have

$$
\frac{d}{dt} X_t(x) = u(t, X_t(x)), \qquad X_0(x) = x,
$$

and in view of the continuity equations for m and n it follows

$$
\begin{aligned}
\frac{dm}{dt}(t, X_t(x)) &= -(m u_x)(t, X_t(x)),\\
\frac{dn}{dt}(t, X_t(x)) &= -(n u_x)(t, X_t(x)).
\end{aligned}
$$

In view of Proposition 1 we conclude that

$$
\frac{1}{m}\frac{dm}{dt}(t, X_t(x)) = \frac{1}{n}\frac{dn}{dt}(t, X_t(x)),
$$

or

$$
\frac{d}{dt}\log(m(t, X_t(x))) = \frac{d}{dt}\log(n(t, X_t(x))).
$$

Thus, we conclude that $m(t,x)$ and $n(t,x)$ are related by

$$
\frac{m(t, X_t(x))}{n(t, X_t(x))} = \Big(\frac{m_0}{n_0}\Big)(x).
$$

In other words,

$$
\frac{n(t,x)}{m(t,x)} = \Big(\frac{n_0}{m_0}\Big)(X_t^{-1}(x)) \overset{\text{def}}{:=} s_0(t,x), \tag{2.3}
$$

for $t \in (0, T_0)$ where $X_t(x)$ is the characteristic emanating from x at time $t = 0$. Particularly, we have

$$
0 < \underline{s}_0 \overset{\text{def}}{:=} \frac{\underline{\iota}_0}{\overline{\kappa}_0} \le \min\Big(\frac{n_0}{m_0}(X_t^{-1}(x))\Big) \le \frac{n}{m}(t,x) \le \max\Big(\frac{n_0}{m_0}(X_t^{-1}(x))\Big) \le \frac{\overline{\iota}_0}{\underline{\kappa}_0} \overset{\text{def}}{=:} \overline{s}_0 < \infty, \tag{2.4}
$$

where $\underline{\iota}_0, \overline{\kappa}_0, \overline{\iota}_0, \underline{\kappa}_0$ refer to the constants in (1.17). Before we proceed to a priori estimates we make some remarks concerning the pressure law $p(m,n)$ given by (1.6).

Remark 1. *Note that we have that*

$$
\begin{aligned}
p_m &= 1 - \frac{b}{\sqrt{b^2 + c}} > 0, \qquad m, n > 0\\
p_n &= a_0 + \frac{a_0}{\sqrt{b^2 + c}}\Big(m + a_0 n + k_0\Big) > 0, \qquad m, n > 0.
\end{aligned} \tag{2.5}
$$

That is, $p(m,n)$ is increasing in m and n for $m, n > 0$.

2.2. **Energy estimates.** We now consider the potential function $Q(m,n)$ given by

$$Q(m,n) = m\int_0^m \frac{p\Big(s, \frac{n}{m}s\Big)}{s^2}\,ds = \int_0^1 \frac{p(sm,sn)}{s^2}\,ds. \tag{2.6}$$

Note that $Q(m,n)\geq 0$ for $m,n\geq 0$. By direct calculation we see that

$$mQ_m(m,n) + nQ_n(m,n) = Q(m,n) + p(m,n). \tag{2.7}$$

Multiplying the first equation of (1.11) by Q_m, the second by Q_n and summing, we arrive at

$$Q(m,n)_t + uQ(m,n)_x = -u_x[mQ_m + nQ_n].$$

That is,

$$Q(m,n)_t + (uQ(m,n))_x - u_xQ(m,n) = -u_x[mQ_m + nQ_n],$$

or

$$Q(m,n)_t + (uQ(m,n))_x = -u_x[mQ_m + nQ_n - Q] = -u_xp(m,n), \tag{2.8}$$

in view of (2.7). Multiplying the third equation of (1.11) by u and doing integration by parts we get

$$m(\frac{1}{2}u^2)_t + (mu)(\frac{1}{2}u^2)_x + u^2[m_t + (mu)_x] + (pu)_x - u_xp - \mu(uu_x)_x + \mu(u_x)^2 = 0.$$

That is,

$$(\frac{1}{2}mu^2)_t + (\frac{1}{2}[mu]u^2)_x + (pu)_x - u_xp - \mu(uu_x)_x + \mu(u_x)^2 = 0$$

or

$$(\frac{1}{2}mu^2)_t + (\frac{1}{2}[mu]u^2 + pu - \mu uu_x)_x - u_xp + \mu(u_x)^2 = 0. \tag{2.9}$$

Summing (2.8) and (2.9) we get

$$(\frac{1}{2}mu^2 + Q(m,n))_t + (\frac{1}{2}[mu]u^2 + u[p+Q] - \mu uu_x)_x + \mu(u_x)^2 = 0. \tag{2.10}$$

In other words, defining $G(U)$ and $F(U)$ as

$$G(U) = \frac{1}{2}mu^2 + Q(m,n), \qquad F(U) = \frac{1}{2}[mu]u^2 + u[p+Q], \qquad U = (m,n,mu),$$

we have the equation

$$\partial_t G(U) + \partial_x[F(U) - \mu uu_x]_x + \mu(u_x)^2 = 0.$$

Integrating with respect to x and assuming that U vanishes at $x = \pm\infty$ we get

$$\frac{d}{dt}\int_{\mathbb{R}}\Big[\frac{1}{2}mu^2 + Q(m,n)\Big]\,dx + \mu\int_{\mathbb{R}}(u_x)^2\,dx = 0, \qquad (t,x)\in[0,T]\times\mathbb{R}. \tag{2.11}$$

In other words,

$$\int_{\mathbb{R}}\Big[\frac{1}{2}mu^2 + Q(m,n)\Big](x,T)\,dx + \mu\int_0^T\int_{\mathbb{R}}(u_x)^2\,dx = \int_{\mathbb{R}}\Big[\frac{1}{2}m_0(u_0)^2 + Q(m_0,n_0)\Big](x)\,dx. \tag{2.12}$$

Note that the above approach in general does not work when we let the fluid velocities become *unequal*, and new techniques seem to be required for that case. Based on the above energy estimate we may derive results for the two-phase model by imposing more restrictive initial data. However, in this work we seek to work within the general setting used by Hoff for single-phase Navier-Stokes which allows general Riemann data with possible different states at $x = \pm\infty$.

2.3. **More energy estimates.** Now, we focus on a general potential function in the form

$$Q(m,n|\overline{m},\overline{n}) := m\int_{\overline{m}}^{m}\Big[\frac{p\left(s,\frac{n}{m}s\right)-p\left(\overline{m},\overline{n}\right)}{s^2}\Big]\,ds + mg\Big(\frac{n}{m},\overline{m}\Big), \tag{2.13}$$

where $\overline{m}=\overline{m}(x)$ and $\overline{n}=\overline{n}(x)$ are the functions occuring in (1.16), whereas $g\in C^1$ is a function to be chosen later. Most importantly, we note that we have the relation

$$mQ_m + nQ_n = Q(m,n|\overline{m},\overline{n}) + \Delta p(m,n|\overline{m},\overline{n}), \tag{2.14}$$

where

$$\Delta p(m,n|\overline{m},\overline{n}) = p(m,n) - p(\overline{m},\overline{n}). \tag{2.15}$$

Unfortunately, we do not know yet that $Q(m,n|\overline{m},\overline{n})$ defined by (2.13) is a non-negative function. To ensure this we shall consider a slight modification of $Q(m,n|\overline{m},\overline{n})$. First, in light of (2.4), we see that for the chosen smooth functions $\overline{m}$ and $\overline{n}$ such that $\underline{\kappa}_0\le\overline{m}\le\overline{\kappa}_0$ and $\underline{\iota}_0\le\overline{n}\le\overline{\iota}_0$, it follows that $\overline{n}=\overline{m}t_0(x)$ for $t_0(x)\in[\underline{s}_0,\overline{s}_0]$. Motivated by this we suggest to consider $Q(m,n|\overline{m},\xi\overline{m})$ for a constant ξ contained in $[\underline{s}_0,\overline{s}_0]$. Then we can obtain the following energy estimate.

Lemma 1. *Let (m,n,u) be a smooth solution of* (1.11) *as described by Proposition 1. If we assume that we can find a function $g(\cdot,\cdot)$ in* (2.13) *and a constant $\xi\in[\underline{s}_0,\overline{s}_0]$ such that*

$$Q(m,n|\overline{m},\overline{m}\xi)\ge 0, \tag{2.16}$$

and

$$m + p(m,n) \le C[1+Q(m,n|\overline{m},\overline{m}\xi)], \tag{2.17}$$

it follows that we have the energy estimate

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\left[\frac{1}{2}m(u-\overline{u})^2 + Q(m,n|\overline{m},\overline{m}\xi)\right]dx + \int_0^T\int_{\mathbb{R}}(u_x)^2\,dxdt \le C(T). \tag{2.18}$$

Proof. Let $A(t)$ denote the integral $A(t)=\int_{\mathbb{R}}\left[\frac{1}{2}m(u-\overline{u})^2+Q(m,n|\overline{m},\overline{n})\right]dx$, where $\overline{n}$ now is given by $\overline{n}=\overline{m}\xi$. Then we get

$$\begin{aligned}
A'(t) &= \int_{\mathbb{R}}\Big(\frac{1}{2}(u-\overline{u})^2 m_t + m(u-\overline{u})u_t + Q_m(m,n|\cdot)m_t + Q_n(m,n|\cdot)n_t\Big)\,dx\\
&= \int_{\mathbb{R}}\Big(\frac{1}{2}(\Delta u)^2(-mu)_x + (\Delta u)(mu_t) + Q_m(m,n|\cdot)(-mu)_x + Q_n(m,n|\cdot)(-nu)_x\Big)\,dx\\
&= \int_{\mathbb{R}}\Big(-\frac{1}{2}(\Delta u)^2(mu)_x + (\Delta u)\Big[-(mu)u_x - (p(m,n)-u_x)_x\Big]\\
&\qquad\qquad + Q_m(m,n|\overline{m},\overline{n})(-mu)_x + Q_n(m,n|\overline{m},\overline{n})(-nu)_x\Big)\,dx.
\end{aligned} \tag{2.19}$$

Next, we observe in light of (2.14) that we have

$$\begin{aligned}
-\Big(Q_m(mu)_x + Q_n(nu)_x\Big) &= -\Big(mQ_m u_x + nQ_n u_x + uQ_m m_x + uQ_n n_x\Big)\\
&= -\Big(u_x[Q+\Delta p] + uQ_x - uQ_{\overline{m}}\overline{m}_x\Big) = -\Big(u_x\Delta p + (uQ)_x - uQ_{\overline{m}}\overline{m}_x\Big).
\end{aligned} \tag{2.20}$$

Using this in (2.19) we get

$$\begin{aligned}
A'(t) = \int_{\mathbb{R}}\Big(&-\frac{1}{2}(\Delta u)^2(mu)_x + (\Delta u)\Big[-(mu)u_x - \Big(p(m,n)-u_x\Big)_x\Big]\\
&- u_x\Delta p - (uQ)_x + uQ_{\overline{m}}\overline{m}_x\Big)\,dx.
\end{aligned} \tag{2.21}$$

Since $Q(m,n|\overline{m},\overline{n})|_{x=\pm\infty} = \overline{m}g(\overline{n}/\overline{m},\overline{m})|_{x=\pm\infty}$ we see that $(uQ)_x$ produces a term that can be absorbed in the constant C appearing on the right side of (2.26). Consequently, we get

$$\begin{aligned} &A'(t) + \int_{\mathbb{R}} [(\Delta u)_x]^2\,dx \\ &= \int_{\mathbb{R}} \Big(-\frac{1}{2}(\Delta u)^2(mu)_x - (\Delta u)(mu)u_x + ([\Delta u]_x)^2 + (\Delta u)u_{xx} \\ &\qquad - (\Delta u)p(m,n)_x - u_x\Delta p + uQ_{\overline{m}}\overline{m}_x \Big)\,dx + C. \end{aligned} \tag{2.22}$$

Now, for the first two terms of second line we have (using that $\Delta u = 0$ at $x = \pm\infty$):

$$\begin{aligned} -\frac{1}{2}(\Delta u)^2(mu)_x - (\Delta u)(mu)u_x &= \frac{1}{2}[(\Delta u)^2]_x(mu) - (\Delta u)u_x(mu) \\ &= -m\overline{u}_x(\Delta u)u = -m\overline{u}_x(\Delta u)^2 - m\overline{u}\,\overline{u}_x(\Delta u). \end{aligned}$$

For the third and fourth term of second line we have

$$(\Delta u)(u_x)_x + ([\Delta u]_x)^2 = -(\Delta u)_x u_x + u_x(\Delta u)_x - \overline{u}_x(\Delta u)_x = -\overline{u}_x(\Delta u)_x.$$

We also have

$$-(\Delta u)p_x - u_x\Delta p = p(\Delta u)_x - u_x\Delta p = -p\overline{u}_x + \overline{p}u_x = -\overline{u}_x\Delta p - (\Delta u)\overline{p}_x,$$

where $\overline{p} = p(\overline{m},\overline{m}\xi)$ and $\Delta u = u - \overline{u}$. Consequently, we get

$$\begin{aligned} &A'(t) + \int_{\mathbb{R}} [(\Delta u)_x]^2\,dx \\ &= \int_{\mathbb{R}} \Big(-m\overline{u}_x(\Delta u)^2 - m\overline{u}\,\overline{u}_x(\Delta u) - \overline{u}_x(\Delta u)_x - \overline{u}_x\Delta p - (\Delta u)\overline{p}_x + uQ_{\overline{m}}\overline{m}_x \Big)\,dx + C. \end{aligned} \tag{2.23}$$

Next, we note that

$$Q_{\overline{m}}\overline{m}_x = -m\frac{[p(\overline{m},\frac{n}{m}\overline{m}) - p(\overline{m},\overline{m}\xi)]}{\overline{m}^2}\overline{m}_x + \frac{d\overline{p}}{d\overline{m}}\overline{m}_x - \frac{m}{\overline{m}}\frac{d\overline{p}}{d\overline{m}}\overline{m}_x + mg_{\overline{m}}(\frac{n}{m},\overline{m})\overline{m}_x.$$

Thus,

$$\begin{aligned} &-(\Delta u)\overline{p}_x + uQ_{\overline{m}}\overline{m}_x \\ &\quad= -(\Delta u)\overline{p}_x + u\overline{p}_x - u\overline{p}_x\frac{m}{\overline{m}} - mu\frac{[p(\overline{m},\frac{n}{m}\overline{m}) - p(\overline{m},\overline{m}\xi)]}{\overline{m}^2}\overline{m}_x + mug_{\overline{m}}(\frac{n}{m},\overline{m})\overline{m}_x \\ &\quad= -\frac{\overline{p}_x}{\overline{m}}(m\Delta u + \overline{u}\Delta m) - muC_1(\overline{m},\xi,\frac{n}{m})\overline{m}_x + muC_2(\overline{m},\frac{n}{m})\overline{m}_x, \end{aligned}$$

where $\Delta m = m - \overline{m}$ and

$$C_1(\overline{m},\xi,\frac{n}{m}) = \frac{p(\overline{m},\frac{n}{m}\overline{m}) - p(\overline{m},\overline{m}\xi)}{\overline{m}^2}, \qquad C_2(\overline{m},\frac{n}{m}) = g_{\overline{m}}(\frac{n}{m},\overline{m}).$$

To conclude, we have

$$\begin{aligned} A'(t) + \int_{\mathbb{R}} [(\Delta u)_x]^2\,dx &= \int_{\mathbb{R}} \Big(-m\overline{u}_x(\Delta u)^2 - m\overline{u}\,\overline{u}_x(\Delta u) - \overline{u}_x(\Delta u)_x - \overline{u}_x\Delta p \\ &\quad - \frac{\overline{p}_x}{\overline{m}}(m\Delta u + \overline{u}\Delta m) - mu\overline{m}_xC_1(\overline{m},\xi,\frac{n}{m}) + mu\overline{m}_xC_2(\overline{m},\frac{n}{m}) \Big)\,dx + C. \end{aligned} \tag{2.24}$$

Smoothness properties of $\overline{m}$ and the fact that $\frac{d\overline{p}}{d\overline{m}}$ is bounded are then used to bound terms on the right hand side of (2.24). Particularly, to bound C_1 we use that

$$|p(\overline{m},\frac{n}{m}\overline{m}) - p(\overline{m},\overline{m}\xi)| \le \max_{v\in[\underline{\iota}_0,\overline{\iota}_0]} |p_v(\overline{m},v)|\Big|\frac{n}{m}\overline{m} - \overline{m}\xi\Big| \le C|s_0 - \overline{s}_0| \le C(\overline{s}_0 - \underline{s}_0),$$

since $n = s_0 m$ and $\overline{n} = \xi\overline{m}$ for $s_0, \xi \in [\underline{s}_0, \overline{s}_0]$. Similarly, we can bound C_2. In the following we make use of (2.16) and (2.17), in particular, we rely on the the fact that $Q(m,n|\overline{m},\xi\overline{m})$ is a non-negative function and consequently also $A(t)$ given by

$$A(t) = \int_{\mathbb{R}} \left[\frac{1}{2}m(u-\overline{u})^2 + Q(m,n|\overline{m},\overline{m}\xi) \right]\,dx. \tag{2.25}$$

We then estimate the various terms on the right hand side of (2.24) by $A(t)$ or $C(T)$ as follows:

- $\int_{\mathbb{R}} m\overline{u}_x(\Delta u)^2\,dx \le C\int_{-1}^{+1}[m^{1/2}\Delta u][m^{1/2}\Delta u]\,dx \le C\int_{-1}^{+1} m(\Delta u)^2\,dx \le 2CA(t)$, where we use that $\overline{u}_x = 0$ outside $[-1,+1]$.
- Similarly, by applying (2.17) we get

$$\begin{aligned}\int_{\mathbb{R}} m\overline{u}\,\overline{u}_x(\Delta u)\,dx &\le C\int_{-1}^{+1} m^{1/2}[m^{1/2}\Delta u]\,dx \\ &\le C\Big(\int_{-1}^{+1} m\,dx\Big)^{1/2}\Big(\int_{-1}^{+1} m(\Delta u)^2\,dx\Big)^{1/2} \\ &\le C\Big(\int_{-1}^{+1}[1+Q(m,n|\overline{m},\overline{m}\xi)]\,dx\Big)^{1/2}\Big(\int_{-1}^{+1} m(\Delta u)^2\,dx\Big)^{1/2} \\ &\le C[1+A(t)]^{1/2}A(t)^{1/2} \le C[1+A(t)].\end{aligned}$$

- $\int_{\mathbb{R}} \overline{u}_x(\Delta u)_x\,dx \le \delta\int_{\mathbb{R}}(\Delta u)_x^2\,dx + \frac{1}{4}C\delta^{-1}\int_{-1}^{+1} 1\,dx, \quad \delta > 0$. Thus, the first term on the right hand side can be absorbed on the left hand side of (2.24);

- Again, by referring to (2.17) we get

$$\int_{\mathbb{R}} \overline{u}_x\Delta p\,dx \le C(1+\int_{-1}^{+1} p(m,n)\,dx) \le C(1+\int_{-1}^{+1}[1+Q(m,n|\overline{m},\overline{m}\xi)]\,dx) \le C[1+A(t)].$$

In a complete analogous manner we can estimate the terms $\int \frac{\overline{p}_x}{\overline{m}}(m\Delta u+\overline{u}\Delta m)\,dx$ and $\int mu\overline{m}_x\,dx$. Consequently, we see that we end up with

$$A'(t) + \int_{\mathbb{R}}[(\Delta u)_x]^2\,dx \le C + CA(t), \tag{2.26}$$

for a suitable choice of C. Application of Gronwall's lemma then gives the desired result. □

Equipped with the fundamental energy estimate (2.18) we can extract estimates of $m(u-\overline{u})^2$, $Q(m,n|\overline{m},\overline{m}\xi)$ and $(u_x)^2$. What remains is to show that a proper choice of g and ξ in fact can be made such that (2.16) and (2.17) hold.

2.4. **Properties of the potential** $Q(m,n|\overline{m},\overline{m}\xi)$**.** In this section we shall carefully investigate properties of the potential function $Q(m,n|\overline{m},\overline{m}\xi)$ defined in (2.13). The goal is to conclude that the quantity $m+p(m,n)$ can be controlled by a non-negative potential $Q(m,n|\overline{m},\overline{m}\xi)$ as described by (2.17).

In view of (2.3), we see that m and n are related as

$$n = ms_0(t,x), \qquad s_0(t,x) \in [\underline{s}_0, \overline{s}_0].$$

Now, we consider the function $G(m,s_0;\overline{m},\xi)$ defined by

$$G(m,s_0;\overline{m},\xi) := Q(m,s_0m|\overline{m},\xi\overline{m}), \tag{2.27}$$

for $m > 0$ and $s_0,\xi \in [\underline{s}_0,\overline{s}_0]$. Consequently,

$$G(m,s_0;\overline{m},\xi) = m\int_{\overline{m}}^{m}\frac{p(s,s_0s)-p(\overline{m},\xi\overline{m})}{s^2}\,ds + mg(s_0,\overline{m}), \quad m>0, \quad s_0,\xi \in [\underline{s}_0,\overline{s}_0].$$

We may write this function in the form

$$G(m,s_0;\overline{m},\xi) = mQ(m,\overline{m},s_0) + p(\overline{m},\xi\overline{m})\Big[1-\frac{m}{\overline{m}}\Big] + mg(s_0,\overline{m}), \tag{2.28}$$

where

$$Q(m,\overline{m},s_0) = \int_{\overline{m}}^{m}\frac{p(s,s_0s)}{s^2}\,ds.$$

First, we note that

$$G(\overline{m},s_0;\overline{m},\xi) = \overline{m}g(s_0,\overline{m}), \tag{2.29}$$

which we want to be non-negative by an appropriate choice of g to be determined below. Moreover,

$$G_m(m, s_0; \overline{m}, \xi) = \frac{1}{m}\Big(G(m, s_0; \overline{m}, \xi) + [p(m, s_0 m) - p(\overline{m}, \xi\overline{m})]\Big). \tag{2.30}$$

Consequently,

$$G_m(\overline{m}, s_0; \overline{m}, \xi) = g(s_0, \overline{m}) + \frac{1}{\overline{m}}[p(\overline{m}, s_0\overline{m}) - p(\overline{m}, \xi\overline{m})], \qquad s_0, \xi \in [\underline{s}_0, \overline{s}_0],$$

which can take both signs, i.e., $G(m, s_0; \overline{m}, \xi)$ can be both increasing and decreasing at $\overline{m}$ depending on the choice of g, s_0, and ξ. Furthermore,

$$G_{mm}(m, s_0; \overline{m}, \xi) = \frac{1}{m} p_m(m, s_0 m) > 0, \qquad m > 0, \qquad s_0, \xi \in [\underline{s}_0, \overline{s}_0]. \tag{2.31}$$

Consequently, $G(m, s_0; \overline{m}, \xi)$ is a convex function in m for $m > 0$ and $s_0, \xi \in [\underline{s}_0, \overline{s}_0]$.

Let us focus on the situation when $\xi = \underline{s}_0$. That is, we consider $G(m, s_0; \overline{m}, \underline{s}_0)$. For this case we have

$$G_m(\overline{m}, s_0; \overline{m}, \underline{s}_0) = g(s_0, \overline{m}) + \frac{1}{\overline{m}}[p(\overline{m}, s_0\overline{m}) - p(\overline{m}, \underline{s}_0\overline{m})] = 0, \qquad s_0 \in [\underline{s}_0, \overline{s}_0],$$

if we choose that $g(s_0, \overline{m})$ is given by

$$g(s_0, \overline{m}) = -\frac{1}{\overline{m}}[p(\overline{m}, s_0\overline{m}) - p(\overline{m}, \underline{s}_0\overline{m})], \qquad s_0 \in [\underline{s}_0, \overline{s}_0].$$

Thus, $\overline{m}$ is an absolute minimum for $G(m, s_0; \overline{m}, \underline{s}_0)$ for all $s_0 \in [\underline{s}_0, \overline{s}_0]$. However, for this choice, unfortunately, we have that

$$g(s_0, \overline{m}) \le 0,$$

and (2.29) is not ensured to be non-negative, since $p(m, n)$ is increasing as a function of n, see Remark 1. Obviously, we should instead consider the choice $\xi = \overline{s}_0$, that is,

$$G(m, s_0; \overline{m}, \overline{s}_0),$$

together with the choice

$$g(s_0, \overline{m}) = -\frac{1}{\overline{m}}[p(\overline{m}, s_0\overline{m}) - p(\overline{m}, \overline{s}_0\overline{m})], \qquad s_0 \in [\underline{s}_0, \overline{s}_0], \tag{2.32}$$

ensuring that $g(s_0, \overline{m}) \ge 0$ for all $s_0 \in [\underline{s}_0, \overline{s}_0]$. Thus, we may conclude that $G(m, s_0; \overline{m}, \overline{s}_0)$ is convex in m and possesses an absolute minimum at $\overline{m}$ and $G(\overline{m}, s_0; \overline{m}, \overline{s}_0) \ge 0$ for all $s_0 \in [\underline{s}_0, \overline{s}_0]$.

We also note that (2.28), by the choice (2.32), takes the form

$$G(m, s_0; \overline{m}, \overline{s}_0) = mQ(m, \overline{m}, s_0) + p(\overline{m}, \overline{s}_0\overline{m}) - \frac{m}{\overline{m}} p(\overline{m}, s_0\overline{m}). \tag{2.33}$$

From this it follows that

$$\liminf_{m\to 0} G(m, s_0; \overline{m}, \overline{s}_0) = p(\overline{m}, \overline{s}_0\overline{m}) \ge p(\underline{\kappa}_0, \overline{s}_0\underline{\kappa}_0) > 0, \tag{2.34}$$

since $Q(m, \overline{m}, s_0) = \int_{\overline{m}}^{m} p(s, s_0 s)/s^2\, ds$ is finite as m approaches zero.

Proposition 2. *For each interval $[\underline{s}_0, \overline{s}_0]$ and for each given function $\overline{m} \in [\underline{\kappa}_0, \overline{\kappa}_0]$, the following holds for $G(m, s_0; \overline{m}, \overline{s}_0)$ given by (2.33):*

$$G(m, s_0; \overline{m}, \overline{s}_0) \ge 0, \qquad m > 0, \qquad s_0 \in [\underline{s}_0, \overline{s}_0], \tag{2.35}$$

$$m + p(m, s_0 m) \le C[1 + G(m, s_0; \overline{m}, \overline{s}_0)], \qquad m > 0, \qquad s_0 \in [\underline{s}_0, \overline{s}_0], \tag{2.36}$$

$$\liminf_{m\to 0} G(m, s_0; \overline{m}, \overline{s}_0) \ge D, \qquad s_0 \in [\underline{s}_0, \overline{s}_0], \tag{2.37}$$

for appropriate choices of $C = C(\underline{s}_0, \overline{s}_0, \overline{m})$ and $D = D(\overline{s}_0, \underline{\kappa}_0)$ which also depend on the pressure law $p(m, n)$.

Proof. Claim (2.35) follows directly from the above discussion whereas (2.37) follows from (2.34). What remains is to verify that (2.36) holds.

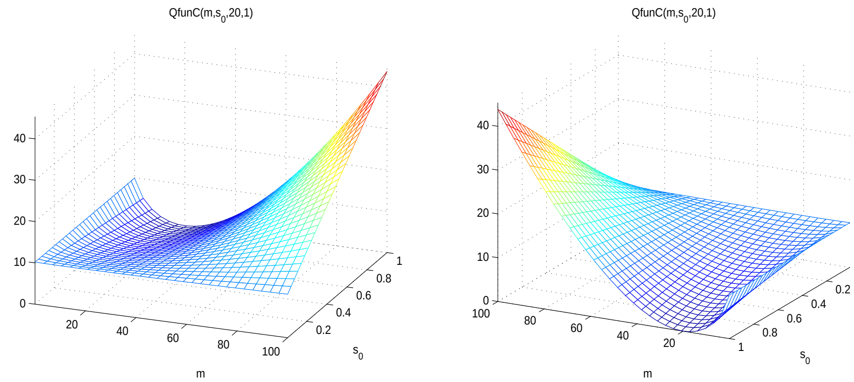

FIGURE 1. Concrete example of the function $G(m, s_0; \overline{m}, \overline{s}_0)$ for $\overline{m} = 20$ and $\underline{s}_0 = 0.01$, $\overline{s}_0 = 1$ considered from two different positions.

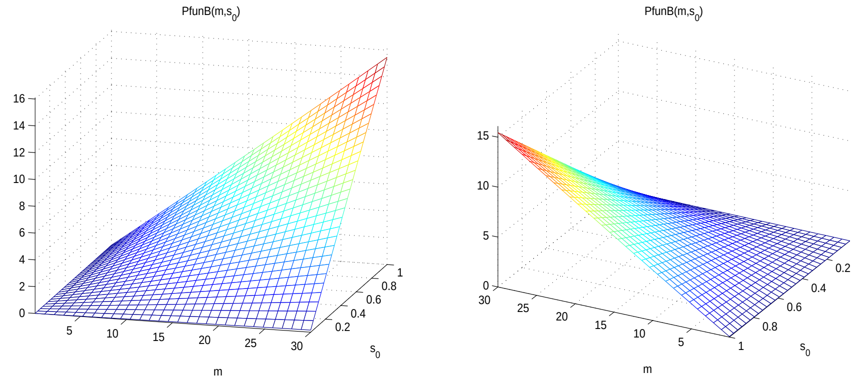

FIGURE 2. Concrete example of the function $p(m, s_0 m)$ for $m > 0$ and $s_0 \in [\underline{s}_0, \overline{s}_0]$ with $\underline{s}_0 = 0.01$ and $\overline{s}_0 = 1$ considered from two different positions.

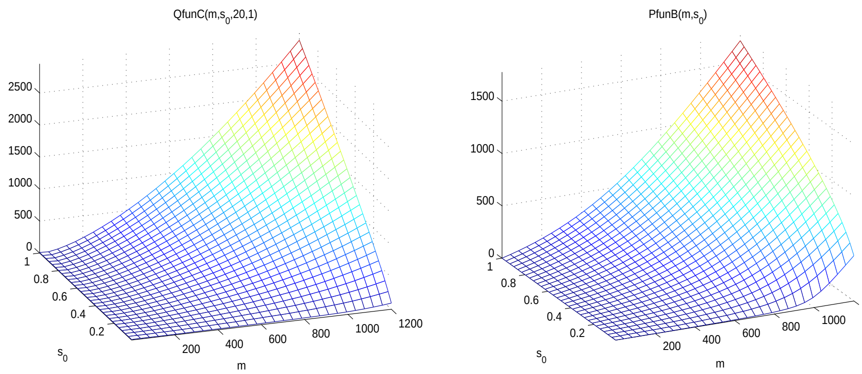

FIGURE 3. **Left:** Concrete example of the function $G(m, s_0; \overline{m}, \overline{s}_0)$ for $\overline{m} = 20$ and $\underline{s}_0 = 0.01$, $\overline{s}_0 = 1$. **Right:** Corresponding plot of $p(m, s_0 m)$.

The case $0 < m \leq \overline{m}$. First, we observe that

$$m \leq \overline{m} = C_1(\overline{m}), \qquad \text{for } 0 < m \leq \overline{m} \tag{2.38}$$

and

$$p(m, s_0 m) \leq \max_{s_0 \in [\underline{s}_0, \overline{s}_0]} p(\overline{m}, s_0 \overline{m}) = C_1(\overline{m}), \qquad \text{for } 0 < m \leq \overline{m}. \tag{2.39}$$

Consequently, (2.36) holds for $0 < m \le \overline{m}$.

The case $m > \overline{m}$. First, we note that (see Figs. 1–3 for a visualization of $G(m, s_0; \overline{m}, \overline{s}_0)$ and $p(m, s_0 m)$)

$$J(m) \stackrel{\text{def}}{:=} G(m, \underline{s}_0; \overline{m}, \overline{s}_0) \le G(\overline{m}, \underline{s}_0; \overline{m}, \overline{s}_0) + G(m, s_0; \overline{m}, \overline{s}_0), \tag{2.40}$$

for all $(s_0, m) \in [\underline{s}_0, \overline{s}_0] \times [\overline{m}, \infty)$. Now we intend to try to bound m and $p(m, s_0 m)$ by the function $J(m) = G(m, \underline{s}_0; \overline{m}, \overline{s}_0)$ in the region $[\underline{s}_0, \overline{s}_0] \times [\overline{m}, \infty)$.

Clearly, in view of (2.29)–(2.31) and (2.32), it follows that $J(\overline{m}) > 0$, $J'(\overline{m}) = 0$ and $J''(m) > 0$ for $m \ge \overline{m}$. Consequently,

$$m \le C_2 J(m), \qquad \text{for } m \ge \overline{m}, \tag{2.41}$$

for an appropriate choice of $C_2 = C_2(\overline{m}, \underline{s}_0, \overline{s}_0)$.

Next, we focus on $p(m, s_0 m)$. First, we observe that

$$p(m, s_0 m) \le p(m, \overline{s}_0 m) := I(m). \tag{2.42}$$

We want to demonstrate that

$$I(m) \le C_1 + C_2 J(m), \tag{2.43}$$

for appropriate choices of C_1 and C_2. More precisely, we have that $I(m)$ is given by

$$\begin{aligned} I(m) &= p(m, \overline{s}_0 m) \\ &= -k_0 + m(1 + a_0 \overline{s}_0) + \sqrt{k_0^2 - 2k_0(1 + a_0\overline{s}_0)m + (1 + a_0\overline{s}_0)^2 m^2 + 4k_0 a_0 \overline{s}_0 m} \\ &:= I_1(m) + I_2(m), \end{aligned}$$

where $I_1(m)$ is a linear function in m and $I_2(m)$ is the square root of a second order polynomial in m. Clearly, both can be bounded by the function $C_1 + C_2 J(m)$ where $C_1 = C_1(\underline{s}_0, \overline{s}_0, \overline{m})$ and $C_2 = C_2(\underline{s}_0, \overline{s}_0, \overline{m})$, thus, (2.43) follows.

To conclude, in view of (2.40) and (2.41) and (2.42) and (2.43), we see that (2.36) also holds for $m > \overline{m}$. □

Now, we are in a position to conclude that the following holds:

Corollary 1. *The potential function $Q(m, n|\overline{m}, \overline{m}\xi)$ defined by* (2.13) *satisfies the properties* (2.16) *and* (2.17) *by using the choice*

$$\xi = \overline{s}_0, \qquad g\Big(\frac{n}{m}, \overline{m}\Big) = -\frac{1}{\overline{m}}\Big[p(\overline{m}, \frac{n}{m}\overline{m}) - p(\overline{m}, \overline{s}_0 \overline{m})\Big].$$

Proof. We just have to observe, in view of (2.3) and (2.27), that

$$Q(m, n|\overline{m}, \overline{m}\overline{s}_0) = Q(m, m s_0(t,x)|\overline{m}, \overline{m}\overline{s}_0) = G(m, s_0(t,x); \overline{m}, \overline{s}_0),$$

and the result follows from (2.35) and (2.36). □

To sum up, we see that Lemma 1 and Corollary 1 imply that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}} G(m, s_0(t,x); \overline{m}, \overline{s}_0)\, dx \le C(T), \tag{2.44}$$

$$\int_0^T \int_{\mathbb{R}} (u_x)^2 \, dxdt \le C(T), \tag{2.45}$$

$$\sup_{0\le t\le T} \int_{\mathbb{R}} \left[\frac{1}{2} m (u - \overline{u})^2\right] dx \le C(T). \tag{2.46}$$

2.5. **Pointwise estimates.** The next task is to explore how we can obtain pointwise bounds on $m(t,x)$ and $n(t,x)$. Equipped with the results of Lemma 2 we closely follow the proof presented in [23] and then observe how the more complicated pressure term $p(m,n)$ naturally can be handled in this approach by making use of the fact that the masses m and n are related to each other by one and the same momentum equation.

Lemma 2. *Let (n,m,u) be as in Lemma 1. Then there is a constant $C(T)$ such that*

$$C(T)^{-1} \le m(t,x) \le C(T), \qquad (t,x) \in [0,T] \times \mathbb{R}. \tag{2.47}$$

Moreover,

$$\underline{s}_0 C(T)^{-1} \le n(t,x) \le C(T)\overline{s}_0, \qquad (t,x) \in [0,T] \times \mathbb{R}. \tag{2.48}$$

Proof. We consider an arbitrary point (t_0,x_0) and our aim is to obtain the bounds of (2.47) at this point. First, we use Hoff's observation that there must be a nearby point (t_0,x_1) and a constant $C(T)$ such that

$$C(T)^{-1} \le m(t_0,x_1) \le C(T) \tag{2.49}$$

$$|x_1 - x_0| \le C(T). \tag{2.50}$$

This follows by making use of (2.37) and (2.44), we refer to the proof of Lemma 4.1 in [33] for more details. The purpose now is to show that this also holds for $m(t_0,x_0)$. As a step toward that aim, we shall estimate the differences

$$\Delta M(t_0) = \log m(t_0,x_1) - \log m(t_0,x_0), \qquad \Delta N(t_0) = \log n(t_0,x_1) - \log n(t_0,x_0)$$

by considering its evolution along the particle trajectories. Thus, we define $X_j(t)$ by

$$\begin{aligned} &\dot{X}_j = u(t,X_j) \\ &X_j(t_0,x_j) = x_j, \quad j = 0,1, \qquad t \in [0,T]. \end{aligned} \tag{2.51}$$

Clearly, from the two mass equations it follows that

$$n(t,X_j(t)) = m(t,X_j(t))\frac{n(t_0,x_j)}{m(t_0,x_j)}.$$

Generally, we have from (2.3)

$$\frac{n(t,x)}{m(t,x)} = \frac{n_0}{m_0}(X_t^{-1}(x)) = s_0(t,x),$$

for the characteristic $X_t(x)$ starting out from x at time $t = 0$. Consequently, we have

$$n(t,X_j(t)) = m(t,X_j(t))\frac{n(t_0,x_j)}{m(t_0,x_j)} = m(t,X_j(t))s_0(t_0,x_j), \qquad s_0 \in [\underline{s}_0, \overline{s}_0]. \tag{2.52}$$

Furthermore, let

$$\Delta M(t) = \log m(t,X_1(t)) - \log m(t,X_0(t)), \qquad \Delta N(t) = \log n(t,X_1(t)) - \log n(t,X_0(t)). \tag{2.53}$$

We then obtain from (1.11) that

$$\begin{aligned} \frac{d}{dt}\Delta M &= \frac{m_x\dot{X}_1 + m_t}{m}(t,X_1(t)) - \frac{m_x\dot{X}_0 + m_t}{m}(t,X_0(t)) \\ &= -u_x|_{X_0}^{X_1} = -\int_{X_0}^{X_1} u_{xx}(t,x)\,dx = -\int_{X_0}^{X_1} (m\dot{u} + p(m,n)_x)\,dx \\ &= -\frac{dI}{dt} - \Delta p, \end{aligned} \tag{2.54}$$

where

$$I(t) = \int_{X_0}^{X_1} mu\,dx \qquad \text{and} \qquad \Delta p = p(m(t,\cdot),n(t,\cdot))\Big|_{X_0}^{X_1}.$$

In particular, we have used that

$$\frac{d}{dt}I(X_0(t),X_1(t),t) = mu^2(X_1(t),t) - mu^2(X_0(t),t) + \int_{X_0}^{X_1}\frac{\partial}{\partial t}(mu(\cdot,t))\,dx$$

$$= \int_{X_0}^{X_1} \left(\partial_t[mu] + \partial_x[mu^2]\right)\,dx = \int_{X_0}^{X_1} (m\dot{u})\,dx.$$

For the pressure term Δp we decompose into two components corresponding to the two variables m and n as follows:

$$\begin{aligned}\Delta p(t) &= p\Big(m(X_1), n(X_1)\Big) - p\Big(m(X_0), n(X_0)\Big)\\ &= p\Big(m(X_1), n(X_1)\Big) - p\Big(m(X_0), n(X_1)\Big)\\ &\qquad\qquad + p\Big(m(X_0), n(X_1)\Big) - p\Big(m(X_0), n(X_0)\Big)\\ &\overset{\text{def}}{:=} (\Delta p)_1(t) + (\Delta p)_2(t).\end{aligned}$$

For the $(\Delta p)_1(t)$ and $(\Delta p)_2(t)$ terms we extend the idea of [23] and introduce

$$\alpha(t) = (\Delta p)_1(t)/\Delta M(t), \qquad \beta(t) = (\Delta p)_2(t)/\Delta N(t),$$

and observe, in view of (2.53), that $\alpha(t)$ and $\beta(t)$ are both positive since $p(m,n)$ is increasing both in m and n and $\log(\cdot)$ is an increasing function. Using this in (2.54) we obtain the following linear ode

$$\frac{d}{dt}\Delta M + \alpha(t)\Delta M + \beta(t)\Delta N = -\frac{d}{dt}I. \tag{2.55}$$

The calculations (2.54) can be performed for $\Delta N(t)$, as well, since fluid velocity is the same for both phases and they share the same momentum equation. Thus, we obtain a corresponding equation for the $\Delta N(t)$ variable.

$$\frac{d}{dt}\Delta N + \alpha(t)\Delta M + \beta(t)\Delta N = -\frac{d}{dt}I. \tag{2.56}$$

Immediately we can conclude that

$$\frac{d}{dt}\Delta M = \frac{d}{dt}\Delta N,$$

that is,

$$\Delta M(t) = \Delta N(t) + C_0,$$

where C_0 is (invoking (2.52) and (2.53))

$$\log(\underline{s}_0/\overline{s}_0) \le C_0 = \log(s_0(t_0,x_0)/s_0(t_0,x_1)) \le \log(\overline{s}_0/\underline{s}_0).$$

Inserting this in (2.55) we get

$$\frac{d}{dt}\Delta M + [\alpha(t)+\beta(t)]\Delta M = -\frac{d}{dt}I + C_0\beta(t). \tag{2.57}$$

The solution is of the form

$$\Delta M(t)\mu(t) = \Delta M(0)\mu(0) + \int_0^t \mu(s)\Big[-\frac{d}{ds}I(s) + C_0\beta(s)\Big]\,ds, \quad \mu(t) = \exp\Big(\int_0^t \gamma(s)\,ds\Big),$$

where $\gamma(s) = \alpha(s) + \beta(s)$. Noting that $\mu(0) = 1$ we have

$$\begin{aligned}\Delta M(t) &= \frac{\Delta M(0)}{\mu(t)} + \int_0^t e^{\left(-\int_s^t \gamma(\tau)\,d\tau\right)}\left[-\frac{d}{ds}I(s) + C_0\beta(s)\right]ds\\ &= \frac{\Delta M(0)}{\mu(t)} - \left[e^{\left(-\int_s^t \gamma(\tau)\,d\tau\right)} I(s)\Big|_0^t - \int_0^t I(s)e^{\left(-\int_s^t \gamma(\tau)\,d\tau\right)}\gamma(s)\,ds\right]\\ &\qquad\qquad + \int_0^t e^{\left(-\int_s^t \gamma(\tau)\,d\tau\right)} C_0\beta(s)\,ds\\ &= \frac{\Delta M(0)}{\mu(t)} - \left[I(t) - I(0)e^{\left(-\int_0^t \gamma(\tau)\,d\tau\right)}\right] + \int_0^t \gamma(s)I(s)e^{\left(-\int_s^t \gamma(\tau)\,d\tau\right)}\,ds\\ &\qquad\qquad + \int_0^t e^{\left(-\int_s^t \gamma(\tau)\,d\tau\right)} C_0\beta(s)\,ds.\end{aligned}$$

Noting that $1/\mu(t) = \exp\left(-\int_0^t \gamma(\tau)\,d\tau\right) \le 1$, we have

$$
\begin{aligned}
|\Delta M(t)| &\le |\Delta M(0)| + |I(0)| + |I(t)| \\
&\qquad + \int_0^t e^{-\int_s^t \gamma(\tau)\,d\tau}\Big[\gamma(s)|I(s)| + |C_0|\beta(s)\Big]\,ds. \\
&\le |\Delta M(0)| + |I(0)| + |I(t)| \\
&\qquad + \int_0^t e^{-\int_s^t \gamma(\tau)\,d\tau}\gamma(s)\Big[|I(s)| + |C_0|\Big]\,ds,
\end{aligned}
\tag{2.58}
$$

since $\beta(s) \le \gamma(s)$. The term $I(t)$ can be bounded by observing that

$$
|I(t)| = \Big|\int_{X_0}^{X_1} mu\,dx\Big| \le \left(\int_{X_0}^{X_1} m\,dx\right)^{1/2}\left(\int_{X_0}^{X_1} mu^2\,dx\right)^{1/2}.
$$

Clearly, (2.36) implies that

$$
m \le C\left[1 + G(m, s_0, \overline{m}, \overline{s}_0)\right], \qquad \forall s_0 \in [\underline{s}_0, \overline{s}_0].
$$

So, provided that $|X_1(t) - X_0(t)| \le C(T)$, it follows from (2.44) that $\int_{X_0}^{X_1} m\,dx \le C(T)$. Similarly, in view of estimate (2.46), we conclude that $\int_{X_0}^{X_1} mu^2\,dx \le C(T)$. Thus, it follows that $|I(t)| \le C(T)$. The estimate $|X_1(t) - X_0(t)| \le C(T)$ can be obtained as in [23]. The main observation is that

$$
\frac{d}{dt}(X_1 - X_0) = u\Big|_{X_0}^{X_1} = \int_{X_0}^{X_1} u_x\,dx \ge -(X_1 - X_0) - \int_{X_0}^{X_1} u_x^2\,dx,
$$

where we have assumed without loss of generality that $X_1 - X_0 > 0$ and used that

$$
\begin{aligned}
\int_{X_0}^{X_1} -u_x\,dx &= \int_{X_0}^{X_1} \chi_{\{-u_x \le 1\}}(x)(-u_x)\,dx + \int_{X_0}^{X_1} \chi_{\{-u_x > 1\}}(x)(-u_x)\,dx \\
&\le (X_1 - X_0) + \int_{X_0}^{X_1} \chi_{\{-u_x > 1\}}(x)(-u_x)\,dx \le (X_1 - X_0) + \int_{X_0}^{X_1} u_x^2\,dx.
\end{aligned}
$$

Application of estimate (2.50) and the control on u_x via (2.45) then gives the desirable estimate. Moreover,

$$
\int_0^t e^{-\int_s^t \gamma(\tau)\,d\tau}\gamma(s)|I(s)|\,ds \le C(T)\int_0^t e^{-\int_s^t \gamma(\tau)\,d\tau}\gamma(s)\,ds,
$$

which clearly must be bounded by a constant $C(T)$ since $\gamma(t) \ge 0$.

From (2.58) we then conclude that $|\Delta M(t)|$ is bounded for $t = t_0$, thus, we have

$$
|\Delta M(t_0)| = |\log m(x_1, t_0) - \log m(x_0, t_0)| \le C(T).
$$

As a consequence, we have

$$
|\log m(x_0, t_0)| \le |\log m(x_0, t_0) - \log m(x_1, t_0)| + |\log m(x_1, t_0)| \le C(T),
$$

where we have used (2.49). From this estimate we can conclude that (2.47) holds for the fixed but arbitrary point (x_0, t_0), and by that also for all points. Finally, we observe that (2.48) is a consequence of (2.47) and the fact that $n = ms_0$ where $s_0 \in [\underline{s}_0, \overline{s}_0]$. □

2.6. **Additional regularity results.** Equipped with the upper and lower limit for m and n (Lemma 2) and the energy estimate of Lemma 1, we will derive more regularity results for various quantities. The main issue here is to make use of the parabolicity of the momentum equation to obtain certain higher order regularity estimates for the fluid velocity. Particularly, the following lemma is obtained by the same arguments as used in [23, 22]. For completeness and the convenience of the reader we sketch the main calculations and observe that the more complicated pressure term $p(m, n)$ gives no new problems, chiefly since we are armed with (2.47) and (2.48).

Lemma 3. *Let* (m,n,u) *and* T *be as in in Lemmas 1 and 2, and let* $\sigma(t)=\min\{1,t\}=1\vee t$. *Then there is a constant* $C(T)$ *such that*

$$\sup_{0\le t\le T}\int_{\mathbb{R}}\left[(\Delta u)^2+(\Delta m)^2+(\Delta n)^2\right]\,dx+\int_0^T\int_{\mathbb{R}}(u_x)^2\,dxdt\le C(T), \tag{2.59}$$

$$\sup_{0\le t\le T}\sigma(t)\int_{\mathbb{R}}(u_x)^2\,dx+\int_0^T\int_{\mathbb{R}}\sigma(\dot u)^2\,dxdt\le C(T), \tag{2.60}$$

$$\sup_{0\le t\le T}\sigma(t)^2\int_{\mathbb{R}}(\dot u)^2\,dx+\int_0^T\int_{\mathbb{R}}\sigma^2(\dot u_x)^2\,dxdt\le C(T), \tag{2.61}$$

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})}\le C(T)\sigma(t)^{-1/4}, \tag{2.62}$$

$$\langle u\rangle^{1/2,1/4}_{[\tau,T]\times\mathbb{R}}\le C(T)\sigma(\tau)^{-1/2}, \tag{2.63}$$

$$\|u(t_2,\cdot)-u(t_1,\cdot)\|_{L^2(\mathbb{R})}\le C(T)\sigma(t_1)^{-1/2}(t_2-t_1)^{1/2}, \tag{2.64}$$

$$\|m(t_2,\cdot)-m(t_1,\cdot)\|_{H^{-1}(\mathbb{R})}+\|(mu)(t_2,\cdot)-(mu)(t_1,\cdot)\|_{H^{-1}(\mathbb{R})}\le C(T)(t_2-t_1)^{1/2}, \tag{2.65}$$

where $0\le t_1\le t_2\le T$. *Here* $\dot u=\frac{du}{dt}=u_t+uu_x$ *and* $\langle u\rangle^{1/2,1/4}_{[\tau,T]\times\mathbb{R}}$ *is the usual Hölder norm with coefficient* $1/2$ *in space and* $1/4$ *in time.*

Proof. First, by Lemma 1 (in view of Corollary 1) and Lemma 2, (2.59) follows directly. The estimates (2.62)–(2.65) follow directly from (2.60) and (2.61) and the model itself (1.11) (weak form), see Section 5 in [22] for details.

We focus on estimates (2.60) and (2.61). We assume that $\overline{u}=0$ and $\overline{m},\overline{n}$ are constants. The extension to the more general case as specified in (1.16) is rather straightforward.

Step 1. First, we focus on an estimate for the terms $\sigma(t)\int_{\mathbb{R}}|u_x|^2\,dx$ and $\int_0^T\int_{\mathbb{R}}\sigma|\dot u|^2\,dxdt$ appearing in (2.60). We have

$$m\dot u+p(m,n)_x=u_{xx}. \tag{2.66}$$

We multiply (2.66) with $\sigma(t)\dot u$ and integrate:

$$\int_0^t\int_{\mathbb{R}}\sigma m|\dot u|^2\,dxds=\int_0^t\int_{\mathbb{R}}\left[-\sigma\dot u p(m,n)_x+\sigma\dot u u_{xx}\right]\,dxds. \tag{2.67}$$

For the first term on the right-hand side we have

$$\begin{aligned}
-&\int_0^t\int_{\mathbb{R}}\sigma(s)(u_s+uu_x)p(m,n)_x\,dxds\\
&=\int_0^t\int_{\mathbb{R}}\sigma(s)(u_s)_x\Delta p(m,n)\,dxds-\int_0^t\int_{\mathbb{R}}\sigma(s)(uu_x)p(m,n)_x\,dxds\\
&=\int_0^t\int_{\mathbb{R}}\sigma(s)(u_x)_s\Delta p(m,n)\,dxds-\int_0^t\int_{\mathbb{R}}\sigma(s)(uu_x)p(m,n)_x\,dxds\\
&=\sigma(t)\int_{\mathbb{R}}[u_x\Delta p(m,n)](t,x)\,dx-\int_0^t\int_{\mathbb{R}}(u_x)(\sigma\Delta p(m,n))_s\,dxds\\
&\qquad-\int_0^t\int_{\mathbb{R}}\sigma(s)(uu_x)p(m,n)_x\,dxds
\end{aligned}$$

$$= \sigma(t)\int_{\mathbb{R}}[u_x\Delta p](t,x)\,dx - \int_0^t\int_{\mathbb{R}}(u_x)\sigma'(s)\Delta p(m,n)\,dxds$$
$$- \int_0^t\int_{\mathbb{R}}\sigma(s)\left[p(m,n)_s u_x + p(m,n)_x u u_x\right]\,dxds,$$

where $\Delta p = p(m,n) - p(\overline{m},\overline{n})$. The integrand of the last term is

$$\sigma(s)p_m\left(-m(u_x)^2 - m_x u u_x + m_x u u_x\right) + \sigma(s)p_n\left(-n(u_x)^2 - n_x u u_x + n_x u u_x\right)$$
$$= -\sigma(s)[p_m(m,n)m + p_n(m,n)n](u_x)^2.$$

Therefore, by using that $\Delta p = [p(m,n) - p(\overline{m},n)] + [p(\overline{m},n) - p(\overline{m},\overline{n})]$, we conclude that

$$\left|\int_0^t\int_{\mathbb{R}}\sigma\dot{u}p(m,n)_x\,dxds\right| \le C(T)\Big[\sigma(t)\int_{\mathbb{R}}|u_x|(|\Delta m| + |\Delta n|)(t,x)\,dx + \int_0^{1\vee t}\int_{\mathbb{R}}|u_x|(|\Delta m| + |\Delta n|)(s,x)\,dxds + \int_0^t\int_{\mathbb{R}}|u_x|^2\,dxds\Big]. \tag{2.68}$$

The second term of the right-hand side of (2.67) is given by

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}\sigma(u_s + uu_x)u_{xx}\,dxds\\
&= \int_0^t\int_{\mathbb{R}}\sigma u_s u_{xx}\,dxds + \frac{1}{2}\int_0^t\int_{\mathbb{R}}\sigma u[(u_x)^2]_x\,dxds\\
&= -\int_0^t\int_{\mathbb{R}}\sigma(u_x)_s u_x\,dxds - \frac{1}{2}\int_0^t\int_{\mathbb{R}}\sigma(u_x)(u_x)^2\,dxds\\
&= -\frac{1}{2}\int_0^t\int_{\mathbb{R}}\sigma[(u_x)^2]_s\,dxds - \frac{1}{2}\int_0^t\int_{\mathbb{R}}\sigma(u_x)^3\,dxds\\
&= -\frac{\sigma(t)}{2}\int_{\mathbb{R}}(u_x)^2\,dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}}\sigma'(s)(u_x)^2\,dxds - \frac{1}{2}\int_0^t\int_{\mathbb{R}}\sigma(u_x)^3\,dxds\\
&= -\frac{\sigma(t)}{2}\int_{\mathbb{R}}(u_x)^2\,dx + \frac{1}{2}\int_0^{1\vee t}\int(u_x)^2\,dxds + \mathcal{O}_1,
\end{aligned} \tag{2.69}$$

where $\mathcal{O}_1 = -\frac{1}{2}\int_0^t\int_{\mathbb{R}}\sigma(u_x)^3\,dxds$. Substituting (2.68) and (2.69) back into (2.67), we obtain

$$\begin{aligned}
&\sigma(t)\int_{\mathbb{R}}(u_x)^2\,dx + \int_0^t\int_{\mathbb{R}}\sigma|\dot{u}|^2\,dxds\\
&\quad\le C(T)\Big[\sigma(t)\int_{\mathbb{R}}|u_x|(|\Delta m| + |\Delta n|)\,dx\\
&\qquad + \int_0^{1\vee t}\int_{\mathbb{R}}|u_x|(|\Delta m| + |\Delta n|)\,dxds + \int_0^t\int_{\mathbb{R}}|u_x|^2\,dxds + \mathcal{O}_1\Big].
\end{aligned} \tag{2.70}$$

This together with (2.59) and the inequality

$$\int_{\mathbb{R}}|u_x|(|\Delta m| + |\Delta n|)\,dx \le \delta\int_{\mathbb{R}}|u_x|^2\,dx + \frac{1}{4\delta}\int_{\mathbb{R}}(|\Delta m| + |\Delta n|)^2\,dx, \qquad \delta > 0,$$

show that

$$\sigma(t)\int_{\mathbb{R}}|u_x|^2\,dx + \int_0^t\int_{\mathbb{R}}\sigma|\dot{u}|^2\,dxds \le C(T)\Big[C_0 + \int_0^t\int_{\mathbb{R}}\sigma|u_x|^3\,dxds\Big]. \tag{2.71}$$

We note that the two-phase nature through the appearance of both Δm and Δn in (2.70) can naturally be handled through the estimate (2.59).

Step 2. Next, we derive an estimate for the terms $\sigma(t)^2\int_{\mathbb{R}}(\dot{u})^2\,dx$ and $\int_0^T\int_{\mathbb{R}}\sigma^2(\dot{u}_x)^2\,dxdt$ appearing in (2.61). To that end we apply the operator $(\cdot)_t+(u\cdot)_x$ to (2.66) and obtain

$$m\frac{d}{dt}\dot{u}+p(m,n)_{xt}+(up(m,n)_x)_x=(u_{xxt}+[uu_{xx}]_x)\,, \tag{2.72}$$

where $\frac{d}{dt}(\cdot)=(\cdot)_t+u(\cdot)_x$. We shall make use of the following transport theorem: if $\rho\dot{w}=g$ and if $h=h(t)$, then

$$\int\frac{1}{2}h\rho w^2\Big|_0^t\,dx=\int_0^t\int\left(\frac{1}{2}h'\rho w^2+hwg\right)dxds$$

Applying this to (2.72) with $h(t)=\sigma^2(t)$, $\rho=m$, and $w=\dot{u}$ (which implies that $g=m\frac{d}{dt}(\dot{u})$) we obtain

$$\begin{aligned}\frac{1}{2}\sigma(t)^2\int_{\mathbb{R}}m|\dot{u}|^2\,dx&=\int_0^t\int_{\mathbb{R}}\sigma(s)\sigma'(s)m|\dot{u}|^2\,dxds\\&+\int_0^t\int_{\mathbb{R}}\sigma^2\dot{u}\Big[-(p(m,n)_t+up(m,n)_x)_x\Big]\,dxds+\int_0^t\int_{\mathbb{R}}\sigma^2\dot{u}\Big[(u_{xt}+[u_{xx}u])_x\Big]\,dxds.\end{aligned} \tag{2.73}$$

We apply (2.71) to bound the first term on the right-hand side noting that $\sigma'\le 1$ and m is bounded. The second term on the right may be written in the form

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\sigma^2\dot{u}\Big(-[p_t+up_x]\Big)_x\,dxds\\&\quad=-\int_0^t\int_{\mathbb{R}}\sigma^2(\dot{u})_x\Big(-p_m[m_t+um_x]-p_n[n_t+un_x]\Big)\,dxds\\&\quad=-\int_0^t\int_{\mathbb{R}}\sigma^2[p_mm+p_nn]u_x(\dot{u})_x\,dxds.\end{aligned}$$

The term is therefore bounded in absolute value as follows:

$$\begin{aligned}&\Big|\int_0^t\int_{\mathbb{R}}\sigma^2[p_mm+p_nn]u_x(\dot{u})_x\,dxds\Big|\\&\le\frac{1}{4\delta}C(T)\int_0^t\int_{\mathbb{R}}|u_x|^2\,dxdt+\delta\int_0^t\int_{\mathbb{R}}\sigma^2|(\dot{u})_x|^2\,dxdt\le C(T)C_0+\delta\int_0^t\int_{\mathbb{R}}\sigma^2|(\dot{u})_x|^2\,dxdt,\end{aligned}$$

by (2.59). For the third term on the right-hand side of (2.73) we have

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\sigma^2\dot{u}[(u_x)_t+u(u_x)_x]_x\,dxds\\&\quad=-\int_0^t\int_{\mathbb{R}}\sigma^2(\dot{u})_x[(u_x)_t+u(u_x)_x]\,dxds=-\int_0^t\int_{\mathbb{R}}\sigma^2[(u_t)_x+(uu_x)_x-(u_x)^2](\dot{u})_x\,dxds\\&\quad=-\int_0^t\int_{\mathbb{R}}\sigma^2[|(\dot{u})_x|^2-(\dot{u})_x(u_x)^2]\,dxds=-\int_0^t\int_{\mathbb{R}}\sigma^2|(\dot{u})_x|^2\,dxds+\mathcal{O}_2,\end{aligned}$$

where $\mathcal{O}_2=\int_0^t\int_{\mathbb{R}}\sigma^2(\dot{u})_x(u_x)^2\,dxds\le\delta\int_0^t\int_{\mathbb{R}}\sigma^2[(\dot{u})_x]^2\,dxds+1/4\delta^{-1}\int_0^t\int_{\mathbb{R}}\sigma^2(u_x)^4\,dxds$. Substituting these estimates back in (2.73) yields

$$\begin{aligned}&\sigma(t)^2\int_{\mathbb{R}}|\dot{u}(t,x)|^2\,dx+\int_0^t\int_{\mathbb{R}}\sigma^2|(\dot{u})_x|^2\,dxds\\&\qquad\le C(T)\left[C_0+\int_0^t\int_{\mathbb{R}}\sigma|u_x|^3\,dxds+\int_0^t\int_{\mathbb{R}}\sigma^2|u_x|^4\,dxds\right].\end{aligned} \tag{2.74}$$

Step 3. Finally, we must verify that the terms $\int_0^t \int_{\mathbb{R}} \sigma |u_x|^3 \, dxds$ and $\int_0^t \int_{\mathbb{R}} \sigma^2 |u_x|^4 \, dxds$, can be controlled. Clearly, $\int_0^t \int_{\mathbb{R}} \sigma |u_x|^3 \, dxds \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} |u_x|^2 \, dxds + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \sigma^2 |u_x|^4 \, dxds$, by application of Cauchy's inequality, so in view of (2.59) we only need to focus on the last term. For that purpose we consider the effective viscous flux F given by

$$F = u_x - p(m,n) + p(\overline{m},\overline{n}) = u_x - \Delta p.$$

Thus, it follows that

$$\int_{\mathbb{R}} |u_x|^4 \, dx \leq C\Big[\int_{\mathbb{R}} |F|^4 \, dx + \int_{\mathbb{R}} |\Delta p|^4 \, dx\Big] \leq C\Big[\int_{\mathbb{R}} |F|^4 \, dx + 1\Big], \tag{2.75}$$

where we have used (2.59) and the pointwise bound on m and n in Lemma 2. Clearly, by virtue of Sobolev inequality, we have that $\|F\|^2_{L^\infty(\mathbb{R})} \leq C(\|F\|^2_{L^2(\mathbb{R})} + \|F_x\|^2_{L^2(\mathbb{R})})$. Thus,

$$\begin{aligned}\int_{\mathbb{R}} |F|^4 \, dx &\leq \|F\|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |F|^2 \, dx \\ &\leq C\|F\|^2_{L^2(\mathbb{R})}\Big(\|F\|^2_{L^2(\mathbb{R})} + \|F_x\|^2_{L^2(\mathbb{R})}\Big) = C\|F\|^2_{L^2(\mathbb{R})} h(s) + C\|F_x\|^2_{L^2(\mathbb{R})} h(s),\end{aligned}$$

where $h(s) = \|F(s,\cdot)\|^2_{L^2(\mathbb{R})}$. In particular, it follows from (2.59) that h is integrable, i.e., $\int_0^T h(s)\, ds \leq C(T)$. We note that $F_x = m\dot{u} + p(\overline{m},\overline{n})_x$, therefore, in light of the pointwise bound on m and n in Lemma 2 and the smoothness of $\overline{m},\overline{n}$ which become constant outside $[-1,1]$, we get

$$\|F_x(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq C(\|\dot{u}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|p(\overline{m},\overline{n})_x\|^2_{L^2(\mathbb{R})}) \leq C(\|\dot{u}(t,\cdot)\|^2_{L^2(\mathbb{R})} + 1).$$

Consequently, we have (using that $\sigma(s) \leq 1$)

$$\begin{aligned}&\int_0^t \sigma^2(s) \int_{\mathbb{R}} |F|^4 \, dxds \\ &\qquad \leq C\int_0^t \sigma^2(s)\|F_x\|^2_{L^2(\mathbb{R})} h(s)\, ds + C\int_0^t \sigma(s)\|F\|^2_{L^2(\mathbb{R})} h(s)\, ds \\ &\qquad \leq C\int_0^t \sigma^2(s)[\|\dot{u}\|^2_{L^2(\mathbb{R})} + 1] h(s)\, ds + C\int_0^t \sigma(s)[\|u_x\|^2_{L^2(\mathbb{R})} + 1] h(s)\, ds \\ &\qquad \leq C_0 + C_1\int_0^t \sigma^2(s)\|\dot{u}\|^2_{L^2(\mathbb{R})} h(s)\, ds + C_2\int_0^t \sigma(s)\|u_x\|^2_{L^2(\mathbb{R})} h(s)\, ds.\end{aligned} \tag{2.76}$$

Here we again have used that $\|(p-\overline{p})(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq C\Big(\|\Delta m(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\Delta n(t,\cdot)\|_{L^2(\mathbb{R})}2\Big) \leq C(T)$, in view of (2.59) and the pointwise bound on m and n in Lemma 2. Summing (2.71) and (2.74), together with (2.75) and (2.76) (and redefining constants), yields the integral inequality

$$\begin{aligned}&\sigma(t)\|u_x(t,\cdot)\|^2_{L^2(\mathbb{R})} + \sigma^2(t)\|\dot{u}(t,\cdot)\|^2_{L^2(\mathbb{R})} + \int_0^t \int_{\mathbb{R}} \sigma|\dot{u}|^2 \, dx\, ds + \int_0^t \int_{\mathbb{R}} \sigma^2|(\dot{u})_x|^2 \, dx\, ds \\ &\qquad \leq C_0 + C_1\int_0^t \sigma^2(s)\|\dot{u}(s,\cdot)\|^2_{L^2(\mathbb{R})} h(s)\, ds + C_2\int_0^t \sigma(s)\|u_x(s,\cdot)\|^2_{L^2(\mathbb{R})} h(s)\, ds.\end{aligned}$$

Application of Gronwall's inequality then gives

$$\sup_{0\leq t\leq T} \sigma(t)\|u_x\|^2_{L^2(\mathbb{R})} + \sup_{0\leq t\leq T} \sigma^2(t)\|\dot{u}\|^2_{L^2(\mathbb{R})} + \int_0^T \int_{\mathbb{R}} \sigma|\dot{u}|^2 \, dx\, ds + \int_0^T \int_{\mathbb{R}} \sigma^2|(\dot{u})_x|^2 \, dx\, ds \leq C(T),$$

and (2.60) and (2.61) follow. □

3. Compactness and convergence to weak solutions

Let (m_0, n_0, u_0) be as described in the theorem (particularly, they can be discontinuous). We regularize the initial data by defining $m_0^\delta = j_\delta \star m_0$, $u_0^\delta = j_\delta \star u_0$, and where $j_\delta(x) = \delta^{-1} j(x/\delta)$. The estimates of Lemmas 1-3 then apply to show that the corresponding smooth solutions $(m_\delta, n_\delta, u_\delta)$ of (1.11) with initial data $(m_0^\delta, n_0^\delta, u_0^\delta)$ exist for all time and satisfy all the estimates of Lemmas 1-3 with constants $C(T)$ which are independent of δ.

By using standard compactness arguments we know that there is a subsequence $\delta \to 0$ for which

$$\begin{aligned} &u^\delta \to u \text{ uniformly on compact sets in } \mathbb{R}\times(0,\infty),\\ &u^\delta(t,\cdot)-u(t,\cdot)\to 0 \text{ strongly in } L^2(\mathbb{R}),\ t\ge 0,\\ &u_x^\delta(t,\cdot)-u_x(t,\cdot)\rightharpoonup 0 \text{ weakly in } L^2(\mathbb{R}),\ t>0, \end{aligned} \tag{3.1}$$

and such that the limiting function u inherits all the bounds in Lemmas 1 and 3 which pertain to u and u_x. We also find a subsequence for which (m^δ, n^δ) converges weakly, say to functions (m, n). However, this does not guarantee that $p(m^\delta, n^\delta)$ converges to $p(m, n)$. Consequently, we can not conclude that the limiting pair (m, n, u) is a weak solution of the second equation in (1.11). We shall have to obtain the masses m and n as a strong limit of m^δ, n^δ.

3.1. **Identification of limit functions.** The purpose of this section is to verify that the following lemma holds:

Lemma 4. *Let $m^\delta, n^\delta, u^\delta$ and u be as above. Then there is a further subsequence $\delta \to 0$ and limiting functions m, n such that*

$$\begin{aligned} m^\delta(t,\cdot)-m(t,\cdot)\to 0 \ \textit{strongly in } L^2_{loc}(\mathbb{R}),\ t\ge 0,\\ n^\delta(t,\cdot)-n(t,\cdot)\to 0 \ \textit{strongly in } L^2_{loc}(\mathbb{R}),\ t\ge 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &u_x^\delta(t,\cdot)-p(m^\delta(t,\cdot),n^\delta(t,\cdot))\\ &\to u_x(t,\cdot)-p(m(t,\cdot),n(t,\cdot)) \ \textit{strongly in } L^2_{loc}(\mathbb{R}),\ t>0. \end{aligned} \tag{3.3}$$

and the limit functions n, m satisfy $$\frac{n}{m}(t,x)=\frac{n_0}{m_0}(X_t^{-1}(x)), \tag{3.4}$$

where $X_t(x)$ is the characteristic which takes x as its starting point at time $t = 0$.

From this lemma, it follows then rather directly that the limiting pair (m, n, u) is indeed a weak solution of (1.11) with initial data (m_0, n_0, u_0), and that the regularity results of Theorem 1 hold. We refer to Section 5 of [22] for more details.

Proof. The approach is to first deduce Holder continuity type of estimates for the effective viscous flux $u_x^\delta - p(m^\delta, n^\delta)$ for a sequence of approximate solutions $(m^\delta, n^\delta, u^\delta)$. This estimate relies on continuity estimates for particle trajectories. Equipped with the regularity of the effective viscous flux we then derive continuity estimates for the quantity $\log(m^\delta)$. From these estimates we can conclude that there is a limit function m such that m^δ and $(m^\delta)^2$ converge weakly to m and m^2 in $L^2(\mathbb{R})$ for each $t > 0$ from which strong convergence follows. Similarly arguments are used for n^δ.

Step 1. We obtain strong convergence first in Lagrangian coordinates. Thus define the particle trajectories $X^\delta(t, y)$ given by

$$\frac{dX^\delta}{dt}=u^\delta\left(t,X^\delta\right),\qquad X^\delta(0,y)=y. \tag{3.5}$$

Furthermore, we set

$$\partial_y X^\delta(t,y)=J(t,y)=\exp\left(\int_0^t u_x^\delta\left(s,X^\delta(s,y)\right)\,ds\right).$$

The last equality follows since we have that

$$\partial_y \dot{X}^\delta = u_x^\delta(t,X^\delta(t,y))\partial_y X^\delta,\qquad \text{i.e.}\quad \frac{d}{dt}X_y^\delta=u_x^\delta(t,X^\delta(t,y))X_y^\delta.$$

In particular, we get

$$\partial_t J = J u_x^\delta(t,X^\delta(t,y)).$$

Consequently, we see that

$$\partial_t(J\tilde{m}^\delta)=\partial_t J\tilde{m}^\delta+J\partial_t\tilde{m}^\delta$$

$$= J\tilde{u}^\delta_x \tilde{m}^\delta + J[\tilde{m}^\delta_x \partial_t X^\delta + \tilde{m}^\delta_t] = J\Big(\partial_t \tilde{m}^\delta + \partial_x[\tilde{u}^\delta \tilde{m}^\delta]\Big),$$

where $\tilde{u}^\delta(t,y) = u^\delta\left(t, X^\delta(t,y)\right)$ and $\tilde{m}^\delta(t,y) = m^\delta\left(t, X^\delta(t,y)\right)$. Thus, in terms of the Lagrangian variables (skipping the "tilde" notation), we see that the continuity equation for m can be written as

$$\partial_t[m^\delta J] = 0, \qquad \text{i.e.} \qquad J = \frac{m^\delta_0(y)}{m^\delta\left(t, X^\delta(t,y)\right)}. \tag{3.6}$$

Hence, from the estimate (2.47) we have

$$C(T)^{-1} \le \frac{\partial X^\delta}{\partial y}(t,y) \le C(T), \qquad (t,y) \in [0,T]\times\mathbb{R}. \tag{3.7}$$

Uniform Holder continuity of the X^δ in t follows from (3.5) and (2.62). There is therefore a subsequence $\delta \to 0$ for which $X^\delta \to X$ uniformly on compact sets in $[0,\infty)\times\mathbb{R}$. Furthermore, (3.7) guarantees the existence of the inverse function $Y^\delta(t,\cdot)$ of $X^\delta(t,\cdot)$, and we can again assert that $Y^\delta \to Y$ uniformly on compact sets in $[0,\infty)\times\mathbb{R}$, where for each t, $Y(t,\cdot) = X^{-1}(t,\cdot)$.

Next, define

$$\begin{aligned} M^\delta(t,y) &= \log m^\delta\left(t, X^\delta(t,y)\right) \\ N^\delta(t,y) &= \log n^\delta\left(t, X^\delta(t,y)\right) \\ F^\delta(t,y) &= \left[u^\delta_x - p(m^\delta,n^\delta)\right]\left(t, X^\delta(t,y)\right). \end{aligned}$$

We claim that there is a subsequence $\delta \to 0$ for which

$$F^\delta(t,\cdot) \to F(t,\cdot), \quad t > 0 \tag{3.8}$$

$$M^\delta(t,\cdot) \to M(t,\cdot), \quad t \ge 0, \tag{3.9}$$

$$N^\delta(t,\cdot) \to N(t,\cdot), \quad t \ge 0, \tag{3.10}$$

strongly in $L^2_{loc}(\mathbb{R})$. To prove (3.8) we simply compute

$$F^\delta_y = \frac{\partial F^\delta}{\partial X^\delta}\frac{\partial X^\delta}{\partial y} = \left[u^\delta_{xx} - p(m^\delta,n^\delta)_x\right]\left(t, X^\delta(t,y)\right)\frac{\partial X^\delta}{\partial y} = [m^\delta \dot{u^\delta}]\left(t, X^\delta(t,y)\right)\frac{\partial X^\delta}{\partial y},$$

which by (2.47), (2.61), and (3.7) is bounded in $L^2_{\text{loc}}(\mathbb{R})$, uniformly in δ, for fixed $t > 0$. Also,

$$\begin{aligned} F^\delta_t = \frac{\partial F^\delta}{\partial X^\delta}\frac{\partial X^\delta}{\partial t} + \frac{\partial F^\delta}{\partial t} &= u^\delta[u^\delta_{xx} - p(m^\delta,n^\delta)_x] + [u^\delta_{xt} - p(m^\delta,n^\delta)_t] \\ &= [u^\delta_{xt} + u^\delta_{xx}u^\delta] - p_m(m^\delta,n^\delta)[m^\delta_t + m^\delta_x u^\delta] - p_n(m^\delta,n^\delta)[n^\delta_t + n^\delta_x u^\delta] \\ &= [(\dot{u^\delta})_x - (u^\delta_x)^2] + p_m(m^\delta,n^\delta)m^\delta u^\delta_x + p_n(m^\delta,n^\delta)n^\delta u^\delta_x. \end{aligned}$$

Here we have used that

$$(\dot{u^\delta})_x - (u^\delta_x)^2 = \partial_x\left(u^\delta_t + u^\delta u^\delta_x\right) - (u^\delta_x)^2 = u^\delta_{xt} + u^\delta u^\delta_{xx}.$$

By using (2.60) and (2.61), together with (2.47) and (2.48), we can conclude that F^δ_t is bounded in $L^2(\mathbb{R}\times[\tau,T])$, uniformly in δ, for each $\tau \in (0,T]$. These estimates prove (3.8).

To prove (3.9) we compute from (1.11) that

$$M^\delta_t = \frac{\dot{m^\delta}}{m^\delta} = -u^\delta_x = -(F^\delta + p^\delta) = N^\delta_t, \tag{3.11}$$

where $p^\delta = p(m^\delta,n^\delta)$. We fix δ_1 and δ_2 and define

$$\begin{aligned} \alpha^M &= (p(m^{\delta_2},n^{\delta_2}) - p(m^{\delta_1},n^{\delta_2}))/(M^{\delta_2} - M^{\delta_1}) \\ \alpha^N &= (p(m^{\delta_1},n^{\delta_2}) - p(m^{\delta_1},n^{\delta_1}))/(N^{\delta_2} - N^{\delta_1}), \end{aligned}$$

which both are positive since $p(m,n)$ is increasing in m and n together with the fact that $\log(m)$ and $\log(n)$ are increasing, respectively in m and n. Next, we apply (3.11) and obtain that

$$\frac{\partial y_1}{\partial t} + \alpha^M y_1 + \alpha^N y_2 = \beta,$$

$$\frac{\partial y_2}{\partial t} + \alpha^M y_1 + \alpha^N y_2 = \beta,$$

where

$$y_1 = (M^{\delta_2} - M^{\delta_1}), \qquad y_2 = (N^{\delta_2} - N^{\delta_1}), \qquad \beta = -(F^{\delta_2} - F^{\delta_1}).$$

This system of ODEs is easy to resolve since we directly see that

$$y_1 = y_2 + C_0, \qquad C_0 = y_1(0) - y_2(0),$$

and

$$\frac{\partial y_1}{\partial t} + [\alpha^M + \alpha^N] y_1 = \beta + \alpha^N C_0.$$

Its solution is given by

$$\frac{d}{dt}\left(e^{\int_0^t \alpha(s)\,ds} y_1\right) = [\beta(t) + \alpha^N(t) C_0] e^{\int_0^t \alpha(s)\,ds}, \qquad \alpha(t) = \alpha^M(t) + \alpha^N(t),$$

that is,

$$y_1(t) = e^{-\int_0^t \alpha(s)\,ds} y_1(0) + \int_0^t [\beta(s) + \alpha^N(s) C_0] e^{-\int_s^t \alpha(\tau)\,d\tau}\,ds.$$

Consequently, using that $\alpha^N(t) \leq \alpha(t)$, we get

$$|y_1(t)| \leq |y_1(0)| + \int_0^t |\beta(s)|\,ds + (|y_1(0)| + |y_2(0)|) \int_0^t e^{-\int_s^t \alpha(\tau)\,d\tau} \alpha(s)\,ds.$$

Hence, we can conclude that for a finite interval $K \subset \mathbb{R}$ we have

$$\begin{aligned}\|M^{\delta_2}(\cdot,t) - M^{\delta_1}(\cdot,t)\|_{L^2(K)} \leq \|M^{\delta_2}(\cdot,0) - M^{\delta_1}(\cdot,0)\|_{L^2(K)} + \int_0^t \|F^{\delta_2}(\cdot,s) - F^{\delta_1}(\cdot,s)\|_{L^2(K)}\,ds \\ + (\|M^{\delta_2}(\cdot,0) - M^{\delta_1}(\cdot,0)\|_{L^2(K)} + \|N^{\delta_2}(\cdot,0) - N^{\delta_1}(\cdot,0)\|_{L^2(K)}) C(T).\end{aligned}$$

This together with (3.8) proves that $\{M^\delta(t,\cdot)\}$ is a Cauchy sequence in $L^2_{\mathrm{loc}}(\mathbb{R})$ for each $t \geq 0$, and (3.9) follows. Finally, the relation between y_1 and y_2 implies (3.10).

Step 2. According to the definition of M^δ we have

$$m^\delta(t,x) = \exp\left(M^\delta(t, Y^\delta(t,x))\right).$$

We therefore define a limit mass m by

$$m(t,x) \overset{\text{def}}{:=} \exp\left(M(t, Y(t,x))\right),$$

where M is as in (3.9) and $Y(t,x)$ is the limiting inverse particle trajectory obtained earlier. Concerning properties of m, first, it is clear that $|M^\delta| \leq C$ and $|M| \leq C$, and it follows that

$$C(T)^{-1} \leq m \leq C(T).$$

Similarly, it follows that m inherits the L^2 estimate (2.59). The same results of course also hold for the pair (n^δ, N^δ) and the corresponding limit functions (n, N).

We observe that the uniform convergence $Y^\delta \to Y$ and the strong L^2 convergence $M^\delta \to M$ are insufficient to guarantee the strong L^2 convergence of the composition $m^\delta = \exp\left(M^\delta(Y^\delta)\right)$ to the limit function m. We shall instead show that for $k = 1$ and 2, $[m^\delta(t,\cdot)]^k \rightharpoonup [m(t,\cdot)]^k$ weakly in $L^2(\mathbb{R})$. Then, in turn, this can be used to deduce the strong convergence

$$m^\delta(t,\cdot) \to m(t,\cdot) \qquad \text{in } L^2_{loc}(\mathbb{R}). \tag{3.12}$$

First, we assume that $\phi(x)$ is a smooth test function having compact support in $\mathbb{R}$, and then we compute at a fixed time t. First, we have

$$\begin{aligned}\int_{\mathbb{R}} [(m^\delta)^k - m^k]\phi\,dx &= \int_{\mathbb{R}} \left[e^{kM^\delta(t,Y^\delta(t,x))}\phi(x) - e^{kM(t,Y(t,x))}\phi(x)\right]\,dx \\ &= \int_{\mathbb{R}} e^{kM^\delta(t,y)} \phi(X^\delta(t,y)) \frac{\partial X^\delta}{\partial y}\,dy - \int_{\mathbb{R}} e^{kM(t,y)} \phi(X(t,y)) \frac{\partial X}{\partial y}\,dy.\end{aligned}$$

For the first term in the integrand we use the substitution

$$y = Y^\delta(t,x), \qquad \text{i.e.} \quad x = X^\delta(t,y), \quad \frac{dx}{dy} = \frac{\partial X^\delta}{\partial y},$$

while for the second term we use

$$y = Y(t,x), \qquad \text{i.e.} \quad x = X(t,y), \quad \frac{dx}{dy} = \frac{\partial X}{\partial y}.$$

Using this we get

$$\begin{aligned}\int_{\mathbb{R}} [(m^\delta)^k - m^k]\phi\,dx = &\int_{\mathbb{R}} \left[e^{kM^\delta(t,y)} - e^{kM(t,y)}\right] \phi(X^\delta(t,y)) \frac{\partial X^\delta}{\partial y}(t,y)\,dy \\ &+ \int_{\mathbb{R}} e^{kM(t,y)} \left[\phi(X^\delta(t,y)) - \phi(X(t,y))\right] \frac{\partial X^\delta}{\partial y}(t,y)\,dy \\ &+ \int_{\mathbb{R}} e^{kM(t,y)} \phi(X(t,y)) \left[\frac{\partial X^\delta}{\partial y}(t,y) - \frac{\partial X}{\partial y}(t,y)\right] dy \\ :=& A + B + C.\end{aligned}$$

To estimate these terms we use:

$$\begin{aligned}|A| &\le \|\phi\|_{L^\infty(\mathbb{R})} \left\| \frac{\partial X^\delta}{\partial y} \right\|_{L^\infty(\mathbb{R})} \int_{\text{supp}(\phi)} |e^{kM^\delta} - e^{kM}|\,dy \\ &\le ||\phi||_{L^\infty(\mathbb{R})} \left\| \frac{\partial X^\delta}{\partial y} \right\|_{L^\infty(\mathbb{R})} \text{supp}(\phi)^{1/2} \left\| e^{kM^\delta} - e^{kM} \right\|_{L^2_{\text{loc}}(\mathbb{R})}.\end{aligned}$$

The convergence follows in light of (3.7) and the fact that

$$\left\| e^{kM^\delta} - e^{kM} \right\|_{L^2_{\text{loc}}(\mathbb{R})} \le \max(ke^{kM(y,t)}) \left\| M^\delta - M \right\|_{L^2_{\text{loc}}(\mathbb{R})}.$$

The convergence of B follows by (3.7) and the uniform convergence $X^\delta \to X$ uniformly on compact sets. The convergence of C follows since

$$\frac{\partial X^\delta}{\partial y} \rightharpoonup \frac{\partial X}{\partial y}, \qquad \text{weakly in } L^2_{loc}(\mathbb{R}).$$

By an approximation argument we can show that the above convergence holds for $\phi \in L^2(\mathbb{R})$ by making use of the L^∞ and L^2 uniform bounds which hold for both m^δ and m. By standard functional analysis, see for example Theorem 2.11 in [14] for one variant of this result, (3.12) now follows. By considering

$$n^\delta(t,x) = \exp\left(N^\delta(t, Y^\delta(t,x))\right).$$

and

$$n(t,x) \stackrel{\text{def}}{:=} \exp\left(N(t, Y(t,x))\right),$$

we can show by the same arguments that

$$n^\delta(t,\cdot) \to n(t,\cdot) \qquad \text{in } L^2_{loc}(\mathbb{R}). \tag{3.13}$$

Consequently, (3.2) has been shown. Next, by (3.1) and (3.2)

$$\begin{aligned} &u^\delta_x(t,\cdot) - p(m^\delta(t,\cdot), n^\delta(t,\cdot)) + p(\overline{m},\overline{n}) \\ &\rightharpoonup u_x(t,\cdot) - p(m(t,\cdot), n(t,\cdot)) + p(\overline{m},\overline{n}), \qquad t > 0,\end{aligned} \tag{3.14}$$

weakly in $L^2(\mathbb{R})$. On the other hand,

$$(u^\delta_x - p(m^\delta, n^\delta))_x = m^\delta(\dot{u^\delta})$$

which is bounded in $L^2(\mathbb{R})$, uniformly in δ, by (2.61). The convergence in (3.14) is therefore strong in $L^2_{\text{loc}}(\mathbb{R})$.

Step 3. Finally, we note from (3.6) and the corresponding equation for n that we have the relation

$$\frac{m_0^\delta(y)}{m^\delta(t, X^\delta(t,y))} = \frac{n_0^\delta(y)}{n^\delta(t, X^\delta(t,y))},$$

or equivalently,

$$\frac{n^\delta(t,x)}{m^\delta(t,x)} = \frac{n_0^\delta}{m_0^\delta}(Y_t^\delta(x)).$$

It's clear that

$$\frac{n_0^\delta}{m_0^\delta}(Y_t^\delta(x)) \to \frac{n_0}{m_0}(Y_t(x)) \qquad \text{in } L^2(\mathbb{R})$$

due to the strong convergence associated with Y^δ and m_0^δ, n_0^δ. Moreover, by decomposing

$$\frac{n^\delta}{m^\delta} - \frac{n}{m} = \Big(\frac{n^\delta}{m^\delta} - \frac{n}{m^\delta}\Big) + \Big(\frac{n}{m^\delta} - \frac{n}{m}\Big)$$

we see that

$$\frac{n^\delta(x,t)}{m^\delta(x,t)} \to \frac{n(x,t)}{m(x,t)} \qquad \text{in } L^2(\mathbb{R})$$

by using the strong convergence $n^\delta \to n$, $1/m^\delta \to 1/m$ in $L^2(\mathbb{R})$ and the boundedness of m^δ and n. Consequently, (3.4) holds.

□

4. Some numerical examples demonstrating characteristic behavior

The purpose of this section is to demonstrate characteristic behaviour of the viscous two-phase model. We write the model in the vector form

$$w_t + f(w)_x = \begin{pmatrix} 0 \\ 0 \\ \mu u_{xx} \end{pmatrix},$$

where $w = (w_1, w_2, w_3)^T = (n, m, mu)^T$ and $f(w) = (nu, mu, mu^2 + p(m,n))^T$. We have used the no-slip condition. We apply Riemann data where we initially have a jump in one or several variables. It is interesting then to study the development of these jumps at later times. We recall from [20, 21, 23] that initial discontinuities in the density for single-phase Navier-Stokes equations in 1-D persist for all time but decrease exponentially fast in the time variable. In Example 1 we have chosen initial data such that n_0/m_0 is constant. Our two-phase model then gives a "single-phase" type of behaviour of the initial discontinuity. In Example 2 and 3 we consider other initial data, respectively, corresponding to an initial increasing (Example 2) and decreasing (Example 3) jump in n_0/m_0. The evolution of the initial jump then shows a behaviour fundamentally different from the single-phase case. For comparison we also have included solutions when the physical viscosity is neglected.

Example 1 (vanishing discontinuity). We consider the compressible liquid-gas two-phase model with Riemann data

$$(w_{1,L}, w_{1,R}) = (1/18, 1/20), \quad (w_{2,L}, w_{2,R}) = (500/9, 500/10),$$
$$(w_{3,L}, w_{3,R}) = (5000/9, 5000/10).$$

In Lagrangian coordinates $c = w_1/w_2$ and $v = 1/w_2$ this corresponds to

$$(c_L, c_R) = (1/1000, 1/1000), \quad (v_L, v_R) = (9/500, 10/500), \quad (u_L, u_R) = (10, 10).$$

Results are presented in Figures 4 and 5 and show results after time $t = 0.5$ and $t = 1.0$, respectively.

We observe from Figure 4 and 5 that initial jumps are dying out as time runs. This can be understood from the fact that $n_0/m_0 = c$ is constant, thus, the two-phase model behaves similar to a single-phase model since $n/m = n_0/m_0$ for all later times, and the single-phase analysis directly carries over to the two-phase model. For example, the pressure $p(m,n) = p(m, mc) = P(m)$ where c is constant, and the behavior of $P(m)$ is similar to the single-phase case.

Example 2 (persisting discontinuity). We consider the viscous two-phase model with Riemann data

$$(w_{1,L}, w_{1,R}) = (0.95/18, 1/20), \quad (w_{2,L}, w_{2,R}) = (500/9, 500/10),$$
$$(w_{3,L}, w_{3,R}) = (5000/9, 5000/10).$$

In Lagrangian coordinates this corresponds to

$$(c_L, c_R) = (0.95/1000, 1/1000), \quad (v_L, v_R) = (9/500, 10/500), \quad (u_L, u_R) = (10, 10),$$

Results are presented in Figure 6 and we observe that as the initial jump in w_1 decreases through time as in Example 1, the jump in w_2 shows a totally different behavior. However, we see that both jumps go in the same direction as the corresponding initial data $(w_{1,L}, w_{1,R})$ and $(w_{2,L}, w_{2,R})$. This behaviour should be understood in view of the relation

$$\frac{n}{m}(t, x) = \frac{n_0}{m_0}(X_t^{-1}(x)). \tag{4.1}$$

Example 3 (vanishing discontinuity + new persisting discontinuty). We consider the viscous two-phase model with Riemann data

$$(w_{1,L}, w_{1,R}) = (1.075/18, 1/20), \quad (w_{2,L}, w_{2,R}) = (500/9, 500/10),$$
$$(w_{3,L}, w_{3,R}) = (5000/9, 5000/10).$$

In Lagrangian coordinates this corresponds to

$$(c_L, c_R) = (1.075/1000, 1/1000), \quad (v_L, v_R) = (9/500, 10/500), \quad (u_L, u_R) = (10, 10).$$

Results are presented in Figures 7 and 8. Figure 7 shows that for both masses w_1 and w_2 the initial jumps decrease due to the smoothing of the velocity u. However, the smoothing out effect from u together with the fundamental relation (4.1) enforce a new persisting discontinuity to be formed in w_2 after some time whose jump is opposite of the initial jump in w_2, see Figure 8.

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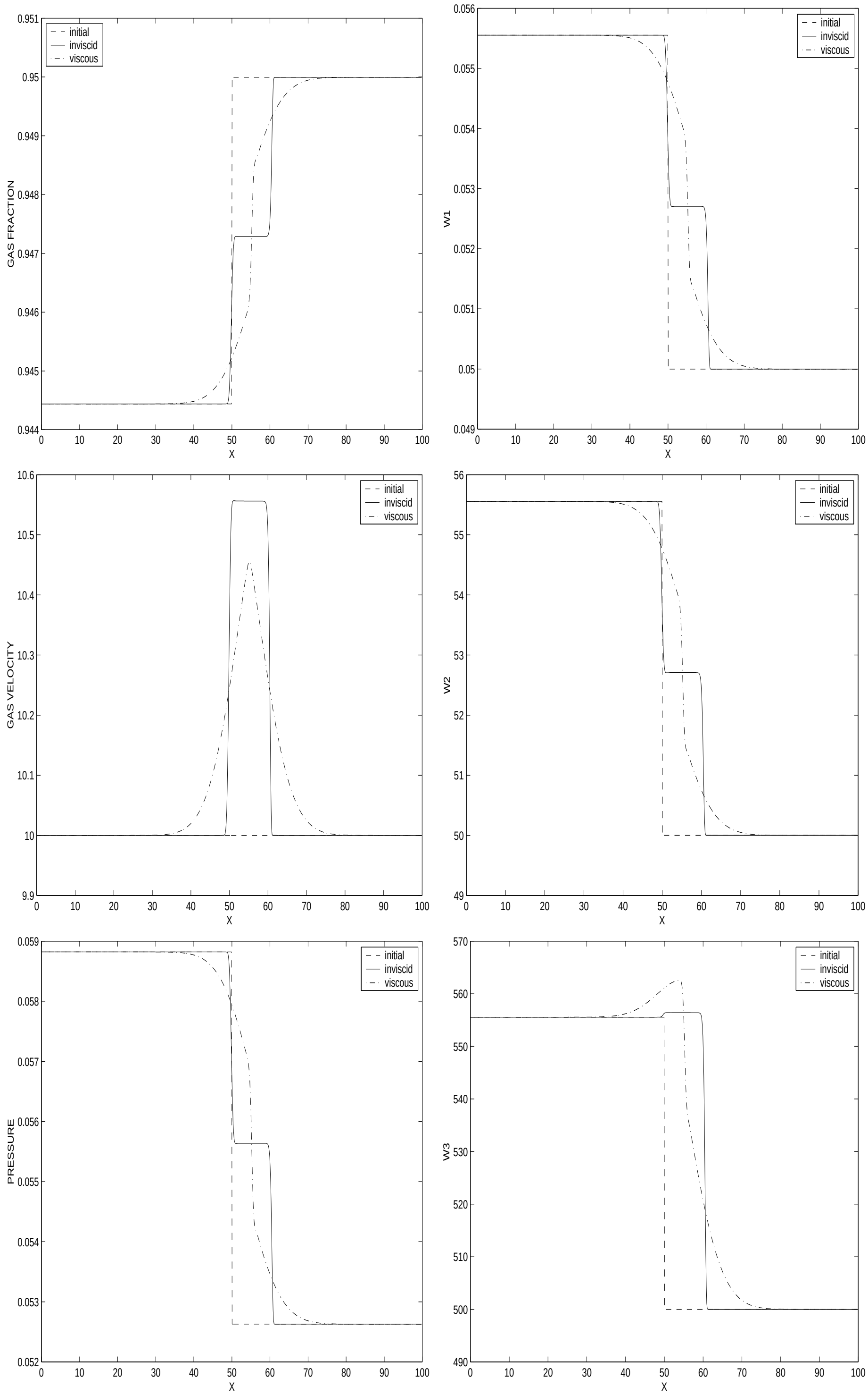

FIGURE 4. Snapshot of α_g, $v_g = v_l$, p (left column) and w_1, w_2, and w_3 at time $t = 0.5$ (right column). We have used 800 nodes and the viscous coefficient is given by $\mu = 3 \cdot 10^3$.

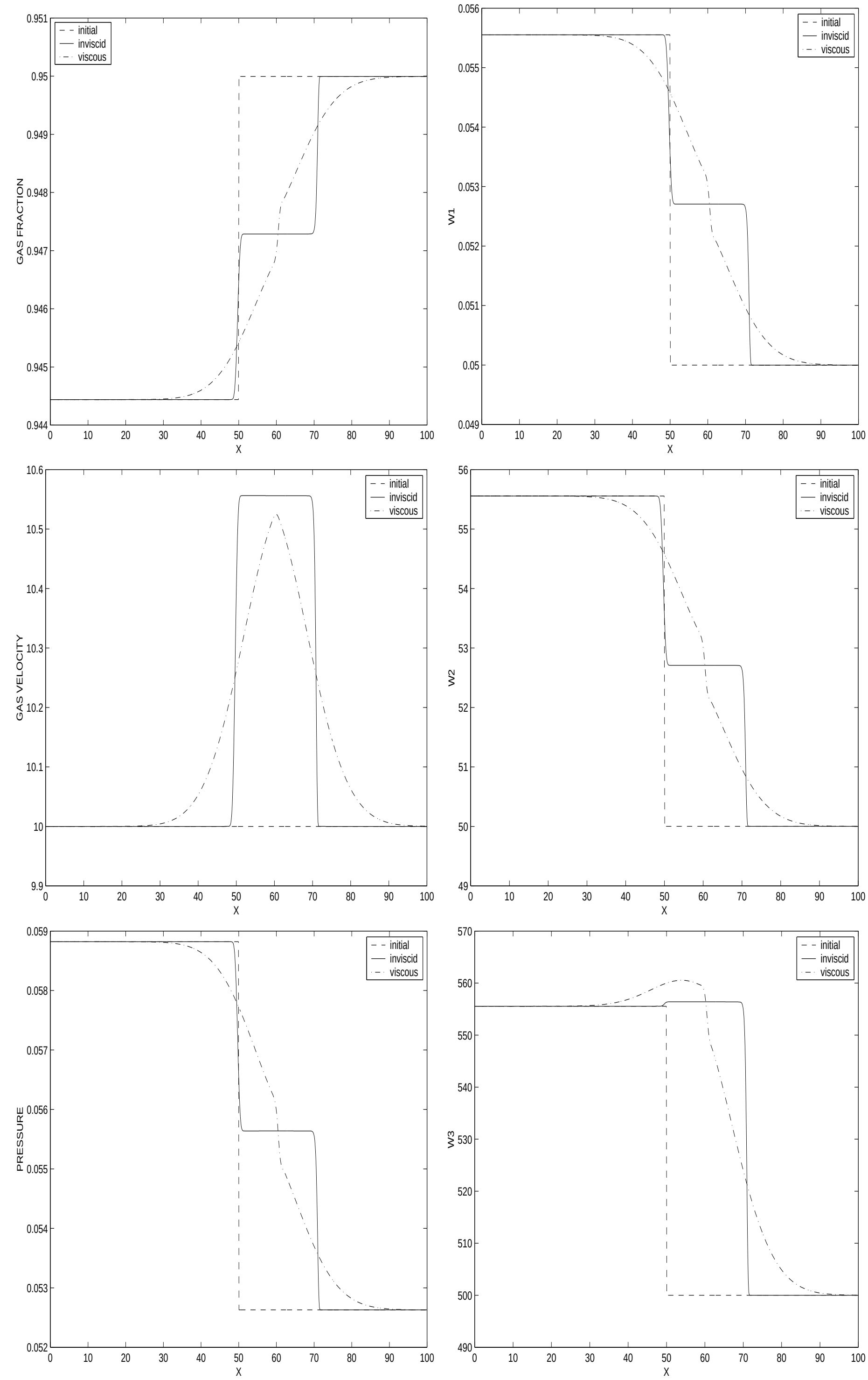

FIGURE 5. Snapshot of α_g, $v_g = v_l$, p (left column) and w_1, w_2, and w_3 (right column) at time $t = 1.0$. We have used 800 nodes and the viscous coefficient is given by $\mu = 3 \cdot 10^3$.

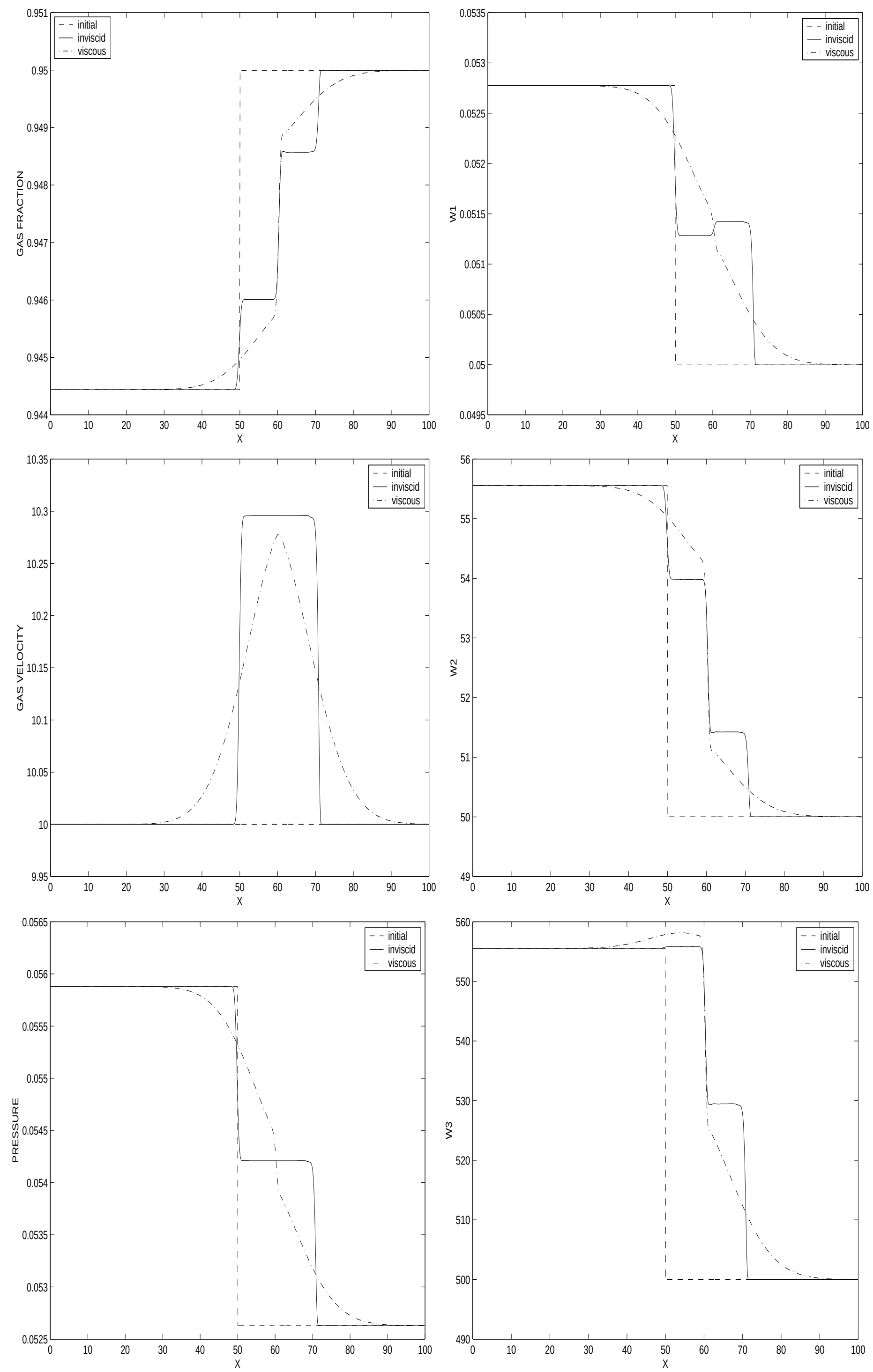

FIGURE 6. Snapshot of α_g, $v_g = v_l$, p (left column) and w_1, w_2, and w_3 (right column) at time $t = 1.0$. We have used 800 nodes and the viscous coefficient is given by $\mu = 3 \cdot 10^3$.

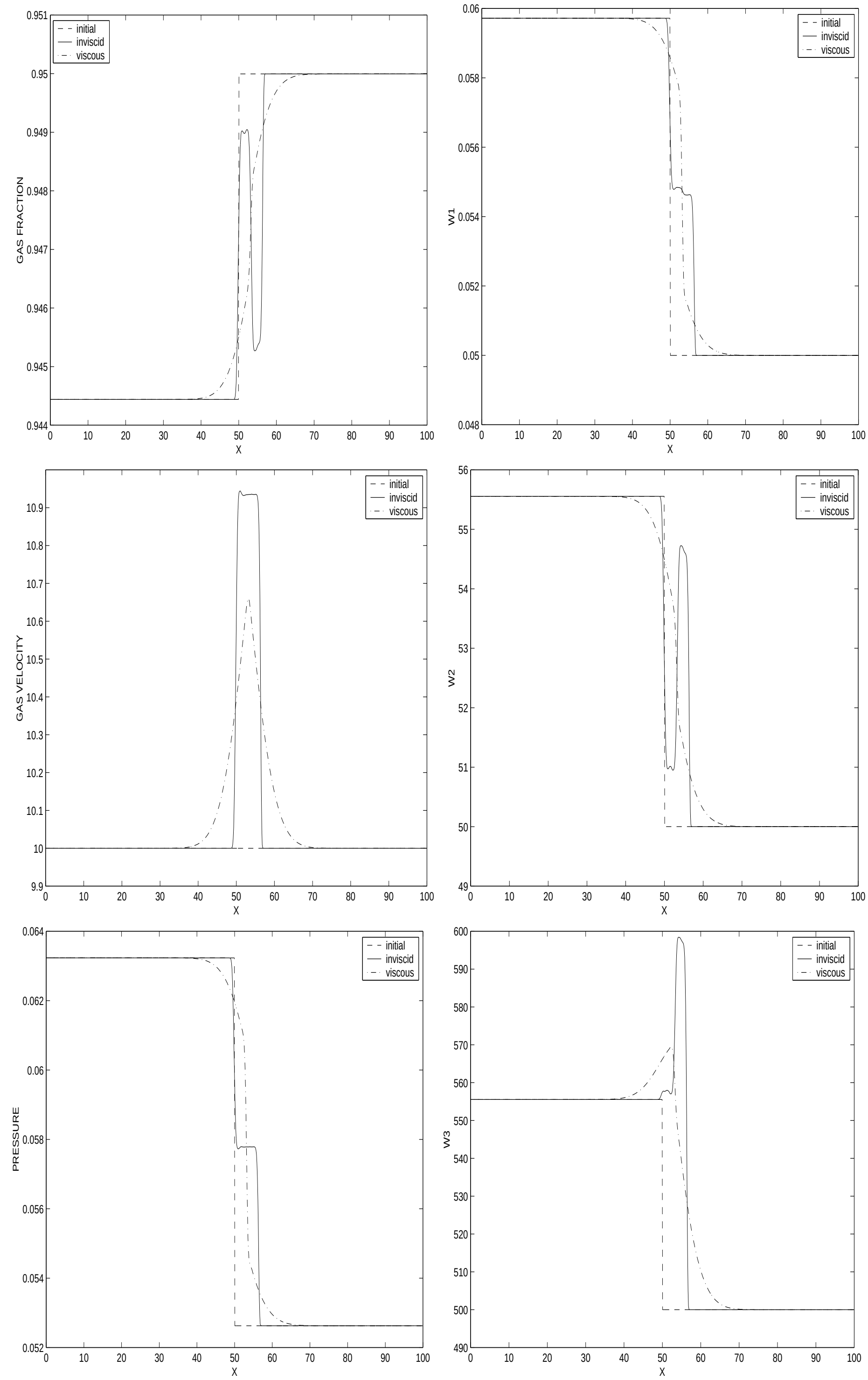

FIGURE 7. Snapshot of α_g, $v_g = v_l$, p (left column) and w_1, w_2, and w_3 (right column) at time $t = 0.3$. We have used 800 nodes and the viscous coefficient is given by $\mu = 3 \cdot 10^3$.

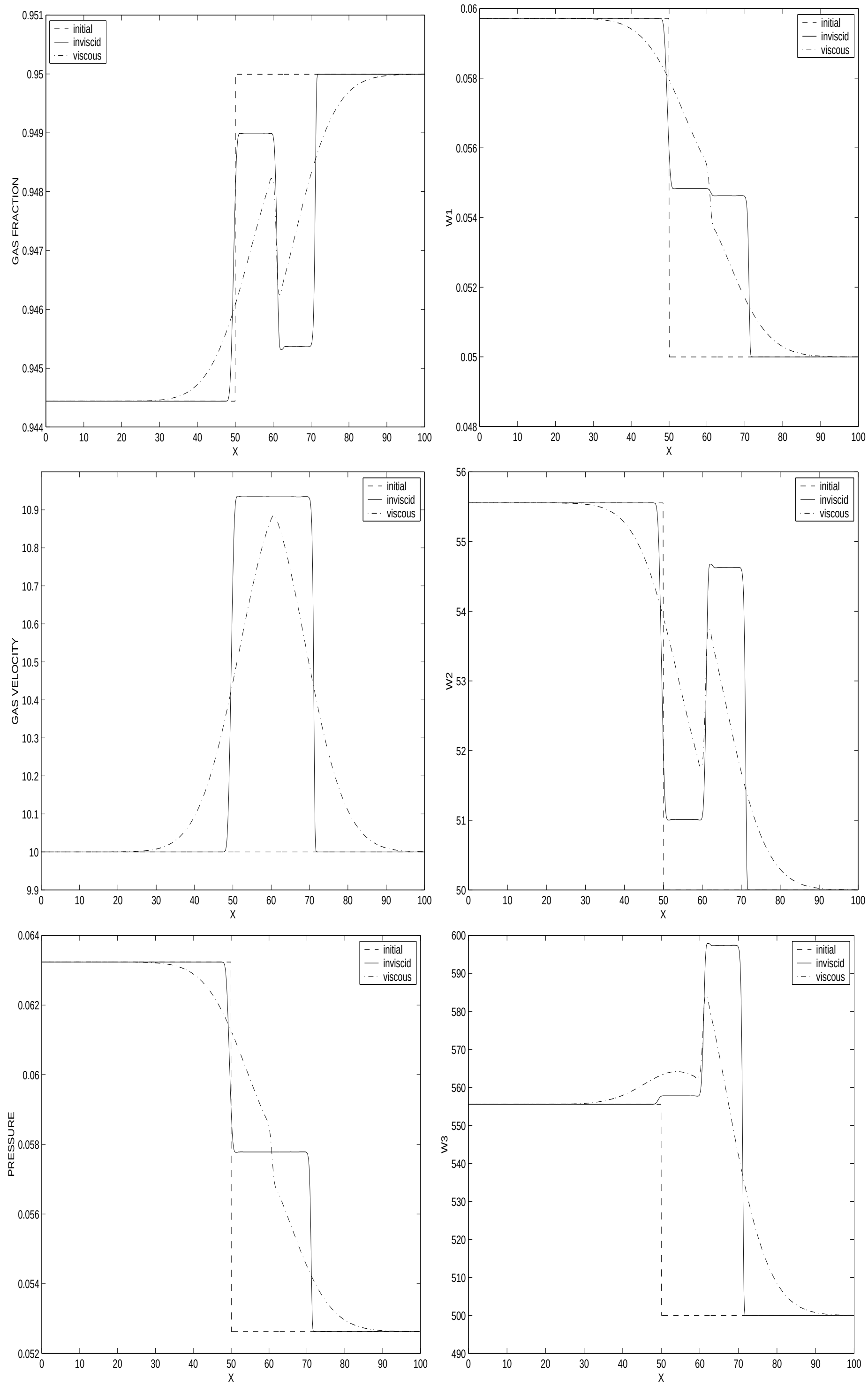

FIGURE 8. Snapshot of α_g, $v_g = v_l$, p (left column) and w_1, w_2, and w_3 (right column) at time $t = 1.0$. We have used 800 nodes and the viscous coefficient is given by $\mu = 3 \cdot 10^3$.